

Spectral escape and mean crossover for concentrating signed potential forms

Scott Schubr

Luminary Oddities, La Grange Highlands, IL, United States of America

scottschubr@luminaryoddities.com

ORCID: 0009-0009-7357-4439

October 25, 2026

Abstract

We study the spectrum of concentrating signed potential forms relative to derivative energy on a bounded interval. Classical weak-binding analysis identifies the role of profile primitives; here the interaction of finitely many fixed separated centers is retained through the full Dirichlet Green matrix. For nonzero even zero-mean profiles with inverse-square tails, we prove that exactly one positive and one negative eigenvalue per center diverge, with Green-matrix leading coefficients and mode-dependent logarithmic corrections. The entire remaining spectrum is uniformly bounded. Perturbing the profile means at the square-root scale gives a finite Hermitian crossover model, a two-term expansion for every escaping eigenvalue, and a quantitative operator-norm approximation. After energy normalization and a choice of phase, escaping modes converge uniformly to trace functions while their derivative energy develops explicit atomic concentrations; in the unperturbed case the diffuse and atomic parts each carry one half of the energy. A two-center example shows that ignoring off-diagonal Green interactions gives incorrect leading coefficients, and unequal tails produce different logarithmic drifts. Applications to localized log-curvature forms include an entire function with a strictly positive inverse Fourier density.

2020 Mathematics Subject Classification: 47A55, 34L20, 47B50.

Keywords: signed potential forms, spectral concentration, Dirichlet Green matrix, eigenvalue asymptotics, Hardy inequality.

1 Introduction and principal results

1.1 The question and a concrete energy quotient

Let f be a nonvanishing real C^2 function on $[a, b]$. Its log-curvature is

$$V_f = -(\log |f|)'' = \frac{f'^2 - f f''}{f^2}.$$

We study this signed curvature through the variational quotient

$$R_f(u) = \frac{\frac{1}{2} \int_a^b V_f |u|^2 dx}{\int_a^b |u'|^2 dx}, \quad 0 \neq u \in H_0^1(a, b). \quad (1)$$

The denominator measures the derivative energy of a test function; the quotient measures curvature relative to that energy. In particular, for polynomial tests $u = f(x)(x - a)(b - x)P(x)$ it is exactly

the quotient of the localized log-curvature and derivative-energy forms developed in Section 3. The associated energy-Riesz operator describes the extremal values of this explicitly specified quotient.

A conjugate pair approaching the real axis supplies a basic singular regime. Fix $r \in (a, b)$. For $f_\varepsilon(x) = (x - r)^2 + \varepsilon^2$, direct differentiation gives

$$V_{f_\varepsilon}(x) = \varepsilon^{-2}v((x - r)/\varepsilon), \quad v(t) = \frac{2(t^2 - 1)}{(1 + t^2)^2}, \quad \int_{\mathbb{R}} v = 0.$$

The negative core and positive wings cancel in total mass. Nevertheless, the extremal energy quotients diverge with both signs as $\varepsilon \downarrow 0$. For several such pairs, how many eigenvalues escape, how do their coefficients depend on the locations, and how does a small failure of zero mean change the answer? We answer these questions for a class of even zero-mean profiles with inverse-square tails, including the profile above.

The variational object is the compact selfadjoint operator C_ε defined by

$$e(C_\varepsilon u, z) = q_\varepsilon(u, z), \quad e(u, z) = \int_a^b u' \overline{z'} dx, \quad u, z \in H := H_0^1(a, b). \quad (2)$$

Here q_ε is the concentrating potential form specified below. The main result is a two-term description of every escaping eigenvalue of the full energy operator, together with a uniform bound for its remaining spectrum. The full Dirichlet Green matrix encodes interaction between centers: each center contributes one escaping eigenvalue of each sign, and the collective modes determine their leading coefficients and logarithmic drifts. Mean changes of order $\sqrt{\varepsilon}$ in the profiles alter this leading finite-dimensional model and have their own explicit two-term law.

1.2 Two centers already interact

On $(-1, 1)$ take $r_1 = -1/2$, $r_2 = 1/2$ and $f_\varepsilon(x) = \prod_{j=1}^2 ((x - r_j)^2 + \varepsilon^2)$. Both profiles are the zero-mean v above. The two-center Dirichlet Green matrix is

$$G = \begin{pmatrix} 3/8 & 1/8 \\ 1/8 & 3/8 \end{pmatrix},$$

with symmetric and antisymmetric eigenvalues $1/2$ and $1/4$. Theorem 1.1 therefore gives

$$\lambda_{\text{even}}^\pm(\varepsilon) = \pm \frac{\sqrt{\pi}}{2\sqrt{\varepsilon}} - \log(1/\varepsilon) + O(1), \quad \lambda_{\text{odd}}^\pm(\varepsilon) = \pm \frac{\sqrt{\pi}}{2\sqrt{2\varepsilon}} - \log(1/\varepsilon) + O(1).$$

Reflection preserves the form, so these are its even and odd branches. Replacing G by its diagonal would predict equal coefficients for the two modes. The off-diagonal interaction is therefore visible in the leading asymptotics. Section 3.2 realizes the same split in an entire family with a strictly positive inverse Fourier density; Section 3.1 exhibits different logarithmic drifts as well.

1.3 Hypotheses and principal laws

Fix $a < r_1 < \dots < r_k < b$, with $1 \leq k < \infty$. For each j let v_j be real, measurable and even, nonzero on a set of positive measure, and suppose that for constants $C_j < \infty$, $\eta_j > 0$, $\alpha_j \in \mathbb{R}$,

$$\int_{\mathbb{R}} v_j = 0, \quad |v_j(t)| \leq \frac{C_j}{1 + t^2}, \quad v_j(t) = \frac{\alpha_j}{t^2} + O(|t|^{-2-\eta_j}) \quad (|t| \rightarrow \infty). \quad (3)$$

Put

$$\begin{aligned} L_j(t) &= \int_{-\infty}^t v_j(s) ds, \quad d_j = \int_{\mathbb{R}} |L_j|^2 > 0, \quad D = \text{diag}(d_1, \dots, d_k), \\ G_{ij} &= \frac{(\min(r_i, r_j) - a)(b - \max(r_i, r_j))}{b - a}. \end{aligned} \quad (4)$$

Let W_ε be real measurable with a common $L^\infty(a, b)$ bound, and set

$$q_\varepsilon(u, z) = \frac{1}{2} \int_a^b \left\{ \sum_{j=1}^k \varepsilon^{-2} v_j \left(\frac{x - r_j}{\varepsilon} \right) + W_\varepsilon(x) \right\} u \bar{z} dx, \quad 0 < \varepsilon < \frac{1}{2}. \quad (5)$$

Every C_ε is compact: for fixed ε the potential is bounded, and the embedding $H_0^1(a, b) \hookrightarrow L^2(a, b)$ is compact. Compactness is asserted separately for each ε .

Theorem 1.1 (Full Green-matrix escape law). *The eigenvalues of $N = D^{1/2} G D^{1/2}$ are simple. Write them as $\rho_1 > \dots > \rho_k > 0$, and choose real unit eigenvectors $\xi_1, \dots, \xi_k$. There is $M < \infty$ such that, for all sufficiently small ε , C_ε has exactly k eigenvalues above M and exactly k below $-M$. Each of these $2k$ eigenvalues is simple, and every other eigenvalue lies in $[-M, M]$. The escaping branches, the negative ones ordered by decreasing absolute value, satisfy*

$$\lambda_j^\pm(\varepsilon) = \pm \frac{\sqrt{\rho_j}}{2\sqrt{\varepsilon}} + o(\varepsilon^{-1/2}), \quad 1 \leq j \leq k. \quad (6)$$

For arbitrary critical-tail coefficients $\alpha_1, \dots, \alpha_k$, the sharper law is

$$\lambda_j^\pm(\varepsilon) = \pm \frac{\sqrt{\rho_j}}{2\sqrt{\varepsilon}} - \frac{1}{2} \left(\sum_{l=1}^k \alpha_l |\xi_{j,l}|^2 \right) \log(1/\varepsilon) + O(1). \quad (7)$$

In particular, equal coefficients $\alpha_l = \alpha$ give the common logarithmic correction $-\alpha \log(1/\varepsilon)/2$. The bounds need not be uniform as centers approach one another or as k increases. No convergence of W_ε is needed.

