# Small-ball estimates for Hölder centers of the Kloosterman process

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**Abstract**

We prove uniform-norm small-ball bounds of logarithmic order $\epsilon^{-2}$ about every admissible 1/2-Hölder center of the Kloosterman limit process, including boundary coefficient and endpoint values. A graded Fejér approximation removes two logarithmic losses from an inspected earlier lower estimate. For every $0 < \gamma \leq 1$, the worst probability over a fixed admissible $\gamma$-Hölder class has negative logarithm of order $\epsilon^{-\max(2,1/\gamma)}$. We also prove an explicit $O(\log\log p/\sqrt{\log p})$ Wasserstein convergence rate, in the uniform path metric, for prime Kloosterman paths with both parameters averaged; this yields uniform shrinking-target counts. The argument quantifies classical shifted-correlation and approximation tools. For other bounded symmetric unimodal coefficient laws, a fractional Fejér method gives a sufficient boundary-integrability condition. For densities proportional to $\exp[-A(2-|x|)^{-a}]$, the uniform lower order $\epsilon^{-1/\gamma}$, $0 < \gamma \leq 1/2$, holds exactly when $a < 1/\gamma$. A fixed admissible multiscale boundary center proves failure at equality. At $\gamma = 1/2, a = 2$, we give a uniform $\epsilon^{-2}\log^3(\mathrm{e}/\epsilon)$ lower-probability exponent, complementing a fixed-center $\epsilon^{-2}\log^2(1/\epsilon)$ upper-probability exponent. For $0 < \gamma < 1$, the noncritical worst-case power order is $\epsilon^{-\max(2,1/\gamma,a)}$; for $\gamma > 1/2$ this holds for every $a > 0$, including $a = 2$. All comparison, summability and arithmetic machinery is credited to its classical sources.

## 1 Statement and context

Small-ball probabilities around a prescribed path involve two competing costs: confining the centered random Fourier tail and forcing coefficients toward the boundary of their allowed support. Graded and fractional Fejér approximations balance these costs. We first treat the Sato–Tate Kloosterman process, then relate the resulting path estimates to finite arithmetic paths and extend the argument to other bounded symmetric coefficient laws.

Let $(X_h)_{h\in\mathbb{Z}}$ be independent real random variables of density

$$f(x) = \frac{\sqrt{4-x^2}}{2\pi}\,\mathbf{1}_{[-2,2]}(x). \tag{1.1}$$

Set $v_0(t) = t$, and, for $h \neq 0$, set

$$v_h(t) = \frac{\mathrm{e}^{2\pi i h t} - 1}{2\pi i h}, \qquad 0 \leq t \leq 1.$$

The Kloosterman process is

$$K(t) = \sum_{h\in\mathbb{Z}} X_h v_h(t), \tag{1.2}$$

where the nonzero frequencies are summed symmetrically. Almost sure uniform convergence is known from the work of Kowalski–Sawin [13]; a proof in the form needed here is included in

Section 2. We use $\|w\|_\infty = \sup_{0\le t\le 1}|w(t)|$ for complex-valued paths, and $\widehat{u}(h) = \int_0^1 u(t)\mathrm{e}^{-2\pi i h t}\,dt$ for Fourier coefficients.

For $L \ge 1$, let $\mathcal{H}_L$ be the class of continuous $g : [0,1] \to \mathbb{C}$ with the following properties:

1. $g(0) = 0$ and $a = g(1)$ belongs to the real interval $[-2, 2]$;
2. $|g(t) - g(s)| \le L\sqrt{|t-s|}$ for all $s, t \in [0,1]$;
3. with $u(t) = g(t) - at$, the numbers
$$\beta_h = 2\pi i h \widehat{u}(h), \qquad h \in \mathbb{Z}\setminus\{0\}, \tag{1.3}$$
are real and belong to $[-2, 2]$.

The endpoint and Fourier conditions are the established support characterization of Kowalski–Sawin [14, Theorem 1.1]; we add a uniform Hölder bound. The proof below uses the displayed conditions directly and does not assume the support theorem.

**Theorem 1.1.** *For every $L \ge 1$, every $g \in \mathcal{H}_L$, and $0 < \epsilon \le 1/4$,*
$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \ge \exp\{-26000(L+2)^2\epsilon^{-2}\}. \tag{1.4}$$
*Let*
$$p = 1 - \pi^{-1}, \qquad \delta = \frac{p}{128\pi^2}, \qquad \epsilon_0 = \sqrt{\delta/2}, \qquad c = \frac{p\delta}{16}. \tag{1.5}$$
*Then, for every deterministic $g \in C([0,1],\mathbb{C})$,*
$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \le \exp\{-c\epsilon^{-2}\}, \qquad 0 < \epsilon \le \epsilon_0. \tag{1.6}$$

Thus, for fixed $L$, the negative logarithm of the probability has order $\epsilon^{-2}$, uniformly on $\mathcal{H}_L$. The constants are deliberately inefficient; the theorem determines the order, not an asymptotic leading constant. Both $a = \pm 2$ and individual $\beta_h = \pm 2$ are allowed. The upper bound needs none of the endpoint, Fourier or Hölder restrictions imposed on the lower bound.

The second main result determines the worst-case scale for every Hölder exponent. Let $\mathcal{H}_{\gamma,L}$ retain the same endpoint and Fourier restrictions, with Hölder exponent $\gamma$ in place of $1/2$. Theorem 7.1 proves that
$$-\log \inf_{g\in\mathcal{H}_{\gamma,L}} \mathbb{P}(\|K - g\|_\infty < \epsilon) \asymp_{\gamma,L} \epsilon^{-\max(2,1/\gamma)}.$$
For $\gamma < 1/2$, a Rudin–Shapiro signed train of narrow triangular spikes supplies the matching upper estimate. This is a worst-case family whose center can vary with $\epsilon$, rather than a sharp law asserted at every fixed rough center. The signs and their use in Kloosterman support are established tools, including in [15]; the claim here is their constrained approximation-probability consequence.

A third result, Theorem 6.1, bounds Wasserstein distance between the entire two-parameter prime path law and its limit by $O(\log\log p/\sqrt{\log p})$, for the uniform path metric. Corollary 6.3 combines this with the small-ball lower bound to count neighborhoods whose radii decrease with the prime. The arithmetic input is the classical normalized Kloosterman sheaf and trace/Euler/weight estimates, with all representation-degree dependence retained. Growing moments control dyadic Fourier blocks; positive Jackson approximation allocates degrees according to the $1/|h|$ Fourier weights. Quantitative path work by S. Pitte is noted in the current author manuscript of Kowalski–Untrau [10]. Recent scalar endpoint Wasserstein estimates of Rahimi [19] concern the scalar law; the present estimates concern the joint full path in the uniform metric.

