

Source-Resolved Resolvent Growth under Reciprocal-Theta Confinement

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Luminary Oddities

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Abstract

We prove two-sided bounds for the complete inverse-energy Gram of prescribed theta-derivative sources in the original polar-constrained archimedean realization of the Weil form. For every finite source size $d + 1$, diagonal normalization by explicit powers of $U = \log(1/\epsilon)$ gives positive degree-dependent lower and upper bounds whenever $U \geq 64(d + 2)^2$. For each fixed family the normalized Gram is bounded above and below by constants, and source i has exact logarithmic exponent $4i + 13/2$. The upper estimate uses exact half-line polar cancellation and a uniformly priced Fourier primitive, retaining the complete exterior polar tail. The lower estimate combines an exact polynomial coefficient map, all-atom theta bounds and a common-weight shell covariance with exact polar shorting. A two-metric residual theorem receives non-square-integrable divided sources, and explicit finite receiving bounds accompany the analytic estimates.

1 The source-resolved inverse problem

Observation estimates for confined Fourier forms control a weighted resolvent uniformly over its source. For specified sources, the actual inverse energy can be substantially smaller than an estimate obtained from the unweighted resolvent norm. In the reciprocal-theta realization, a natural source family consists of derivatives of the complete theta density with one polar moment removed. Their quotients by the physical weight grow polynomially in $\pi e^{2|u|}$. Consequently, a global square-integrable factorization does not give a receiving estimate for this family.

This paper resolves that source-specific problem. Let A be the positive constrained archimedean operator defined in Section 2, and put $C_\epsilon = A + \epsilon I$. For the prescribed sources z_i of (6), write

$$Z_d = (z_0, \dots, z_d), \quad \Gamma_{\epsilon,d} = Z_d^* C_\epsilon^{-1} Z_d, \quad U = \log(1/\epsilon).$$

The source-energy Gram $\Gamma_{\epsilon,d}$ is formed with this associated constrained inverse.

Theorem 1.1 (Uniform source-resolved resolvent geometry). *For every integer $d \geq 0$ put*

$$n = d + 2, \quad L = 2n - 1, \quad m = 4n + 3, \quad b_i = \frac{(-2)^{2i+3}}{2\pi^{1/4}}, \quad \tilde{D}_U = \text{diag}(b_i U^{2i+13/4})_{i=0}^d,$$

and define the explicit constants

$$h_n = \frac{2^{-26}}{(m+1)^2 75^{2m}}, \quad g_n = \frac{2^{-67} h_n^4}{n^2 (m+1)^2},$$

$$\bar{K}_n = 1024(2L + 5)^L(L + 2)!, \quad C_n = 512n\bar{K}_n^2.$$

For all real $U \geq 64n^2$ and $\epsilon = e^{-U}$,

$$g_n I \preceq \tilde{D}_U^{-1} \Gamma_{\epsilon, d} \tilde{D}_U^{-1} \preceq C_n I. \quad (1)$$

Thus d may vary with U on the explicit schedule $d + 2 \leq \sqrt{U/64}$. For each fixed d , writing

$$D_U = \text{diag}(U^{2i+13/4})_{i=0}^d, \quad (2)$$

there are positive constants c_d, C'_d such that $c_d I \preceq D_U^{-1} \Gamma_{\epsilon, d} D_U^{-1} \preceq C'_d I$ on the stated range. In particular, for every fixed $i \geq 0$,

$$\lim_{\epsilon \downarrow 0} \frac{\log \langle z_i, C_\epsilon^{-1} z_i \rangle}{\log \log(1/\epsilon)} = 4i + \frac{13}{2}. \quad (3)$$

The joint inequality retains every source cross term, the exact polar constraint, all theta atoms and the original minimal realization. The upper bound in Section 5 subtracts a polar multiple so that the compact divided source has zero mean on each half-line. A Fourier dual estimate then replaces a worst-case observation factor, while the complete exterior source and polar tails are paid in the original Hilbert metric. The lower bound uses exact polynomial coefficient geometry under one common theta denominator; it does not require a growing family to be uniformly close to its limiting monomials. An outward-shell test returns the covariance to the original domain with exact polar shorting.

The constants obey $\log(g_n^{-1}) = O(d)$ and $\log C_n = O(d \log(d + 2))$. They bound normalized source geometry, without asserting an evaluated inverse or an operation count. The first source has exponent $13/2$. Section 6 gives a literal upper at $\epsilon = 10^{-18}$. The two-metric residual theorem is useful independently of the sharper source-specific upper, and Section 7 records a general complete-profile shell criterion for other finite-family comparisons.

1.1 Relationship with preceding results

The reciprocal-theta Hilbert realization and its genuine derivative-null family occur in the quantitative null-space compression construction [1]. Compression estimates concern the action of the archimedean operator on an entire unknown null space after finite observations. Here the object is instead the inverse-energy Gram of a prescribed finite family.

The observation theorem under double-exponential confinement [2] controls the norm of $M_s C_\epsilon^{-1/2}$ and provides a factored residual estimate for sources $P_X M_s b$ with $b \in L^2$. We use its elementary finite upper and give its proof in Lemma 2.2. Localizing in two metrics makes the non-square-integrable divided derivative profiles accessible. The joint-cutoff theorem [3] controls complete signed approximation ranks and negative-spectrum decay; the present result computes the logarithmic scale of specific positive-reference inverse energies.

Variational residual identities, positive moment Grams and polynomial norm comparisons are classical tools [5, 6]. Their role here is to preserve the original polar constraint while paying the complete theta source. Suzuki's spectral convergence theorem [4] concerns inverse compressions of a differentiation operator in an RH-conditional Weil Hilbert space. The operator, domain and approximation parameter there differ from the positive archimedean regularization and prescribed sources considered here.