The off-diagonal entries of G are part of the leading answer. Discarding them generally gives incorrect coefficients, as the exact parity example in Section 3.2 shows.

Theorem 1.2 (Critical mean crossover). *In (5), replace v_j by $v_j + \sqrt{\varepsilon} w_j$, where each w_j is fixed, real, bounded, even and compactly supported. Put $\kappa_j = \int_{\mathbb{R}} w_j$, $\mathcal{K} = \text{diag}(\kappa_j)$, $\mathcal{A} = \text{diag}(\alpha_j)$, and*

$$\begin{aligned} S &= \frac{1}{2} (G^{1/2} D G^{1/2})^{1/2}, \quad A_{\det} = \frac{1}{2} G^{1/2} \mathcal{K} G^{1/2}, \quad \mathcal{L} = \begin{pmatrix} 0 & S \\ S & A_{\det} \end{pmatrix}, \\ \mathcal{C} &= -\frac{1}{2} \left(G^{-1/2} \mathcal{A} G^{1/2} + G^{1/2} \mathcal{A} G^{-1/2} \right). \end{aligned} \quad (8)$$

There are isometries $J_\varepsilon : \mathbb{C}^{2k} \rightarrow H$ such that

$$\|\sqrt{\varepsilon} C_\varepsilon - J_\varepsilon \mathcal{L} J_\varepsilon^*\|_{\text{op}} = O(\sqrt{\varepsilon} \log(1/\varepsilon)). \quad (9)$$

The matrix $\mathcal{L}$ has k positive and k negative eigenvalues, all simple. If μ_a is one of them and $(u_a, z_a) \in \mathbb{C}^k \oplus \mathbb{C}^k$ is a corresponding unit eigenvector, then its escaping branch satisfies

$$\Lambda_a(\varepsilon) = \frac{\mu_a}{\sqrt{\varepsilon}} + (z_a^* \mathcal{C} z_a) \log(1/\varepsilon) + O(1). \quad (10)$$

All $2k$ escaping branches are simple for sufficiently small ε ; every remaining unscaled eigenvalue stays uniformly bounded. The embeddings J_ε vary with ε .

The same reduction also describes the escaping modes themselves. Proposition 2.7 gives explicit primitive-layer profiles and a quantitative limit for their derivative-energy measures. For a zero-mean escaping pair, one half of the normalized energy remains in a Green-tent mode and one half concentrates at the centers, although the eigenfunctions converge uniformly to the trace component.

1.4 Mechanism and organization

The key decomposition separates test functions vanishing at every center from their finite-dimensional trace complement. Hardy’s inequality bounds the entire potential form on the trace-zero subspace. The complement is spanned by Dirichlet Green tents, so the geometry of the interval enters through their full Gram matrix G .

Zero mass makes each profile primitive decay at infinity, but its squared integral remains positive. The coupling between the trace space and trace-zero space consequently has squared size of order ε^{-1} . Its singular values produce paired escape at order $\varepsilon^{-1/2}$. The inverse-square tails supply the logarithmic trace term; a mean perturbation of size $\sqrt{\varepsilon}$ contributes at the same scale as the leading coupling. A separate codimension argument bounds every residual eigenvalue, completing the spectral statement.

Section 2 proves these estimates and the two principal laws. Section 3 returns to the original log-curvature quotient and develops its examples. Appendix A supplies the supporting real-root comparison: once the zeros lie on the real axis, the polynomial completion imposes value traces and Hardy channels describe the essential spectrum. Its fusion theorem concerns moving real centers, whereas the principal concentration theorems keep the centers fixed.

1.5 Relation to earlier work

Simon [7, Theorems 2.4–2.5] identifies primitive-energy effects for one-dimensional weak binding. Blankenbecler, Goldberger and Simon [1, Sections 3–4] use a Dirichlet cut and Hardy bounds, and examine inverse-square-tail logarithmic corrections. These results concern Schrödinger eigenvalues; the cited critical-tail logarithm has nonzero negative mean. They are substantive antecedents to our proof. Golovaty and Hryniv [5] study spectral escape for singular ε^{-2} potentials despite norm-resolvent limits. The general phenomenon of escaping spectrum under concentration is established in the cited literature. Our stated results use the original energy-Riesz normalization and establish its complete Green geometry, paired counts, bounded bulk, critical crossover and two-term unequal-tail laws.

The parameter comparison makes the distinction precise. The energy eigenvalue equation is $-u'' = V_\varepsilon u / (2\lambda)$. For a single center with $W_\varepsilon = 0$, $x = r + \varepsilon t$ gives exactly $-y'' = v(t)y / (2\lambda)$ on a Dirichlet interval whose endpoints recede at order ε^{-1} . The escaping branches have reciprocal coupling of order $\sqrt{\varepsilon}$; the expanding boundary and interactions enter the leading threshold calculation together. Full-line weak-binding asymptotics therefore supply relevant primitive energies, while the present proof retains the finite-interval Green geometry and controls the complete complementary space. With several centers, the rescaled equation retains every translated profile. Kielty [6] studies singular limits of indefinite weighted eigenproblems for a fixed monotone form family $b - tc$, with $c \geq 0$. Here the shrinking signed profiles and their vanishing means lead to paired escape and mode-dependent logarithmic coefficients.

The real-zero comparison below uses classical Hardy–Mellin analysis [4, 3] and Kato–Rosenblum theory [8]. The author’s research preprint *Weighted polynomial localization and theta correction inertia* (Luminary Research Archive, edition 2, 2026; <https://research.luminaryoddities.com/weighted-polynomial-localization-theta-correction-inertia/v/2>) studies finite-degree

localization and correction inertia for a full-line theta form. Its successor *Critical-tail localization and growing theta correction inertia* (same Archive, edition 2, 2026; <https://research.luminaryoddities.com/beyond-logarithmic-powers-critical-tail-theta-correction-inertia/v/2>) improves that finite-degree rate. Our log-curvature applications instead identify the bounded-interval quotient with the energy operator through the exact form identity in Section 3.

2 The trace decomposition and the escape law

All estimates below have constants depending on the fixed data in (3), the interval, center locations and common bound for W_ε . We write $\ell_\varepsilon = \log(1/\varepsilon)$.

Lemma 2.1 (Primitives). *Each L_j is odd and absolutely continuous, with*

$$L_j(t) = O(\min(|t|, |t|^{-1})), \quad L_j(t) = -\frac{\alpha_j}{t} + O(|t|^{-1-\eta_j}) \quad (|t| \rightarrow \infty).$$

Moreover $d_j > 0$, $\sup |tL_j(t)| < \infty$, and

$$J_j(t) := \int_0^t L_j(s) ds = -\alpha_j \log(1 + |t|) + O(1).$$

Proof. Evenness and zero total mass give $L_j(0) = 0$ and oddness. Boundedness of v_j gives the estimate at zero. On the positive half-line use $L_j(t) = -\int_t^\infty v_j$, and integrate the stated tail; oddness gives the negative half-line. Integrating this tail after subtracting $-\alpha_j/t$ proves the J_j formula, because $\eta_j > 0$. Finally, $d_j = 0$ would give $L_j = 0$ almost everywhere and hence $v_j = L'_j = 0$ almost everywhere, contrary to the hypothesis. $\square$

Let $G_r \in H$ be the Riesz representer of evaluation at r : $e(u, G_r) = u(r)$. It is the Dirichlet Green tent, and $e(G_{r_i}, G_{r_j}) = G_{ij}$ with G from (4). Define

$$H_* := \{u \in H : u(r_j) = 0 \ (1 \leq j \leq k)\}, \quad T := H_*^\perp.$$

Point evaluations at distinct points are independent, as disjoint compactly supported smooth test functions show. Thus $\dim T = k$, G is positive definite, and T is the span of the Green tents. Its elements are continuous functions linear between successive marked points $a, r_1, \dots, r_k, b$.