The closest exact quantitative predecessor inspected is Kyselov [15, Theorem 1.2], whose version 1 gives

$$\mathbb{P}(\|K-g\|_\infty < \epsilon) \geq \exp\{-C_L \epsilon^{-2} \log^2(2/\epsilon)\}$$

for closed centers $g(0) = g(1) = 0$ with $\beta_h \in [-1, 1]$ and the same Hölder exponent. The lower estimate (1.4) removes both logarithmic factors, extends the coefficient interval to its full permitted range, and allows every permitted endpoint. Estimate (1.6) shows that the power $\epsilon^{-2}$ cannot be improved uniformly, or even at an individual admissible center, to a smaller logarithmic order.

Our contribution is specific to this bounded-coefficient Fourier process and this path norm. Shifted small-deviation estimates for non-Gaussian series form an established subject; for example, the $p$-exponential framework in [1] uses unbounded coefficient laws. The classical comparison ingredients below are instances of Prékopa marginal log-concavity [17], product peakedness associated with Kanter [11, 16], and Gaussian translation comparison [2]. The point at which coefficient boundaries matter is Section 4: the Fejér weights supply different interior margins at different frequencies, and their product retains a factorial.

Other superficially similar small-deviation results do not directly imply the estimate here. Uniform-series priors in [5, Section 4.2] use a coordinate-supremum sequence topology rather than the uniform norm of this Fourier image. The non-Gaussian Fourier estimates in [3, Section 2.6.4] concern $L^2$ small deviations; an $L^2$ lower bound does not by itself give a uniform-norm lower bound. Section 8 isolates a sufficient boundary condition for other bounded symmetric unimodal coefficient laws. This direct extension clarifies the scope of the matching argument for the stated bounded symmetric unimodal coefficient laws. For the family $f(x) \propto \exp[-A(2-|x|)^{-a}]$, Sections 9 and 10 determine the lower-order property for $0 < \gamma \leq 1/2$: it holds precisely when $a < 1/\gamma$. The failure at equality uses a single fixed admissible boundary center. Theorem 9.2 also supplies a uniform critical lower estimate at $\gamma = 1/2, a = 2$, leaving one logarithm between the two proved bounds at its fixed witness. Section 11 completes the noncritical power orders for $0 < \gamma < 1$: the worst-case negative logarithm has order $\epsilon^{-\max(2,1/\gamma,a)}$, excluding only $a = 1/\gamma$ when $\gamma \leq 1/2$. In particular the smoother classes $1/2 < \gamma < 1$ have order $\epsilon^{-\max(2,a)}$ for every $a > 0$; $a = 2$ causes no extra logarithm there. These are extensions within the same result family. Supremum-norm small-ball estimates for independent symmetric unimodal wavelet series were established by Reiss and Schmidt-Hieber [18, Lemma 4.1]; their localized-wavelet proof does not directly give the full-boundary harmonic Fourier result considered here, and a change of basis need not preserve coefficient independence. Regularity-dependent shifted concentration for Gaussian processes is also established through approximation in their reproducing-kernel Hilbert spaces [21, 6]. Those precedents motivate comparison, but do not give the bounded Sato–Tate harmonic-class worst-case theorem above.

A three-author preprint of Bober–Kowalski–Sawin [4], listed in Kowalski's book [12], was not available in full text during the literature comparison. The quantitative comparison above is with the identified, directly inspected theorem of [15].

## 2 Convergence and a centered comparison

**Lemma 2.1.** *If the independent coefficients $W_h$ are either all distributed as (1.1) or all distributed as $N(0, 4)$, the symmetric partial sums $\sum_{|h|\leq H} W_h v_h$ converge almost surely uniformly on $[0, 1]$, and in expected uniform norm. The limit is continuous. A finite set of coefficients is independent of the continuous series formed from its complement.*

*Proof.* For integers $n \geq 1$ and $n \leq H \leq 2n$, put

$$P_{n,H}(t) = \sum_{n<|h|\leq H} \frac{W_h \mathrm{e}^{2\pi i h t}}{2\pi i h}.$$

A centered variable in $[-2,2]$ satisfies $\mathbb{E}\mathrm{e}^{\lambda X} \leq \cosh(2\lambda) \leq \mathrm{e}^{2\lambda^2}$: convexity bounds $\mathrm{e}^{\lambda x}$ by its endpoint chord, whose expectation is $\cosh(2\lambda)$, and $\log\cosh y \leq y^2/2$ follows by integrating $|\tanh y| \leq |y|$. This is the elementary bounded-variable exponential estimate. The last bound is an equality for $N(0,4)$. Independence and $\sum_{n<|h|\leq 2n} h^{-2} \leq C/n$ therefore show that both real and imaginary parts of $P_{n,H}(t)$ have subgaussian tails of variance parameter at most $C/n$. Exponential Markov inequality, followed by a union over the two parts, gives

$$\mathbb{P}(|P_{n,H}(t)| > v) \leq 4\mathrm{e}^{-c_1 n v^2} \tag{2.1}$$

with absolute constants, uniformly in $t, H$.

For completeness, a deterministic grid may be chosen with only polynomially many points. If $P$ is a trigonometric polynomial of degree $d$, each Fourier coefficient is at most $\|P\|_\infty$ in absolute value, whence

$$\|P'\|_\infty \leq 2\pi \sum_{|k|\leq d} |k| \, \|P\|_\infty = 2\pi d(d+1) \, \|P\|_\infty .$$

A grid of spacing at most $[4\pi d(d+1)]^{-1}$ gives $\|P\|_\infty \leq 2 \max_{t \text{ on grid}} |P(t)|$, by applying the derivative bound between a maximizer and a grid point. For $d = 2n$ this uses $O(n^2)$ points. Combining (2.1) for these points and all $H$ gives

$$\mathbb{P}\left(\max_{n\leq H\leq 2n} \|P_{n,H}\|_\infty > 2v\right) \leq \min(1, C_1 n^3 \mathrm{e}^{-c_1 n v^2}).$$

Integrating this inequality in $v$ yields

$$\mathbb{E} \max_{n\leq H\leq 2n} \|P_{n,H}\|_\infty \leq C_2 \sqrt{\frac{\log(2n)}{n}}.$$

Subtracting $P_{n,H}(0)$ changes the bound by at most a factor two and produces the blocks of $\sum W_h v_h$. On $n = 1, 2, 4, \ldots$, the expected maximal block norms have a finite sum. Tonelli's theorem gives almost sure summability of these maxima, hence convergence of complete and incomplete symmetric blocks. The same estimates give convergence in expected norm. The finite low-frequency terms cause no difficulty. Measurability as a uniform limit and independence of disjoint coefficient families give the last assertion. □

Write $Z \succeq Y$ for symmetric scalar variables when $\mathbb{P}(|Z| \leq r) \geq \mathbb{P}(|Y| \leq r)$ for all $r \geq 0$. This is the usual central peakedness order in one dimension.