2 The constrained reciprocal-theta form

Use $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u) e^{-iu\xi} du$ and Plancherel measure $d\xi/(2\pi)$. Inner products are conjugate-linear in the first variable. For $u \geq 0$ set

$$\psi(u) = e^{u/2} \sum_{m \geq 1} (4r_m^2 - 6r_m) e^{-r_m}, \quad r_m = \pi m^2 e^{2u}, \quad s(u)^2 = \frac{\psi(u)}{\cosh(u/2)}. \quad (4)$$

Theta inversion gives the smooth even continuation of ψ . The positive smooth function s is also even. Put $p(u) = \sinh(u/2)$ and let

$$X = \{h \in L^2(\mathbb{R}) : h \text{ is odd, } \langle sp, h \rangle = 0\}.$$

Let $\mathcal{V}_0$ be the smooth compact odd functions satisfying this moment, and let V be its completion for

$$\|h\|_V^2 = \|h\|_2^2 + \mathfrak{a}[h], \quad \mathfrak{a}[h] = \int_{\mathbb{R}} a(\xi) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi},$$

where, with $\Psi = \Gamma'/\Gamma$,

$$a(\xi) = \Re \Psi(1/4 + i\xi/2) - \Psi(1/4). \quad (5)$$

The closed nonnegative form on X determines its associated selfadjoint operator A . Throughout, $C_\epsilon = A + \epsilon I$ is formed on this same constrained space, and P_X denotes its orthogonal projection. The representation of closed forms and variational inverse identities have their usual meaning [5].

For $i \geq 0$ define

$$\ell_i = 2i + 3, \quad z_i = \frac{\psi^{(\ell_i)} - \psi^{(\ell_i-2)}/4}{s}. \quad (6)$$

These are the standard odd derivative-null sources of the complete weighted Weil realization [1]. Their membership in X can also be seen directly: integrate twice against p , use $p'' = p/4$, and use the complete theta tails to remove all boundary terms. The same tails give $z_i \in L^2$. The inverse-growth proof uses this source formula and the polar moment directly.

Lemma 2.1 (Elementary source and multiplier bounds). *For $r = \pi e^{2|u|}$,*

$$2r^2 e^{-r} \leq s(u)^2 \leq 169 e^{4|u|-r}, \quad 0 \leq a(\xi) \leq 17\xi^2. \quad (7)$$

For $|\xi| \leq 1$, $a(\xi) \geq (16/5)\xi^2$.

Proof. The first atom is positive and $4r^2 - 6r \geq 2r^2$ for $r \geq \pi$, while $e^{|u|/2}/\cosh(u/2) \geq 1$. For the upper bound, $\sum_{m \geq 1} m^4 e^{-(m^2-1)r} < 2$: successive ratios are at most $16e^{-3r} < 1/2$. Since $e^{|u|/2}/\cosh(u/2) \leq 2$, this gives $s^2 < 16r^2 e^{-r} < 169 e^{4|u|-r}$. The digamma difference is

$$a(\xi) = \sum_{n \geq 0} \frac{(\xi/2)^2}{(n+1/4)((n+1/4)^2 + (\xi/2)^2)}.$$

The zeroth term is $16\xi^2/(1+4\xi^2)$. Its upper is $16\xi^2$, and the other terms sum to at most $\xi^2 \sum_{n \geq 1} n^{-3}/4 < \xi^2$. The zeroth term also proves the stated lower bound. $\square$

Lemma 2.2 (A complete observation upper). *For $0 < \epsilon \leq 1$ and $U = \log(1/\epsilon)$,*

$$J_\epsilon := \|M_s C_\epsilon^{-1/2}\| \leq 4 \log(U+4). \quad (8)$$

Proof. We reproduce the finite observation estimate of [2]. Let P_R cut physical space to $[-R, R]$ and Q_b cut Fourier space to $[-b, b]$. Their product has Hilbert–Schmidt norm $\rho = (2Rb/\pi)^{1/2}$. For $f = sh$, the identity $f = P_R Q_b f + (I - P_R)f + P_R(I - Q_b)f$ gives, when $\rho < 1$,

$$(1 - \rho)\|f\| \leq \sigma_R\|h\| + a(b)^{-1/2}\sqrt{\mathfrak{a}[h]}, \quad \sigma_R = \sup_{|u|>R} s(u).$$

The series in Lemma 2.1 makes a increasing in $|\xi|$, justifying the frequency-tail term. Cauchy–Schwarz against $\mathfrak{a}[h] + \epsilon\|h\|^2$ yields

$$\|sh\|^2 \leq \frac{\sigma_R^2/\epsilon + 1/a(b)}{(1 - \rho)^2}(\mathfrak{a}[h] + \epsilon\|h\|^2).$$

Take $R = \log(U + 4)$, $b = \pi/(8R) < 1$, so that $\rho = 1/2$. With $t = U + 4 \geq 4$, the spatial envelope gives $\sigma_R^2/\epsilon \leq 169t^4 e^{-3t^2+t-4} < 1$. Indeed its logarithmic derivative is $4/t - 6t + 1 < 0$, and its value at 4 is $43264e^{-48} < 1$. The frequency lower gives $1/a(b) \leq 20R^2/\pi^2$. Thus

$$J_\epsilon^2 \leq 4(1 + 20R^2/\pi^2) < 16R^2.$$

The argument extends from the compact polar core to V , and spectral calculus identifies the energy quotient with the displayed norm. $\square$

3 A complete two-metric residual receiver

Let $Q : \mathbb{C}^m \rightarrow X$ be a finite source map and let $Y : \mathbb{C}^m \rightarrow \text{Dom } A$ be a trial map. Set

$$E = Q - C_\epsilon Y, \quad G = Q^* C_\epsilon^{-1} Q, \quad D = Q^* Y + Y^* Q - Y^* C_\epsilon Y.$$

Expansion gives the exact variational identity

$$G - D = E^* C_\epsilon^{-1} E \succeq 0. \tag{9}$$

Its source covariance is a joint matrix, including every column cross term.

Theorem 3.1 (Localized two-metric receiving). *Let $0 \leq \chi \leq 1$ be an even measurable cutoff. Suppose the finite source maps*

$$b = \chi E/s, \quad R = (1 - \chi)E$$

have square-integrable columns. For every $\theta > 0$,

$$0 \preceq G - D \preceq (1 + \theta)J_\epsilon^2 b^* b + (1 + \theta^{-1})\epsilon^{-1} R^* R. \tag{10}$$

For a single source the corresponding upper is

$$\langle E, C_\epsilon^{-1} E \rangle \leq (J_\epsilon \|b\| + \epsilon^{-1/2} \|R\|)^2. \tag{11}$$

Proof. Since E takes values in X , $E = P_X M_s b + P_X R$. The adjoint of $M_s C_\epsilon^{-1/2}$ is $C_\epsilon^{-1/2} P_X M_s$, with the projection present because the inverse acts on X . Apply this identity to an arbitrary common coefficient vector. The first term is bounded by $J_\epsilon \|bc\|$ and the second by $\epsilon^{-1/2} \|Rc\|$. Young’s inequality with parameter θ proves (10); the triangle inequality proves (11). Equation (9) proves the lower. $\square$

The two pieces need not have orthogonal inverse images, even when their physical supports are disjoint. Formula (10) pays that mixed term for every common coefficient combination. Its ingredients are variational residual algebra and mixed-norm Cauchy–Schwarz; the application below supplies the complete derivative-source estimates. For a nonzero trial Y , the exterior residual includes the tail of its constrained projection. In the main growth theorem we take $Y = 0$, so that $E = Q$ is the exact source itself.

