Trace-perturbed minimizers of the mixed discriminant

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Abstract

For each fixed matrix size, we prove that the scalar tuple uniquely minimizes the mixed discriminant of positive semidefinite Hermitian matrices whose sum is the identity and whose individual traces are sufficiently close to one. This answers Question 1.7 of Gurvits–Leake, arXiv:2609.21083v2. The argument combines Gurvits's doubly stochastic uniqueness theorem with a positive constrained Hessian and compactness. We also record the exact region of positive definiteness of that Hessian at a scalar tuple. The global neighborhood in the theorem is qualitative and may depend on the size.

1 Statement and relation to earlier work

The two marginal perturbations and the proof tasks The matrix sum remains the identity while the individual traces vary near one. This differs from perturbing the total matrix while fixing every trace. The proof first obtains a uniform local quadratic gap on one parameter-independent tangent space, then uses compactness and the known balanced uniqueness theorem to localize every global minimizer. The later explicit Hessian region is a local classification; the two-dimensional identity separately gives a global special case including endpoint traces.

For $n \times n$ matrices define the normalized mixed discriminant by

$$d(X_1, \dots, X_n) = \frac{1}{n!} \frac{\partial^n}{\partial z_1 \cdots \partial z_n} \det \left(\sum_{i=1}^{n} z_i X_i \right) \Bigg|_{z=0}.$$

Thus d is symmetric and multilinear, and $d(I, \dots, I) = 1$. All positive semidefinite matrices below are complex Hermitian. Write $\mathbf{1} = (1, \dots, 1)$.

Gurvits proved that, among PSD tuples satisfying $\operatorname{Tr} X_i = 1$ and $\sum_i X_i = I$, the unique minimizer is $(I/n, \dots, I/n)$ [1, Theorem 4.1]. His unnormalized discriminant is $n!d$. Gurvits–Leake ask whether this uniqueness persists when the individual traces vary near one while the sum stays equal to the identity [2, Question 1.7]. The following gives an affirmative answer.

Theorem 1. *For every integer $n \geq 1$, there exists $\varepsilon_n > 0$ such that the following holds. If $t \in \mathbb{R}^n$ satisfies*

$$\sum_{i=1}^{n} t_i = n, \qquad \|t - \mathbf{1}\|_1 < \varepsilon_n,$$

then

$$d(X_1, \dots, X_n) \;\geq\; \frac{\prod_{i=1}^{n} t_i}{n^n}$$

for all $X_i \succeq 0$ with $\operatorname{Tr} X_i = t_i$ and $\sum_i X_i = I$. Equality holds if and only if $X_i = (t_i/n)I$ for every i.

The perturbation mechanism is standard. In particular, the proof of Gurvits's Corollary 4.2 already uses uniqueness and compactness to localize minimizers when the total matrix varies and each trace stays one; his Lemma 3.6 gives a related quadratic gap [1]. Theorem 1 applies this type of argument to the other marginal constraint, with the constrained Hessian computed directly. Theorem 1.2 of [2] concerns perturbations of the total matrix with fixed individual traces. Normalizing X_i by t_i in our problem instead produces the weighted constraint $\sum_i t_i(X_i/t_i) = I$, so it does not directly put the problem into that form.

For simultaneously diagonalizable tuples, $n!d$ is a permanent. The corresponding near-uniform marginal conclusion in that subcase is already known from [3, Section 6], using transposition to exchange the two margins. The conclusion here concerns the full Hermitian tuple domain.

We use Gurvits's global uniqueness theorem as an input, rather than give a new proof of it. No explicit or dimension-uniform value of ε_n is obtained. The argument is for matrix tuples. All question and theorem numbers for that reference refer to version 2, dated October 15, 2026.

2 Proof of trace stability

The case $n = 1$ is immediate, so assume $n \geq 2$. Use the real Euclidean space of Hermitian tuples, with

$$\|A\|^2 = \sum_i \operatorname{Tr}(A_i^2).$$

Let

$$\mathcal{T} = \left\{ (A_1, \dots, A_n) : A_i = A_i^*, \quad \operatorname{Tr} A_i = 0 \text{ for every } i, \quad \sum_i A_i = 0 \right\}.$$

For a positive trace vector t of sum n, set

$$C_i(t) = \frac{t_i}{n} I, \qquad f(t, A) = d(C_1(t) + A_1, \dots, C_n(t) + A_n), \quad A \in \mathcal{T}.$$

The space $\mathcal{T}$ is independent of t, and every tuple satisfying the affine constraints of Theorem 1 is uniquely $C(t) + A$ with $A \in \mathcal{T}$.

