Weighted polynomial localization and theta correction inertia

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Luminary Oddities

October 25, 2026

Abstract

Let $F(z) = \xi((1+iz)/2)/4$, $G = F'^2 - FF''$, and let $D_0 = E_0 - Q_0$ be the original endpoint correction, where $E_0(P) = (4\pi)^{-1}\int |(FP)'|^2$ and $Q_0(P) = (8\pi)^{-1}\int GP^2$. For every fixed $0 < \theta < 1$ and $A > 0$, every sufficiently large degree N, and each polynomial parity, we construct a real subspace with a rational basis, dimension at least $c_{A,\theta}(\log N)^A(\log\log N)^2$, and $Q_0 \geq (2-\theta)E_0$ on its entire span. The proof combines classical unconditional short-interval zero moments with bounded-population microscopic seed selection, a two-scale root-product budget, seed-averaged logarithmic root energy and higher fixed-order weighted approximation in a derivative and absolute-potential norm. All local complex zeros, multiplicities, exterior roots and mixed polynomial Gram entries are retained. Companion spaces have $0 < Q_0/E_0 \leq \theta$ at the same dimension scale, yielding the corresponding necessary repair-rank bounds for scalar-centered fixed strict contractions. A finite Gaussian zero-ideal approximation theorem and an exact generic current certificate delineate the scope of the method. The constructed correction-negative spaces have positive original Q_0.

1 Introduction

Polynomial positivity questions for theta products involve both a positive derivative energy and a correction. Their difference must be studied in the original normalization: an indefinite correction does not by itself make the original polynomial-square form negative. The predecessor [1] makes this distinction explicit. It proves finite-height positivity for the original form and shows that finitely many selected real zeros contribute additively to negative correction inertia. Allowing an arbitrarily large but fixed selection makes the negative index divided by $\log\log N$ tend to infinity. Its constants depend on that selection; it does not give a uniform increasing-center rate.

Here the centers and their local root populations are controlled while the degree increases. The main quantitative result is Theorem 2.1. In either parity the negative correction index grows faster than every fixed power of $\log N$, with a full-subspace relative bound arbitrarily close to the simple inverse-square edge. Each prescribed power uses its own fixed approximation order and onset.

There are two different localization scales. A fixed-size complex disk provides a factorization anchored off the critical line, where an Euler product lower bound is available. Its complete zero population can be $O(\log T)$ at height T. A microscopic neighborhood of size $1/\log T$ is used for cap placement. Classical unconditional increment moments of Fujii [4] allow us to discard crowded microscopic bins while preserving a positive proportion of the real seeds supplied by Bui, Conrey and Young [2]. The retained inner populations are bounded by one fixed integer. No lower bound on a zero derivative or on separation within an inner cluster is required.

The two scales must not be conflated. Roots outside the microscopic cluster still affect both the potential and the size of F. We pay their full product cost, and assign a separate logarithmic budget

to the inner roots. A second weighted selection bounds the actual exterior logarithmic root energy; it avoids charging every exterior root at the smallest allowed distance. Logarithmic caps supply many disjoint targets with controlled joint norms. A compact-support specialization of classical critical-exponential approximation [7] and an anchored primitive pay the complete derivative and absolute-potential error. Exact local amplitude normalization removes an unnecessary exponential height cost. Differentiating the reciprocal through its logarithm pays the product modulus once, independently of the fixed derivative order. These estimates and all mixed Gram errors yield one common ordinary-polynomial degree cutoff.

The inverse-square instability and logarithmic substitution have classical Hardy ancestry [9, 10]. The local analytic products, Harnack estimates, orthogonal-polynomial generating functions and finite Gram perturbation arguments are likewise established tools. The contribution here is their two-scale resource accounting and its quantitative conclusion for the original theta correction. Neither Fujii's zero statistic nor the positive-proportion simple-zero theorem is new. The predecessor's whole-line lifting and opposite packet results remain attributed earlier contributions; the present weighted receiver retains their original joint-norm requirement.

For comparison, the local-anchor mechanisms in the informal preprint [17] concern a different complete Weil form, with a hypothetical off-line-zero premise. Collective collision interpolation in [18] concerns an RH-conditional Nyman–Beurling approximation problem.

The main argument includes the precise lifting interface, uniform complex-zero factors, root budgeting, microscopic selection and the rank consequence. Appendix B proves a finite Gaussian ideal comparison: coherent confluent repair can avoid separation and amplitude penalties when the zero ideal is finite, while location and support resolution still impose costs. Appendix C gives a portable generic example of the unpaid positive-width current and mixed-block obstruction. Appendix D explains why scalar signed-measure certificate results cannot simply be relabeled as original correction theorems. These supporting results do not remove the actual theta all-order sign obligation.

The two channels and the supporting interfaces The principal localization theorem constructs common polynomial spans with opposite correction signs and positive original Q_0 on both. Each prescribed logarithmic power is fixed before the approximation order, constants and onset are selected. The supporting interfaces have separate tasks: short caps enter the original joint-norm return; finite Gaussian zero-ideal approximation treats merged jets at bounded locations with fixed total multiplicity; and the non-Hankel correction example identifies why an original signed receiver must retain its full matrix and mixed entries. The generic current certificate is a comparison of receiving structures.

2 The endpoint forms and the localization theorem

Throughout, log is the natural logarithm. Put

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad F(z) = \tfrac{1}{4}\xi((1+iz)/2), \qquad G = F'^2 - FF''.$$

The functional equation and conjugation symmetry make F even and real on the real axis. Its zeros, counted with their multiplicities, lie in $|\operatorname{Im} z| < 1$. For real polynomials define

$$E_0(P) = \frac{1}{4\pi}\int_{\mathbb{R}} |(FP)'|^2, \qquad Q_0(P) = \frac{1}{8\pi}\int_{\mathbb{R}} GP^2, \qquad D_0 = E_0 - Q_0. \tag{1}$$

These are the original endpoint forms of [1]. In particular, no cutoff in the theta center variable has been made. Their integrals converge by the exponential decay of the completed zeta function and its derivatives in fixed horizontal strips. The energy E_0 is positive definite: $(FP)' = 0$ would make FP constant, and its decay would then make it zero.

Off the real zeros put $V = G/F^2$. For compact real targets supported away from those zeros, use the polarized forms

$$E(u,v) = \frac{1}{4\pi}\int u'v', \qquad Q(u,v) = \frac{1}{8\pi}\int Vuv, \qquad \|u\|_*^2 = \int (|u'|^2 + (1+|V|)|u|^2).$$

For $u = FP$ these are precisely (1). At real zeros the potential expression means $V(FP)^2 = GP^2$ almost everywhere; the latter has no singularity. Write

$$\mathcal{P}_N^\epsilon = \{P \in \mathbb{R}[x] : \deg P \le N,\ P(-x) = (-1)^\epsilon P(x)\}, \qquad \epsilon = 0, 1.$$

A rational basis always means a basis in $\mathbb{Q}[x]$ for a real vector space; arbitrary real linear combinations are allowed.

Theorem 2.1 (Growing-center correction inertia)**.** *For every fixed $\theta \in (0,1)$ and every fixed $A > 0$ there are $c_{A,\theta} > 0$ and $N_{A,\theta}$ such that, for every integer $N \ge N_{A,\theta}$ and each $\epsilon \in \{0,1\}$, a real subspace $\mathcal{L}_N^{\epsilon,A,\theta} \subset \mathcal{P}_N^\epsilon$ has a rational basis and satisfies*

$$\dim \mathcal{L}_N^{\epsilon,A,\theta} \ge c_{A,\theta}(\log N)^A(\log\log N)^2, \qquad Q_0(P) \ge (2-\theta)E_0(P) \quad (P \in \mathcal{L}_N^{\epsilon,A,\theta}). \tag{2}$$

Consequently D_0 is strictly negative and Q_0 strictly positive on the nonzero vectors of this whole subspace.

This replaces the finite-center $\omega(\log\log N)$ conclusion of [1, Section 4] by a specified lower rate. The new issue is uniformity as the number and height of the centers grow. Classical short-interval zero statistics supply many microscopic seed bins with bounded total zero population and bounded exterior logarithmic product loss. Fixed-order compact weighted approximation then transfers high-order caps at the original joint norm. The reciprocal's logarithmic derivatives pay its small-modulus cost once, independently of the fixed approximation order. The argument assumes neither a lower bound for $|F'(r)|$ nor a separation bound inside a cluster. The constants may depend on A and θ. In particular, each negative correction index divided by any fixed power of $\log N$ tends to infinity. This is a family of fixed-power estimates.

3 Compact approximation at the critical exponential weight

The approximation receiver below uses the critical exponential weight itself. Its support condition is essential: the corresponding global Jackson–Favard inequality is false. The input is Lubinsky's classical Jackson theorem, rather than a new weighted approximation theorem.