4 Complete source geometry

The sources in Theorem 1.1 are prescribed derivative rows, rather than arbitrary vectors in the kernel. Their complete theta expansion makes both the upper estimate and the matching logarithmic power accessible. Write $p(u) = \sinh(u/2)$ and $r = \pi e^{2u}$ on $u \geq 0$.

Lemma 4.1 (Derivative atoms and scaled profiles). *Define polynomials by*

$$P_0(x) = 4x^2 - 6x, \quad P_{n+1}(x) = \left(\frac{1}{2} - 2x\right)P_n(x) + 2xP'_n(x). \quad (12)$$

Then

$$\psi^{(n)}(u) = e^{u/2} \sum_{m \geq 1} P_n(rm^2) e^{-rm^2}, \quad \deg P_n = n + 2,$$

and the leading coefficient of P_n is $4(-2)^n$. For $\ell_i = 2i + 3$ put

$$R_i = P_{2i+3} - \frac{1}{4}P_{2i+1}, \quad k_i = z_i/s, \quad D_i = \deg R_i = \ell_i + 2.$$

The exact divided source is

$$k_i(u) = \cosh(u/2) \frac{\sum_{m \geq 1} R_i(rm^2) e^{-rm^2}}{\sum_{m \geq 1} P_0(rm^2) e^{-rm^2}}. \quad (13)$$

For each fixed i , uniformly in C^1 on every compact subinterval of $(0, \infty)$,

$$U^{-\ell_i-1/4} k_i\left(\frac{1}{2} \log(Ux/\pi)\right) \longrightarrow b_i x^{\ell_i+1/4}, \quad b_i = \frac{(-2)^{\ell_i}}{2\pi^{1/4}}, \quad (14)$$

$$U^{-1/4} p\left(\frac{1}{2} \log(Ux/\pi)\right) \longrightarrow b_p x^{1/4}, \quad b_p = \frac{1}{2\pi^{1/4}}. \quad (15)$$

Proof. Differentiating one atom uses $r' = 2r$ and gives (12). The series and any fixed number of its derivatives converge locally uniformly, so the differentiation is valid. The leading term in the recurrence multiplies the previous leading coefficient by -2 . Substituting $s^2 = \psi / \cosh(u/2)$ gives (13).

For $r \geq \pi$ all denominator atoms are positive and $P_0(r) \geq 2r^2$. If R has fixed degree D and coefficient absolute sum B , then

$$\sum_{m \geq 2} |R(rm^2)| e^{-rm^2} \leq Br^D e^{-4r} \sum_{m \geq 2} m^{2D} e^{-\pi(m^2-4)}.$$

The last sum is finite. One r derivative has a bound of the same form with a larger fixed polynomial power. Hence, relative to the positive first denominator atom, the remaining atoms and one fixed derivative are bounded by $O_i(r^{C_i} e^{-3r})$. Polynomial division gives

$$\frac{R_i(r)}{P_0(r)} = (-2)^{\ell_i} r^{\ell_i} \{1 + O_i(r^{-1})\}.$$

The expansion and its derivative can be used uniformly when $r = Ux$ and x stays in the stated compact interval. Finally

$$\cosh(u/2) = \frac{1}{2}\pi^{-1/4}r^{1/4}(1 + \sqrt{\pi/r}), \quad \sinh(u/2) = \frac{1}{2}\pi^{-1/4}r^{1/4}(1 - \sqrt{\pi/r}),$$

which prove both C^1 limits. $\square$

In particular k_i grows as a nonzero power of r on each arm, and $k_i \notin L^2(\mathbb{R})$. The whole-line factorization $z_i = M_s k_i$ therefore cannot directly use an L^2 observation-factor bound. The localization in Theorem 3.1 resolves this source-specific obstruction.

Lemma 4.2 (Two complete source budgets). *For each i there is an effectively specified $K_i > 0$ such that*

$$\left| \sum_{m \geq 1} R_i(rm^2)e^{-rm^2} \right| \leq K_i r^{D_i} e^{-r} \quad (r \geq \pi).$$

For $T \geq \pi$, both physical arms together satisfy

$$\int_{r \leq T} |k_i(u)|^2 du \leq \frac{K_i^2 T^{2D_i-7/2}}{4\sqrt{\pi}(2D_i-7/2)}. \quad (16)$$

For $T \geq 4D_i$,

$$\int_{r > T} |z_i(u)|^2 du \leq \frac{K_i^2}{2\sqrt{\pi}} \int_T^\infty r^{2D_i-5/2} e^{-r} dr \leq \frac{K_i^2}{\sqrt{\pi}} T^{2D_i-5/2} e^{-T}. \quad (17)$$

Proof. If B_i is the coefficient absolute sum of R_i , one possible choice is

$$K_i = B_i \sum_{m \geq 1} m^{2D_i} e^{-\pi(m^2-1)}.$$

This convergent scalar sum admits finite enclosures: after sufficiently many terms its successive ratios are bounded by a geometric ratio strictly below one. Since $r \geq \pi > 1$, it bounds every lower polynomial power as well as the leading one.