**Lemma 2.2** (Classical peakedness comparison in the required form)**.** *Let $Z_1, \ldots, Z_m$ be independent symmetric variables with log-concave densities, and let $Y_j$ be independent $N(0,4)$ variables. If $Z_j \succeq Y_j$ for each $j$, then for every real linear map $T : \mathbb{R}^m \to \mathbb{R}^d$ and convex symmetric Borel set $A \subseteq \mathbb{R}^d$,*

$$\mathbb{P}(TZ \in A) \geq \mathbb{P}(TY \in A).$$

*The conclusion applies to the inverse image of a closed uniform-norm ball under a finite continuous-function map.*

*Proof.* Replace one coordinate at a time. For the coordinate being replaced, the conditional acceptance function is

$$F(a) = \int \mathbf{1}_A \left( aT(e_j) + \sum_{k \neq j} z_k T(e_k) \right) \prod_{k \neq j} f_k(z_k)\, dz.$$

The integrand is jointly log-concave: its indicator is that of a convex inverse image, with the extended convention for zero values. Prékopa's marginal theorem [17, Theorem 6] makes $F$ log-concave. Reflection of all coordinates makes $F$ even. An even log-concave function is nonincreasing in absolute value, since every $a \in [-b, b]$ is a convex combination of $b$ and $-b$. Its strict superlevel sets are central intervals. Scalar peakedness and layer cake therefore give $\mathbb{E}F(Z_j) \geq \mathbb{E}F(Y_j)$. Each intermediate product still has symmetric log-concave factors, so iteration proves the result. No full-rank assumption on $T$ is used. For a function map, its closed-ball inverse image in coefficient space is convex and symmetric, so the same proof applies directly. □

If $X$ has density (1.1), its central mass on $[0, 2]$ is concave in the radius and joins 0 to 1. Thus

$$\mathbb{P}(|X| \leq r) \geq r/2 \quad (0 \leq r \leq 2).$$

For $Y \sim N(0, 4)$, the density bound gives $\mathbb{P}(|Y| \leq r) \leq r/\sqrt{2\pi} \leq r/2$. For $r \geq 2$ the first probability is one. Also

$$\frac{d^2}{dx^2} \log f(x) = -\frac{4 + x^2}{(4 - x^2)^2} < 0 \quad (|x| < 2),$$

so $X$ has the symmetry, log-concavity and peakedness required in Lemma 2.2. A uniform variable on $[-r_j, r_j]$, $0 < r_j \leq 1/2$, has the same properties and dominates $Y$: for $0 \leq s \leq r_j$ its central mass is $s/r_j \geq 2s$, and beyond $r_j$ it is one. All these laws have densities, so open or closed central intervals give the same probabilities.

**Lemma 2.3** (The bounded symmetric unimodal case)**.** *Let $Z_1, \ldots, Z_m$ be independent variables whose densities are even, bounded, supported on $[-2, 2]$, and nonincreasing on $[0, 2)$. Let $Y_j$ be independent $N(0, 4)$ variables. For every real linear map $T$ and convex symmetric Borel set $A$,*

$$\mathbb{P}(TZ \in A) \geq \mathbb{P}(TY \in A).$$

*The conclusion also applies to the inverse image of a closed uniform-norm ball under a finite continuous-function map.*

*Proof.* This is a special case of classical unimodal product peakedness [16, Theorem 3.6]; the following mixture argument records the hypotheses needed here. Choose the right-continuous monotone representative of one density $f$ on $[0, 2)$ and set $f(2) = 0$. Let $\mu$ be its positive Stieltjes measure on $(0, 2]$, so that $\mu((x, 2]) = f(x)$ for $0 \leq x < 2$. Put $\nu(ds) = 2s\mu(ds)$. Tonelli's theorem gives

$$\nu((0, 2]) = 2 \int_0^2 \mu((x, 2])\, dx = 2 \int_0^2 f(x)\, dx = 1.$$

Furthermore,

$$f(x) = \int_{(0,2]} \frac{\mathbf{1}_{\{|x| < s\}}}{2s}\, \nu(ds) \quad \text{for almost every } x.$$

Thus each coordinate is a scale mixture of centered uniforms of half-width at most two. Choose its mixing scale independently of the other coordinates. Conditional on all scales, the variables are independent, symmetric and log-concave. A uniform variable on $[-s, s]$, $0 < s \leq 2$, is more peaked than $N(0, 4)$: its central mass is at least $r/2$ for $0 \leq r \leq 2$, with value one once $r \geq s$, whereas the Gaussian central mass is at most $r/\sqrt{2\pi}$. Lemma 2.2 therefore gives the claimed inequality conditionally on every scale vector. Integrating over that vector proves the result. The finite path-ball assertion follows from its convex symmetric inverse image exactly as before. □

## 3 A Gaussian reference and uniform tail control

Let $Y_h$ be independent $N(0,4)$ variables and put $G = \sum_h Y_h v_h$, using Lemma 2.1.

**Lemma 3.1.** *For $0 < r \le 1$,*

$$\mathbb{P}(\|G\|_\infty \le r) \ge \exp(-64/r^2). \tag{3.1}$$

*For every finite $I \subset \mathbb{Z}$, with $G_I = \sum_{h\in I} Y_h v_h$,*

$$\mathbb{P}(\|G_I\|_\infty \le r) \ge \mathbb{P}(\|G\|_\infty \le r). \tag{3.2}$$

*Consequently, for every $M \ge 0$ and $R_M = \sum_{|h|>M} X_h v_h$,*

$$\mathbb{P}(\|R_M\|_\infty \le r) \ge \exp(-256/r^2), \qquad 0 < r \le 1. \tag{3.3}$$

*Proof.* For $h \ge 1$ put $U_h = (Y_h + Y_{-h})/\sqrt{2}$ and $V_h = (Y_h - Y_{-h})/\sqrt{2}$. These are mutually independent $N(0,4)$ variables, independent of $Y_0$. Direct expansion gives

$$\operatorname{Re} G(t) = tY_0 + \sum_{h\ge1} \frac{U_h \sin(2\pi ht)}{\sqrt{2}\pi h},$$
$$\operatorname{Im} G(t) = \sum_{h\ge1} \frac{V_h(1-\cos(2\pi ht))}{\sqrt{2}\pi h}.$$

Consider the continuous centered Gaussian process

$$B(t) = \sum_{h\ge1} \frac{U_h \sin(2\pi ht) + V_h(\cos(2\pi ht) - 1)}{\sqrt{2}\pi h}.$$

The preceding convergence proof also applies to its partial sums. Its covariance is

$$\frac{2}{\pi^2} \sum_{h\ge1} \frac{\cos(2\pi h(t-s)) - \cos(2\pi hs) - \cos(2\pi ht) + 1}{h^2}.$$