Lemma 3.1 (Compact weighted primitive approximation)**.** *Fix $0 < b_0 < b_1 < \infty$ and put $W_b(x) = e^{-b|x|}$. Fix an integer $m \ge 1$. There are $C_m, n_{0,m}$ such that the following holds uniformly for $b \in [b_0, b_1]$. Let $w \in H^{m+1}(\mathbb{R})$ be real, of one parity, supported in $[-A, A]$, with $w(0) = 0$. For every integer $n \ge n_{0,m}$ with $\sqrt{n} > bA + 2$ there is a real polynomial P of the same parity, of degree at most $n+1$, such that*

$$\|(P-w)W_b\|_2 + \|(P'-w')W_b\|_2 \le \frac{C_m}{(\log n)^m}\|w^{(m+1)}W_b\|_2. \tag{3}$$

Proof. We specify the imported theorem and its tail. For $W_1 = e^{-|x|}$, Lubinsky [7, Theorem 1 and (1.3)–(1.4)], at $r = m, p = 2$ and fixed $\varepsilon = 1/2$, gives

$$E_n[f; W_1]_2 \le C_m \omega_{m,2}(f, W_1, 1/\log n).$$

Here E_n is the best weighted L^2 error among polynomials of degree at most n. The source explicitly permits dropping the constant inside the logarithm for $p \ge 1$. If $X = e^{1/(2t)}$, the modulus used is (with $\mathcal{P}_{m-1}$ denoting real polynomials of degree at most $m-1$)

$$\begin{aligned}\omega_{m,2}(f, W_1, t) = &\sup_{0<s\le t} \|W_1 \Delta_s^m f\|_{L^2([-X,X])} \\ &+ \inf_{q\in\mathcal{P}_{m-1}} \|(f-q)W_1\|_{L^2(\mathbb{R}\setminus[-X+1,X-1])}, \\ \Delta_s^m f(x) = &\sum_{j=0}^{m} (-1)^j \binom{m}{j} f(x + (m/2 - j)s).\end{aligned}$$

For compact f supported in $[-A, A]$ and $\sqrt{n} > A + 2$, the second term at $t = 1/\log n$ is zero: choose $q = 0$. For $f \in H^m$, the iterated fundamental theorem of calculus, Minkowski's inequality and $W_1(x) \le e^{|v|} W_1(x + v)$ give

$$\|W_1 \Delta_s^m f\|_2 \le s^m e^{ms/2} \|W_1 f^{(m)}\|_2.$$

Consequently

$$E_n[f; W_1]_2 \le \frac{C_m}{(\log n)^m} \|f^{(m)} W_1\|_2 \quad (\sqrt{n} > A + 2). \tag{4}$$

The main-interval restriction only decreases the difference norm. The tail, rather than a putative global derivative-only theorem, is what supplies this restricted consequence. The same theorem and the warning about the global failure are restated in [8, Section 5, (5.2) and Theorem 5.1].

The substitution $y = bx$ changes support radius to bA, keeps polynomial degree unchanged, and introduces a derivative factor b^{-m}. Thus (4) holds with W_b in place of W_1, uniformly for the stated compact range of b. Apply it to $f = w'$. A real polynomial q of degree at most n then satisfies

$$\|(q - w')W_b\|_2 \le \frac{C_m}{(\log n)^m} \|w^{(m+1)} W_b\|_2.$$

Projection onto the parity of w' is contractive because W_b is even. Take q of that parity and set $P(x) = \int_0^x q(t)\,dt$. This is a polynomial of degree at most $n + 1$, with the parity of w, and $P(0) = w(0) = 0$.

On the positive half-line,

$$W_b(x)(P - w)(x) = \int_0^x e^{-b(x-t)} W_b(t)(q - w')(t)\,dt.$$

Young's convolution inequality bounds its L^2 norm by $1/b$ times that of $W_b(q - w')$ on the same half-line. Reflection proves the identical bound on the negative half-line. Combining them with the derivative estimate proves (3). The primitive need not be compact: the convolution bound includes its complete tails. This constructs one polynomial for the value and derivative errors, with no uncontrolled integration constant. □

For the general resource statement below we also record the inherited strip-decay interface. Suppose f is nonzero, even, real on $\mathbb{R}$, holomorphic in a strip, and on each closed substrip

$$|f(x+iv)| \le C(1+|x|)^p e^{-a_f|x|}, \qquad a_f > 0. \tag{5}$$

Set $G_f = f'^2 - ff''$ and $V_f = G_f/f^2$, and define the joint norm $\|\cdot\|_f$ with V_f. The published joint-lifting lemma [1, Section 4, Lemma "Quantitative joint lifting"] says that, for a compact real one-parity target u supported away from the zeros, $u/f \in H^2$ after zero extension, $B \ge \max(1, \|u/f\|_{H^2})$ and $0 < \eta \le 1$, there is a same-parity rational polynomial with

$$\|fP - u\|_f < \eta, \qquad \log(\deg P + 2) \le C_f[B/\eta + \log(B/\eta + 2)]. \tag{6}$$

The constant is independent of the target and support. That result includes derivatives, absolute potential, all real-line tails and rational rounding. It will give the generic resource consequence; the sharper theta theorem uses the compact weighted receiver proved here instead. The theta transform satisfies the strip hypothesis. Indeed, the xi formula, uniform Stirling estimates and polynomial bounds for zeta in any fixed vertical strip imply, for every fixed b_0,

$$|F(x+iv)| \le C_{b_0}(1+|x|)^{p_{b_0}} e^{-\pi|x|/8}, \qquad |v| \le b_0. \tag{7}$$

One can obtain the zeta bound first in $\operatorname{Re} s > 0$ from the continuation formula

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{u\}u^{-s-1}\,du$$

and then use the functional equation on the remaining fixed strip; the apparent singularity is removed in the xi product. Bounded heights are absorbed by continuity. These standard estimates are also discussed in [3, Chapters II and IX]. In particular F, F', F'' are bounded on the whole real line.

4 Uniform local products without root separation

The following lemma includes every nonreal zero in its local disk. This is necessary: the individual nonreal pole terms need not be positive on the real line.

Lemma 4.1 (Uniform local factor)**.** *There are fixed positive constants C_0, C_1, C_2 with the following property for all sufficiently large T. Set $K = \lceil C_0 \log(T+40)\rceil \ge 2$. For any real $c \in [10, T]$ choose any $b_c \in (8, 9)$ for which the circle $|z-c| = b_c$ contains no zero, and let Z_c be the multiset of all zeros inside. Then $\#Z_c \le K$, and on the real interval $|x-c| \le 1$, away from zeros,*

$$V(x) = \sum_{\rho\in Z_c} \frac{1}{(x-\rho)^2} + R_c(x), \qquad |R_c(x)| \le C_1 K. \tag{8}$$

On the same interval, for any positive numbers $d_\rho \le |x-\rho|$,

$$|F(x)| \ge e^{-\pi c/8}(T+40)^{-C_2} e^{-C_2 K} \prod_{\rho\in Z_c} \min(1, d_\rho). \tag{9}$$

All products and sums count multiplicities.

Proof. The Riemann–von Mangoldt formula with its $O(\log T)$ error [3, Chapter IX] bounds the number of all zeros, with multiplicities, in every fixed-width real window up to height $T+20$ by $C\log(T+40)$. All zeros have imaginary part of magnitude less than one. This bounds the zeros in every disk used in the statement, after fixing C_0.

Normalize $f_c(w) = e^{\pi c/8}F(c+w)$. Formula (7), used on the wider fixed strip needed for $|w| \le 20$, implies $|f_c(w)| \le C(T+40)^p$ there. The off-axis point $w = -3i$ is a uniform anchor: its zeta argument is $2 + ic/2$, and

$$|\zeta(2+ic/2)| \ge \frac{1}{\zeta(2)}, \qquad |\Gamma(1+ic/4)|^2 = \frac{\pi c/4}{\sinh(\pi c/4)}.$$

The first bound follows by taking absolute values in the absolutely convergent reciprocal Euler series. The xi prefactors and the last identity give $|f_c(-3i)| \ge a_3 > 0$ for $c \ge 10$.

In the disk $|w| < b = b_c$, divide f_c by the full finite disk Blaschke product

$$B_c(w) = \prod_{\rho \in Z_c} \frac{b(w-(\rho-c))}{b^2 - \overline{(\rho-c)}w},$$

where unimodular factors may be inserted without changing any bound. The quotient $H_c = f_c/B_c$ is holomorphic and zero-free in that disk. Since $|B_c| = 1$ on its boundary, the maximum principle gives $|H_c| \le C(T+40)^p$ inside. Also $|B_c(-3i)| \le 1$, so $|H_c(-3i)| \ge a_3$. For a suitably enlarged polynomial bound M, $U = \log M - \log|H_c|$ is positive harmonic. Harnack's inequality first compares $-3i$ with zero, then zero with $|w| \le 2$. All comparison constants are uniform for $8 < b < 9$. Thus

$$0 \le U(w) \le C\log(T+40), \qquad |H_c(w)| \ge e^{-C\log(T+40)} \quad (|w| \le 2).$$

The harmonic derivative estimate on concentric disks (equivalently Borel–Caratheodory followed by Cauchy's estimate for a holomorphic logarithm) gives $|(\log H_c)''(w)| \le C\log(T+40)$ for $|w| \le 1$.

Write $\alpha = \rho - c$. The second logarithmic derivative of its Blaschke factor is

$$-\frac{1}{(w-\alpha)^2} + \frac{\overline{\alpha}^2}{(b^2 - \overline{\alpha}w)^2}.$$

The last term has magnitude at most $(b-1)^{-2} \le 1/49$ on $|w| \le 1$. Since $V = -(\log F)''$ locally away from zeros, summing these identities proves (8). For real $|w| \le 1$, each factor also satisfies

$$\left|\frac{b(w-\alpha)}{b^2 - \overline{\alpha}w}\right| \ge \frac{|w-\alpha|}{b+1} > \frac{|w-\alpha|}{10}.$$

Multiply these inequalities and the lower bound for H_c, and undo the normalization $e^{\pi c/8}$. The factors $10^{-\#Z_c}$ are absorbed in e^{-C_2K}, while the quotient loss is a power of $T+40$. Clipping each assigned distance at one only weakens the bound. This proves (9) uniformly even when roots approach one another or the chosen boundary circle. □

5 Microscopic windows with bounded population

The additional zero-statistics input is classical and unconditional. We use only a fixed moment, with a shift of order the mean spacing. The count includes nonreal zeros and all multiplicities.

Lemma 5.1 (Many bounded-population seed bins)**.** *There are fixed $a \in (0,1)$, $b, B_0 > 0$ and an integer $J \geq 1$ such that, for every sufficiently large T, with $h = 1/\log(T+40)$, one can select at least $bT\log T$ distinct real zeros r of F in $[aT, T]$ with the following properties: their pairwise distances are at least $3h$, and for every selected seed the disk $|z-r| < h$ contains at most J zeros of F, with all complex zeros and multiplicities counted. Moreover, with $\log_+ u = \max(0, \log u)$,*

$$\sum_{\rho:\ F(\rho)=0,\ |\rho-r|\geq h} \log_+ \frac{2}{|\rho - r|} \leq B_0/h. \tag{10}$$

The sum counts multiplicities; only distances less than two contribute. The constants do not depend on T.