The positive first denominator atom in (13) and $\cosh(u/2) \leq e^{u/2}$ give

$$|k_i(u)| \leq \frac{1}{2} K_i e^{u/2} r^{D_i-2}.$$

Here $e^u = \sqrt{r/\pi}$, and integration on both arms changes du to dr/r . Extending the lower integration endpoint from π to zero proves (16). Alternatively, the numerator of z_i is bounded by $K_i e^{u/2} r^{D_i} e^{-r}$ and the source lower bound is $s^2 \geq 2r^2 e^{-r}$. Consequently

$$|z_i(u)|^2 \leq \frac{K_i^2}{2\sqrt{\pi}} r^{2D_i-3/2} e^{-r},$$

which gives the first inequality in (17). For $\nu = 2D_i - 5/2$ and $r \geq T \geq 4D_i$, $\frac{d}{dr} \log(r^\nu e^{-r}) = \nu/r - 1 \leq -1/2$. Integrating this exponential majorant bounds the remaining integral by $2T^\nu e^{-T}$. $\square$

5 Uniform two-sided source geometry

We prove Theorem 1.1 with explicit dependence on the source count. Throughout this section $n = d + 2$, $L = 2n - 1$, $m = 4n + 3$ and $U \geq 64n^2$. The normalization includes the signed leading coefficients:

$$\tilde{D}_U = \text{diag}(b_i U^{\ell_i+1/4})_{i=0}^d, \quad E = Z_d \tilde{D}_U^{-1}.$$

5.1 Half-line cancellation and the complete upper

Choose a common wall $T \geq \max\{4(L+2), \pi e^2\}$, put $w = \frac{1}{2} \log(T/\pi)$ and $\chi = 1_{[-w, w]}$. Write

$$\alpha_i = \ell_i + \frac{1}{4}, \quad \beta_i = 2\alpha_i, \quad A_i = \frac{K_i}{2^{\ell_i}} (T/U)^{\alpha_i}, \quad H_w = (1 - e^{-w/2})^{-2}.$$

Here K_i is any complete source coefficient from Lemma 4.2. For $k_i = E_i/s$, that lemma gives

$$|k_i(u)| \leq A_i e^{-\beta_i(w-u)} \quad (0 \leq u \leq w). \quad (18)$$

Set $I_i(u) = \int_0^u k_i(v) dv$, $I_p(u) = 2(\cosh(u/2) - 1)$ and

$$c_i = I_i(w)/I_p(w), \quad f_i = \chi(k_i - c_i p), \quad F_i(u) = \int_{-\infty}^u f_i(v) dv.$$

Oddness makes the integral of f_i on each half-line vanish exactly. Thus F_i vanishes outside $[-w, w]$ and at zero; there is no central primitive plateau. For $0 \leq u \leq w$, elementary hyperbolic identities give

$$\frac{I_p(u)}{I_p(w)} \leq H_w e^{-(w-u)/2}, \quad \frac{p(u)}{I_p(w)} \leq \frac{1}{2} H_w e^{-(w-u)/2}.$$

Also $|I_i(u)| \leq (A_i/\beta_i) e^{-\beta_i(w-u)}$. Integrating these bounds over both arms, and using the squared triangle inequality, yields

$$\begin{aligned} \|f_i\|_2^2 &\leq 2A_i^2/\beta_i + H_w^2 A_i^2/\beta_i^2, \\ \|F_i\|_2^2 &\leq 2A_i^2(1 + H_w)^2/\beta_i^2. \end{aligned}$$

Consequently

$$\frac{1}{4} \|f_i\|_2^2 + \frac{1}{16} \|F_i\|_2^2 \leq \mathcal{K}_i := A_i^2 \left\{ \frac{1}{2\beta_i} + \frac{3H_w^2 + 2H_w + 1}{8\beta_i^2} \right\}. \quad (19)$$

The zeroth positive digamma summand in Lemma 2.1 gives $1/a(\xi) \leq 1/4 + 1/(16\xi^2)$ for $\xi \neq 0$. For a compact L^2 function f with L^2 primitive F , Parseval and $\widehat{f} = i\xi \widehat{F}$ therefore imply

$$\int_{\mathbb{R}} \frac{|\widehat{f}|^2}{a} \frac{d\xi}{2\pi} \leq \frac{1}{4} \|f\|_2^2 + \frac{1}{16} \|F\|_2^2.$$

Weighted Fourier Cauchy–Schwarz proves, for every $h \in V$,

$$|\langle P_X M_s f_i, h \rangle|^2 \leq \mathcal{K}_i \mathfrak{a}[h]. \quad (20)$$

This extends from the chosen compact core by its form closure; the hard cutoff is an L^2 forcing factor, never an operator-domain trial.

The exact exterior return is

$$E_i = P_X M_s f_i + P_X V_i, \quad V_i = (1 - \chi)(E_i - c_i s p). \quad (21)$$

Indeed $P_X E_i = E_i$ and $P_X(sp) = 0$. In particular the exterior polar term cannot be dropped. The complete atom envelope also gives $s^2 p^2 \leq \psi \cosh(u/2) \leq 8r^{5/2} e^{-r}/\sqrt{\pi}$. Integrating both arms and using the logarithmic slope of $r^{3/2} e^{-r}$ gives

$$\|1_{r>T} s p\|_2^2 \leq 16T^{3/2} e^{-T}/\sqrt{\pi} \quad (T \geq 3).$$

For completeness, $\sum_{a \geq 1} a^4 e^{-r(a^2-1)} < 2$ for $r \geq \pi$, so the envelope follows directly by bounding $P_0(ra^2) \leq 4r^2 a^4$ and $e^{u/2} \cosh(u/2) \leq e^u = \sqrt{r/\pi}$. Since $|c_i| \leq H_w A_i e^{-w/2}/\beta_i$, the source tail in (17) and the polar tail give

$$\epsilon^{-1} \|V_i\|_2^2 \leq \mathcal{T}_i := A_i^2 (8 + 32H_w^2/\beta_i^2) T e^{U-T}. \quad (22)$$

Before applying the squared triangle inequality, the two payments are

$$4A_i^2 T e^{U-T} \quad \text{and} \quad 16H_w^2 A_i^2 T e^{U-T} / \beta_i^2.$$

Thus their mixed term has also been paid.

For one common complex coefficient vector a , combine the columns f_i, F_i, V_i before estimating. Cauchy–Schwarz in the column index, (20), variational duality and Minkowski in the inverse energy norm give the joint inequality

$$E^* C_\epsilon^{-1} E \preceq \left(\sqrt{\sum_i \mathcal{K}_i} + \sqrt{\sum_i \mathcal{T}_i} \right)^2 I. \quad (23)$$

Every mixed source entry is included; there is no diagonal substitution. In a single coordinate, cancellation of the normalization gives the useful literal finite bound

$$\langle z_i, C_\epsilon^{-1} z_i \rangle \leq \frac{K_i^2 T^{2\alpha_i}}{4\sqrt{\pi}} \left[\sqrt{\kappa_i(H_w)} + \sqrt{(8 + 32H_w^2/\beta_i^2) T e^{U-T}} \right]^2, \quad (24)$$

where $\kappa_i(H) = 1/(2\beta_i) + (3H^2 + 2H + 1)/(8\beta_i^2)$. The wall assumptions are the same as above.