The identity

$$\sum_{h\ge1} \frac{\cos(2\pi hx)}{h^2} = \pi^2(x^2 - x + 1/6), \qquad 0 \le x \le 1, \tag{3.4}$$

follows by computing the Fourier coefficients of the displayed quadratic; the absolutely convergent series represents that continuous periodic function. For $0 \le s \le t \le 1$, substitution gives $\operatorname{Cov}(B(s), B(t)) = 4s(1-t)$. Thus $B$ is a variance-four Brownian bridge, since a centered Gaussian law is determined by its covariances. Its sine and cosine-minus-one parts are respectively $(B(t) - B(1-t))/2$ and $(B(t) + B(1-t))/2$, each of norm at most $\|B\|_\infty$. Hence

$$\|G\|_\infty \le |Y_0| + 2\,\|B\|_\infty\,. \tag{3.5}$$

Here is an explicit confinement estimate, with its normalization recorded. Variance-four Brownian motion has generator $2\,d^2/dx^2$ and free density $1/\sqrt{8\pi}$ at $(1;0,0)$. The Dirichlet eigenfunction expansion on $(-b,b)$ gives its killed density there as

$$p_D(1;0,0) = \frac{1}{b} \sum_{j\ge0} \exp\left\{-\frac{(2j+1)^2\pi^2}{2b^2}\right\}.$$

Indeed the odd indexed eigenfunctions have squared value $1/b$ at zero, and the other ones vanish there. Dividing by the free endpoint density is the probability that the bridge stays in $(-b, b)$. Keeping the first positive term yields

$$\mathbb{P}(\|B\|_\infty < b) \geq \frac{\sqrt{8\pi}}{b} \exp\{-\pi^2/(2b^2)\} \geq \exp\{-\pi^2/(2b^2)\}, \qquad 0 < b \leq 1.$$

This is a bound from a positive kernel expansion, not an asymptotic formula. On $[-r/3, r/3]$, the $N(0,4)$ density is at least $\mathrm{e}^{-1/72}/(2\sqrt{2\pi})$ for $r \leq 1$. Independence, with $b = r/3$, and (3.5) give

$$\mathbb{P}(\|G\|_\infty \leq r) \geq c_0 r \exp\{-9\pi^2/(2r^2)\}, \qquad c_0 = \frac{\mathrm{e}^{-1/72}}{3\sqrt{2\pi}}.$$

Since $9\pi^2/2 < 45$, $\log(1/c_0) < 4$, and $\log(1/r) \leq r^{-2}$ on this range, (3.1) follows.

To prove (3.2), first evaluate paths on a finite time grid and identify complex values with a real vector. Its radius-$r$ acceptance set $A$ is a product of disks, hence convex and symmetric. If $z$ is the finite Gaussian coefficient vector of $G_I$ with evaluation map $T$, then

$$F(y) = \int \mathbf{1}_A(Tz + y)\phi(z)\, dz$$

is log-concave by Prékopa and even by reflection. Therefore $F(y) \leq F(0)$. Conditioning on the independent evaluation vector of $G - G_I$ proves that adding this complement cannot increase acceptance. Increasing the grid to a countable dense set proves the assertion for closed uniform balls, by path continuity and continuity of probability from above. This argument allows singular evaluation maps and an infinite independent complement; it is the required instance of the classical Gaussian translation comparison.

Finally let $S_{M,H} = \sum_{M<|h|\leq H} X_h v_h$. Lemma 2.2 and (3.2) show

$$\mathbb{P}(\|S_{M,H}\|_\infty \leq r') \geq \mathbb{P}(\|G\|_\infty \leq r').$$

For $r' < r$, almost sure uniform convergence implies

$$\limsup_{H\to\infty} \{\|S_{M,H}\|_\infty \leq r'\} \subseteq \{\|R_M\|_\infty \leq r\}$$

outside a null set. Reverse Fatou for bounded indicators gives the probability of this limsup at least the limsup of the probabilities. Take $r' = r/2$ and apply (3.1) to obtain (3.3). The radius slack avoids any assumption about atoms on a norm-sphere. □

# 4 Fejér approximation with graded margins

Fix $g \in \mathcal{H}_L$, $a = g(1)$ and $u = g - at$. The function $u$ vanishes at both endpoints and has Hölder constant at most $L + 2$ on $[0, 1]$. As a periodic function, it satisfies

$$|u(t) - u(s)| \leq \sqrt{2}(L + 2)\sqrt{d(t - s, \mathbb{Z})}. \tag{4.1}$$

For a wrapped pair this follows by passing through the common zero endpoint and using $\sqrt{x} + \sqrt{y} \leq \sqrt{2(x + y)}$; otherwise use the ordinary Hölder estimate.

The degree-$M$ Fejér kernel has integral one, is nonnegative, and obeys

$$F_M(t) \leq M + 1, \qquad F_M(t) \leq \frac{1}{4(M + 1)d(t, \mathbb{Z})^2}.$$

These bounds follow from $F_M(t) = \sin^2((M+1)\pi t)/((M+1)\sin^2(\pi t))$ and $\sin(\pi d) \geq 2d$ for $0 \leq d \leq 1/2$. Splitting the integral at $1/(M+1)$ gives

$$\int_0^1 F_M(t)\sqrt{d(t,\mathbb{Z})}\,dt \leq \frac{7}{3\sqrt{M+1}}, \qquad M \geq 1. \tag{4.2}$$

In fact, the inner part is at most $4/(3\sqrt{M+1})$, and the outer part is at most $1/\sqrt{M+1}$, by direct integration of the two kernel bounds. If $\sigma_M u = F_M * u$, equations (4.1) and (4.2) imply

$$\|\sigma_M u - \sigma_M u(0) - u\|_\infty \leq \frac{14\sqrt{2}(L+2)}{3\sqrt{M+1}}.$$

For $0 < \epsilon \leq 1/4$ choose

$$M = \left\lceil 4096(L+2)^2\epsilon^{-2} \right\rceil. \tag{4.3}$$

The last approximation error is at most $\epsilon/8$. Set

$$b_0 = (1-\epsilon/8)a, \qquad x_h = \frac{|h|}{M+1}, \qquad b_h = (1-x_h)\beta_h \quad (0 < |h| \leq M). \tag{4.4}$$

The pinned polynomial

$$q_M(t) = b_0 t + \sigma_M u(t) - \sigma_M u(0) = \sum_{|h|\leq M} b_h v_h(t)$$

satisfies

$$\|q_M - g\|_\infty \leq \epsilon/4 + \epsilon/8 = 3\epsilon/8. \tag{4.5}$$

The equality follows directly from the Fourier formula for a Fejér mean and (1.3).