Proof. Let $N_\zeta(u)$ count all nontrivial zeta zeros with ordinates in $(0, u]$, with multiplicity, and set $N_F(t) = N_\zeta(t/2)$. Fujii's unconditional increment-moment estimate [4, Lemma 1, p. 742], at the fixed fourth moment, implies, uniformly for U comparable to T and fixed $\ell > 0$,

$$\int_U^{2U} |S_\zeta(u+\delta) - S_\zeta(u)|^4\, du \leq C_\ell U, \qquad \delta = \frac{\ell}{2\log(T+40)},$$

where $S_\zeta(u) = \pi^{-1}\arg\zeta(1/2+iu)$ has the usual continuous argument convention. Indeed, in that lemma $\log(3+\delta\log U)$ is bounded, so both its fixed-moment main term and error are $O_\ell(U)$. Cauchy–Schwarz gives the same upper bound for the second moment. Puchta [5, Theorem 4] also states the uniform unconditional moment upper bound explicitly. We use no hypothesis concerning the real parts of the zeros.

The smooth term in Riemann–von Mangoldt changes by $O_\ell(1)$ over a shift δ on a comparable-height range. A fixed finite dyadic cover and the change of variable $t = 2u$ therefore give

$$\int_{aT/2}^{T+5} [N_F(t+\ell h) - N_F(t)]^2\, dt \leq C_{a,\ell} T \tag{11}$$

for any fixed $a \in (0,1)$ and $\ell > 0$. Choices at zero ordinates affect only a set of measure zero.

By [2, Theorem 1.1], at least $b_0 T\log T$ distinct simple real zeros of F lie below T eventually. Choose one sufficiently small fixed $a > 0$. The total all-zero count below aT is then at most $(b_0/2)T\log T$ for large T, by Riemann–von Mangoldt. There are thus at least $(b_0/2)T\log T$ simple real seeds in $[aT, T]$. This subtraction uses a cumulative lower count; it does not assume a corresponding lower count in $[T/2, T]$.

Discarding the two partial end bins costs only $O(\log T)$ zeros. Partition the real axis into half-open bins $I_j = [jh, (j+1)h)$. Let n_j count all zeros of F with real coordinate in I_j, and let s_j count the distinct simple real seeds in the central bins $I_j \subset [aT, T]$. Put

$$q_j = n_{j-2} + n_{j-1} + n_j + n_{j+1} + n_{j+2}.$$

For $t \in ((j-3)h, (j-2)h)$ every zero counted in q_j is counted by $N_F(t+7h) - N_F(t)$. These integration intervals have disjoint interiors. For the central bins, they lie in the integration range of (11) when T is large. Hence

$$\sum_j q_j^2 \leq CT/h \leq C'T\log T. \tag{12}$$

We also need the finite enlarged coordinate range. Applying (11) with $\ell = 3$ on the disjoint intervals $((j-1)h, jh)$ gives

$$\sum_{j:\ I_j \cap [aT-4, T+4] \neq \varnothing} n_j^2 \leq CT/h. \tag{13}$$

All integration endpoints lie in the stated moment range for large T. Define the symmetric nonnegative sequence

$$w_l = \begin{cases} \log(2/h), & |l| \le 2, \\ \log_+(2/[(|l|-1)h]), & |l| \ge 3, \end{cases} \qquad H_j = \sum_l w_l n_{j+l}.$$

It has finite support and $\sum_l w_l \le C/h$. Indeed its central terms cost $5\log(2/h)$, and the decreasing remaining sum is bounded by its first term plus $h^{-1}\int_0^2 \log(2/x)\,dx = 2/h$ on each side. For each shift with $w_l > 0$, all relevant bins lie in the enlarged range. Cauchy–Schwarz in (13) therefore gives

$$\sum_{j \text{ central}} s_j H_j \le \sum_j n_j H_j = \sum_l w_l \sum_j n_j n_{j+l} \le CT/h^2. \tag{14}$$

All latter sums over j are also over the central bins, using the enlarged range for shifted factors.

Choose a fixed integer J so large that $C'/J < b_0/8$. For the bad bins $q_j > J$,

$$\sum_{q_j > J} s_j \le \sum_{q_j > J} q_j \le J^{-1} \sum_j q_j^2 \le (b_0/8) T \log T.$$

Choose also a fixed B_0 sufficiently large. By (14), discarding $H_j > B_0/h$ loses at most $CT/(B_0 h) \le (b_0/8)T\log T$ further seeds. The bins satisfying both $q_j \le J$ and $H_j \le B_0/h$ therefore contain at least $(b_0/4)T\log T$ seeds for large T. Each contains at most J seeds, so at least $b_1 T \log T$ good bins are occupied. Retain one residue class modulo 4 containing at least a quarter of them, and choose one seed per retained bin. Their spacings are at least $3h$. Every zero with $|\rho - r| < h$ has real coordinate in the seed bin or an immediately adjacent bin. It is included in $q_j \le J$, which proves the full complex population bound. For an outer root $\alpha = \rho - r$ with $|\alpha| \ge h$, the bins $|l| \le 2$ use that lower distance directly. For $|l| \ge 3$, $|\alpha| \ge |\operatorname{Re}\rho - r| \ge (|l|-1)h$. Each term of its logarithmic loss is therefore bounded by the corresponding w_l, with all multiplicities included in n_{j+l}. This proves (10). No spacing among the inner roots and no lower derivative bound has been deduced. □

6 Two-scale root budgets and bounded-norm caps

We prove the construction as a finite-resource proposition. This also records which information is required for a different analytic kernel to receive the argument.

Proposition 6.1 (Two-scale finite-resource localization)**.** *Let f satisfy the strip-decay assumptions of (5), and suppose f, f', f'' are bounded on the real line. Fix integers $J, m \ge 1$. Constants below may depend on the fixed m, with that dependence suppressed in their subscripts. Consider M positive real seeds $r > 2$ separated by at least $3h$, where $0 < h \le 1/2$. To each associate a real center c with $|r - c| \le 1/2$ and a conjugation-invariant multiset Z containing r, of size at most $K \ge \max(2, J)$. Suppose at most J members of Z satisfy $|\rho - r| < h$, with multiplicities retained. Suppose also that, on $|x - c| \le 1$,*

$$V_f(x) = \sum_{\rho \in Z} (x-\rho)^{-2} + R_c(x), \qquad |R_c(x)| \le C_{\mathrm{loc}} K, \tag{15}$$

$$|f(x)| \ge e^{-A_* - C_{\mathrm{loc}} K} \prod_{\rho \in Z} \min(1, |x-\rho|), \qquad A_* \ge 1. \tag{16}$$

The identities are required away from the zeros, with multiplicities retained. For each fixed $\theta \in (0,1)$ there are constants $c_\theta, C_\theta > 0$, depending only on the fixed kernel and C_{loc} and J in addition to θ,

with the following property. If

$$R \geq C_\theta[1 + \log(1/h) + \log K], \tag{17}$$

then one obtains in each parity d disjointly supported compact targets with

$$c_\theta MR \leq d \leq C_\theta MR, \qquad E_f(u_i, u_j) = \delta_{ij}, \tag{18}$$

$$Q_f(u_i, u_j) = q_i\delta_{ij}, \quad 2 - \theta/4 \leq q_i \leq C_\theta, \quad \|u_i\|_f \leq C_\theta, \quad \|u_i/f\|_{H^2} \leq e^{C_\theta(A_* + K\log(2/h) + R)}. \tag{19}$$

Here E_f, Q_f are the target forms defined from V_f as above, and $\mathcal{E}_f(P) = E_f(fP, fP)$, $\mathcal{Q}_f(P) = Q_f(fP, fP) = (8\pi)^{-1}\int G_f P^2$ are their polynomial forms. Their simultaneous lifting gives a rational basis for a real d-dimensional polynomial subspace on which $\mathcal{Q}_f \geq (2-\theta)\mathcal{E}_f$, at every degree cutoff satisfying

$$\log(N+2) \geq C_\theta d \exp\{C_\theta(A_* + K\log(2/h) + R)\}. \tag{20}$$

Constants may be enlarged once in these sufficient conditions.

Proof. Fix $0 < \varepsilon \leq \theta/1000$. Choose constants

$$A \geq \max(10, 4/\varepsilon), \quad B \geq \max(4, \sqrt{8/\varepsilon}), \quad D \geq 10, \quad (4 + C_{\mathrm{loc}})/D^2 \leq \varepsilon/2,$$

and put $S_0 = \log(D\sqrt{K}/h)$, $t_0 = e^{-S_0}$. Since $|r - c| \leq 1/2$ and $t_0 < h$, every $r + t$, $0 < t \leq t_0$, lies in the interval of the local formulas.

Write $\alpha = \rho - r$, counting multiplicities, and split Z into the inner roots $|\alpha| < h$ and the outer roots $|\alpha| \geq h$. For $t \leq t_0$, every outer root has distance at least $h/2$ from $r + t$. Consequently the entire outer pole sum and analytic remainder cost at most

$$t^2\left[\sum_{\text{outer}} |t - \alpha|^{-2} + |R_c(r+t)|\right] \leq \frac{4}{D^2} + \frac{C_{\mathrm{loc}}h^2}{D^2} \leq \varepsilon/2. \tag{21}$$

For a nonzero inner α set $\lambda_\alpha = -\log|\alpha| > 0$, and delete the closed scale interval

$$[\lambda_\alpha - \log A, \lambda_\alpha + \log B + \tfrac{1}{2}\log J]. \tag{22}$$

For $\alpha = 0$ put $\lambda_\alpha = +\infty$, with no deleted interval. If $s \geq S_0$ avoids the deleted union and $t = e^{-s}$, every inner root is either interior, $|\alpha| \leq t/A$, or exterior to the cap scale, $|\alpha| \geq B\sqrt{J}\,t$. On each good component the classes are constant, conjugation invariant, and have an interior multiplicity $1 \leq \nu \leq J$.