We make the order dependence explicit. If V_j is the coefficient ℓ^1 norm of P_j , the recurrence gives

$$V_0 = 10, \quad V_{j+1} \leq (2j + 13/2)V_j, \quad V_j \leq 10(2j + 5)^j.$$

Hence $\|\text{coeff } R_i\|_1 \leq 13(2\ell_i + 5)^{\ell_i}$. Writing $D = \ell_i + 2$, the inequality $x^D \leq D!e^x$ gives

$$\sum_{a \geq 1} a^{2D} e^{-\pi(a^2-1)} \leq e^\pi D! (2/\pi)^D \sum_{a \geq 1} e^{-\pi a^2/2} < 64D!.$$

The last sum is less than 1 by $a^2 \geq a$ and $\pi > 3$. Thus all K_i may be bounded by

$$\bar{K}_n = 1024(2L + 5)^L (L + 2)!. \quad (25)$$

Take $T = 2U$. All wall assumptions hold, $H_w \leq 7$ and $A_i^2 = \sqrt{2}K_i^2$. Since $\beta_i \geq 13/2$,

$$\frac{1}{2\beta_i} + \frac{3 \cdot 49 + 14 + 1}{8\beta_i^2} < 1, \quad 16 + 64 \cdot 49/\beta_i^2 < 128.$$

Using $Ue^{-U} \leq 1$ and fewer than n source columns gives $\sum_i \mathcal{K}_i \leq 2n\bar{K}_n^2$ and $\sum_i \mathcal{T}_i \leq 256n\bar{K}_n^2$. Equation (23) proves the upper bound of Theorem 1.1 with $C_n = 512n\bar{K}_n^2$.

5.2 Exact polynomial geometry on the outward shell

Use the even cutoff

$$\chi_0(x) = (2x - 1)^4 (1 - x)^4 \quad (1/2 \leq x \leq 1), \quad \chi_0(x) = 0 \quad \text{otherwise.} \quad (26)$$

It is C^1 , with zero endpoint derivatives and $|\chi'_0| \leq 16$. On the positive shell put $r = Ut^2$, $1/\sqrt{2} \leq t \leq 1$ and $\delta = \sqrt{\pi/U}$. Define

$$A_k(r) = \frac{P_k(r)}{4(-2)^k}, \quad a_k(t) = \frac{A_k(Ut^2)}{U^{k+2}}, \quad C_k = k^2/2 + 7k/4 + 3/2.$$

The exact recurrence becomes

$$a_0 = t^4 - 3t^2/(2U), \quad a_{k+1} = (t^2 - 1/(4U))a_k - (t/(2U))a'_k, \quad \|\text{coeff } a_k\|_1 \leq e^{C_k/U}. \quad (27)$$

The norm bound follows by multiplying $1 + 3/(2U)$ by the step factors $1 + (k + 9/4)/U$. Put

$$\begin{aligned} Q_p &= 4a_0, & Q_i &= 4(a_{\ell_i} - a_{\ell_i-2}/(16U^2)), \\ T_p &= 4H_0, & T_i &= 4(H_{\ell_i} - H_{\ell_i-2}/(16U^2)), \\ H_k(t) &= \sum_{a \geq 2} a_k(at) e^{-Ut^2(a^2-1)}. \end{aligned}$$

The denominator $D = Q_p + T_p$ is positive. With $F = t^{-1/2}/D$ set

$$p_p = (t - \delta)Q_p, \quad p_i = (t + \delta)Q_i, \quad e_p = (t - \delta)T_p, \quad e_i = (t + \delta)T_i.$$

The actual normalized divided profiles, including every atom, are exactly

$$v_p = \frac{p(u)}{b_p U^{1/4}} = F(p_p + e_p), \quad v_i = \frac{z_i(u)/s(u)}{b_i U^{\ell_i+1/4}} = F(p_i + e_i). \quad (28)$$

The common factor is retained in every subsequent covariance.

We first prove a coefficient floor for the polynomial rows. Writing $A_k(r) = \sum_j (-1)^j a_{k,j} r^{k+2-j}$, the recurrence gives

$$a_{k+1,j} = a_{k,j} + (k + 13/4 - j)a_{k,j-1}, \quad 0 \leq a_{k,j} \leq C_k^j/j!.$$

Terms outside the polynomial degree are zero. The bound follows coefficientwise from $(1 + 3z/2) \prod_{q=0}^{k-1} (1 + (q + 9/4)z)$, whose node sum is C_k . Select the odd coefficients of $p_p/4, p_0/4, \dots, p_d/4$ at degrees $5, 11, 15, \dots, 4d + 11$. The resulting square matrix B has diagonal one. The terms containing δ have even degree and disappear exactly. Let $x = C_L/U$ and $\delta_0 = 1/(16U^2)$. A source-to-source off-diagonal coefficient loses an even number $j \geq 2$ of powers of r ; a source-to-polar coefficient loses an odd $j \geq 3$. Its absolute value is at most

$$x^j/j! + \delta_0 x^{j-2}/(j-2)!.$$

Summing the even and odd losses bounds both row and column sums of $B - I$ by $e^x - 1 - x + \delta_0 e^x$. Since $C_L \leq 3n^2$, even the weaker condition $U \geq 4n^2$ gives $x \leq 3/4$ and $U \geq 16$. The elementary series bound $e^{3/4} < 17/8$ shows that this sum is at most $12305/32768 < 1/2$. Thus $\sigma_{\min}(B) \geq 1/2$, and, for every $\alpha \in \mathbb{C}^n$,

$$\left\| \text{coeff} \left(\alpha_p p_p + \sum_i \alpha_i p_i \right) \right\|_2 \geq 2 \|\alpha\|_2. \quad (29)$$

This is an exact finite polynomial estimate, not proximity to limiting monomials.

