

Three-point positivity for weighted Riemann theta products

Scott Schubert

Abstract

We give an explicit curvature argument for three-point positivity of weighted products of the Riemann theta density. If Φ is the usual positive even density, then for every $\alpha \geq 5/8$ and every nonnegative measure ν with $0 < \int \Phi(s)^{2\alpha} d\nu(s) < \infty$, the function $\int \Phi(s+t/2)^\alpha \Phi(s-t/2)^\alpha d\nu(s)$ has positive definite three-point translation matrices at distinct real positions. Repeated positions give positive semidefinite matrices. The proof combines a general curvature criterion with positive-Laplace estimates for the complete theta-series tail. All infinite-tail bounds and rational interval estimates are provided explicitly. The result concerns three-point matrices; it does not establish positive definiteness of arbitrary order.

1 Statement and context

Use the normalization

$$\Phi(s) = \sum_{n \geq 1} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}, \quad s \in \mathbb{R}. \quad (1)$$

The series defines the classical positive even Riemann theta density; we verify the properties needed here below. A real even function H has three-point positivity if $[H(t_i - t_j)]_{i,j=1}^3$ is positive semidefinite for all real t_1, t_2, t_3 .

Theorem 1. *Let $\alpha \geq 5/8$, and let ν be a nonnegative Borel measure on $\mathbb{R}$ such that $0 < Z = \int_{\mathbb{R}} \Phi(s)^{2\alpha} d\nu(s) < \infty$. Then*

$$H_{\alpha,\nu}(t) = \int_{\mathbb{R}} \Phi(s+t/2)^\alpha \Phi(s-t/2)^\alpha d\nu(s) \quad (2)$$

is finite and even, and its three-point translation matrix is positive definite when the three positions are distinct. For arbitrary positions it is positive semidefinite. The same assertions hold for $H_{\alpha,\nu}/Z$.

In particular, for $y > 0$ the kernel

$$K_y(t) = 2 \int_0^\infty s \sinh(2ys) \Phi(s+t/2) \Phi(s-t/2) ds \quad (3)$$

has this property. At $y = 0$ the kernel K_0 is identically zero; the meaningful nonzero limit is

$$Q_1(t) = \lim_{y \downarrow 0} \frac{K_y(t)}{2y} = 2 \int_0^\infty s^2 \Phi(s+t/2) \Phi(s-t/2) ds. \quad (4)$$

It too is strictly positive on distinct three-point configurations. Arbitrary nonnegative polynomial weights are also covered. The fractional exponent in Theorem 1 is applied to the density *before* integration. We make no assertion about fractional entrywise powers of an already averaged K_y .

Weighted associated kernels and their relationship to Laguerre inequalities are classical; see Csordas [1, Sections 3–4]. Positivity for matrices of all orders is a substantially stronger question than the theorem here. Umihara [3] proves the qualitative inequality $(\log \Phi)'''' < 0$ in this same normalization, by separating three initial theta terms and controlling the remaining tail. We do not claim priority for that sign. Our proof below uses a different positive-Laplace representation and also obtains the quantitative curvature estimate needed for the fractional density range. The cosine characterization of a 3×3 correlation matrix is classical elementary Gram geometry; see also Tomás, Alsina and Rubio-Martinez [4]. Its derivation is included. The general entrywise-power theorem of FitzGerald and Horn [2] gives the critical exponent 1 for arbitrary nonnegative 3×3 positive semidefinite matrices. Our exponent $5/8$ pertains to the specific theta-product family, not to that general class. No optimal exponent is claimed.

2 A curvature criterion

Theorem 2. *Let $\varphi > 0$ be C^4 on $\mathbb{R}$, and suppose $k = -(\log \varphi)''$ satisfies*

$$k > 0, \quad 0 \leq k'' \leq 2k^2 \quad \text{on } \mathbb{R}. \quad (5)$$

For each real s put $p_s(t) = \varphi(s+t/2)\varphi(s-t/2)/\varphi(s)^2$. Every p_s has strictly positive definite three-point translation matrices at distinct positions. Any probability mixture of these functions has the same property. Neither evenness of φ nor a reflection assumption is required in this criterion.