Take a real energy-orthonormal basis $e_1, \dots, e_k$ of T , and let $A_{jl} = e_l(r_j)$. The evaluation identity gives

$$AA^\top = G. \tag{11}$$

The canonical choice obtained from the Green tents has $A = G^{1/2}$. In $H = H_* \oplus T$, write

$$C_\varepsilon = \begin{pmatrix} B_\varepsilon & T_\varepsilon \\ T_\varepsilon^* & A_\varepsilon \end{pmatrix}. \tag{12}$$

Lemma 2.2 (Uniform reserve). *There is a fixed M_0 with $\|B_\varepsilon\| \leq M_0$.*

Proof. The profile envelope gives $|\varepsilon^{-2}v_j((x - r_j)/\varepsilon)| \leq C_j/(x - r_j)^2$ away from r_j . For $u \in H_*$ the two one-sided Hardy inequalities give

$$\int_a^b \frac{|u|^2}{(x - r_j)^2} dx \leq 4e(u, u).$$

The bounded W_ε term is controlled by Dirichlet Poincaré. This bounds $|q_\varepsilon(u, u)|$ on H_* ; polarization gives the operator bound for its selfadjoint compression. All centers are tested on this same subspace and the complete form is retained. $\square$

Lemma 2.3 (The complete coupling Gram). *In any basis satisfying (11),*

$$\|A_\varepsilon\| = O(\ell_\varepsilon), \quad \varepsilon T_\varepsilon^* T_\varepsilon \longrightarrow \frac{1}{4} A^\top D A \quad (13)$$

in finite-dimensional operator norm. More precisely,

$$T_\varepsilon^* T_\varepsilon = \frac{A^\top D A}{4\varepsilon} + O(\ell_\varepsilon^2). \quad (14)$$

In the canonical basis the trace block has expansion

$$A_\varepsilon = -\frac{1}{2} \left(G^{-1/2} \text{diag}(\alpha_j) G^{1/2} + G^{1/2} \text{diag}(\alpha_j) G^{-1/2} \right) \ell_\varepsilon + O(1). \quad (15)$$

Proof. The l th column $b_{l,\varepsilon} \in H_*$ of T_ε is the Riesz representer of $z \mapsto q_\varepsilon(e_l, z)$ on H_* . On each marked subinterval it is the solution, with Dirichlet values at both ends, of

$$-b_{l,\varepsilon}'' = \frac{1}{2} \left\{ \sum_j \varepsilon^{-2} v_j((x - r_j)/\varepsilon) + W_\varepsilon \right\} e_l.$$

For a center at an endpoint of that subinterval, write $e_l(x) = A_{jl} + c_l(x - r_j)$. A derivative primitive of its contribution is

$$-\frac{A_{jl}}{2\varepsilon} L_j(t) - \frac{c_l}{2} t L_j(t) + \frac{c_l}{2} J_j(t), \quad t = (x - r_j)/\varepsilon. \quad (16)$$

Direct differentiation gives the required negative forcing. Subtract the subinterval mean of the sum of these primitives to enforce its two Dirichlet values. Terms from centers outside the closed subinterval and from W_ε have uniformly bounded forcing and bounded primitives: their fixed positive separation is used here. Consequently, throughout I ,

$$b_{l,\varepsilon}' = -\sum_j \frac{A_{jl}}{2\varepsilon} L_j((x - r_j)/\varepsilon) + R_{l,\varepsilon}, \quad \|R_{l,\varepsilon}\|_2 = O(\ell_\varepsilon). \quad (17)$$

Using the common expression for the leading sum in nonadjacent cells changes the remainder only by a bounded function.

The diagonal primitive-square integrals obey

$$\int_a^b \varepsilon^{-2} L_j((x - r_j)/\varepsilon)^2 dx = \frac{d_j}{\varepsilon} + O(1). \quad (18)$$

Indeed $L_j(t) = O(|t|^{-1})$ gives omitted squared L^2 mass $\int_{\mathbb{R} \setminus ((a-r_j)/\varepsilon, (b-r_j)/\varepsilon)} |L_j(t)|^2 dt = O(\varepsilon)$ after changing variables. Equivalently, the omitted-tail L^2 norm is $O(\sqrt{\varepsilon})$; multiplication of the squared-mass error by ε^{-1} gives the stated $O(1)$ remainder. For different centers the unscaled product of the two singular primitives has integral $O(\ell_\varepsilon)$: in a fixed small neighborhood of one center the other factor is bounded, the singular factor has L^1 norm $O(\ell_\varepsilon)$, and outside both neighborhoods both are bounded. These facts and (17) first give the second limit in (13); the squared remainder costs $O(\varepsilon \ell_\varepsilon^2)$ after scaling and its leading cross term costs $O(\sqrt{\varepsilon} \ell_\varepsilon)$.

For the sharper estimate retain (16). Subtract the constant logarithmic part of each J_j before subinterval centering; centering removes that subtraction exactly. By Lemma 2.1, the resulting uncentered remainder is bounded by

$$C \left(1 + \sum_j |\log(|x - r_j| + \varepsilon)| \right). \quad (19)$$

Its L^2 norm and means are uniformly bounded. Its cross integral with a singular primitive is $O(\ell_\varepsilon^2)$: for $|x - r_j| \leq \varepsilon$ use $|L_j(t)| \leq C|t|$, and outside that range use $|\varepsilon^{-1}L_j(t)| \leq C/|x - r_j|$ and integrate $(1 + |\log|x - r_j||)/|x - r_j|$. Centering the leading primitives changes their full squared and mixed Grams by $O(\ell_\varepsilon^2)$ because each mean is $O(\ell_\varepsilon)$. Together with (18) and the different-center estimate this proves (14), including all mixed entries.

For the trace block, integrate the primitive $\varepsilon^{-1}L_j((x - r_j)/\varepsilon)$ against $(e_l e_m)'$. Outer boundary terms vanish, and inner terms cancel because $e_l e_m$ is continuous. The bounded derivative on each side gives $O(\ell_\varepsilon)$. To compute its coefficient, let H_{jl} be the jump of e_l' at r_j . The canonical Green basis has $H = -G^{-1/2}$, since each Green tent has jump -1 at its node. The derivative jump of $e_l e_m$ is $A_{jl}H_{jm} + H_{jl}A_{jm}$. The two tail integrals therefore give the logarithmic coefficient

$$\frac{1}{2} \left(A^\top \text{diag}(\alpha_j) H + H^\top \text{diag}(\alpha_j) A \right).$$

Every remainder is integrable after the $-\alpha_j/t$ primitive tail is subtracted; $\eta_j > 0$ gives an $O(1)$ error. Substituting the canonical A, H proves (15). $\square$