5.3 All-atom errors and the complete shorted covariance

For $k \leq L$, $e^{C_k/U} \leq 2$. Differentiating the absolutely convergent positive majorant for H_k gives

$$\begin{aligned} |H_k| + |H'_k| &\leq 2 \sum_{a \geq 2} a^{2k+4} (1 + 2k + 4 + 2Ua^2) e^{-U(a^2-1)/2} \\ &\leq 6U \sum_{a \geq 2} a^{2L+6} e^{-U(a^2-1)/2} \leq 12U e^{-21U/16}. \end{aligned}$$

Here $2L + 5 \leq U$, $\log a \leq (a^2 - 1)/4$, and $U \geq 16(n + 1)$ imply that each last summand is at most $e^{-7U(a^2-1)/16}$. Consecutive $a^2 - 1$ differ by at least 5, so the geometric factor is at most 2. It follows that

$$\max_j (|T_j| + |T'_j|) \leq 52U e^{-21U/16} \leq e^{-U}. \quad (30)$$

The final scalar inequality holds for $U \geq 64$ by logarithms and monotonicity; the same summable majorants justify termwise differentiation. Since $Q_p = t^4(4 - 6/(Ut^2))$ and $T_p \geq 0$, these bounds imply

$$3/4 \leq D \leq 5, \quad 1/5 \leq F \leq 3, \quad |F'| \leq 70. \quad (31)$$

For the derivative bound use $|Q'_p| \leq 17$, $|T'_p| \leq 1$ and $t^{-1/2}, t^{-3/2} < 2$.

Give the shell the common positive measure

$$d\nu = 2\chi_0(t^2)F(t)^2 dt/t, \quad \nu(\mathbb{R}) < 9.$$

On $I = [4/5, 9/10]$, its density is at least 2^{-24} . For any polynomial P of degree at most m ,

$$\int_I |P|^2 dt \geq \frac{\|\text{coeff } P\|_2^2}{10(m+1)^2 75^{2m}}. \quad (32)$$

Indeed the coefficient ℓ^1 norm of the shifted Legendre polynomial $L_k(20t - 17)$ is at most 75^k . The first norms are 1, 37; the three-term recurrence bounds the next by 74 times the preceding norm plus its predecessor. Orthonormalization on the interval of length $1/10$ and Cauchy-Schwarz then give the coefficient cost in (32), also for complex coefficients. Together with (29), this proves that the full polynomial covariance in ν is at least

$$h_n I_n, \quad h_n = \frac{2^{-26}}{(m+1)^2 75^{2m}}.$$

All polynomial rows have coefficient ℓ^1 norm at most 12; (30) gives $|e_j| \leq 2e^{-U}$. Expanding every product, the norm of the full covariance perturbation is bounded by

$$n\nu(\mathbb{R})(48e^{-U} + 4e^{-2U}) \leq 512ne^{-U} \leq h_n/2.$$

For the last inequality, $\log(1024n/h_n) < 80n$ follows from $m = 4n + 3$, $\log 75 < 9/2$ and $\log n \leq n$, while $U \geq 64n^2 > 80n$ for $n \geq 2$. Thus the actual joint covariance of $v_p, v_0, \dots, v_d$, with measure $\chi_0 du$ on both arms, is at least $h_n I_n/2$. Minimizing over the actual polar coordinate leaves a shorted covariance at least $h_n I_{n-1}/2$.

5.4 Original-domain tests and the lower bound

For $a \in \mathbb{C}^{d+1}$ put $k_a = \sum_i a_i v_i$ and

$$c_a = \frac{\int_{\mathbb{R}} \chi_0(r/U) v_p k_a du}{\int_{\mathbb{R}} \chi_0(r/U) v_p^2 du}, \quad f_a = \chi_0(r/U)(k_a - c_a v_p), \quad h_a = f_a/s.$$

Profiles and f_a extend oddly, with the cutoff even. The exact polar constraint holds because v_p is a constant multiple of p . The shell is compact and avoids zero, and s is positive smooth on it. Approximate f_a in H^1 by odd smooth compact functions on a common larger compact set; subtract a multiple of one fixed compact odd function with nonzero polar moment. The multiples tend to zero. After division by s , convergence holds in the original Hilbert norm, and $\mathbf{a}[h] \leq 17\|(sh)'\|_2^2$ gives convergence in its form norm. Hence $h_a \in V$, the original minimal domain. The exact variance identity and shorted floor give

$$\langle Ea, h_a \rangle = \int_{\mathbb{R}} \chi_0(r/U) |k_a - c_a v_p|^2 du \geq (h_n/2) \|a\|_2^2. \quad (33)$$

The earlier bounds imply $|v_j| \leq 42$, $|v'_j| \leq 256(m+1)$, where the prime here denotes t differentiation. The polar denominator is at least $h_n/2$ and the cross vector has norm at most $2048\sqrt{n}$, so $|c_a| \leq 4096\sqrt{n}h_n^{-1}\|a\|_2$. Since $dt/du = t$ and $|d\chi_0(r/U)/du| \leq 32$, product differentiation gives

$$|f_a|, |f'_a| \leq 2^{27}(m+1)\sqrt{n}h_n^{-1}\|a\|_2.$$

The two-arm shell has measure $\log 2 < 1$; consequently

$$\|f_a\|_2^2 + \|f'_a\|_2^2 \leq E_n \|a\|_2^2, \quad E_n = 2^{60}n^2(m+1)^2h_n^{-2}. \quad (34)$$

On $U/2 \leq r \leq U$, the full first atom gives $\epsilon/s^2 \leq e^{r-U}/(2r^2) \leq 2U^{-2}$. The original constrained energy is therefore at most $19E_n\|a\|_2^2$. Variational duality with (33) yields

$$a^* E^* C_\epsilon^{-1} Ea \geq \frac{h_n^2}{76E_n} \|a\|_2^2 \geq g_n \|a\|_2^2, \quad g_n = \frac{2^{-67}h_n^4}{n^2(m+1)^2}.$$

This completes the lower bound and Theorem 1.1.

For each fixed d , conjugating by the fixed real diagonal $\text{diag}(b_i)$ gives positive constant lower and upper bounds for $D_U^{-1}\Gamma_{\epsilon,d}D_U^{-1}$. In a fixed coordinate this is the constant-factor scale $U^{4i+13/2}$; logarithms give (3). Moreover $\log(g_n^{-1}) = O(d)$ and $\log C_n = O(d \log(d+2))$, so the logarithm of the normalized condition number has the latter order. These are source-geometry estimates: they identify neither a limiting Gram nor the work required to evaluate its entries.

Corollary 5.1 (Slowly growing families). *Fix $\eta > 0$ and let $n = d + 2 = \lfloor \eta \log U / \log \log U \rfloor$ for sufficiently large U . Then*

$$U^{-o(1)} I \preceq \tilde{D}_U^{-1} \Gamma_{\epsilon,d} \tilde{D}_U^{-1} \preceq U^{8\eta+o(1)} I, \quad \text{cond}(\tilde{D}_U^{-1} \Gamma_{\epsilon,d} \tilde{D}_U^{-1}) \leq U^{8\eta+o(1)}.$$

If instead $2 \leq n = o(\log U / \log \log U)$, the normalized Gram, its inverse and its condition number all have upper norm bounds $U^{o(1)}$.