**Lemma 4.1.** *For $T_M z = \sum_{|h|\leq M} z_h v_h$ and the coefficients in (4.4),*

$$\mathbb{P}(\|T_M(X-b)\|_\infty \leq r) \geq c_e \epsilon^{3/2} \exp\{-6M - 64/r^2\}, \qquad 0 < r \leq 1, \tag{4.6}$$

*where $c_e = \sqrt{3/8}/(8\pi) > \mathrm{e}^{-5}$.*

*Proof.* For $0 < |h| \leq M$ put $r_h = \min(1/2, x_h)$ and let $Z_h$ be independent uniform variables on $[-r_h, r_h]$. If $|z| \leq r_h$, then

$$|b_h + z| \leq 2 - 2x_h + r_h \leq 2 - x_h, \qquad f(b_h + z) \geq \frac{\sqrt{3x_h}}{2\pi}.$$

As $2r_h \geq x_h$, comparison of this shifted density with the uniform density pays a factor at least $c_* x_h^{3/2}$, where $c_* = \sqrt{3}/(2\pi)$. Multiplying over both signs of each frequency gives

$$c_*^{2M}\left(\frac{M!}{(M+1)^M}\right)^3 \geq \mathrm{e}^{-6M}. \tag{4.7}$$

Indeed,

$$\begin{aligned}\log\frac{M!}{(M+1)^M} &\geq \int_1^M \log t\,dt - M\log(M+1)\\ &= -M + 1 - M\log(1+1/M) \geq -M,\end{aligned}$$

and $3+2\log(2\pi/\sqrt{3})<6$. Retaining the differing margins $x_h$ in this product is essential: replacing all of them by their smallest value would introduce an unnecessary $M\log M$ loss.

For $h=0$, take $r_0=\epsilon/8$ and $Z_0$ uniform on $[-r_0,r_0]$. The inequalities

$$|b_0+z|\le 2-\epsilon/8, \qquad f(b_0+z)\ge \frac{\sqrt{3\epsilon/8}}{2\pi}$$

hold on this interval, including when $a=\pm 2$. Its density factor is at least $c_e\epsilon^{3/2}$. Integrating the finite product density only on the box $\prod_{|h|\le M}[-r_h,r_h]$ gives

$$\mathbb{P}(\|T_M(X-b)\|_\infty\le r)\ge c_e\epsilon^{3/2}\mathrm{e}^{-6M}\mathbb{P}(\|T_M Z\|_\infty\le r).$$

All uniform widths are positive and at most $1/2$. Lemma 2.2, followed by (3.2) and (3.1), bounds the last probability below by $\mathrm{e}^{-64/r^2}$. This proves (4.6). The shifts were handled by a finite density comparison before applying the centered comparison; no translated peakedness or infinite product of shifted densities was used. □

# 5 Proof of the two-sided bounds

*Proof of the lower bound in Theorem 1.1.* The low-frequency event in Lemma 4.1 with $r=\epsilon/4$ has probability at least

$$c_e\epsilon^{3/2}\exp\{-6M-1024\epsilon^{-2}\}.$$

The independent high-frequency event $\|R_M\|_\infty\le\epsilon/4$ has probability at least $\exp(-4096\epsilon^{-2})$ by (3.3). Together with (4.5), these events imply

$$\|K-g\|_\infty\le\epsilon/4+\epsilon/4+3\epsilon/8=7\epsilon/8<\epsilon.$$

Their product gives the strict-ball lower bound $c_e\epsilon^{3/2}\exp\{-6M-5120\epsilon^{-2}\}$. For the allowed parameters,

$$M\le 4097(L+2)^2\epsilon^{-2}, \qquad \log\frac{1}{c_e\epsilon^{3/2}}\le 7\epsilon^{-2}.$$

Therefore its negative logarithm is at most

$$[24582(L+2)^2+5127]\epsilon^{-2}\le 26000(L+2)^2\epsilon^{-2},$$

which proves (1.4). □

*Proof of the upper bound in Theorem 1.1.* For an arbitrary $g\in C([0,1],\mathbb{C})$, set $a=g(1)$, $u=g-at$ and $\beta_h=2\pi ih\widehat{u}(h)$, which may now be complex. On the strict small-ball event, $K(1)=X_0$ gives $|X_0-a|<\epsilon$. Consequently

$$D(t)=K(t)-tX_0-u(t)=(K(t)-g(t))+t(a-X_0)$$

has norm less than $2\epsilon$. Uniform convergence justifies its nonzero Fourier coefficients $\widehat{D}(h)=(X_h-\beta_h)/(2\pi ih)$. Parseval, retaining only positive frequencies through $N$, gives

$$\sum_{h=1}^{N}|X_h-\beta_h|^2\le N^2\sum_{h=1}^{N}\frac{|X_h-\beta_h|^2}{h^2}\le 16\pi^2N^2\epsilon^2. \tag{5.1}$$

Since $f \le 1/\pi$, an interval of length one has probability at most $1/\pi$. The independent indicators $I_h = \mathbf{1}_{\{|X_h - \operatorname{Re}\beta_h| \ge 1/2\}}$ have success probabilities at least $p = 1 - 1/\pi$. On the small-ball event,

$$\sum_{h=1}^{N} I_h \le 4 \sum_{h=1}^{N} |X_h - \operatorname{Re}\beta_h|^2 \le 4 \sum_{h=1}^{N} |X_h - \beta_h|^2 \le 64\pi^2 N^2 \epsilon^2.$$

With $N = \lfloor \delta\epsilon^{-2} \rfloor$ and $\delta = p/(128\pi^2)$, the right-hand side is at most $pN/2$. For clarity, exponential Markov inequality at $\lambda = \log 2$ yields

$$\begin{aligned} \mathbb{P}\left(\sum I_h \le pN/2\right) &\le \exp\{\lambda pN/2 + pN(\mathrm{e}^{-\lambda} - 1)\} \\ &= \exp\{-pN(1 - \log 2)/2\} \le \exp(-pN/8). \end{aligned}$$

Here each success probability is at least $p$, so $\mathbb{E}\mathrm{e}^{-\lambda I_h} \le \exp(p(\mathrm{e}^{-\lambda} - 1))$. If $\epsilon^2 \le \delta/2$, then $N \ge \delta/(2\epsilon^2)$, giving (1.6) with the constants (1.5). □

## 6 Effective arithmetic and shrinking targets

For an odd prime $p$ and $(A, B) \in (\mathbb{F}_p^\times)^2$, let $K_p(\cdot; A, B)$ be the polygonal path with vertices

$$K_p\left(\frac{j}{p-1}; A, B\right) = p^{-1/2} \sum_{1 \le x \le j} \exp\left(\frac{2\pi i(Ax + B\overline{x})}{p}\right), \qquad 0 \le j \le p-1. \tag{6.1}$$

Here $\overline{x}$ is the inverse of $x$ modulo $p$, and both parameters are chosen independently and uniformly nonzero. Write $\mu_p$ for this path law and $\mu = \operatorname{Law}(K)$. On probability measures with finite first uniform moment, $W_1$ is the Wasserstein distance for the cost $\|x - y\|_\infty$, equivalently the supremum of differences of integrals of real 1-Lipschitz functions.