Lemma 2.4 (Simple finite-matrix perturbations). *Let F and E be fixed selfadjoint matrices, and suppose that every eigenvalue θ_a of F is simple. If v_a is a corresponding unit eigenvector, the eigenvalue of $F + tE$ near θ_a is simple for sufficiently small real t and equals*

$$\theta_a + t v_a^* E v_a + O(t^2).$$

Proof. Weyl's finite-matrix eigenvalue bound isolates one eigenvalue in each of the disjoint intervals about the θ_a when $|t| \|E\|$ is smaller than a fixed fraction of the minimum spectral gap. Let P be projection onto v_a , $Q = I - P$, and write its unit eigenvector as $x = c v_a + y$, $y \in \text{ran } Q$. The restriction $Q(F + tE - \theta(t))Q$ has a uniformly bounded inverse on $\text{ran } Q$. The Q equation therefore gives

$$y = -tc [Q(F + tE - \theta(t))Q]^{-1} Q E v_a = O(t)c.$$

In particular $c \neq 0$. The P equation, divided by c , is $\theta(t) = \theta_a + t v_a^* E v_a + t v_a^* E y/c$, which proves the claimed remainder. A one-dimensional matrix has the exact formula and needs no complementary inverse. $\square$

Lemma 2.5 (Nodal stiffness and simplicity). *The eigenvalues of $D^{1/2} G D^{1/2}$ are simple. For every real diagonal K , the matrix $\mathcal{L}$ in (8) also has simple spectrum and no zero eigenvalue.*

Proof. Put $r_0 = a$, $r_{k+1} = b$. A piecewise-linear trace function with nodal values x_j and endpoint values $x_0 = x_{k+1} = 0$ has energy

$$\sum_{j=0}^k \frac{|x_{j+1} - x_j|^2}{r_{j+1} - r_j}.$$

Consequently the nodal stiffness matrix $K = G^{-1}$ is tridiagonal, with

$$K_{jj} = \frac{1}{r_j - r_{j-1}} + \frac{1}{r_{j+1} - r_j}, \quad K_{j,j+1} = K_{j+1,j} = -\frac{1}{r_{j+1} - r_j}.$$

Indeed the minimum-energy interpolant is the trace-tent combination; its energy in nodal coordinates is $x^* G^{-1} x$. Hence $(D^{1/2} G D^{1/2})^{-1} = D^{-1/2} K D^{-1/2}$ is a real symmetric tridiagonal matrix whose adjacent off-diagonal entries are all nonzero. Its eigenvector recurrence determines every coordinate from the first; a zero first coordinate forces all coordinates to vanish. Each eigenspace has dimension at most one, and selfadjointness gives simple eigenvalues. This also proves the assertion for $k = 1$.

If $\mathcal{L}(u, z) = 0$, invertibility of S gives $z = u = 0$. For a nonzero eigenvalue μ eliminate $u = Sz/\mu$ and put $y = G^{1/2}z$. The remaining equation is

$$\left(\mu^2 G^{-1} - \frac{\mu}{2} \mathcal{K} - \frac{1}{4} D\right) y = 0. \quad (20)$$

The adjacent entries of this tridiagonal matrix are $-\mu^2/(r_{j+1} - r_j)$, which are nonzero. The same recurrence shows that its kernel has dimension at most one. The correspondence with (u, z) is injective, so the selfadjoint $\mathcal{L}$ has simple spectrum. $\square$

Proof of Theorem 1.1. The reference block $\begin{pmatrix} 0 & T_\varepsilon \\ T_\varepsilon^* & 0 \end{pmatrix}$ has eigenvalues equal to plus and minus the k singular values of T_ε , and zero otherwise. By (13) these singular values, multiplied by $\sqrt{\varepsilon}$, tend to $\sqrt{\rho_j}/2$. Here we used (11) and the equality of the nonzero spectra of $A^\top D A$ and $D^{1/2} G D^{1/2}$. The difference from (12) has norm $O(\ell_\varepsilon)$. Minmax/Weyl comparison proves (6).

The compression bound of Lemma 2.2 separately implies that at most k eigenvalues lie above M_0 and at most k below $-M_0$: an allegedly larger spectral subspace would meet H_* nontrivially and violate the Rayleigh bound there. The k divergences of each sign prove exact counts, and every other eigenvalue stays bounded. The $O(\ell_\varepsilon)$ comparison alone would not prove this bounded-bulk assertion.

Use the canonical basis and write S and $\mathcal{C}$ as in (8). The sharp coupling Gram gives

$$(T_\varepsilon^* T_\varepsilon)^{1/2} = \frac{S}{\sqrt{\varepsilon}} + O(\sqrt{\varepsilon} \ell_\varepsilon^2) = \frac{S}{\sqrt{\varepsilon}} + o(1). \quad (21)$$

For completeness, if X, Y are positive definite and $E = X - Y$, then $XE + EY = X^2 - Y^2$ and

$$E = \int_0^\infty e^{-tX} (X^2 - Y^2) e^{-tY} dt, \quad \|E\| \leq \frac{\|X^2 - Y^2\|}{\lambda_{\min}(X) + \lambda_{\min}(Y)}.$$

Here the denominator is of order $\varepsilon^{-1/2}$ and the numerator is $O(\ell_\varepsilon^2)$. Thus no differentiability of matrix eigenvectors is required.

The polar isometry of T_ε identifies its range with $\mathbb{C}^k$. The complete operator differs by uniformly bounded norm from the following finite block, extended by zero on its orthogonal complement:

$$F_\varepsilon = \frac{1}{\sqrt{\varepsilon}} \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} + \ell_\varepsilon \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{C} \end{pmatrix}. \quad (22)$$

The comparison pays the entire reserve block, its mixed components in this polar splitting, the $O(1)$ trace remainder, and the $o(1)$ root error in (21).

Let ξ_j be the normalized eigenvectors in the statement and put

$$x_j = \rho_j^{-1/2} G^{1/2} D^{1/2} \xi_j.$$

Then the x_j are unit eigenvectors of S , with eigenvalues $s_j = \sqrt{\rho_j}/2$. Lemma 2.5 makes these s_j distinct. The normalized eigenvectors of the leading block in (22) are $2^{-1/2}(x_j, \pm x_j)$. Moreover, using $N\xi_j = \rho_j \xi_j$ and the commutation of the two diagonal matrices D and $\mathcal{A}$,

$$x_j^* \mathcal{C} x_j = -\frac{1}{2\rho_j} \xi_j^* (\mathcal{A} N + N \mathcal{A}) \xi_j = -\xi_j^* \mathcal{A} \xi_j.$$

Apply Lemma 2.4 to $\sqrt{\varepsilon} F_\varepsilon$ with $t = \sqrt{\varepsilon} \ell_\varepsilon$. Its eigenvalue error, after division by $\sqrt{\varepsilon}$, is $O(\sqrt{\varepsilon} \ell_\varepsilon^2) = o(1)$, and its first-order coefficient is $x_j^* \mathcal{C} x_j/2$. The bounded full-operator comparison proves (7). The distinct nonzero leading coefficients isolate one eigenvalue in each escaping cluster, so these branches are simple for sufficiently small ε . $\square$

Corollary 2.6 (One center). *For $k = 1$, put $\tau = G_{11} = (r - a)(b - r)/(b - a)$. The two escaping eigenvalues are simple for small ε , all others are uniformly bounded, and*

$$\lambda^\pm(\varepsilon) = \pm \frac{\sqrt{\tau} \|L\|_2}{2\sqrt{\varepsilon}} - \frac{\alpha}{2} \log(1/\varepsilon) + O(1).$$

2.1 Proof of the mean crossover

Proof of Theorem 1.2. The added potential is $\varepsilon^{-3/2} w_j((x - r_j)/\varepsilon)$. Since w_j is bounded and compactly supported, it is bounded by $C/(1 + t^2)$. Hardy therefore bounds its compression to H_* by $O(\sqrt{\varepsilon})$. In a trace-to- H_* pairing,

$$|z(r_j + \varepsilon t)| \leq \sqrt{\varepsilon |t|} \|z'\|_2, \quad z \in H_*.$$

After the change of variables the added coupling has norm $O(1)$. Writing T_ε^0 for the zero-mean coupling, we obtain

$$T_\varepsilon = T_\varepsilon^0 + O(1), \quad T_\varepsilon^* T_\varepsilon = \frac{S^2}{\varepsilon} + O(\varepsilon^{-1/2}), \quad (T_\varepsilon^* T_\varepsilon)^{1/2} = \frac{S}{\sqrt{\varepsilon}} + O(1).$$

The Gram estimate uses $\|T_\varepsilon^0\| = O(\varepsilon^{-1/2})$ and $\ell_\varepsilon^2 = O(\varepsilon^{-1/2})$; the root estimate follows from the positive-matrix bound used in (21).