Proof. The explicit constants give

$$\log(g_n^{-1}) = 32(\log 75)n + O(\log n), \quad \log C_n = 8n \log n + O(n).$$

The first identity follows by substituting $m = 4n + 3$ in h_n ; the second follows from the factorial in (25), using $\log(k!) = k \log k - k + O(\log(k+1))$. Both schedules eventually satisfy $U \geq 64n^2$. For the first, $n \log n / \log U \rightarrow \eta$ and $n / \log U \rightarrow 0$; divide the logarithms of the bounds in Theorem 1.1 by $\log U$. For the second schedule both logarithmic costs are $o(\log U)$. The norm of the inverse is at most g_n^{-1} and the condition number at most C_n/g_n , proving the claims. $\square$

The corollary prices this specific source geometry. It does not replace a complete signed-form comparison or certify an inverse evaluation algorithm.

6 An explicit finite receiving bound

The first source gives a concrete application of the complete estimates. The recurrence (12) yields

$$R_0(r) = -90r + 810r^2 - 1056r^3 + 360r^4 - 32r^5.$$

For $r \geq \pi$ its absolute value is bounded by Br^5 , where

$$B = 32 + \frac{360}{\pi} + \frac{1056}{\pi^2} + \frac{810}{\pi^3} + \frac{90}{\pi^4} < 283.$$

The complete atom factor satisfies

$$\sum_{m \geq 1} m^{10} e^{-\pi(m^2-1)} < \frac{11}{10}.$$

Indeed its $m = 2$ term is $1024e^{-3\pi} < 0.083$, and the successive ratio after that is at most $(3/2)^{10}e^{-5\pi} < 10^{-5}$. Thus $K_0 = 312$ is an admissible complete coefficient.

Corollary 6.1. *For $0 < \epsilon \leq 1$ and every $T \geq 20$,*

$$\langle z_0, C_\epsilon^{-1} z_0 \rangle \leq \left[J_\epsilon \sqrt{\frac{312^2 T^{13/2}}{26\sqrt{\pi}}} + \sqrt{\frac{312^2 \epsilon^{-1} T^{15/2} e^{-T}}{\sqrt{\pi}}} \right]^2. \quad (35)$$

The sharper polar-cancellation estimate of (24), with $\epsilon = 10^{-18}$ and $T = 54$, gives

$$\langle z_0, C_{10^{-18}}^{-1} z_0 \rangle < 7.445061 \cdot 10^{14}.$$

Proof. Apply the scalar mixed-metric bound from Theorem 3.1 to the exact source z_0 , localized at $r = T$. The two squared source budgets are (16) and (17) with $D_0 = 5$ and $K_0 = 312$. The stronger displayed decimal enclosure uses $\alpha_0 = 13/4$, $\beta_0 = 13/2$ and $K_0 = 312$ in (24). Its wall satisfies $54 \geq \max\{20, \pi e^2\}$. The accompanying directed scalar verifier encloses this finite expression. $\square$

The bound is available without solving an inverse equation. It is a complete source-energy upper that can be used in a residual receiving calculation together with the joint covariance estimate.

7 A degree-dependent shell receiver

Theorem 1.1 uses exact common-weight polynomial geometry to handle the actual growing derivative family. For other finite-family comparisons, a direct complete-profile perturbation criterion can be useful. We record such a criterion with explicit dependence on the number of rows. It is a sufficient condition, separate from the coefficient argument that proves the unconditional schedule above. The moment estimate is a classical polynomial norm comparison applied to the polar and odd-derivative exponents.

Let χ_0 be (26), and put

$$e = (0, 3, 5, \dots, 2d+3), \quad n = d+2, \quad N = 2d+3, \quad h_d = \frac{2^{-22}}{(N+1)^2 29^{2N}}.$$

The exponent zero is the polar coordinate. Define full joint matrices

$$\begin{aligned} H_{jk} &= \int_{1/2}^1 \chi_0(x) x^{e_j+e_k-1/2} dx, \\ R_{jk} &= 2^{-20} \frac{(7/8)^{e_j+e_k+1} - (5/8)^{e_j+e_k+1}}{e_j + e_k + 1}. \end{aligned} \quad (36)$$

Proposition 7.1 (Explicit joint and shorted moment floor). *For every integer $d \geq 0$,*

$$H \succeq R \succeq h_d I_n.$$

Eliminating coordinate zero by minimizing either quadratic form over that coordinate leaves a form at least $h_d I_{n-1}$.

Proof. On $I = [5/8, 7/8]$, $\chi_0 \geq 2^{-20}$ and $x^{-1/2} \geq 1$, proving $H \succeq R$. The interval length is $l = 1/4$. Its shifted orthonormal Legendre polynomials are

$$\phi_k(x) = \sqrt{(2k+1)/l} L_k(8x-6),$$

where L_k is the usual Legendre polynomial. The coefficient ℓ^1 norm of $L_k(8x-6)$ is at most 29^k : the first two norms are 1, 14, and the three-term Legendre recurrence bounds the next by 28 times the preceding norm plus the norm before it. The induction closes because $28 + 1/29 < 29$.

For an ordinary polynomial P of degree at most N , expand it in the ϕ_k . Cauchy-Schwarz applied to its monomial coefficient vector gives

$$\|\text{coeff } P\|_2^2 \leq \frac{1}{l} \sum_{k=0}^N (2k+1) 29^{2k} \|P\|_{L^2(I)}^2 \leq \frac{(N+1)^2 29^{2N}}{l} \|P\|_{L^2(I)}^2.$$

Restricting the monomials to the exponents in e preserves the inequality. Multiplication by 2^{-20} gives the asserted h_d . Finally, for any remaining coefficient vector a , $\inf_c (c, a)^* R(c, a) \geq \inf_c h_d (|c|^2 + \|a\|_2^2) = h_d \|a\|_2^2$, and similarly for H . $\square$

We formulate a source comparison that can be checked at a given pair (d, U) . On $[1/2, 1]$ define the actual normalized profiles

$$\begin{aligned} v_{0,U}(x) &= \frac{p(\frac{1}{2} \log(Ux/\pi))}{b_p U^{1/4}}, \\ v_{i+1,U}(x) &= \frac{k_i(\frac{1}{2} \log(Ux/\pi))}{b_i U^{\ell_i+1/4}} \quad (0 \leq i \leq d), \end{aligned}$$

and the target profiles $v_j(x) = x^{e_j+1/4}$.