For an interior root,

$$\left|(1 - \alpha/t)^{-2} - 1\right| \leq 4/A \leq \varepsilon.$$

For all inner roots exterior to the cap scale together,

$$t^2 \sum_{\text{exterior}} |t - \alpha|^{-2} \leq \frac{J}{(B\sqrt{J} - 1)^2} \leq \frac{4}{B^2} \leq \varepsilon/2.$$

Together with (21), this shows that the complete, real potential, rather than each complex pole, satisfies

$$(1 - 2\varepsilon)\nu \leq t^2 V_f(r + t) \leq (1 + 2\varepsilon)\nu. \tag{23}$$

In particular it is positive on the supports to be used.

Define for $s \geq 0$

$$W(s) = \sum_{\rho \in Z,\ |\rho - r| < h} \min(s, \lambda_\alpha). \tag{24}$$

The seed implies $1 \leq W'(s) \leq J$ almost everywhere and $W(s) \geq s$. There is a unique $S > S_0$ with $W(S) = R$ whenever $R > W(S_0)$, and $S \leq R$. On a good component $W' = \nu$, since every finite threshold where this derivative changes has been deleted. The initial cost is at most JS_0; the union of at most J bands has length at most Jw_J, where $w_J = \log(AB) + \frac{1}{2}\log J$. Hence, writing $\mathcal{G}$ for the good part of $[S_0, S]$,

$$\int_{\mathcal{G}} \nu(s)\, ds \geq R - JS_0 - J^2 w_J. \tag{25}$$

There are at most $J + 1$ good components. Since J is fixed, condition (17) pays both losses.

The useful minimum modulus retains the rootwise distances. In both classes among the inner roots, $|t - \alpha| \geq c \max(t, |\alpha|)$ with a fixed positive c. Here $|\alpha| < h < 1$, so $\max(t, |\alpha|) = \exp[-\min(s, \lambda_\alpha)]$. Every outer factor has clipped distance at least $h/2$. Multiplication in (16) therefore gives

$$|f(r + e^{-s})| \geq e^{-A_* - C_\theta K - C_\theta J - W(s)} (h/2)^K \geq e^{-C_\theta [A_* + K \log(2/h) + R]} \qquad (s \in \mathcal{G}). \tag{26}$$

This bound does not raise the smallest distance to the number of roots. It is valid for arbitrarily tight real or complex clusters.

Use the zero-extended cutoff $\chi_m(z) = z^{m+1}(1-z)^{m+1}$ on $[0, 1]$. Its endpoint derivatives through order m vanish, and it belongs to H^{m+1}. Write

$$A_0 = \int \chi_m^2 > 0, \qquad A_1 = \int (\chi_m')^2 > 0, \qquad \gamma_m = A_1/A_0. \tag{27}$$

All its derivative norms through order $m + 1$ are fixed and finite. Choose $L = L_{m,\theta}$ with $L^2 \geq 1000 \max(1, \gamma_m)/\theta$. In a good component of multiplicity ν, pack consecutive intervals of length $\ell_\nu = L/\sqrt{\nu}$. On such a cap $[a, a + \ell_\nu]$ put

$$v(s) = c_\nu \ell_\nu^{-1/2} \chi_m((s - a)/\ell_\nu), \quad c_\nu^2 = \frac{4\pi}{A_1/\ell_\nu^2 + A_0/4}, \quad u(r + t) = \sqrt{t}\, v(-\log t), \tag{28}$$

and set $u = 0$ elsewhere. The change of variable $t = e^{-s}$ gives the exact identities

$$4\pi E_f(u) = \int (v'^2 + v^2/4) = 4\pi, \qquad 8\pi Q_f(u) = \int t^2 V_f(r + t) v^2. \tag{29}$$

The energy cross term vanishes because v is zero at the endpoints. For the pure potential ν/t^2 the quotient is

$$q_\nu = \frac{2\nu}{1 + 4\gamma_m \nu/L^2}. \tag{30}$$

This increases for $\nu \geq 1$ and is at most $L^2/(2\gamma_m)$. Moreover

$$(1 - 2\varepsilon) q_\nu \geq 2 - 4\varepsilon - 8\gamma_m/L^2 \geq 2 - \theta/4.$$

Equations (23)–(30) show $2 - \theta/4 \leq Q_f(u) \leq (1 + 2\varepsilon)L^2/(2\gamma_m)$. The absolute potential integral equals $8\pi Q_f(u)$ because $V_f > 0$ on the cap. Also $\int u^2 \leq t_0^2 \int v^2 \leq 16\pi t_0^2$. The entire joint norm is therefore bounded by a constant M_θ, independent of ν, K, R and the window.

A component of length a contributes $\lfloor a\sqrt{\nu}/L \rfloor$ caps. Since $\sqrt{\nu} \geq \nu/\sqrt{J}$, their total number at one seed is at least

$$\frac{R - JS_0 - J^2 w_J}{L\sqrt{J}} - (J+1).$$

Enlarge the constant in (17) so this is at least $R/(2L\sqrt{J})$. Select exactly $b_R = \lfloor R/(4L\sqrt{J}) \rfloor$ caps per seed; the same threshold ensures $b_R \geq c_\theta R$. Their support interiors are disjoint, even for abutting caps. Distinct seed supports lie within $t_0 < h$ on the right of seeds separated by at least $3h$, so those supports are disjoint too. Reflect each positive target with sign $(-1)^\epsilon$ onto the negative axis and divide by $\sqrt{2}$. Evenness of f, V_f preserves the forms and the joint norm. There are $d = Mb_R$ targets in each chosen parity, and their full raw energy Gram is I and full raw Q_f Gram is diagonal. Reflection supplies one direction of each parity per positive cap and does not double the count.

We next bound division by the same kernel. The exact derivatives are

$$u'(r+t) = t^{-1/2}(v/2 - v'), \qquad u''(r+t) = t^{-3/2}(v'' - v/4).$$

The normalization gives $\|v\|_2^2 \leq 16\pi$, $\|v'\|_2^2 \leq 4\pi$, and $\|v''\|_2^2 \leq C_\theta \nu \leq C_\theta J$. As every cap lies in $[S_0, S]$, changing variables in the last derivative yields

$$\|u\|_{H^2} \leq C_\theta \sqrt{J}\, e^S \leq C_\theta \sqrt{J}\, e^R.$$

Put

$$b_* = e^{-C_\theta[A_* + K\log(2/h) + R]} \leq 1.$$

The fixed global bounds for f, f', f'' and (26) imply

$$\|1/f\|_{W^{2,\infty}} \leq C_f b_*^{-3}$$

on each cap. The product rule now proves the last bound in (19). Both u and u' vanish at every cap endpoint. More generally all derivatives of u through order m vanish there, and so do those of u/f. Its zero extension is therefore in H^{m+1}, with no boundary distributions. The reciprocal is used only on these supports; no reciprocal value at a zero outside the support is required.

For the simultaneous transfer take

$$B_* = \max(1, e^{C_\theta(A_* + K\log(2/h) + R)}), \qquad \eta = \frac{\theta}{1000d(1+M_\theta)^2}, \qquad M_\theta \geq 1.$$

Apply the inherited interface (6) to all targets with this same budget. Cauchy–Schwarz in the derivative and absolute-potential parts shows that replacing two targets by polynomial multiples changes an energy entry by at most $\eta(2M_\theta + \eta)/(4\pi)$ and a Q_f entry by at most $\eta(2M_\theta + \eta)/(8\pi)$. A $d \times d$ matrix has operator norm at most d times its largest entry magnitude. Consequently both complete Gram errors are less than $\theta/100$. In particular, the polynomial energy Gram is positive definite and the polynomials are independent. On their whole real span,

$$\frac{\mathcal{Q}_f}{\mathcal{E}_f} \geq \frac{2 - \theta/4 - \theta/100}{1 + \theta/100} \geq 2 - \theta. \tag{31}$$

Every newly generated mixed entry and the rational rounding error has been paid in this estimate. Finally (6) bounds the logarithm of every degree plus two by $C_\theta d \exp[C_\theta(A_* + K\log(2/h) + R)]$, absorbing its additional logarithmic term. This proves (20) and the proposition. □

Remark 6.2 (Coincident roots and the role of the product). If all J inner roots coincide with the seed then $W(s) = Js$ and $S = R/J$. The cap length is $L/\sqrt{J}$, and the count is of order $R/\sqrt{J}$, hence of order R when J is fixed. Arbitrarily close roots can have the same behavior over a finite logarithmic horizon. The improvement uses a bounded population, not separation inside a cluster. If instead one has only a possibly growing full population K, the same root-budget and cap mechanism incurs a $\sqrt{K}$ packing loss; that is why treating every real seed as an independent macroscopic window would be unjustified. The outer product payment $K\log(2/h)$ is essential in the two-scale version.

7 A height-cancelled receiver for the theta caps

Lemma 7.1 (Compact theta lifting)**.** *Consider the raw caps for F constructed in Proposition 6.1, with seeds in $[a_0T, T]$ for fixed $a_0 > 0$, $K \asymp \log(T + 40)$, fixed inner population J and fixed cutoff/approximation order $m \geq 1$. For this lemma any fixed positive L may be used in (28); its near-edge quotient lower bound is not needed here. Suppose that on every cap its outer roots satisfy the product budget*

$$\prod_{\rho\in Z,\ |\rho-r|\geq h} \min(1, |x-\rho|) \geq e^{-H}. \tag{32}$$

There is an exponent $D > 0$ independent of m, and constants $C_{m,\theta}$, such that for $0 < \eta \leq 1$ each reflected one-parity cap u has a rational polynomial approximant of the same parity satisfying $\|FP - u\|_* < \eta$ and

$$\log(\deg P + 2) \leq C_{m,\theta}\left[\log(T+2) + (\eta^{-1}T^D K^{m+1})^{1/m} e^{H/m+(1+1/m)R}\right]. \tag{33}$$

The exponent D is fixed by the geometry and coefficient estimates before m is chosen. Multiplicative constants may depend on m and the fixed L; none depends on cap location, the number of caps or internal root separations.