The extra (l, m) trace entry multiplied by $\sqrt{\varepsilon}$ is

$$\frac{1}{2} \sum_j \int_{\mathbb{R}} w_j(t) e_l(r_j + \varepsilon t) e_m(r_j + \varepsilon t) dt = \frac{1}{2} (A^\top \mathcal{K} A)_{lm} + O(\varepsilon).$$

The piecewise-Lipschitz trace functions and compact supports justify this error, including across the centers. In the canonical basis, (15) consequently gives the full trace expansion

$$A_\varepsilon = \frac{A_{\det}}{\sqrt{\varepsilon}} + \mathcal{C} \ell_\varepsilon + O(1).$$

The sharper added-trace error is $O(\sqrt{\varepsilon})$, which is absorbed in the stated bounded remainder.

For small ε , T_ε has rank k . Its polar factor

$$U_\varepsilon = T_\varepsilon (T_\varepsilon^* T_\varepsilon)^{-1/2} : \mathbb{C}^k \rightarrow H_*$$

is an isometry. Let J_ε be U_ε on the first k coordinates and the canonical trace basis on the last k . Their ranges are orthogonal. In this moving splitting, the complete operator differs by uniformly bounded norm from

$$J_\varepsilon \left\{ \frac{\mathcal{L}}{\sqrt{\varepsilon}} + \ell_\varepsilon \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{C} \end{pmatrix} \right\} J_\varepsilon^*. \quad (23)$$

Indeed the off-diagonal root error is $O(1)$, the trace remainder is $O(1)$, and the entire reserve block remains bounded. Multiplying this comparison by $\sqrt{\varepsilon}$ proves the quantitative (9).

The change $u \mapsto u + \frac{1}{2} S^{-1} A_{\det} z$ converts $2\Re(u^* S z) + z^* A_{\det} z$ to the pure cross form, so $\mathcal{L}$ has inertia (k, k) . Its eigenvalues are simple by Lemma 2.5. Apply Lemma 2.4 to $\mathcal{L} + t \operatorname{diag}(0, \mathcal{C})$ with $t = \sqrt{\varepsilon} \ell_\varepsilon$. For a unit eigenvector (u_a, z_a) , its first-order coefficient is $z_a^* \mathcal{C} z_a$. Rescaling and using the bounded comparison in (23) proves (10); the finite-matrix second-order error is $O(\sqrt{\varepsilon} \ell_\varepsilon^2) = o(1)$. The distinct leading eigenvalues isolate simple escaping branches, and the codimension- k bounded compression argument again bounds every remaining unscaled eigenvalue. No commutation of $G, D, \mathcal{K}$ or $\mathcal{C}$ was used. $\square$

2.2 Escaping modes and their derivative energy

Proposition 2.7 (Eigenfunction profiles and concentration of derivative energy). *Write $\|u\|_e = e(u, u)^{1/2}$. Use the hypotheses and notation of Theorem 1.2, and let (u_a, z_a) be a unit eigenvector of $\mathcal{L}$ for μ_a . In the canonical trace basis define*

$$\tau_a = \sum_{l=1}^k z_{a,l} e_l, \quad c_a = \frac{1}{2} D^{1/2} G^{1/2} S^{-1} u_a, \quad \delta_\varepsilon = \sqrt{\varepsilon} \log(1/\varepsilon).$$

Put $r_0 = a$, $r_{k+1} = b$ and $I_i = (r_i, r_{i+1})$. Let $\chi_{a,\varepsilon} \in H_*$ be the function vanishing at all marked points whose derivative on I_i is

$$\chi'_{a,\varepsilon}(x) = h_{a,\varepsilon}(x) - \frac{1}{|I_i|} \int_{I_i} h_{a,\varepsilon}(y) dy, \quad h_{a,\varepsilon}(x) = -\frac{1}{\sqrt{\varepsilon}} \sum_{l=1}^k \frac{c_{a,l}}{\sqrt{d_l}} L_l((x - r_l)/\varepsilon).$$

For sufficiently small ε , an energy-unit eigenfunction $\Psi_{a,\varepsilon}$ for $\Lambda_a(\varepsilon)$ can be given a phase such that

$$\|\Psi_{a,\varepsilon} - \tau_a - \chi_{a,\varepsilon}\|_e = O(\delta_\varepsilon), \quad \|\Psi_{a,\varepsilon} - \tau_a\|_\infty = O(\delta_\varepsilon). \quad (24)$$

For every real Lipschitz function g on $[a, b]$,

$$\left| \int_a^b g |\Psi'_{a,\varepsilon}|^2 dx - \int_a^b g |\tau'_a|^2 dx - \sum_{l=1}^k g(r_l) |c_{a,l}|^2 \right| \leq C(\|g\|_\infty + \text{Lip}(g)) \delta_\varepsilon. \quad (25)$$

Here $\|\tau_a\|_e^2 = \|z_a\|^2$ and $\sum_l |c_{a,l}|^2 = \|u_a\|^2$.

In the unperturbed case $w_l = 0$ for every l , put

$$t_j = \rho_j^{-1/2} \sum_{l=1}^k \sqrt{d_l} \xi_{j,l} G_{r_l}, \quad \|t_j\|_e = 1.$$

Both signs of the j th escaping pair have uniform limit $t_j/\sqrt{2}$ with the rate in (24); their derivative-energy measures tend to

$$\frac{1}{2} |t'_j|^2 dx + \frac{1}{2} \sum_{l=1}^k |\xi_{j,l}|^2 \delta_{r_l},$$

with the bounded-Lipschitz rate in (25). Thus one half of the normalized derivative energy remains in the trace function and one half concentrates at the centers. For the same phase choices,

$$\left\| \Psi_{j,\varepsilon}^\pm - t_j/\sqrt{2} \right\|_e^2 = \frac{1}{2} + O(\delta_\varepsilon).$$

Proof. The eigenvalue μ_a is separated from the other eigenvalues of $\mathcal{L}$ and from zero by a fixed positive gap. The norm comparison (9), applied to the orthogonal complement of $J_\varepsilon(u_a, z_a)$, therefore gives

$$\|\Psi_{a,\varepsilon} - U_\varepsilon u_a - \tau_a\|_e = O(\delta_\varepsilon)$$

after choosing the phase. Explicitly, the reference operator is $J_\varepsilon \mathcal{L} J_\varepsilon^*$; its complementary resolvent is bounded by the inverse fixed gap. The actual scaled eigenvalue differs from μ_a by $O(\delta_\varepsilon)$. Projecting its eigenvalue equation onto that complement bounds the complementary component by $O(\delta_\varepsilon)$; unit normalization then bounds the difference of the phase-aligned vectors by the same order.