Proof. Fix s and set $q(t) = -\log p_s(t)$ for $t \geq 0$. The exchange of the two arguments makes q even, with $q(0) = q'(0) = q'''(0) = 0$. Write $k_{\pm} = k(s \pm t/2)$. Differentiation gives

$$q'' = (k_+ + k_-)/4 > 0, \quad q''' = [k'(s+t/2) - k'(s-t/2)]/8 \geq 0, \quad (6)$$

$$q'''' = (k''_+ + k''_-)/16, \quad 2(q'')^2 - q'''' \geq k_+k_-/4 > 0. \quad (7)$$

The third-derivative sign uses convexity of k across the entire ordered interval, including negative arguments. In particular $q, q' > 0$ for $t > 0$. Define

$$F = 2q'q'' - q''', \quad J = (q')^2 - q''(1 - e^{-2q}).$$

Both vanish at zero, and

$$F' = 2(q'')^2 + 2q'q''' - q'''' > 0, \quad J' = (1 - e^{-2q})F > 0.$$

Consequently $\theta(t) = \arccos(e^{-q(t)})$ is increasing and strictly concave:

$$\theta' = \frac{q'}{\sqrt{e^{2q} - 1}} > 0, \quad \theta'' = -\frac{e^{2q}J}{(e^{2q} - 1)^{3/2}} < 0 \quad (t > 0).$$

At zero, $q(t) = k(s)t^2/4 + O(t^4)$, so $\theta(0) = 0$ and $\theta'(0+) = \sqrt{k(s)}/2$. Thus concavity extends to the closed half-line.

We now use the classical three-positive correlation geometry [4]. Sort and translate three distinct positions, with successive gaps $u, v > 0$. Strict concavity with zero origin implies strict subadditivity: apply concavity at u and v as fractions of $u + v$ and add. With $a = \theta(u), c = \theta(v), b = \theta(u + v)$, monotonicity therefore gives

$$|a - c| < b < a + c, \quad 0 < a, b, c < \pi/2, \quad a + c < \pi.$$

Let $A = p_s(u)$, $C = p_s(v)$, $B = p_s(u + v)$. Taking cosines on $[0, \pi]$ yields

$$AC - \sqrt{(1 - A^2)(1 - C^2)} < B < AC + \sqrt{(1 - A^2)(1 - C^2)}.$$

The matrix with successive rows $(1, A, B)$, $(A, 1, C)$, $(B, C, 1)$ has determinant

$$(1 - A^2)(1 - C^2) - (B - AC)^2 > 0,$$

and all its one- and two-point principal minors are positive. It is therefore positive definite, also for complex coefficient vectors. Repeated positions follow by continuity. Finally a common probability measure averages the same quadratic forms in every entry. Each nonzero form is strictly positive at every s when the positions are distinct, so its integral is strictly positive. $\square$

3 The complete theta tail

Put $\ell = \log \Phi$ and $\kappa = -\ell''$. We will prove

$$\ell'' \leq -42, \quad \ell'''' < 0, \quad \ell'''' + 2(\ell'')^2 \geq 3456, \quad (8)$$

and then strengthen the curvature ratio.

Let $\Theta(X) = 1 + 2 \sum_{n \geq 1} e^{-\pi n^2 X}$. Its Poisson identity implies $f(s) = e^s \Theta(e^{4s}) = f(-s)$. Termwise differentiation shows $\Phi = (f'' - f)/16$. The convergence of all derivatives on compact intervals follows from the exponential theta tail. Thus Φ is smooth and even. For $s \geq 0$ every summand of (1) is positive, and reflection proves positivity on $\mathbb{R}$.