The qualitative convergence $\mu_p \Rightarrow \mu$ is known [13, Theorem 1.5]. We quantify it for this two-parameter ensemble. The sheaf-theoretic machinery below is classical: in particular the rank, puncture and Swan trace estimate is the curve estimate explained in [9, Remark 4.2]. The proof records its dependence on growing representation degree. Growing-moment bounds on dyadic Fourier blocks then sharpen the path approximation, before a coordinate-weighted Jackson comparison. The current author version of Kowalski–Untrau [10, Section 1.3(4)] notes related quantitative path work by S. Pitte.

For $M \ge 1$ put

$$a_0 = 1, \qquad a_h = \frac{1}{\pi|h|} \ (0 < |h| \le M), \qquad S_M = \sum_{|h| \le M} a_h = 1 + \frac{2}{\pi} \sum_{h=1}^{M} \frac{1}{h}.$$

For integers $2 \le M \le N < p/2$ and $n_h \ge 1$, $|h| \le M$, put $d = 2M + 1$, $r = \lceil \log(16M) \rceil$ and

$$\begin{aligned} D_C(p, M, N, (n_h)) = {} & 1024\sqrt{r/M} + 2048N^{-1/4} + 384r(N^2/p)^{1/(4r)} \\ & + p^{-1/2} + \frac{8M}{p} + 32 \sum_{|h| \le M} \frac{a_h}{n_h} \\ & + \frac{8S_M(d+1)4^d \prod_{|h| \le M} n_h^2}{\sqrt{p}}. \end{aligned} \tag{6.2}$$

**Theorem 6.1.** *For every odd prime $p$, integers $2 \le M \le N < p/2$ and $n_h \ge 1$,*

$$W_1(\mu_p, \mu) \le D_C(p, M, N, (n_h)).$$

*In particular,*

$$W_1(\mu_p, \mu) = O\left(\frac{\log\log p}{\sqrt{\log p}}\right). \tag{6.3}$$

*A coarse explicit version is $W_1(\mu_p, \mu) \le 20000 \log\log p/\sqrt{\log p}$ for $\log p \ge 10^6$.*

## 6.1 Path approximation with a uniform expected modulus

For $I \subseteq \mathbb{F}_p^\times$, $H = |I|$, set

$$T_I = p^{-1/2} \sum_{x \in I} \exp(2\pi i(Ax + B/x)/p).$$

Orthogonality over all $A, B \in \mathbb{F}_p$ gives

$$\sum_{A,B \in \mathbb{F}_p} |T_I|^4 = N_I,$$

where $N_I$ counts quadruples with equal sums and equal inverse sums. If the common sum is nonzero, the inverse-sum condition forces equal products; each first ordered pair has at most two second pairs. If it is zero, both pairs are opposite and there are at most $H^2$ quadruples. Thus $N_I \le 3H^2$. Restricting this nonnegative sum to nonzero parameters yields

$$\mathbb{E}_{A,B \neq 0} |T_I|^4 \le \frac{3H^2}{(p-1)^2}. \tag{6.4}$$

This use of both parameters is essential to the present argument.

Define $Q_p$ on the mesh $j/p$ using the same partial sums through $j = p - 1$, and make its last mesh segment constant. Then $K_p(t) = Q_p((p-1)t/p)$ and

$$\|K_p - Q_p\|_\infty \le p^{-1/2}, \tag{6.5}$$

because $Q_p$ is deterministically $\sqrt{p}$-Lipschitz. For $\delta = |t - s| < p^{-1}$ this bound gives $|Q_p(t) - Q_p(s)|^4 \le \delta^2$. For $\delta \ge p^{-1}$, decompose an increment into its complete interior mesh intervals and at most two fractional endpoint increments. The interior set has at most $p\delta$ elements, and each endpoint increment is at most $p^{-1/2}$. Using (6.4), $p/(p-1) \le 3/2$ and $|z_1 + z_2 + z_3|^4 \le 27 \sum |z_j|^4$ gives

$$\mathbb{E}|Q_p(t) - Q_p(s)|^4 \le 1024\delta^2. \tag{6.6}$$

The same estimate holds for $K$. Indeed

$$\sum_{h \in \mathbb{Z}} |v_h(t) - v_h(s)|^2 = \delta$$

by (3.4). The bounded-variable exponential estimate in Lemma 2.1, applied to the real and imaginary parts and then integrated, gives the displayed fourth-moment bound. Finite truncations converge in $L^4$ at fixed times, so the estimate passes to the series.

Here is the expected-modulus consequence with constants. For a continuous $X$ satisfying (6.6) and $X(0) = 0$, let $A_j$ be the largest adjacent dyadic increment of level $j$. The fourth moment and a union bound give $\mathbb{E}A_j \le 6\, 2^{-j/4}$. If $\delta \le 2^{-m} < 2\delta$, dyadic chaining and continuity give

$$\omega_X(\delta) \le A_m + 2 \sum_{j > m} A_j, \qquad \mathbb{E}\omega_X(\delta) \le 100\delta^{1/4}.$$

Also $\mathbb{E}|X(1)| < 6$. Pin $u_X(t) = X(t) - tX(1)$ and extend it periodically. For a wrapped pair, split at the common zero endpoint. The expected circular modulus is at most $208\delta^{1/4}$.

For the ordinary Fejér kernel, direct integration of its two bounds in Section 4 gives

$$\int_{-1/2}^{1/2} |t|^{1/4} F_M(t)\, dt \le 3(M+1)^{-1/4}.$$

With $\mathcal{P}_M X = tX(1) + F_M * u_X - (F_M * u_X)(0)$, it follows that

$$\mathbb{E}\, \|X - \mathcal{P}_M X\|_\infty \le 2048 M^{-1/4}, \qquad X = Q_p,\ K. \tag{6.7}$$

The dyadic sums also prove finite first uniform moments. These are expected-modulus estimates.

## 6.2 Arithmetic coefficients and character correlations

Put

$$W_h = p^{-1/2} \sum_{x \in \mathbb{F}_p^\times} \exp(2\pi i((A-h)x + B/x)/p) = \mathrm{Kl}_2(B(A-h); p).$$

These variables are real and $|W_h| \le 2$, including the singular value $W_h = -p^{-1/2}$ when $A = h$. Integrating each mesh derivative against $\mathrm{e}^{-2\pi i h t}$ shows that the normalized pinned coefficient of $Q_p$ is $c_{p,h} W_h$, where

$$c_{p,h} = \mathrm{e}^{\pi i h/p} \frac{\sin(\pi h/p)}{\pi h/p}, \qquad |c_{p,h} - 1| \le 2\pi |h|/p \quad (0 < |h| < p/2).$$

The endpoint is $W_0$. On $[-2,2]^{2M+1}$ define

$$\mathcal{A}_M(x) = x_0 t + \sum_{0<|h|\le M} \left(1 - \frac{|h|}{M+1}\right) x_h v_h(t).$$

Then

$$\|\mathcal{P}_M Q_p - \mathcal{A}_M(W)\|_\infty \le 8M/p, \qquad \mathcal{P}_M K = \mathcal{A}_M((X_h)_{|h|\le M}). \tag{6.8}$$

For the first inequality each nonzero frequency contributes at most $2|c_{p,h} - 1|/(\pi|h|) \le 4/p$.