The mean-crossover proof gives $(T_\varepsilon^* T_\varepsilon)^{1/2} = S/\sqrt{\varepsilon} + O(1)$. Inverting yields $(T_\varepsilon^* T_\varepsilon)^{-1/2} = \sqrt{\varepsilon} S^{-1} + O(\varepsilon)$, and hence $U_\varepsilon u_a = \sqrt{\varepsilon} T_\varepsilon^0 S^{-1} u_a + O_H(\sqrt{\varepsilon})$, where T_ε^0 is the zero-mean coupling. Apply (17). Its leading derivative is exactly $h_{a,\varepsilon}$, with an L^2 remainder $O(\delta_\varepsilon)$. Subtracting the mean on each I_i is an L^2 orthogonal projection; the derivative of $T_\varepsilon^0 S^{-1} u_a$ already has zero cell means. Thus $\|U_\varepsilon u_a - \chi_{a,\varepsilon}\|_e = O(\delta_\varepsilon)$. Since $\int_a^b |h_{a,\varepsilon}| dx = O(\delta_\varepsilon)$, centering gives $\|\chi_{a,\varepsilon}\|_\infty = O(\delta_\varepsilon)$. The first assertion and the one-dimensional energy-to-uniform bound prove (24).

For the energy measures, centering changes $h_{a,\varepsilon}$ in L^2 by $O(\delta_\varepsilon)$, while its L^2 norm is bounded. Consequently it changes the squared-derivative measure in total variation by $O(\delta_\varepsilon)$. The diagonal layer terms obey

$$\frac{1}{\varepsilon d_l} \int_a^b g(x) |L_l((x - r_l)/\varepsilon)|^2 dx = g(r_l) + O((\|g\|_\infty + \text{Lip}(g))\varepsilon \log(1/\varepsilon)).$$

Indeed the omitted tail mass is $O(\varepsilon)$, and the truncated first moment of $|L_l|^2$ is $O(\log(1/\varepsilon))$. Distinct-center layer products have L^1 norm $O(\varepsilon \log(1/\varepsilon))$, by the separated neighborhood estimate in Lemma 2.3. The derivative of τ_a is bounded on each fixed cell, so its cross term with $\chi'_{a,\varepsilon}$ costs $O(\|g\|_\infty \delta_\varepsilon)$. Finally, the energy-norm error in (24) changes the squared-derivative measure in total variation by $O(\delta_\varepsilon)$. These estimates prove (25).

The coefficient map is orthogonal, because

$$\left(\frac{1}{2} D^{1/2} G^{1/2} S^{-1}\right)^* \left(\frac{1}{2} D^{1/2} G^{1/2} S^{-1}\right) = \frac{1}{4} S^{-1} G^{1/2} D G^{1/2} S^{-1} = I.$$

For the unperturbed problem the unit finite-block eigenvectors may be chosen as $(sx_j, x_j)/\sqrt{2}$, $s \in \{+1, -1\}$, with x_j from the main proof. Then $\tau_a = t_j/\sqrt{2}$ and $c_a = s\xi_j/\sqrt{2}$, proving the half-energy statement. Finally, $U_\varepsilon u_a$ is energy-orthogonal to the trace space and has squared energy $\|u_a\|^2 = 1/2$. The phase-aligned energy approximation therefore gives the last identity. $\square$

3 Original log-curvature forms and exact geometry

For a real C^2 function f on a bounded interval, define the localized original forms on vectors $u = f(x)(x - a)(b - x)P(x)$ by

$$E_f(P) = \frac{1}{4\pi} \int_a^b |u'|^2 dx, \quad Q_f(P) = \frac{1}{8\pi} \int_a^b V_f |u|^2 dx, \quad V_f = \frac{f'^2 - f f''}{f^2}.$$

At real zeros the product $V_f |u|^2$ has its removable integrable interpretation. Their quotient is exactly $q_f/e = Q_f/E_f$. The endpoint factors impose the Dirichlet conditions of this localized receiver.

Suppose for $\varepsilon > 0$ that

$$f_\varepsilon(x) = h_\varepsilon(x) \prod_{j=1}^k ((x - r_j)^2 + \varepsilon^2)^{\nu_j}, \quad \|-(\log |h_\varepsilon|)''\|_\infty \leq C,$$

where h_ε is real nonzero and C^2 . Direct logarithmic differentiation gives the exact potential in (5) with

$$v_j(t) = \frac{2\nu_j(t^2 - 1)}{(1 + t^2)^2}, \quad L_j(t) = -\frac{2\nu_j t}{1 + t^2}, \quad d_j = 2\pi\nu_j^2, \quad \alpha_j = 2\nu_j. \quad (26)$$

For every fixed $\varepsilon > 0$ the multiplier f_ε is nonzero, and the same polynomial-approximation argument as Proposition A.1 gives the full completion H . Consequently Theorem 1.1 governs this original quotient. The leading squared-coefficient matrix is

$$\frac{\pi}{2} \text{diag}(\nu_j) G \text{diag}(\nu_j).$$

For one pair it gives

$$\lambda^\pm(\varepsilon) = \pm \nu \sqrt{\frac{\pi G(r, r)}{2\varepsilon}} - \nu \log(1/\varepsilon) + O(1). \quad (27)$$

For simple pairs, all critical tails equal 2, giving the common $-\log(1/\varepsilon)$ drift for every branch. With a nonvanishing C^2 limit of h_ε , the real-root limit has multiplicities $2\nu_j$ and the bounded Hardy essential band from Theorem A.2. The asymptotic statements themselves require only the uniform log-curvature bound.

3.1 Unequal multiplicities and logarithmic weights

On $(-1, 1)$ with centers $-1/2, 1/2$, take multiplicities $\nu_1 = 1, \nu_2 = 2$ in (26). Then

$$\text{diag}(\nu_j)G\text{diag}(\nu_j) = \frac{1}{8} \begin{pmatrix} 3 & 2 \\ 2 & 12 \end{pmatrix}, \quad \beta_\sigma = \frac{15 + \sigma\sqrt{97}}{16}, \quad \sigma \in \{+1, -1\}.$$

For a normalized eigenvector v_σ , its squared second coordinate is $(1 + \sigma 9/\sqrt{97})/2$. Hence

$$v_\sigma^* \text{diag}(2, 4)v_\sigma = 3 + \sigma 9/\sqrt{97}.$$

Theorem 1.1 gives, for $s \in \{+1, -1\}$,

$$\lambda_\sigma^s(\varepsilon) = s \sqrt{\frac{\pi(15 + \sigma\sqrt{97})}{32\varepsilon}} - \frac{3 + \sigma 9/\sqrt{97}}{2} \log(1/\varepsilon) + O(1).$$

The two signs within a leading pair have the same logarithmic drift; the two geometric modes have different drifts.

3.2 A positive-inverse two-center family

On $I = (-1, 1)$ take centers $r_1 = -1/2, r_2 = 1/2$. Their exact Green matrix and symmetric/antisymmetric eigenvalues are

$$G = \begin{pmatrix} 3/8 & 1/8 \\ 1/8 & 3/8 \end{pmatrix}, \quad \lambda_{\text{even}}(G) = 1/2, \quad \lambda_{\text{odd}}(G) = 1/4. \quad (28)$$

Consider the even entire function

$$f_\varepsilon(z) = e^{-z^2} \{\cos(2\pi z) + \cosh(2\pi\varepsilon)\}. \quad (29)$$

With the Fourier convention $f(z) = \int_{\mathbb{R}} \phi(s) e^{izs} ds$, put $B(s) = e^{-s^2/4}/(2\sqrt{\pi})$. Its inverse density is

$$\phi_\varepsilon(s) = \frac{B(s - 2\pi) + B(s + 2\pi)}{2} + \cosh(2\pi\varepsilon)B(s) > 0.$$

It is even and has Gaussian tails. The pairs inside the interval are $\pm 1/2 \pm i\varepsilon$. After their quadratic factors are removed, the remaining real factor is uniformly nonzero on the closed interval and has uniformly bounded logarithmic derivatives through order two for small ε . Locally this follows from

$$\frac{\cosh s - \cos t}{s^2 + t^2} = \frac{1}{2} \text{sinhc}((s + it)/2) \text{sinhc}((s - it)/2), \quad \text{sinhc}(z) = \frac{\sinh z}{z},$$

with its value 1 at zero. The remaining limiting zeros lie at a fixed positive distance from the interval, so the compact remainder is uniformly controlled.