For $s \geq 0$ write

$$x = \pi e^{4s} > 3, \quad a = \frac{3}{2}, \quad \tau_n = n^2 - 1, \quad D = 4x \frac{d}{dx}.$$

The first theta term and the exact remainder are

$$\Phi_1 = 2\pi e^{5s}(x - a)e^{-x}, \quad \Phi = \Phi_1(1 + R), \quad R = \sum_{n \geq 2} \left[n^4 + \frac{an^2\tau_n}{x - a} \right] e^{-\tau_n x}. \quad (9)$$

In particular R is a Laplace transform of a positive measure:

$$R = \int_3^\infty e^{-vx} d\sigma(v), \quad d\sigma = \sum_{n \geq 2} \left[n^4 \delta_{\tau_n} + an^2 \tau_n e^{a(v - \tau_n)} \mathbf{1}_{v \geq \tau_n} dv \right]. \quad (10)$$

Although σ need not have finite total mass, all exponentially weighted polynomial moments used below converge. Expanding the integrated moments gives explicit polynomial-in- n times $e^{-3(n^2 - 1)}$ majorants. This also justifies every differentiation and interchange of sums and integrals below.

Since $a/(x - a) \leq 1$, the n th coefficient in (9) is at most $2n^4$. The first term of the resulting majorant is $32e^{-9}$ and the ratio of successive terms is at most $(81/16)e^{-15}$. The positive Taylor polynomial of e^3 through degree five exceeds 16. Hence

$$0 < R < \frac{32/16^3}{1 - 81/16^6} = \frac{131072}{16777135} < \frac{1}{64}. \quad (11)$$

3.1 A positive logarithmic fourth derivative of the remainder

Set $A_j = \int (vx)^j e^{-vx} d\sigma(v)$, with $A_0 = R$, and let $R_j = D^j R$. Because $vx \geq 9$,

$$\begin{aligned} R_1 &= -4A_1, & R_2 &= 16B_2, & R_3 &= -64B_3, & R_4 &= 256B_4, \\ B_2 &= A_2 - A_1, & B_3 &= A_3 - 3A_2 + A_1, & B_4 &= A_4 - 6A_3 + 7A_2 - A_1. \end{aligned}$$

We have $B_2 \geq 8A_2/9$, $0 < B_3 \leq A_3$, $B_2 \leq A_2$, and $B_4 \geq c_0 A_4$, where $c_0 = 305/729$. Indeed $P_4(z) = z^4 - 6z^3 + 7z^2 - z$ satisfies $P_4(9)/9^4 = c_0$, and $(P_4(z)/z^4)' = (6z^2 - 14z + 3)/z^4 > 0$ for $z \geq 9$. Positive-measure moment inequalities give

$$A_1^2 \leq RA_2, \quad A_2^2 \leq RA_4, \quad A_1 A_3 \leq RA_4. \quad (12)$$

These follow from Cauchy–Schwarz or log-convexity of moments.

Write $U = 1 + R$ and $h = \log U$. The exact chain rule gives

$$h'''' = \frac{R_4}{U} - \frac{4R_1 R_3 + 3R_2^2}{U^2} + \frac{12R_1^2 R_2}{U^3} - \frac{6R_1^4}{U^4}. \quad (13)$$

The final two terms equal $1536A_1^2(2UB_2 - A_1^2)/U^4 \geq 0$, since $2UB_2 - A_1^2 \geq (16 + 7R)A_2/9$. By (12), $4R_1 R_3 + 3R_2^2 \leq 256 \cdot 7RA_4 \leq (7R/c_0)R_4$. Using $R/U < 1/65$ in (13) proves

$$h'''' \geq \left(1 - \frac{7R}{c_0 U}\right) \frac{R_4}{U} > \frac{14722}{19825} \frac{R_4}{U} > 0. \quad (14)$$

3.2 Uniform first and second derivative envelopes

For $k = 1, 2, 4$, the absolute physical derivative of e^{-vx} is bounded by $4^k (vx)^k e^{-vx}$ on $vx \geq 9$. The integrated contribution of the n th term is therefore bounded by

$$M_{n,k}(x) = 4^k x^k e^{-\tau_n x} \left[n^4 \tau_n^k + a n^2 \tau_n \sum_{j=0}^k \binom{k}{j} \frac{\tau_n^{k-j} j!}{(x-a)^{j+1}} \right]. \quad (15)$$