Proof. Put $a = \pi/8$. The exact exponential rate in (7) and fixed-radius Cauchy differentiation imply, for fixed A_0, C and every real x,

$$|F(x)| + |F'(x)| + |F''(x)| + \sqrt{|G(x)|} \leq C(1+|x|)^{A_0} e^{-a|x|}. \tag{34}$$

For large T choose $b = a - 1/T \in [a/2, a]$. Since

$$\sup_{x\geq 0}(1+x)^{A_0} e^{-x/T} \leq C_{A_0} T^{A_0},$$

the coefficients in (34) are bounded globally by $CT^{A_0}W_b(x)$. Thus for an error e,

$$\|Fe\|_*^2 = \int \left[|Fe' + F'e|^2 + (F^2 + |G|)|e|^2\right] \leq CT^{2A_0}\left[\|e'W_b\|_2^2 + \|eW_b\|_2^2\right]. \tag{35}$$

This includes every real-line tail. The identity $|V|F^2 = |G|$ pays the potential at every real zero.

On a positive cap with seed r, set $f_r(x) = e^{ar}F(x)$. The center c differs from r by at most $1/2$, which changes normalization by a fixed factor. The normalized local product (9), the inner root budget $W(s) \leq R$, and (32) give

$$|f_r(x)| \geq C_\theta^{-1} T^{-C_0} e^{-H-R}.$$

Here the factor e^{-C_2K} is absorbed into a power of T; the inner rootwise constant costs only $e^{-C_\theta J}$. In particular C_0 is independent of the derivative order.

To retain that independence, differentiate the reciprocal through its logarithm. The same local factorization is $f_r = B_c H_r$. On a fixed disk with a margin around every cap, the real part of a logarithm of H_r is bounded in absolute value by $C \log T$. Harmonic derivative estimates, or Borel–Caratheodory followed by Cauchy's estimate after subtracting an imaginary constant, give $|(\log H_r)^{(q)}| \le C_q \log T$ for each fixed $q \ge 1$. Every local root has $|x-\rho| \ge c_\theta t$ on a good cap: the inner roots satisfy the interior/exterior alternative, and the outer roots are at distance at least $h/2$. The logarithmic derivatives of the Blaschke denominators cost only C_q per root. Thus, with $t = x - r$,

$$|(\log f_r)^{(q)}(x)| \le C_{q,\theta} K t^{-q}. \tag{36}$$

The derivatives are independent of the logarithm's local branch. The reciprocal derivative is $1/f_r$ times the complete exponential Bell polynomial in $-(\log f_r)', \dots, -(\log f_r)^{(q)}$. Every monomial has total derivative order q and at most q factors. Consequently

$$|(1/f_r)^{(q)}(x)| \le C_{q,\theta} T^{C_0} e^{H+R} K^q t^{-q}, \qquad q \ge 0, \tag{37}$$

where $q = 0$ uses the modulus bound. The power T^{C_0} is paid once, not $q+1$ times. In $u/F = e^{ar} u/f_r$, the height amplitude likewise occurs once.

For $0 \le j \le q \le m+1$, differentiation gives

$$u^{(j)}(r+t) = t^{1/2-j} L_j(d/ds) v(s), \qquad \|t^{-(q-j)} u^{(j)}\|_2 \le C_{m,\theta} e^{(q-1)S} \quad (q \ge 1),$$

where L_j is a fixed constant-coefficient differential polynomial of order j. The squared integral has factor $t^{2-2q}\, ds$, independently of j. All derivatives of v through order $m+1$ are bounded by fixed constants, since J and m are fixed. On the positive support $|x-r| \le h \le 1/2$,

$$e^{ar} W_b(x) \le e^{(a-b)r+bh} \le C,$$

as $r \le T$. Reflection gives the same estimate on the negative support. Leibniz, (37) and $S \le R$ therefore yield, for $w = u/F$ extended by zero,

$$\|w^{(m+1)} W_b\|_2 \le C_{m,\theta} T^{C_0} K^{m+1} e^{H+(m+1)R}. \tag{38}$$

The endpoint traces through order m give $w \in H^{m+1}$. It is supported in $[-T-1, T+1]$ and vanishes at zero. No derivative of W_b at zero is used.

Apply Lemma 3.1 to this w, then use (35) and (38). The resulting degree at most $n+1$ polynomial has error at most

$$\frac{C_{m,\theta} T^D K^{m+1} e^{H+(m+1)R}}{(\log n)^m}, \qquad n \ge n_{0,m}, \quad \sqrt{n} > b(T+1)+2,$$

where $D = C_0 + A_0$ is independent of m. It is enough to take

$$\log n \ge C \log(T+2), \qquad \log n \ge C_{m,\theta} (\eta^{-1} T^D K^{m+1})^{1/m} e^{H/m+(1+1/m)R}.$$

The first condition pays the compact-tail support restriction. Taking the ceiling of the exponential of the maximum of these budgets bounds the logarithm of a constructed sufficient degree by (33), with constants enlarged once. Finally reserve half the tolerance for finite coefficient rounding. All $\|Fx^j\|_*$ are finite by (34); sufficiently close rational rounding of the allowed parity coefficients preserves degree and pays that reserve. This proves the asserted strict error bound. □

8 The all-degree theta consequence

Proof of Theorem 2.1. Use Lemma 5.1 to select exactly $M = \lfloor b_1 T \log T \rfloor$ seeds for a sufficiently small fixed $b_1 > 0$. For each seed take the midpoint of its containing unit interval. Apply Lemma 4.1 at height $T + 1$ to obtain the full local multiset, with

$$K = \lceil C_0 \log(T + 41) \rceil, \qquad h = 1/\log(T + 40), \qquad A_* \le C(T + K).$$

For large T one has $K \ge J$. Every root with distance less than h from the seed belongs to the macroscopic disk, and there are at most the fixed number J of them. These are all hypotheses of Proposition 6.1. Distinct seeds may share a macroscopic factorization disk; disjointness is required only of their microscopic supports.

The selected seeds also satisfy (10). For an outer root, $|x - \rho| \ge |r - \rho|/2$ throughout the cap, so its actual product satisfies

$$\prod_{\text{outer}} \min(1, |x - \rho|) \ge \exp\left[- \sum_{\text{outer}} \log_+ \frac{2}{|r - \rho|} \right] \ge e^{-B_0/h}.$$

Thus Lemma 7.1 applies with $H = B_0/h \le B_1 \log T$, for one fixed B_1 and all large T. In particular the exponent $D + B_1$ is independent of the approximation order.

Fix the requested exponent $A > 0$, and choose a fixed integer m so large that

$$\frac{(D + B_1 + 1)A}{m} < \frac{1}{8}. \tag{39}$$

This choice is noncircular because D, B_1 were fixed before m. Fix $\kappa > 0$ with $(1 + 1/m)\kappa < 1/8$. For a large integer N put

$$Y = \log\log N, \qquad T = e^{AY} = (\log N)^A, \qquad R = \kappa Y.$$

All population and fixed-order cap conditions hold eventually: $\log(1/h) + \log K = O_A(\log Y) = o(R)$. The raw construction gives

$$d \asymp_{m,\theta} T \log T \, R \ge c_{A,\theta} (\log N)^A Y^2. \tag{40}$$

Approximate all d targets using the compact theta receiver, with the common tolerance $\eta = \theta/[1000 d(1 + M_{m,\theta})^2]$. The complete Gram error estimate in the resource proof applies verbatim: it depends only on these joint-norm errors. Thus (31) gives the whole-span quotient and independence.

The new receiver's sufficient degree budget is at most

$$C_{m,\theta} \left[\log(T + 2) + (d T^{D+B_1} K^{m+1})^{1/m} e^{(1+1/m)R} \right]. \tag{41}$$

The logarithm of its second term is bounded by

$$\left[\frac{(D + B_1 + 1)A}{m} + (1 + 1/m)\kappa \right] Y + O_{m,A,\theta}(\log Y) < Y/3$$

eventually. Its first term is only $O_A(Y)$. Hence a constructed common degree has logarithm at most $e^{Y/2} < e^Y = \log N$ for all sufficiently large N. The primitive degree shift, integer ceiling and same-degree rational rounding have already been paid in Lemma 7.1. All selected polynomials therefore fit the same cutoff N, proving the theorem for every sufficiently large integer, not only a subsequence. Finally $D_0 \le (-1 + \theta)E_0 < 0$ and $Q_0 \ge (2 - \theta)E_0 > 0$ on every nonzero vector of the constructed subspace. □

Theorem 8.1 (Short-cap companion channels)**.** *For each fixed $A > 0$ and $\theta \in (0,1)$, and every sufficiently large integer N, each parity contains a real polynomial subspace $\mathcal{T}_N^{\epsilon,A,\theta}$ with rational basis and degree at most N, satisfying*

$$\dim \mathcal{T}_N^{\epsilon,A,\theta} \geq c_{A,\theta}(\log N)^A(\log\log N)^2, \qquad 0 < \frac{Q_0(P)}{E_0(P)} \leq \theta \quad (0 \neq P \in \mathcal{T}_N^{\epsilon,A,\theta}).$$

Thus $D_0(P) \geq (1-\theta)E_0(P)$ on the whole subspace.

Proof. Use the same selected seeds, good scales and cutoff χ_m. The cap formula (30) is exact for every positive L. Choose instead a fixed short length with $L_{\text{short}}^2 \leq \gamma_m\theta/8$. Since q_ν increases in $\nu \geq 1$ and is at most $L_{\text{short}}^2/(2\gamma_m)$, the complete potential bound gives

$$0 < q_* := \frac{2(1-2\varepsilon)}{1+4\gamma_m/L_{\text{short}}^2} \leq Q(u) \leq (1+2\varepsilon)\theta/16 \leq \theta/8, \qquad E(u) = 1.$$

Here q_* is a fixed strictly positive number, allowed to depend on m, θ. Shortening a fixed cap changes only fixed constants: packing still supplies a fixed multiple of R caps per seed, all derivatives through order m still vanish at the endpoints, and the divided derivative and weighted approximation estimates in Lemma 7.1 still hold. Joint norms are uniformly bounded, since the derivative integral is 4π, the absolute-potential integral is $8\pi Q(u)$, and the unweighted mass is bounded. After reflection each raw whole energy Gram is I and each raw whole Q Gram has spectrum in $[q_*, \theta/8]$.