Reflection preserves the complete form. The symmetric and antisymmetric trace vectors in (28) and their coupling vectors belong to the even and odd spaces, respectively. Applying the proof of Theorem 1.1 on each invariant parity gives exactly one escaping branch of each sign in each parity:

$$\lambda_{\text{even}}^{\pm}(\varepsilon) = \pm \frac{\sqrt{\pi}}{2\sqrt{\varepsilon}} - \log(1/\varepsilon) + O(1), \quad (30)$$

$$\lambda_{\text{odd}}^{\pm}(\varepsilon) = \pm \frac{\sqrt{\pi}}{2\sqrt{2\varepsilon}} - \log(1/\varepsilon) + O(1). \quad (31)$$

Replacing G by its diagonal would give two equal eigenvalues $3/8$, contradicting the actual coefficients. The clean exact control in `verify_green.py` checks this finite algebra, the Green inverse/jump identity and the polar model's characteristic polynomial.

3.3 Fixed degree and completion have different limits

At $\varepsilon = 0$,

$$f_0(x) = 2e^{-x^2} \cos^2(\pi x), \quad V_0(x) = 2 + 2\pi^2 \sec^2(\pi x) > 0$$

away from the two real zeros. Thus every nonzero localized finite-degree original Q form is strictly positive. The energy Gram is also positive definite. For each fixed degree and parity, the E and Q entries are analytic in ε^2 : write Q without division using $(f_{\varepsilon}'^2 - f_{\varepsilon}f_{\varepsilon}'')$ and the endpoint factors, and integrate on the compact interval. The same holds for E by differentiating the complete product. Finite-dimensional continuity therefore preserves their strict positivity for sufficiently small ε .

For every fixed small $\varepsilon > 0$, however, (30)–(31) give a negative completed-space direction in each parity. Polynomial density and continuity of q_{ε} in energy yield a finite-degree negative polynomial direction of that parity. It follows that the least detecting degree tends to infinity in each parity as $\varepsilon \downarrow 0$.

This order-two entire family exhibits the exact geometry-dependent spectral transition and the demonstrated difference between finite-degree and completed limits.

4 Further questions

The concentration results keep the number and positive separations of the centers fixed. Joint coalescence, a growing number of centers and uniform estimates for such regimes require additional arguments. The mode-dependent logarithmic formula suggests a corresponding analysis when geometric gaps and profile widths shrink together. The positive-inverse example also poses a quantitative question: how does the least polynomial degree detecting a negative direction grow as the complex pairs approach the real axis?

A Real roots: Hardy channels and fusion

This section describes the limiting trace-domain geometry. The main spectral input is classical; its role is to make the distinction between real fusion and complex-pair approach precise.

Suppose on the closed interval that

$$f(x) = h(x) \prod_{j=1}^k (x - r_j)^{\nu_j}, \quad \nu_j \in \{1, 2, \dots\}, \quad (32)$$

where h is real C^2 and nowhere zero. Then

$$V_f := \frac{f'^2 - f f''}{f^2} = \sum_j \frac{\nu_j}{(x - r_j)^2} + W, \quad W = -(\log |h|)'' \in L^\infty. \quad (33)$$

The form $q_f(u, z) = \frac{1}{2} \int V_f u \bar{z}$ is bounded on $H_r = \{u \in H : u(r_j) = 0 \text{ for all } j\}$. Indeed

$$|q_f(u, u)| \leq \left(2 \sum_j \nu_j + \frac{\|W\|_\infty (b-a)^2}{2\pi^2} \right) e(u, u). \quad (34)$$

Let C_f be its energy-Riesz representative on H_r .

Proposition A.1 (Polynomial completion). *The energy closure of*

$$\{f(x)(x-a)(b-x)P(x) : P \text{ a polynomial}\}$$

is exactly H_r . The norm formed by adding $\int (1 + |V_f|)|u|^2$ to derivative energy is equivalent to energy on H_r . With real data real polynomials suffice for the real space.

Proof. Every displayed function vanishes at the endpoints and roots, so its closure is contained in H_r . Conversely functions smooth and compactly supported away from all roots and endpoints are dense in H_r . To justify cutting off near a root, a cutoff with derivative $O(1/\delta)$ on $\delta < |x - r_j| < 2\delta$ has derivative-error term bounded by a constant times

$$\int_{|x-r_j|<2\delta} \frac{|u(x)|^2}{|x-r_j|^2} dx \longrightarrow 0;$$

Hardy ensures integrability. The other terms tend to zero by absolute continuity of the derivative integral. Apply ordinary Sobolev approximation on the remaining separated intervals. For such a smooth target u , the quotient $u/[f(x)(x-a)(b-x)]$ is C^1 after extending it by zero near every excluded point. Approximate that quotient and its first derivative uniformly by polynomials: approximate the derivative uniformly and integrate, fixing the value at one endpoint. Multiplication by the fixed C^1 factor gives energy convergence. Hardy and Poincaré give the asserted norm equivalence. $\square$

In particular, a multiple zero imposes a value trace in the H^1 completion; additional pointwise derivative jets are not closed H^1 constraints. This does not conflict with stronger smooth-topology domains carrying higher jets.

Theorem A.2 (Real-root spectral classification). *For $k \geq 1$, C_f is unitarily equivalent, up to a trace-class perturbation, to two copies for every j of*

$$\frac{\nu_j}{2} \left(-\frac{d^2}{dz^2} + \frac{1}{4} \right)_D^{-1} \quad \text{on } L^2(0, \infty), \quad (35)$$

together with finitely many zero directions. Consequently

$$\sigma_{\text{ess}}(C_f) = [0, 2 \max_j \nu_j], \quad m_{\text{ac}}(q) = 2\#\{j : 0 < q < 2\nu_j\} \quad \text{for a.e. } q > 0. \quad (36)$$

There is no absolutely continuous spectrum off that interval. Point or singular continuous spectrum is not excluded. For $I - C_f$ the spectral variable is reflected by $q \mapsto 1 - q$.

Proof. Place an additional Dirichlet cut at each midpoint between consecutive roots. Together with the existing root traces and interval endpoints, these cuts give exactly $2k$ one-sided root half-cells. The simultaneous extra-trace-zero subspace has codimension $k - 1$ in H_r . In that subspace the complete derivative and potential forms split by disjoint supports. Restoring its finite-dimensional orthogonal complement adds a finite-rank operator, including its mixed block.