Each positive summand has logarithmic derivative at most $k/x - \tau_n < 0$. For $k = 2$, multiplication by $x + 2$ still leaves a negative derivative, bounded above by $1/(x+2) + 2/x - \tau_n \leq -2$. At $x = 3$ direct simplification gives

$$\begin{aligned} M_{n,1}(3) &= 4n^2 \tau_n (6n^2 - 1) e^{-3\tau_n} \leq 24n^6 e^{-3\tau_n}, \\ M_{n,2}(3) &= 16n^2 \tau_n (18n^4 - 15n^2 + 5) e^{-3\tau_n} \leq 288n^8 e^{-3\tau_n}. \end{aligned}$$

For $n = 2$ the exact coefficients are respectively 1104 and 44736. For $n \geq 3$, successive majorant terms have ratio less than $(4/3)^{10} 2^{-21} < 1/100$, and $e^{-24} < 2^{-24}$. The Taylor polynomial for e^3 through degree eight exceeds 20, so $e^9 > 8000$. Thus

$$|R_1| \leq \frac{1104}{8000} + \frac{100 \cdot 24 \cdot 3^6}{99 \cdot 2^{24}} < \frac{1}{4}, \quad (16)$$

$$0 < R_2 \leq U_2 := \frac{44736}{8000} + \frac{100 \cdot 288 \cdot 3^8}{99 \cdot 2^{24}} = \frac{1028315733}{180224000} < 6, \quad (17)$$

$$(x+2)R_2 \leq 5U_2 < 30. \quad (18)$$

4 The global curvature band

Put $\ell_1 = \log \Phi_1$. Direct physical differentiation gives

$$\ell_1'' = -16A(x), \quad A(x) = x + \frac{6x}{(2x-3)^2} \leq x + 2. \quad (19)$$

Also

$$\frac{\ell_1'''' + 2(\ell_1'')^2}{256} = \frac{xP_2(x)}{(2x-3)^4},$$

where $P_2 = 32x^5 - 208x^4 + 624x^3 - 960x^2 + 522x - 135$. The identity

$$\begin{aligned} (3+u)P_2(3+u) - 21(3+2u)^4 \\ = 2349u + 3762u^2 + 3072u^3 + 1488u^4 + 368u^5 + 32u^6 \end{aligned}$$

proves $\ell_1'''' + 2(\ell_1'')^2 \geq 5376$ for $x \geq 3$. Since $h'' = R_2/U - R_1^2/U^2 \leq R_2$, the negativity of ℓ_1'' and (14) imply

$$\begin{aligned} \ell_1'''' + 2(\ell_1'')^2 &= \ell_1'''' + 2(\ell_1'')^2 + h'''' + 4\ell_1''h'' + 2(h'')^2 \\ &\geq 5376 - 64AR_2 \geq 5376 - 1920 = 3456. \end{aligned}$$

There is no assumption that h'' is positive. Furthermore, $\ell'' \leq -16x + R_2 \leq -42$.

4.1 Strict negativity of the fourth derivative

The polynomial P_4 above obeys

$$(P_4 - P_4')(9+v) = 1162 + 921v + 241v^2 + 26v^3 + v^4 > 0.$$

Hence $256P_4(vx)e^{-vx}$ is positive and decreasing in $x \geq 3$. Integration against the fixed measure σ proves that $R_4(x)$ decreases. Its exact $n = 2$ contribution is $e^{-3x}C(x)$, with

$$C(x) = 256 \left[16P_4(3x) + \frac{18}{x-a} \sum_{j=0}^4 P_4^{(j)}(3x) \left(\frac{x}{x-a} \right)^j \right]. \quad (20)$$

Here the derivatives are ordinary derivatives. The factorials from the Taylor expansion cancel those from the integrated exponential moments. The four exact endpoint values are

$$\begin{aligned} C(3) &= 43140096, & C(16/5) &= \frac{42636828818522112}{887410625}, \\ C(7/2) &= 59421432, & C(4) &= \frac{278436163584}{3125}. \end{aligned} \quad (21)$$