For $d \asymp_{m,\theta} T\log T\, R$ targets, take one common

$$\eta = \frac{\min(q_*, \theta)}{Cd(1+M_{m,\theta})^2},$$

reserving half for rational rounding. With C fixed sufficiently large, the full Gram error estimate gives

$$\frac{99}{100}I \preceq \mathbf{E}_{\text{poly}} \preceq \frac{101}{100}I, \qquad \frac{3q_*}{4}I \preceq \mathbf{Q}_{\text{poly}} \preceq (\theta/8 + \theta/100)I.$$

Thus the polynomials are independent, Q_0 is strictly positive on their whole span, and its quotient by energy is at most $[(1/8+1/100)/(99/100)]\theta < \theta$. The factor q_*^{-1} in the tolerance is fixed once the approximation order is fixed; it does not introduce an order-dependent power of T. For a prescribed A, choose the fixed order by (39) and then $T = e^{AY}, R = \kappa Y$ as before. The same degree estimate (41) applies with changed multiplicative constants, so the common degree fits N for every sufficiently large N. This proves the dimension and the whole-span assertion. No mixed Gram between the two different channel families is required. □

9 A necessary rank for scalar-centered contractions

The published opposite packet channels have dimension $\Omega(\log N)$ and positive quotient $Q_0/E_0 = O((\log N)^{-2})$ on their whole span [1, Section 5]. That dimension alone would limit the rank deduction to $\Omega(\log N)$. The short-cap companion supplies the larger dimension required below.

Corollary 9.1 (Repair-rank lower bound)**.** *Represent E_0, D_0 in any real basis of $\mathcal{P}_N$, or of a fixed parity sector, by matrices $\mathbf{E}_N, \mathbf{D}_N$, and put*

$$K_N = \mathbf{E}_N^{-1/2}\mathbf{D}_N\mathbf{E}_N^{-1/2}.$$

Fix $c \in [0,1)$. If real scalars α_N and real symmetric matrices S_N satisfy

$$\|K_N - \alpha_N I + S_N\|_{\mathrm{op}} \le c$$

for all sufficiently large N, then for every fixed $A > 0$

$$\operatorname{rank} S_N \ge C_{A,c}(\log N)^A (\log\log N)^2$$

for all sufficiently large N, with $C_{A,c} > 0$. In the full polynomial space S_N need not preserve parity.

Proof. Fix $A > 0$ and choose $0 < \theta < 1 - c$. In energy coordinates, Theorem 2.1 supplies a subspace of dimension d_- on which $K_N \le -1 + \theta$ as a quadratic form, while Theorem 8.1 supplies one of dimension d_+ on which $K_N \ge 1 - \theta$ eventually. If $\operatorname{rank} S_N < \min(d_-, d_+)$, each subspace meets $\ker S_N$ nontrivially. For energy-unit vectors in these intersections, the norm bound implies, respectively,

$$\alpha_N \le c - 1 + \theta, \qquad \alpha_N \ge 1 - \theta - c.$$

These inequalities are incompatible with $\theta < 1 - c$. Thus the rank is at least $\min(d_-, d_+)$. One parity alone gives the required order even if S_N mixes parities, since both dimensions are at least fixed multiples of $(\log N)^A (\log\log N)^2$. Congruence by an invertible energy matrix preserves the rank of an equivalent symmetric correction form. □

The corollary is a necessary condition for this specified contraction architecture. Both receiving spaces have positive original Q_0.

The main theorem concerns the endpoint $y = 0$ and its actual energy. The physical forms at positive averaging parameter have a different strip energy [1, Section 5]; no free transfer of the logarithmic endpoint caps to that metric is implied.

10 Scope and remaining questions

The lower bound concerns strictly negative inertia of D_0, with $Q_0 > 0$ on every constructed nonzero vector. It coexists with the predecessor's finite-degree positivity theorem. The repair-rank consequence is a necessary condition for the stated scalar-centered strict contraction architecture in the energy metric.

The present construction still uses division by F to invoke the whole-line approximation interface. A resource-sensitive projector for the actual infinite-zero theta space might improve that step, but no such estimate follows from qualitative density or the finite Gaussian ideal theorem. Likewise, the exact positive-width current identity does not pay its boundary current for the actual theta receiver. A same-state estimate retaining the full complementary block and mixed terms remains a separate sign problem.

Computational and assistance disclosure. The principal inertia theorem is analytic. Supplementary exact rational controls check finite identities, and two implementations enclose the generic current certificate using ball arithmetic. AI assistants were used in mathematical development, checking, source comparison, code and manuscript preparation.

A An alternative first-order weighted receiver

There is a distinct published input for the first-order part of the weighted receiver. Bizeul and Klartag [11, Theorem 1, equation (3)] prove that, for $\mu_1(dy) = e^{-|y|}dy/2$ and the associated orthonormal-polynomial coefficients f_k of an absolutely continuous function f,

$$\sum_{k\geq 1} \log^2(e+k)|f_k|^2 \leq C\left(\|\log(e+|y|)f\|^2_{L^2(\mu_1)} + \|f'\|^2_{L^2(\mu_1)}\right).$$

We use this as an imported theorem at its stated finite-right-hand-side scope, rather than claim a new weighted Jackson inequality. Taking its spectral tail gives a bound for the approximation NORM with denominator $\log(n+2)$; the error convention in that paper is the squared norm.

For $W_b(x) = e^{-b|x|}$ with b in a fixed compact positive interval, apply this theorem to $f(y) = h'(y/(2b))$. The factor $2b$, rather than b, matches μ_1 to the squared weight W_b^2. It gives a real polynomial q of degree at most n such that

$$\|(q-h')W_b\|_2 \leq \frac{C}{\log(n+2)}\left(\|\log(e+|x|)h'W_b\|_2 + \|h''W_b\|_2\right),$$

with a constant uniform in that interval of b. Assume the displayed norms are finite, $h \in H^2(\mathbb{R})$ has one parity and $h(0) = 0$. Parity projection of q, followed by $P(x) = \int_0^x q(t)dt$, preserves degree at most $n+1$ and the target parity. The same half-line exponential convolution estimate used in the main text pays $\|(P-h)W_b\|_2$ by $b^{-1}\|(q-h')W_b\|_2$. Thus the estimate controls both weighted value and derivative errors, including their complete tails.

For the compact theta targets the support lies in $[-T-1, T+1]$, so the logarithmic location norm adds at most a factor $\log(T+3)$ to the normalized target estimates. The global F, F', G weights then give the same kind of first-order original joint-norm receiver as the compact-support Lubinsky argument. This alternative has an explicit location norm instead of an n-dependent compact-tail cutoff. It is a second analytic certificate of this FIRST-ORDER interface.

The arbitrary-fixed-power inertia theorem in the main text uses the higher fixed-order Lubinsky input and its separately paid derivative budgets.

B A finite Gaussian zero ideal

This appendix gives a bounded comparison model for the original joint norm. Its zero ideal is finite and its spectral reference is Gaussian. Hermite projection and ladder identities, confluent divided differences and Hardy's inequality are classical ingredients; see [12, 13, 14] for the relevant approximation background. The statement below specifies the composition and the singular norm used here.

Let $m \geq 1$ and $s > m$ be integers and let $B \geq 0$. Set

$$A(x) = \prod_{j=1}^{J}(x-r_j)^{\nu_j}, \qquad \sum_j \nu_j = m, \quad r_j \in [-B, B], \qquad f(x) = \gamma A(x)e^{-x^2/2}, \quad 0 \neq \gamma \in \mathbb{R}.$$

Repeated nodes are combined into their total multiplicity. On $L^2(\mathbb{R})$ write $H = -\partial_x^2 + x^2$ and $\|u\|_{(s)} = \|(1+H)^{s/2}u\|_2$. Away from the roots define

$$V_f = \frac{f'^2 - ff''}{f^2} = 1 + \sum_j \frac{\nu_j}{(x-r_j)^2}, \qquad \|v\|_f^2 = \int_{\mathbb{R}}\left(|v'|^2 + (1+|V_f|)|v|^2\right)dx.$$

At singularities the latter is the nonnegative Lebesgue integral, with its natural finite-norm domain. Use the associated real bilinear forms

$$E_f(u,v) = \frac{1}{4\pi}\int u'v', \qquad Q_f(u,v) = \frac{1}{8\pi}\int V_f uv, \qquad D_f = E_f - Q_f.$$

For polynomial arguments, $E_f(P,R)$ below means $E_f(fP, fR)$, and likewise for Q_f, D_f.

Theorem B.1 (Fixed finite ideal, uniformly through collisions)**.** *Suppose u is real, $\|u\|_{(s)} < \infty$, and $u^{(q)}(r_j) = 0$ for $0 \le q < \nu_j$. For every integer $K \ge m$ there is a real polynomial P of degree at most $K - m$ such that*

$$\|fP - u\|_f \le C_{m,B,s}(K+1)^{-(s-m)/2}\|u\|_{(s)}. \tag{42}$$

The constant is independent of γ, the root positions in $[-B, B]$, their separations and the multiplicity partition. The estimate also applies at a collision when u satisfies the merged jets. Rational coefficients may be required of P, at twice the bound. If f is even and u has a specified parity, P can have that parity.

Proof. Let ψ_n be the normalized Hermite functions, so that $H\psi_n = (2n+1)\psi_n$, and put

$$S_K u = \sum_{n=0}^{K} \langle u, \psi_n\rangle \psi_n = e^{-x^2/2} R_K(x), \qquad w = u - S_K u.$$

The ladder identities $\partial_x = (a - a^*)/\sqrt{2}$, $a\psi_n = \sqrt{n}\psi_{n-1}$ and $a^*\psi_n = \sqrt{n+1}\psi_{n+1}$ give

$$\|v\|_{H^m} \le C_m \|(1+H)^{m/2} v\|_2.$$

Indeed each derivative of order $q \le m$ is a sum of 2^q ladder words, each shifting the index by at most q with coefficient bounded by $C_q(n+1)^{q/2}$. The inequality first holds for finite sums and then by closure. The spectral tail therefore satisfies

$$\|w\|_{H^m} \le C_m(2K+4)^{-(s-m)/2}\|u\|_{(s)}. \tag{43}$$

The integer hypothesis $s > m$ also supplies all the traces in the theorem.