Lemma 2. *For every positive t of sum n, $D_A f(t, 0)[A] = 0$ for all $A \in \mathcal{T}$. At $t = \mathbf{1}$,*

$$D_A^2 f(\mathbf{1}, 0)[A, A] = \lambda_n \|A\|^2, \qquad \lambda_n = \frac{n^{-(n-2)}}{n(n-1)} > 0.$$

Proof. Polarization of the determinant gives

$$d(B, I, \dots, I) = \frac{\operatorname{Tr} B}{n}, \tag{1}$$

$$d(B, H, I, \dots, I) = \frac{\operatorname{Tr} B \operatorname{Tr} H - \operatorname{Tr}(BH)}{n(n-1)}. \tag{2}$$

For example, these follow by comparing the linear and mixed quadratic terms of $\det(I + sB + uH)$. These identities do not require commutativity. By multilinearity, every summand in the first derivative of $f(t, A)$ at $A = 0$ has a factor $d(A_i, I, \dots, I)$, which is zero. At the uniform trace vector, (2) gives

$$\frac{1}{2} D_A^2 f(\mathbf{1}, 0)[A, A] = -\frac{n^{-(n-2)}}{n(n-1)} \sum_{i<j} \operatorname{Tr}(A_i A_j).$$

Since $\sum_i A_i = 0$, the sum on the right equals $-\frac{1}{2} \sum_i \operatorname{Tr}(A_i^2)$. This proves the claim. □

By continuity of the restricted Hessian in this finite-dimensional space, there exist $r > 0$ and a neighborhood U of $\mathbf{1}$ in the hyperplane $\sum_i t_i = n$, consisting of positive trace vectors, such that

$$D_A^2 f(t, B)[A, A] \geq \frac{\lambda_n}{2}\|A\|^2 \quad (t \in U,\ B \in \mathcal{T},\ \|B\| \leq r,\ A \in \mathcal{T}).$$

Stationarity from Lemma 2 and Taylor's integral formula therefore imply

$$\begin{aligned} f(t, A) - f(t, 0) &= \int_0^1 (1 - s) D_A^2 f(t, sA)[A, A]\, ds \\ &\geq \frac{\lambda_n}{4}\|A\|^2 \qquad (t \in U,\ \|A\| \leq r). \end{aligned} \tag{3}$$

This is a uniform local estimate.

Lemma 3. *After shrinking the trace neighborhood, every global minimizer in each feasible trace fiber satisfies $\|X - C(t)\| < r$.*

Proof. The set

$$K = \{(X_1, \ldots, X_n) : X_i \succeq 0,\ \sum_i X_i = I\}$$

is compact, since $0 \preceq X_i \preceq I$. Each positive trace fiber is a nonempty closed subset of K, containing $C(t)$, so a minimizer exists.

If the assertion failed, there would be positive trace vectors $t^{(j)} \to \mathbf{1}$ and global minimizers $X^{(j)}$ with $\|X^{(j)} - C(t^{(j)})\| \geq r$. After taking a subsequence, $X^{(j)} \to X^* \in K$, and $\operatorname{Tr} X_i^* = 1$ for every i. Comparing each minimizer with its scalar feasible tuple gives

$$d(X_1^{(j)}, \ldots, X_n^{(j)}) \leq d(C_1(t^{(j)}), \ldots, C_n(t^{(j)})) = \frac{\prod_i t_i^{(j)}}{n^n} \longrightarrow n^{-n}.$$

Consequently $d(X^*) \leq n^{-n}$. Gurvits's theorem [1, Theorem 4.1] gives the reverse inequality and forces $X^* = (I/n, \ldots, I/n)$. Thus $X^{(j)} - C(t^{(j)}) \to 0$, a contradiction. □

Completion of the proof of Theorem 1. Choose $\varepsilon_n < 1$ small enough that both the local estimate (3) and Lemma 3 apply whenever $\|t - \mathbf{1}\|_1 < \varepsilon_n$ and $\sum_i t_i = n$. The bound $\varepsilon_n < 1$ ensures $t_i > 0$. Every global minimizer can then be written $C(t) + A$ with $\|A\| < r$. Its value is at most $f(t, 0)$, while (3) forces $A = 0$. Existence of a minimizer now proves that $C(t)$ is the unique global minimizer. Multilinearity gives $d(C(t)) = \prod_i t_i / n^n$. □

3 The Hessian at a general scalar tuple

The preceding theorem only needs the Hessian at $\mathbf{1}$. For completeness, its positive-definiteness region can be computed exactly. This is a local statement and does not determine the global minimizer region.