For all $n \geq 3$, (15) gives

$$M_{n,4}(3) \leq C_4 n^{12} e^{-3\tau_n}, \quad C_4 = 12^4 \left[1 + \sum_{j=0}^4 \binom{4}{j} j! (2/27)^j \right] = \frac{107597312}{2187} < 62208.$$

The ratio of the majorant terms is less than $(4/3)^{12}/20^7 < 1/100$. Therefore the entire $n \geq 3$ fourth tail is less than

$$\frac{100 \cdot 62208 \cdot 3^{12}}{99 \cdot 20^8} = \frac{14348907}{11000000} < 2. \quad (22)$$

Dropping the negative terms of (13) and using (16)–(17) gives $h'''' \leq R_4 + 9/2$.

The leading fourth derivative is $\ell_1'''' = -256B(x)$, where

$$B(x) = x + T(a/x), \quad T(r) = \frac{r(1 + 4r + r^2)}{(1 - r)^4}.$$

The derivative of T is positive, with numerator $1 + 11r + 11r^2 + r^3$. On $[L, H]$ we may use $B(x) \geq L + T(a/H)$; on $x \geq 4$ use $B(x) \geq 4$. The following rational bounds complete the sign proof. The middle column is a lower bound for e^{3L} ; the next is a strict upper bound for $C(L)e^{-3L}$.

Interval	exponential bound	$n = 2$ fourth term	lower bound for $256B$
$[3, 16/5]$	8000	5393	5430
$[16/5, 7/2]$	14400	3337	3801
$[7/2, 4]$	34000	1748	2557
$[4, \infty)$	160000	557	1024

The finite exponential certificates, beyond $e^3 > 20$, are $e^{3/5} > 1 + 3/5 + 9/50 + 27/750 > 9/5$ and $e^{3/2} > \sum_{j=0}^4 (3/2)^j/j! > 17/4$. They imply the four displayed exponential bounds. All other table entries follow by direct substitution in (21) and T . In every row the last entry exceeds the preceding entry plus $2 + 9/2$. Together with (22), this proves $\ell'''' < 0$ on the half-line. Evenness extends all estimates to $\mathbb{R}$, proving (8). In particular $\kappa \geq 42$, $\kappa'' > 0$, and $\kappa'' \leq 2\kappa^2$. Since $\ell'(0) = 0$, twice integrating the second-derivative bound gives

$$\Phi(s) \leq \Phi(0)e^{-21s^2} \quad (s \in \mathbb{R}). \quad (23)$$

5 A stronger ratio and fractional densities

Proposition 3. *For every real s , the theta density satisfies*

$$0 < \frac{\kappa''(s)}{2\kappa(s)^2} < \frac{5}{8}. \quad (24)$$

Proof. It suffices to consider $s \geq 0$. The inequalities $h'''' > 0$ and $h'' < 6$ give $\kappa'' < 256B$ and $\kappa \geq 16A - 6$. First $A \geq 24/5$ for $x \geq 3$. Multiplication by $5(2x - 3)^2$ reduces this to $P(x) = 20x^3 - 156x^2 + 363x - 216 \geq 0$. On $x = 3 + u/2$, $0 \leq u \leq 1$,

$$P(3 + u/2) = 9(1 - u)^3 + \frac{21}{2}u(1 - u)^2 + u^3 > 0.$$

On $x = 7/2 + v$, $v \geq 0$, it equals $20v^3 + 54v^2 + 6v + 1$. Thus $\kappa \geq 354/5$.