**Lemma 6.2.** *Define $\chi_k(2\cos\theta) = \sin((k+1)\theta)/\sin\theta$ for $k \ge 0$, using continuous endpoint values. Let $h_i$ be distinct modulo an odd prime $p$, and let $k_i \ge 0$ not all vanish. Put $s = \#\{i : k_i > 0\}$ and $D = \prod_i (k_i + 1)$. For each fixed $B \ne 0$,*

$$\left| \mathbb{E}_{A \ne 0} \prod_i \chi_{k_i}(W_{h_i}) \right| \le \frac{4(s+1)D}{\sqrt{p}}. \tag{6.9}$$

*The corresponding expectation for independent Sato–Tate variables is zero. The trivial character has zero discrepancy.*

*Proof.* The polynomial $\chi_k$ is the trace character of the $k$th symmetric power of the standard $\mathrm{SL}_2$ representation. It has dimension $k+1$ and $|\chi_k(x)| \le k+1$ on $[-2,2]$. Also

$$\mathbb{E}\chi_k(X) = \frac{2}{\pi} \int_0^\pi \sin((k+1)\theta) \sin\theta\, d\theta = \mathbf{1}_{\{k=0\}}.$$

We now specify the classical sheaf inputs and their quantitative use. The normalized rank-two Kloosterman sheaf is pure of weight zero, has trivial determinant, trace $-\mathrm{Kl}_2$, geometric and arithmetic monodromy $\mathrm{SL}_2$, tame nontrivial unipotent ramification at zero, and sole infinity break $1/2$, of Swan conductor one; see [7, proof of Proposition 3.2]. The change $y = BA$ reduces the active factors to distinct translations $y \mapsto y - Bh_i$. Their direct sum has full product geometric monodromy $\mathrm{SL}_2^s$ by the Goursat–Kolchin–Ribet argument in [13, Lemma 2.4]. Distinct finite singularities exclude rank-one-twisted geometric isomorphisms: nontrivial unipotent local monodromy cannot become unramified by a scalar twist.

Tensor the symmetric powers with positive $k_i$. The resulting rank-$D$ sheaf $\mathcal{F}$ on $U = \mathbb{A}^1 \setminus \{Bh_i : k_i > 0\}$ has no geometric invariants: each nontrivial symmetric power is an irreducible nontrivial representation of its own factor of the product. Irreducibility here is in characteristic zero; the usual raising and lowering operators connect its monomial basis. At finite punctures it is tame. Every infinity upper ramification group of index greater than $1/2$ acts trivially on every rank-two factor, hence on tensor powers, symmetric-power subquotients and their tensor product. Thus $\mathrm{Swan}_\infty(\mathcal{F}) \le D/2$, independently of the degrees except for the explicit rank.

The trace/Euler/weight formulas [8, Theorems 4.1, 4.2, 4.6 and (4.2)], also summarized in the curve estimate [9, Remark 4.2], give

$$H_c^0 = H_c^2 = 0, \qquad \dim H_c^1 = D(s-1) + \mathrm{Swan}_\infty(\mathcal{F}) \le Ds.$$

The sheaf is pure of weight zero, so the full $U$ trace sum has absolute value at most $Ds\sqrt{p}$ by Deligne's theorem. Its trace is $(-1)^{\sum k_i} \prod_i \chi_{k_i}(W_{h_i})$; this sign does not affect the zero-main-term bound.

At most $s+1$ points are changed when passing from $U(\mathbb{F}_p)$ to $\mathbb{F}_p^\times$. Each product character there is at most $D$ in absolute value, including at singular values $W_h = -p^{-1/2}$. Therefore

$$\left| \mathbb{E}_{A\neq 0} \prod_i \chi_{k_i}(W_{h_i}) \right| \le D\,\frac{s\sqrt{p}+s+1}{p-1} \le \frac{4(s+1)D}{\sqrt{p}}.$$

The proof is uniform in $B$ and the distinct shifts. This is an explicit use of the classical character trace bound, rather than an extrapolation of a fixed-degree implicit constant. □

### 6.3 Growing moments and a sharper path approximation

For distinct residues $h$, complex scalars $b_h$, and an integer $r \ge 1$, put $\sigma^2 = \sum_h |b_h|^2$ and $\ell = \sum_h |b_h|$. Then

$$\left( \mathbb{E} \left| \sum_h b_h W_h \right|^{2r} \right)^{1/(2r)} \le 8\sqrt{r}\,\sigma + 8\ell p^{-1/(4r)}. \tag{6.10}$$

Here and below the arithmetic expectation averages both nonzero parameters. To see the degree dependence explicitly, the character $x^m = \chi_1(x)^m$ decomposes as $\sum_k c_{m,k}\chi_k(x)$ with nonnegative integer multiplicities and $\sum_k c_{m,k}(k+1) = 2^m$. This follows by induction from $\chi_1\chi_k = \chi_{k+1} + \chi_{k-1}$ for $k \ge 1$ and $\chi_1\chi_0 = \chi_1$. Thus Lemma 6.2 bounds the discrepancy of any monomial of total degree $2r$ by $4(2r+1)2^{2r}p^{-1/2}$: the trivial character cancels, and every other term involves at most $2r$ active shifts. Expanding the absolute $2r$th power gives

$$\mathbb{E}|\sum_h b_h W_h|^{2r} \le \mathbb{E}|\sum_h b_h X_h|^{2r} + 4(2r+1)(2\ell)^{2r}p^{-1/2}.$$

For the independent variables, the bounded-variable exponential inequality gives $\mathbb{P}(|\sum b_h X_h| > u) \le 4\exp[-u^2/(16\sigma^2)]$. Integration bounds its $2r$th moment by $4(16\sigma^2)^r r!$. Taking roots, using $r! \le r^r$ and $2[4(2r+1)]^{1/(2r)} \le 8$, proves (6.10). The second term is absent for independent coefficients. This argument does not require a sheaf with all available shifts active simultaneously.