Proposition 4. *Let $n \geq 2$, $t_i > 0$, and $\sum_i t_i = n$. The Hessian $D_A^2 f(t, 0)$ is positive definite on $\mathcal{T}$ if and only if*

$$\sum_i t_i^2 < \frac{n^2}{n-1}, \quad \textit{equivalently} \quad \|t - \mathbf{1}\|_2^2 < \frac{n}{n-1}. \tag{4}$$

At equality it is positive semidefinite with a nonzero null direction. If the inequality is reversed strictly, the scalar tuple admits a feasible direction of local descent.

Proof. Put $B_i = A_i/t_i$. The trace conditions give $\operatorname{Tr} B_i = 0$, and the sum condition becomes $\sum_i t_i B_i = 0$. Using (2) in the multilinear expansion yields

$$D_A^2 f(t,0)[A,A] = \frac{\prod_i t_i}{n^{n-1}(n-1)} \left(\sum_i \|B_i\|_{\mathrm{HS}}^2 - \left\| \sum_i B_i \right\|_{\mathrm{HS}}^2 \right), \tag{5}$$

where $\|H\|_{\mathrm{HS}}^2 = \operatorname{Tr}(H^2)$ for Hermitian H.

Expand the B_i in any real orthonormal basis of traceless Hermitian matrices. For each basis coordinate the coefficient vector $b \in \mathbb{R}^n$ lies in $t^\perp$, and the quadratic form in parentheses becomes

$$\|b\|_2^2 - (\mathbf{1}^\top b)^2.$$

The squared norm of the summation functional on $t^\perp$ is

$$\| \operatorname{Proj}_{t^\perp} \mathbf{1} \|_2^2 = n - \frac{(\sum_i t_i)^2}{\sum_i t_i^2} = n - \frac{n^2}{\sum_i t_i^2}.$$

Thus the form is positive definite precisely when this quantity is less than one, which is (4). Equality produces a null direction, and a larger value produces a negative direction. These directions are realized in $\mathcal{T}$ by taking $B_i = b_i H$ for a fixed nonzero traceless Hermitian matrix H. Since each $C_i(t)$ is positive definite, sufficiently small perturbations along such directions are feasible. Stationarity and the negative second derivative prove the local descent assertion. □

Inside (4), stationarity and the positive Hessian give strict local minimality. Proposition 4 does not extend the global conclusion of Theorem 1 to all of that region. At equality the Hessian alone does not decide local minimality.

For $n = 2$ there is also a direct global calculation. Write $X_1 = t_1 I/2 + A$, $X_2 = t_2 I/2 - A$, where $\operatorname{Tr} A = 0$ and $t_1 + t_2 = 2$. Then

$$d(X_1, X_2) = \frac{t_1 t_2}{4} + \frac{\operatorname{Tr}(A^2)}{2}.$$

Hence the scalar tuple is the unique minimizer for every feasible trace vector $0 \le t_1 \le 2$, including the endpoints.

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References

[1] L. Gurvits, *Van der Waerden conjecture for mixed discriminants*, Advances in Mathematics **200** (2006), no. 2, 435–454. doi:10.1016/j.aim.2004.12.002. Preprint arXiv:math/0406420v1, June 21, 2004. Theorem, lemma, and corollary numbers cited here refer to this preprint.

[2] L. Gurvits and J. Leake, *Unique minimizers for permanents, mixed discriminants, and log-concave polynomials*, arXiv:2609.21083v2, October 15, 2026.

[3] L. Gurvits, N. Klein, and J. Leake, *From trees to polynomials and back again: New capacity bounds with applications to TSP*, arXiv:2311.09072v2, May 9, 2024.