On $[-B-1, B+1]$ set $z = e^{x^2/2} w$. Multiplication by this smooth factor and the one-dimensional H^1 pointwise estimate give

$$\max_{0 \le q < m} \|z^{(q)}\|_{L^\infty([-B,B])} \le C_{m,B}\|w\|_{H^m}. \tag{44}$$

Order the multiset of roots as $y_1, \dots, y_m$ and let $I_A z$ be its confluent Newton interpolant of degree less than m. For real C^{m-1} data the divided-difference mean-value formula, including its confluent limit, gives

$$|z[y_1, \dots, y_{q+1}]| \le \frac{\|z^{(q)}\|_\infty}{q!}, \qquad 0 \le q < m.$$

It follows either from repeated Rolle's theorem and passage to the limit or from the simplex integral formula. No nodal separation occurs. Since the coefficient ℓ^1 norm of a Newton basis product is at most $(1+B)^q$,

$$\|\operatorname{coeff}(I_A z)\|_1 \le \left(\sum_{q=0}^{m-1} \frac{(1+B)^q}{q!}\right) \max_{q<m} \|z^{(q)}\|_\infty.$$

All prescribed jets of $e^{x^2/2}u = R_K + z$ vanish. Thus $R_K + I_A z$ is exactly divisible by A. Define P by the algebraic identity $\gamma AP = R_K + I_A z$; its degree is at most $K - m$. Gaussian polynomials of degree less than m have H^1 norm bounded by a constant times their coefficient norm. Consequently

$$\|fP - u\|_{H^1} = \|e^{-x^2/2} I_A z - w\|_{H^1} \le C_{m,B} \|w\|_{H^m}. \tag{45}$$

This is an interpolation estimate on smooth coherent error data.

The error $v = fP - u$ has zero trace at every r_j. Hardy's inequality on both half-lines at each root yields

$$\int \frac{|v(x)|^2}{(x - r_j)^2}\, dx \le 4 \int |v'|^2.$$

It applies to H^1 zero-trace functions by half-line approximation. Hence the entire singular potential and the derivative are paid:

$$\|v\|_f^2 \le (1 + 4m)\|v'\|_2^2 + 2\|v\|_2^2 \le (2 + 4m)\|v\|_{H^1}^2.$$

Together with (43)–(45) this proves (42). The construction uses no norm of u/f. Multiplication of f by γ leaves both its potential and its polynomial-multiple space unchanged.

Every fixed basis element fx^j has finite joint norm. Approximate the finitely many coefficients of P by rationals so closely that the added joint-norm error is at most the original allowance. The factor f retains all exact zeros, even for irrational roots. When f is even, reflection is an isometry of the norm; projection onto the desired parity is contractive, and rounding within that parity completes the proof. □

One conservative choice of constants in this proof is

$$C_{m,B,s} \le C_{m,s} e^{(B+1)^2/2} (1 + B)^{2m}. \tag{46}$$

Derivatives through order m of the exponential on the interpolation interval cost $C_m e^{(B+1)^2/2}(1 + B)^m$; the finite Newton sum costs at most $C_m(1+B)^{m-1}$. A local H^1 pointwise estimate on intervals of length two has an absolute constant. The remaining Gaussian polynomial norms depend only on m. Thus at fixed m, s a sufficient degree has logarithm $O_{m,s}(B^2 + \log(1 + \|u\|_{(s)}/\eta) + \log(B + 2))$.

B.1 A common subspace and logarithmic model inertia

Suppose d targets have joint norms at most M_0 and oscillator norms at most Z. Approximate each with joint error at most η at one common degree. Cauchy–Schwarz gives entrywise errors at most

$$\frac{\eta(2M_0 + \eta)}{4\pi} \quad (E_f), \qquad \frac{\eta(2M_0 + \eta)}{8\pi} \quad (Q_f), \qquad \frac{3\eta(2M_0 + \eta)}{8\pi} \quad (D_f).$$

The operator error is at most d times the largest entry error. In particular, for any fixed raw-Gram margin, taking $\eta = c/d$ with c sufficiently small preserves the full span and is possible with

$$K + 1 \le C_{m,B,s,M_0,c} \left[1 + (Zd/c)^{2/(s-m)}\right]. \tag{47}$$

Here and below enlarge the constant and take an integer ceiling to ensure $K \ge m$. All mixed entries are retained.

Corollary B.2 (A fixed Gaussian model)**.** *Let A be fixed, even and real-rooted, with $M \geq 1$ distinct positive roots and degree m. Choose a nonzero real $\chi \in C_c^\infty(0,1)$ and a fixed L such that*

$$C_\chi = \frac{\|\chi'\|_2^2}{\|\chi\|_2^2}, \qquad L^2 > 4C_\chi.$$

For every fixed integer $s > m$, in each polynomial parity,

$$i_-(D_f; \mathcal{P}_N^{\text{parity}}) \geq \frac{M(s-m)}{2(s-1)L} \log N - O_{A,s,\chi,L}(\log\log N). \tag{48}$$

The negative spaces have rational polynomial bases and $Q_f > 0$ on every nonzero vector. Constants concern this fixed A.

Proof. At each positive root r choose one side and write $t = |x - r| = e^{-z}$. Use collars with $u(x) = \sqrt{t}\, v(z)$, where v is a translate of $\chi(z/L)$, and reflect the collars at $-r$ to impose either parity. Take their initial logarithmic positions sufficiently large that all root neighborhoods are disjoint. Consecutive translated cutoffs have disjoint supports. An integration by parts gives

$$\int |u'|^2\, dx = \int \left(|v'|^2 + \tfrac{1}{4}|v|^2\right) dz, \qquad \int \frac{|u|^2}{t^2}\, dx = \int |v|^2\, dz.$$

Thus the singular part at a root of multiplicity ν has quotient $2\nu/(1 + 4C_\chi/L^2) > 1$. All remaining terms of V_f are nonnegative. Normalize the collars to have E_f norm one. Disjoint supports give E_f Gram matrix I and D_f Gram bounded above by $-\delta I$ for a fixed $\delta > 0$. Their joint norms are uniformly bounded: the own-root integral above is fixed and the other root terms are bounded on each fixed neighborhood. These smooth collars vanish near every root, so they satisfy every required jet condition.

For the lth collar at any fixed center,

$$\|u_{j,l}\|_{(s)} \leq C_{A,s,\chi,L} e^{(s-1)Ll}. \tag{49}$$

A derivative of order q contributes a factor at worst $t^{1/2-q}$; using $dx = t\, dz$ gives an L^2 cost at most $Ce^{(q-1)Ll}$ for $q \geq 1$. All supports lie in a fixed bounded interval, so powers of x add fixed constants. For even $s = 2j$, expand $(1 + H)^j$ into differential monomials of total order at most s. For odd $s = 2j + 1$, apply $\|(1 + H)^{1/2} w\|_2^2 = \|w\|_2^2 + \|w'\|_2^2 + \|xw\|_2^2$ to $w = (1 + H)^j u$. This proves (49).

Take l collars at each of the M positive centers, so $d = Ml$. Use (47) with $\eta = c/d$. The full Gram margin remains negative and the E_f Gram remains positive definite, so the approximating polynomials are independent. Their sufficient degree obeys

$$\log(K + 1) \leq \frac{2(s-1)L}{s-m}\, l + O_{A,s,\chi,L}(\log(l + 2)).$$

Choose the largest integer l below the inverted bound for $N + m$. Since the polynomial degree is $K - m$, this gives (48) for every sufficiently large N. Finally $V_f > 0$ away from the roots, so $Q_f(P) > 0$ for every nonzero polynomial. □

Taking successively larger fixed s makes the lower leading coefficient approach $M/(2L)$, with different constants each time. No uniform limit in s or growing-multiplicity theorem is implied.

B.2 Location and resolution cannot both disappear

For $f = e^{-x^2/2}$, let u have unit L^2 norm and support in $[R-1, R+1]$, $R > 1$. If $v = fP$, $\deg P \le K$, and $\|v-u\|_2 \le \varepsilon < 1$, then $\|v\|_2 \le 1+\varepsilon$ and its norm on that interval is at least $1-\varepsilon$. The oscillator spectral identity gives $\int x^2|v|^2 \le (2K+1)\|v\|_2^2$, whence necessarily

$$K \ge \frac{1}{2}\left[\frac{(R-1)(1-\varepsilon)}{1+\varepsilon}\right]^2 - \frac{1}{2}. \tag{50}$$

There is also a normalized moving-zero version: $f_R = (x^2 - R^2)e^{-x^2/2}/(1+R^2)$, $R \ge 1$. It has potential $1 + (x-R)^{-2} + (x+R)^{-2}$ and uniform strip upper bounds because $|z^2 - R^2|/(1+R^2) \le 1 + |z|^2$. Take fixed reflected smooth bumps in $(R+1, R+2)$ and its negative, normalized into either parity. Their Sobolev and joint norms are uniformly bounded and they vanish near both roots. An approximating $f_R P$ is a Gaussian polynomial of degree at most $K+2$, giving

$$K \ge \frac{1}{2}\left[\frac{(R+1)(1-\varepsilon)}{1+\varepsilon}\right]^2 - \frac{5}{2}. \tag{51}$$

Thus those upper strip bounds, fixed simple multiplicities and amplitude normalization do not imply a location-free degree bound.

For a separate resolution bound let u be smooth, supported in an interval I of length δ, with $\|u'\|_2 = 1$. If $\|v' - u'\|_2 \le \varepsilon < 1$ and $v = e^{-x^2/2}P$ has degree K, then v' belongs to Hermite modes through $K+1$. For modes through L, the diagonal Mehler formula gives, for $t > 0$,

$$\sum_{n=0}^{L} |\psi_n(x)|^2 \le e^{(2L+1)t}[2\pi \sinh(2t)]^{-1/2}.$$

The omitted factor $e^{-x^2 \tanh t}$ is at most one. With $t = (2L+1)^{-1}$ this is at most $e\sqrt{L+1}/\sqrt{2\pi}$. Pointwise Cauchy–Schwarz therefore gives $\|w\|_{L^2(I)}^2 \le \delta e\sqrt{L+1}\|w\|_2^2/\sqrt{2\pi}$ for every such w. Apply it to v' with $L = K+1$, using the norm bounds $1+\varepsilon$ globally and $1-\varepsilon$ on I. Necessarily

$$K + 2 \ge \frac{2\pi}{e^2}\left[\frac{1-\varepsilon}{1+\varepsilon}\right]^4 \delta^{-2}. \tag{52}$$

These bounds pay location and support resolution, not inverse nodal amplitudes. They leave open reference-scaled or theta-specific estimates with explicit resource budgets.