On a root half-cell $0 < t < l$ the potential is $\nu/t^2 + R(t)$ with R bounded. Set $t = le^{-z}$ and $u(t) = \sqrt{t}v(z)$. For compact targets, then by completion,

$$\begin{aligned} \int_0^l |u'|^2 dt &= \int_0^\infty \left(|v'|^2 + \frac{1}{4}|v|^2 \right) dz, \\ \frac{1}{2} \int_0^l V_f |u|^2 dt &= \frac{1}{2} \int_0^\infty \left(\nu + l^2 e^{-2z} R(le^{-z}) \right) |v|^2 dz. \end{aligned}$$

The condition at $t = l$ is Dirichlet at $z = 0$; the completed space is the form domain of $A = -d^2/dz^2 + 1/4$ on the half-line. The further energy-to- L^2 unitary $v \mapsto A^{1/2}v$ gives

$$\frac{\nu}{2} A^{-1} + \frac{1}{2} A^{-1/2} M_w A^{-1/2}, \quad w(z) = l^2 e^{-2z} R(le^{-z}).$$

The second term is trace class. The Dirichlet Green diagonal is $A^{-1}(z, z) = 1 - e^{-z}$, so factoring positive and negative parts through their Hilbert–Schmidt square roots gives

$$\left\| \frac{1}{2} A^{-1/2} M_w A^{-1/2} \right\|_1 \leq \frac{1}{2} \int_0^\infty |w(z)| (1 - e^{-z}) dz \leq \frac{l^2 \|R\|_\infty}{12}.$$

This proves the trace-class comparison, including the finite-rank restoration. The sine transform diagonalizes A with multiplier $\xi^2 + 1/4$. Each reference operator has simple purely absolutely continuous spectrum $[0, 2\nu]$. Weyl’s essential-spectrum invariance and Kato–Rosenblum [8, Theorems 13.2–13.3] give (36) and its multiplicities. The argument is the one-dimensional Hardy–Mellin mechanism, as in [4, 3]. $\square$

The jumps in m_{ac} record the multiplicity histogram: the jump at $2n$ is twice the number of roots of multiplicity n . Also $\int_0^\infty m_{\text{ac}}(q) dq = 4 \sum_j \nu_j$.

Theorem A.3 (Genuine real fusion). *Let distinct $r_j(\varepsilon)$ tend to one interior point r_0 , with fixed multiplicities ν_j , and let $W_\varepsilon \rightarrow W_0$ uniformly with a common bound. Put $m = \sum_j \nu_j$ and let C_ε act on $H_\varepsilon = \{u : u(r_j(\varepsilon)) = 0\}$ for the corresponding real-root potential. At zero use $V_0 = m/(x - r_0)^2 + W_0$ and $H_0 = \{u : u(r_0) = 0\}$. Extend each operator by zero on the energy orthogonal complement of its domain, writing $\tilde{C}_\varepsilon$. Then $\tilde{C}_\varepsilon \rightarrow \tilde{C}_0$ strongly on H . Nevertheless every unitary $U_\varepsilon : H_\varepsilon \rightarrow H_0$ and compact selfadjoint T_ε on H_0 satisfy*

$$\|U_\varepsilon C_\varepsilon U_\varepsilon^* + T_\varepsilon - C_0\| \geq 2(m - \max_j \nu_j). \quad (37)$$

For $k \geq 2$ this is a positive norm obstruction. For $k = 1$ it is zero.

Proof. Let P_ε be the energy projection onto H_ε . Evaluation representers satisfy $\|G_r - G_s\|_e \leq \sqrt{|r - s|}$, by the elementary evaluation inequality. Smooth functions compactly supported away from r_0 are dense in H_0 and eventually belong to every H_ε ; thus $P_\varepsilon u \rightarrow u$ on H_0 . Also G_{r_0} has distance tending to zero from the span of the $G_{r_j(\varepsilon)}$, so $P_\varepsilon G_{r_0} \rightarrow 0$. Hence $P_\varepsilon \rightarrow P_0$ strongly on H .

For a compact target u away from r_0 , the ordinary functions $V_\varepsilon u$ converge in L^2 to $V_0 u$. Their unrestricted energy-Riesz representers converge, and projecting them with P_ε proves convergence of

the extended operators on such targets. The uniform Hardy bound (34) extends convergence by density on H_0 . On its one-dimensional orthogonal complement use $\|\tilde{C}_\varepsilon u\| \leq C \|P_\varepsilon u\|_e \rightarrow 0$.

For (37), Theorem A.2 gives essential upper edges $2 \max_j \nu_j$ before fusion and $2m$ afterward. Unitary conjugation and compact perturbation preserve essential spectrum. For bounded selfadjoint operators the Hausdorff distance of the essential spectra is at most the operator norm distance, by the corresponding spectral bound in the Calkin algebra. The missing upper interval gives (37). $\square$

For example, $(x^2 - \varepsilon^2)e^{-x^2/2}$ on $(-1, 1)$ converges smoothly to $x^2 e^{-x^2/2}$. The essential bands are respectively $[0, 2]$ and $[0, 4]$, and the norm gap is at least 2 under every identification and compact correction allowed above. Pole-collision sensitivity has substantial prior literature, including higher-dimensional inverse-square settings [2]; the assertion here is the precise one-dimensional energy-completion statement and its norm bound.

References

- [1] R. Blankenbecler, M. L. Goldberger and B. Simon, *The bound states of weakly coupled long-range one-dimensional quantum Hamiltonians*, Ann. Physics **108**(1) (1977), 69–78, [https://doi.org/10.1016/0003-4916\(77\)90351-7](https://doi.org/10.1016/0003-4916(77)90351-7); author-hosted text.
- [2] R. Bosi, J. Dolbeault and M. J. Esteban, *Estimates for the optimal constants in multipolar Hardy inequalities for Schrödinger and Dirac operators*, Commun. Pure Appl. Anal. **7**(3) (2008), 533–562, <https://doi.org/10.3934/cpaa.2008.7.533>.
- [3] O. F. Brevig, K.-M. Perfekt and A. Pushnitski, *The spectrum of some Hardy kernel matrices*, Ann. Inst. Fourier **74**(3) (2024), 1061–1094, <https://doi.org/10.5802/aif.3589>; see the continuous Hardy kernel and its Mellin multiplier in Sections 1.1–1.2.
- [4] F. Gesztesy, L. L. Littlejohn, I. Michael and R. Wellman, *On Birman’s sequence of Hardy–Rellich-type inequalities*, J. Differential Equations **264**(4) (2018), 2761–2801, <https://doi.org/10.1016/j.jde.2017.11.002>; author version [arXiv:1710.06955v3](https://arxiv.org/abs/1710.06955v3), Section 6, Theorem 6.6.
- [5] Y. Golovaty and R. Hryniv, *On negative eigenvalues of 1D Schrödinger operators with δ' -like potentials*, [arXiv:2504.14934v2](https://arxiv.org/abs/2504.14934v2) (2025), Front. Appl. Math. Stat. **11** (2025), article 1615447, <https://doi.org/10.3389/fams.2025.1615447>.
- [6] D. KIELTY, *Singular limits of sign-changing weighted eigenproblems*, Asymptotic Analysis **122** (2021), nos. 1–2, 165–200, <https://doi.org/10.3233/ASY-201615>.
- [7] B. Simon, *The bound state of weakly coupled Schrödinger operators in one and two dimensions*, Ann. Physics **97**(2) (1976), 279–288, [https://doi.org/10.1016/0003-4916\(76\)90038-5](https://doi.org/10.1016/0003-4916(76)90038-5); author-hosted text.
- [8] B. Simon, *Tosio Kato’s work on non-relativistic quantum mechanics, Part 2*, Bull. Math. Sci. **9**(1) (2019), article 1950005, <https://doi.org/10.1142/S166436071950005X>; author-hosted text, Theorems 13.2–13.3.

Statements and Declarations

Funding. This work received no external funding.

Competing interests. The author’s organization, Luminary Oddities, operates the Luminary Research Archive hosting the preprint. The author declares no other relevant competing interests.

AI assistance. ChatGPT (OpenAI) was used to generate and revise manuscript drafts, propose proof developments and strengthenings, compare literature arguments, develop verification code, and assist with internal proof review. The author directed the research and revision decisions and is responsible for the mathematical content and presentation.

Data availability. No empirical datasets were analysed. The finite algebraic verification results can be reproduced using the code in the accompanying source package.

Code availability. The manuscript contains the complete mathematical proofs. Editable LaTeX sources and standard-library finite-verification code are included in the accompanying source package.