The classical Archimedean lower bound $\pi > 223/71 > 157/50$ allows a compact split. If $157/50 \leq x \leq 4$, monotonicity of T gives

$$\frac{\kappa''}{2\kappa^2} < \frac{128(4 + T(75/157))}{(354/5)^2} = \frac{54596377700}{88528266369} < \frac{5}{8}.$$

The difference from $5/8$ is $5870310245/708226130952 > 0$. For $x \geq 4$, we have $T(3/(2x)) \leq T(3/8) = 4056/625 < 13/2$ and $\kappa \geq 16x - 6$. The ratio $128(x + 13/2)/(16x - 6)^2$ decreases there: the derivative has the sign of $-16x - 214$. Its value at 4 is $336/841 < 5/8$. This proves (24). $\square$

Proof of Theorem 1. For $\varphi = \Phi^\alpha$ its logarithmic curvature is $k = \alpha\kappa$. If $\alpha \geq 5/8$, Proposition 3 gives $k > 0$ and $0 < k'' < 2k^2$. Apply Theorem 2. The mixture measure is $d\mu(s) = Z^{-1}\Phi(s)^{2\alpha} d\nu(s)$, which is a probability measure. Log-concavity of Φ implies $0 < p_s(t) \leq 1$, so all integrals in the theorem are finite. The normalized weighted kernel is precisely the common mixture of $p_s(t)^\alpha$, proving all assertions. $\square$

For (3) and its fractional-density counterparts, Gaussian domination from (23) makes Z finite for every fixed $y > 0$ and $\alpha \geq 5/8$. For (4), the bound $\sinh(2ys)/(2y) \leq s \cosh(2y_0s)$ for $0 < y \leq y_0$ supplies integrable domination, uniformly for t in compact sets. The normalization at $y = 0$ is therefore exactly the limit displayed in (4); it does not use the vanishing value $K_0(0)$.

6 Limitations and explicit controls

The power threshold above is sufficient, not optimal. The distinction from a necessary local threshold can be seen directly. At a fixed s ,

$$p_s(t)^\alpha = 1 - at^2 + bt^4 + O(t^6), \quad a = \frac{\alpha\kappa(s)}{4}, \quad b = \frac{\alpha^2\kappa(s)^2}{32} - \frac{\alpha\kappa''(s)}{192}.$$

The determinant on $0, t, 2t$ has leading term $8a(6b - a^2)t^6$. Thus a necessary local condition is $\alpha \geq \kappa''(s)/(4\kappa(s)^2)$. It is not the sufficient ratio condition used in Theorem 2, and no equality of the two thresholds is asserted.

Even the complete original curvature band does not imply four-point positivity. Indeed $\varphi(u) = \exp(-u^2 - u^4/3)$ has $k = 2 + 4u^2$ and $k'' = 8 \leq 2k^2$. At $s = 0$ its normalized product is $p(t) = \exp(-t^2/2 - t^4/24)$. The odd-reversal block for the four equally spaced positions $0, t, 2t, 3t$ has determinant

$$(1 - p(3t))(1 - p(t)) - (p(t) - p(2t))^2 = -t^8 + \frac{35}{8}t^{10} + O(t^{12}).$$

It is negative for sufficiently small nonzero t . This is a generic pointwise obstruction to extending the curvature method. It is not a counterexample for the actual theta density or its averaged kernels, and it does not settle their higher-order positivity.

The proofs consist of analytic domination and moment inequalities plus finite polynomial and rational identities. Those identities can be verified directly from the formulas displayed here; numerical sampling is not a premise. No claim about Fourier positivity or the Riemann hypothesis follows from these finite-order conclusions.

References

- [1] G. Csordas, *Fourier transforms of positive definite kernels and the Riemann ξ -function*, Comput. Methods Funct. Theory **15** (2015), 373–391, arXiv:1309.0055.
- [2] C. H. FitzGerald and R. A. Horn, *On fractional Hadamard powers of positive definite matrices*, J. Math. Anal. Appl. **61** (1977), 633–642.
- [3] H. Umihara, *A Fourth-Derivative Inequality for the Riemann Xi Kernel*, preprint, 12 August 2026, doi:10.5281/zenodo.21902312.
- [4] M. S. Tomás, C. Alsina and J. Rubio-Martinez, *Pseudometrics from three-positive semidefinite similarities*, Fuzzy Sets and Systems **157** (2006), 2347–2355.