The finite-ideal proof does not extend merely by replacing f with a finite zero product of an infinite-zero function. A Gaussian polynomial with finitely many repaired jets need not be a polynomial multiple of $e^{-x^2/2}\cos x$, let alone of the theta transform. The remaining zero structure, zero-free factor, derivatives and global tails require additional estimates.

C A portable generic current certificate

The following scope example uses a positive even inverse kernel of order two. It is not the Riemann theta kernel. An admissible order-one endpoint-to-interior nonimplication family was already constructed in [15]; the present explicit polynomial is a reproducible same-state certificate. The all-depth root-hierarchy obstruction in [16] concerns another input hierarchy.

Set

$$b(s) = \frac{e^{-s^2/4}}{2\sqrt{\pi}}, \quad \phi(s) = \frac{b(s-2)+b(s+2)}{2} + \frac{b(s-4)+b(s+4)}{10}, \quad f(z) = e^{-z^2}\left(\cos 2z + \frac{\cos 4z}{5}\right).$$

The positive-sign Fourier convention gives $f(z) = \int \phi(s)e^{izs}\,ds$. All physical center integrals converge for every finite y by Gaussian decay. With the original operation

$$w_y(s) = \frac{s\sinh(2ys)}{2y}, \qquad J_y(t) = \int_0^\infty w_y(s)\phi(s+t/2)\phi(s-t/2)\,ds,$$

put

$$g_y(x) = \frac{\partial_y|f(x+iy)|^2}{8y}, \qquad Q_y(P) = \frac{1}{2\pi}\int_{\mathbb{R}} P(x)^2 g_y(x)\,dx.$$

Indeed the substitutions $s = (a+b)/2$, $t = a-b$ in the product of the two Fourier integrals give $|f(x+iy)|^2 = \int e^{ixt}\,2\int_0^\infty \phi(s+t/2)\phi(s-t/2)\cosh(2ys)\,ds\,dt$. Differentiation proves $g_y(x) = \int J_y(t)e^{ixt}\,dt$ with the printed factors. Gaussian domination justifies all interchanges.

For $h(x) = \cos 2x + \cos 4x/5$, writing $c = \cos 2x$ gives

$$h'^2 - hh'' = \frac{40c^3 + 60c + 116}{25} \geq \frac{16}{25}, \qquad g_0 = \frac{e^{-2x^2}}{4}(h'^2 - hh'' + 2h^2) > 0.$$

The cubic is increasing on $[-1,1]$. Consequently $Q_0(P) > 0$ for every nonzero polynomial, at every degree.

C.1 Exact finite moments and the complete Gram matrix

The public file `support_current_witness.json` specifies the even degree-80 rational polynomial

$$P(x) = \sum_{i=0,2,\ldots,80} \frac{a_i}{10^{70}} \frac{\mathrm{He}_i(2x)}{i!}.$$

All 41 integer numerators a_i are supplied; He_i are the probabilists' Hermite polynomials. This exact rational polynomial, rather than a numerical spectral search, is the certificate input. At $y = 3/4$ the complete product expansion gives

$$g_y(x) = \frac{e^{-2x^2+2y^2}}{8y}\sum_{k=0}^{4} b_k(y)\cos(2kx),$$

where

$$\begin{aligned}
b_0 &= 4y\left(\tfrac{1}{2}\cosh 4y + \tfrac{1}{50}\cosh 8y\right) + 2\sinh 4y + \tfrac{4}{25}\sinh 8y, \\
b_1 &= (4y\cosh 6y + 6\sinh 6y)/5, & b_2 &= 2y, \\
b_3 &= (4y\cosh 2y + 2\sinh 2y)/5, & b_4 &= 2y/25.
\end{aligned}$$

If X is standard normal, define $L_y(R) = \mathbb{E}[R(X/2)\sum_{k=0}^4 b_k(y)\cos(kX)]$. Differentiation of the Gaussian characteristic function gives

$$\mathbb{E}[(X/2)^{2n}\cos(kX)] = \frac{(-1)^n\,\mathrm{He}_{2n}(k)e^{-k^2/2}}{2^{2n}}, \qquad Q_y(P) = \frac{e^{2y^2}}{16y\sqrt{2\pi}}L_y(P^2).$$

Odd cosine moments vanish. Thus all polynomial operations and all moment coefficients are rational; only a finite list of exponential expressions requires interval evaluation. There is no quadrature cutoff or empirical tail estimate.

Both supplied current checkers use exact rational arithmetic and Arb ball enclosures at 512 and 768 bits. They certify

$$-0.92 < L_{3/4}(P^2) < -0.90, \qquad -0.094 < Q_{3/4}(P) < -0.092.$$

In the same basis $\{1, P+1\}$ the full normalized Gram matrix is

$$\begin{pmatrix} L_y(1) & L_y(P) + L_y(1) \\ L_y(P) + L_y(1) & L_y((P+1)^2) \end{pmatrix}.$$

The diagonal entries are approximately 128.8787643 and 127.8574288; the determinant is approximately -117.0222564. The certified signs are positive, positive and negative. Its determinant is exactly $L_y(1)L_y(P^2) - L_y(P)^2$. Thus two positive diagonal blocks do not pay their mixed term. This illustrates the missing hypothesis of a Schur-complement argument, not a failure of its criterion.

C.2 Translated strip area and boundary current

For real P and fixed $y > 0$ write

$$H_y(z) = f(z)P(z - iy), \qquad B(v) = \int_{\mathbb{R}} |H_y(x + iv)|^2 \, dx, \quad 0 \le v \le y.$$

Gaussian decay on closed strips justifies all derivatives and integrations by parts. Holomorphicity gives $B''(v) = 4 \int |H_y'(x + iv)|^2 dx$. At the top boundary $P(x), P'(x)$ are real, so their cross term in $B'(y)$ has zero real part, and $Q_y(P) = B'(y)/(16\pi y)$. At the bottom the $f'f|P|^2$ term likewise contributes zero, giving

$$B'(0) = -2 \int f(x)^2 \operatorname{Im}\big(P'(x - iy)\overline{P(x - iy)}\big) \, dx.$$

Integrating B'' proves the exact identity

$$\begin{aligned} Q_y(P) &= T_y(P) - C_y(P), \\ T_y(P) &= \frac{1}{4\pi y} \int_0^y \int_{\mathbb{R}} |H_y'(x + iv)|^2 \, dx \, dv \ge 0, \\ C_y(P) &= \frac{1}{8\pi y} \int_{\mathbb{R}} f(x)^2 \operatorname{Im}\big(P'(x - iy)\overline{P(x - iy)}\big) \, dx. \end{aligned}$$

For the rational witness, the checkers enclose

$$C_{3/4}(P) \simeq 63408346.64544519546, \qquad T_{3/4}(P) \simeq 63408346.55246610488,$$

and certify $C_{3/4}(P) > T_{3/4}(P) > 0$. The current is evaluated from exact rational translation of P and complete Gaussian moments of f^2; the cosine weights of h^2 are $13/25, 1/5, 1/2, 1/5, 1/50$. The area is evaluated as $T = C + Q$ using the proved identity, not by independent area quadrature. The two checkers independently reconstruct the finite expressions: one uses a Hermite recurrence and the five cosine weights, while the other uses the closed coefficient formula, Gaussian integration by parts, all sixteen ordered Fourier pairs and Taylor translation.

As $y \downarrow 0$, dominated polynomial–Gaussian estimates give $T_y(P) \to E_f(P)$ and $C_y(P) \to D_f(P)$: the latter follows also from $\operatorname{Im}(P'(x - iy)\overline{P(x - iy)}) = y(P'^2 - PP'') + O(y^2)$ under the integral. For positive y, however, T_y and C_y are these translated strip forms, and are not individually the original positive-width E_y, D_y. The identity concerns the same original Q_y throughout. Endpoint positivity, positive inverse density and positive strip energy alone therefore do not establish $C_y \le T_y$.

D The correction is not a scalar moment form

For any nonzero real smooth strip-decaying f for which the polynomial integrations by parts are justified, polarizing the original endpoint forms gives

$$D_f(P,R) = \frac{1}{8\pi}\int f^2\left[P'R' - \frac{P''R + PR''}{2}\right]. \tag{53}$$

Indeed expand $2(fP)'(fR)' - (f'^2 - ff'')PR$, integrate $(ff')'PR$ once and then $ff'(PR)'$ once. All boundary terms vanish. For $i + j \geq 2$ this yields

$$D_f(x^i, x^j) = \frac{i + j - (i - j)^2}{16\pi}\int f(x)^2 x^{i+j-2}\,dx.$$

A scalar moment form $L(PR)$ would have equal entries for equal $i + j$. Here already $D_f(1, x^2) = -D_f(x, x)$, with nonzero absolute value. The contradiction persists in each parity separately. Writing $\mu_q = \int f^2 x^q$, one has

$$D_f(1, x^4) = -\frac{6\mu_2}{8\pi}, \quad D_f(x^2, x^2) = \frac{2\mu_2}{8\pi}, \qquad D_f(x, x^5) = -\frac{5\mu_4}{8\pi}, \quad D_f(x^3, x^3) = \frac{3\mu_4}{8\pi}.$$

The moments μ_2, μ_4 are strictly positive. Thus a scalar signed-measure collision theorem cannot be made a theorem about D_f by relabeling its moments. Matrix-valued, jet, differential or transformed representations remain possible only with a proved receiving map. In contrast Q_f is a scalar moment form with density $(f'^2 - ff'')/(8\pi)$; applying scalar negative-mass detection to it requires an actual negative-mass input for that same density. A negative correction space with $Q_f > E_f > 0$ supplies no such input.

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