Theorem 7.2 (Complete-profile shell criterion). *Let $d \geq 0$, $U > 2\pi$ and $\epsilon = e^{-U}$. Suppose the complete profiles, including every theta atom, satisfy*

$$\max_{0 \leq j < n} \|v_{j,U} - v_j\|_{C^1([1/2,1])} \leq \eta, \quad \eta \leq \min\{1, h_d/(8n)\}. \quad (37)$$

Put $\tilde{D}_U = \text{diag}(b_i U^{\ell_i+1/4})_{i=0}^d$ and

$$E_d = 10^6 n^2 (N+3)^2 / h_d^2.$$

Then the actual constrained source Gram obeys

$$\tilde{D}_U^{-1} \Gamma_\epsilon \tilde{D}_U^{-1} \succeq \frac{h_d^2}{4(17+2U^{-2})E_d} I_{d+1}. \quad (38)$$

Proof. Give the profile space the measure $\chi_0(x) dx/x$. Every target profile has modulus at most one. The hypothesis implies that the complete actual joint covariance differs from H in operator norm by at most

$$n \log 2 (2\eta + \eta^2) < h_d/2.$$

For example, each of its entries has that bound without the factor n , by expanding the product of two actual profiles and retaining both cross terms. The maximum row sum then proves the matrix bound. Proposition 7.1 and minimization over the actual polar coordinate therefore give an actual shorted floor $h_d/2$.

For $a \in \mathbb{C}^{d+1}$ write

$$g = \sum_{i=0}^d a_i v_{i+1,U}, \quad \beta = \frac{\int_{1/2}^1 \chi_0 v_{0,U} g dx/x}{\int_{1/2}^1 \chi_0 v_{0,U}^2 dx/x}, \quad f(u) = \chi_0(r/U) \{g(r/U) - \beta v_{0,U}(r/U)\},$$

with odd reflection for f . The proportionality of $v_{0,U}$ and p makes $h = f/s$ exactly polar-free; the compact-domain approximation in Section 5 gives $h \in V$. For $E = Z_d \tilde{D}_U^{-1}$, the exact variance identity gives

$$\langle Ea, h \rangle = \int_{1/2}^1 \chi_0 |g - \beta v_{0,U}|^2 dx/x \geq (h_d/2) \|a\|_2^2. \quad (39)$$

Here are explicit derivative payments. The hypothesis gives $|v_{j,U}| \leq 2$ and $|v'_{j,U}| \leq N+2$. Hence

$$|g| \leq 2\sqrt{n} \|a\|_2, \quad |g'| \leq (N+2)\sqrt{n} \|a\|_2, \quad |\beta| \leq 8\sqrt{n} h_d^{-1} \|a\|_2.$$

The last inequality uses the polar denominator at least $h_d/2$, $\chi_0 \leq 1$ and $\int_{1/2}^1 dx/x < 1$. Since $h_d \leq 1$, putting $w = g - \beta v_{0,U}$ gives

$$|w| \leq 18\sqrt{n} h_d^{-1} \|a\|_2, \quad |w'| \leq 9(N+2)\sqrt{n} h_d^{-1} \|a\|_2.$$

Direct differentiation of (26) gives $|\chi'_0| \leq 16$. Using $d/du = 2x d/dx$, $N \geq 3$ and $N+34 \leq 7(N+3)$ yields

$$|f| \leq 18\sqrt{n} h_d^{-1} \|a\|_2, \quad |f'| \leq 126(N+3)\sqrt{n} h_d^{-1} \|a\|_2.$$

Both arms together have measure dx/x on this shell, whose mass is less than one. Consequently

$$\|f\|_2^2 + \|f'\|_2^2 \leq 20000n(N+3)^2 h_d^{-2} \|a\|_2^2 \leq E_d \|a\|_2^2.$$

As before $\epsilon/s^2 \leq 2/U^2$ on the shell. The full original energy is therefore at most $(17+2U^{-2})E_d \|a\|_2^2$. Combining this with (39) in the variational quotient proves (38). The real signed diagonal $\tilde{D}_U$ is an exact invertible congruence; its leading coefficients have not been removed from the source covariance. $\square$

The criterion isolates two separate quantitative inputs. The moment floor is explicit for every d , with $\log(h_d^{-1}) = O(d)$. The complete profile comparison is the hypothesis that must be paid when applying this criterion at a chosen (d, U) . For fixed d it follows eventually from Lemma 4.1; for a specified changing family the same theorem applies whenever (37) has been established. Theorem 1.1 does not assert this particular closeness condition on its growing schedule: its exact common-weight construction bypasses that condition. The exact rational matrices in (36), their polar shorts and directed scalar source bounds are included in the accompanying verification programs.

8 Finite receiving bounds and reproducibility

The accompanying program checks the exact derivative-polynomial recurrence, the complete scalar atom envelope, the explicit first-source finite bounds, and the full rational shell moment floors. It retains the polar row and every joint and shorted matrix entry. Directed scalar enclosures use Arb through python-flint [7]; the moment matrices and their factorization checks use exact rational arithmetic. Proposed finite inequalities are accepted only after the stated positive margins are enclosed.

The three regularizations in Table 1 use fixed integer walls. The complete first-source upper is obtained by substituting those walls into the polar-cancellation bound (24).

Table 1: Complete upper bounds for $\langle z_0, C_\epsilon^{-1} z_0 \rangle$; displayed bounds are rounded upward.

ϵ	wall in r	receiving upper
10^{-18}	54	$7.445061 \cdot 10^{14}$
2^{-256}	200	$1.808308 \cdot 10^{18}$
2^{-1024}	736	$7.080630 \cdot 10^{21}$

The growing-family controls check coefficient-map scalar envelopes at 33 quadratic-schedule boundaries, polynomial coefficient norms, complete-tail and covariance-perturbation budgets, scalar upper constants and the three new finite receiving bounds, for 1,361 additional gates. The coefficient-map and Legendre norm inequalities are proved analytically; these finite controls supplement their all-degree and complete-tail arguments.

The code also verifies full joint and polar-shortened lower matrices through highest index $d = 8$, with exact factor reconstruction. These finite checks complement the all-degree polynomial norm proof. The analytic theorem determines the growth exponents; the program encloses the stated source budgets at finite parameters. The reader instructions specify the dependency version, entry point and complete reference outputs.

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