For a trigonometric polynomial $T$ of degree at most $H$, the Bernstein inequality $\|T'\|_\infty \le 2\pi H\|T\|_\infty$ and the grid of $J = \lceil 4\pi H\rceil \le 14H$ equally spaced points give

$$\|T\|_\infty \le 2 \max_{0\le j<J} |T(j/J)|.$$

For complex $T$, the Bernstein inequality follows from its real form by rotating the value of the derivative at a maximizing point and taking the real part. Applying (6.10) at the grid points, with no independence assumption on those points, yields

$$\mathbb{E}\|T - T(0)\|_\infty \le 32(14H)^{1/(2r)}(\sqrt{r}\,\sigma + \ell p^{-1/(4r)}), \quad T(t) = \sum_{|h|\le H} b_h W_h \mathrm{e}^{2\pi i h t}. \tag{6.11}$$

Write $w_J(h) = (1 - |h|/(J+1))_+$. The difference $\mathcal{P}_N Q_p - \mathcal{P}_M Q_p$ has zero endpoint term and coefficients

$$b_h = \frac{[w_N(h) - w_M(h)]c_{p,h}}{2\pi i h}, \qquad 0 < |h| \le N,$$

in front of $W_h(\mathrm{e}^{2\pi i h t} - 1)$. In particular $|c_{p,h}| \le 1$. Split the frequencies into $1 \le |h| \le M$ and the subsequent blocks $H_{j-1} < |h| \le \min(H_j, N)$, where $H_j = 2^j M$ and $1 \le j \le J_* := \lceil \log_2(N/M)\rceil$. The first block has $|b_h| \le 1/(2\pi M)$, hence $\ell \le 1/\pi < 1$ and $\sigma^2 \le 1/(2\pi^2 M) < 1/M$. Each later block has $|b_h| \le 1/(\pi H_j)$ and at most $2H_j$ coefficients, so $\ell < 1$ and $\sigma^2 < 1/H_j$ again. The last block has degree at most $H_{J_*} < 2N$, but its nonzero frequencies still lie in $[-N, N]$ and are distinct modulo $p$.

Set $r = \lceil \log(16M)\rceil \ge 4$. The first terms in (6.11) sum to at most

$$32\sqrt{r}\,(14M)^{1/(2r)} M^{-1/2} \sum_{j\ge 0} 2^{j(-1/2+1/(2r))} \le 512\sqrt{r/M},$$

since $(14M)^{1/(2r)} \le \sqrt{\mathrm{e}} < 2$ and the sum is at most $(1 - 2^{-3/8})^{-1} < 5$. The second terms sum to at most $384r(N^2/p)^{1/(4r)}$, using $(1 - 2^{-1/(2r)})^{-1} \le 6r$ and $28^{1/(2r)} < 2$. The former inequality follows from $1 - \mathrm{e}^{-x} \ge x/2$ for $0 \le x \le 1$, with $x = \log 2/(2r)$. Apply (6.7) only at the larger cutoff $N$ to obtain

$$\begin{aligned}\mathbb{E}\|Q_p - \mathcal{P}_M Q_p\| \le{}& 512\sqrt{r/M} + 2048N^{-1/4} \\ &+ 384r(N^2/p)^{1/(4r)}.\end{aligned} \tag{6.12}$$

For $K$ the block proof has no arithmetic error. Letting $N \to \infty$ and using the $L^1$ convergence in (6.7) gives

$$\mathbb{E}\|K - \mathcal{P}_M K\| \le 512\sqrt{r/M}. \tag{6.13}$$

### 6.4 A coordinate-weighted Jackson approximation

For $n \ge 1$, on the angular torus with measure $d\theta/(2\pi)$, put

$$S_n(\theta) = \frac{\sin(n\theta/2)}{\sin(\theta/2)}, \qquad I_n = \frac{2n^3 + n}{3}, \qquad J_n = S_n^4/I_n.$$

The continuous value is used at zero. Since $S_n^2 = \sum_{|k|<n}(n-|k|)e^{ik\theta}$, Parseval gives $\int S_n^4 = I_n$. Thus $J_n$ is an even nonnegative probability kernel of degree $2(n-1)$. The bounds $|S_n(\theta)| \le \min(n, \pi/|\theta|)$ and $I_n \ge 2n^3/3$ imply

$$\int_{-\pi}^{\pi} \theta^2 J_n(\theta) \frac{d\theta}{2\pi} \le 64n^{-2}. \tag{6.14}$$

Indeed splitting the numerator at $1/n$ bounds it by $n/(3\pi) + \pi^3 n$; division gives a constant below 47. This also covers $n = 1$.

Fix a real 1-Lipschitz path functional $\varphi$, with $\varphi(0) = 0$, and set $G(x) = \varphi(\mathcal{A}_M(x))$. Then

$$|G(x) - G(y)| \le \sum_{|h|\le M} a_h |x_h - y_h|, \qquad |G(x)| \le 2S_M.$$

Lift $x_h = 2\cos\theta_h$ and convolve with the product kernels $J_{n_h}$. By (6.14), each angular displacement has expected absolute value at most $8/n_h$. The convolved even function therefore represents an algebraic polynomial $P$ in $y_h = x_h/2$ with

$$\|G - P\|_{\infty,[-2,2]^{2M+1}} \le 16 \sum_{|h|\le M} a_h/n_h. \tag{6.15}$$

Its cosine degree in coordinate $h$ is at most $q_h = 2(n_h - 1)$. Each complex Fourier coefficient has absolute value at most $2S_M$, since a probability kernel has multipliers bounded by one. Grouping sign orbits into cosine products costs no more than summing the original complex Fourier modes.

Define $C_j(2\cos\theta) = \cos(j\theta)$. The exact identities

$$C_0 = \chi_0, \qquad C_1 = \chi_1/2, \qquad C_j = (\chi_j - \chi_{j-2})/2 \quad (j \ge 2)$$

show that the absolute character mass weighted by representation dimension is at most $j + 1$ for $C_j$ (in fact it is $j$ for $j \ge 2$). If $P(x) = \sum_{\mathbf{k}} b_{\mathbf{k}} \prod_h \chi_{k_h}(x_h)$, the dimension-weighted mass therefore satisfies

$$\sum_{\mathbf{k}} |b_{\mathbf{k}}| \prod_h (k_h + 1) \le 2S_M \prod_h \sum_{j=-q_h}^{q_h} (|j| + 1) \le 2S_M \prod_h (4n_h^2). \tag{6.16}$$

Indeed $q_h^2 + 3q_h + 1 = 4n_h^2 - 2n_h - 1 \le 4n_h^2$. This use of standard characters retains polynomial rather than exponential dependence on each coordinate degree.

The trivial character has zero discrepancy. Apply Lemma 6.2 to every other term, with $s \le d$, and average over $B$. Together with the two errors (6.15), (6.16) gives

$$|\mathbb{E}G(W) - \mathbb{E}G(X)| \le 32 \sum_h a_h/n_h + \frac{8S_M(d+1)4^d \prod_h n_h^2}{\sqrt{p}}.$$

This is a direct path-functional estimate, so no further coefficient-map Lipschitz factor is needed. Adding (6.12), (6.13), (6.5) and (6.8), and taking the supremum over $\varphi$, proves the finite bound of Theorem 6.1. The finite first moments already established justify Kantorovich–Rubinstein duality on the separable path space.

For the rate put $t = \log p \ge 10^6$ and choose

$$M = \left\lfloor \frac{t}{128 \log t} \right\rfloor, \quad N = \lfloor p^{1/8} \rfloor, \quad r = \lceil \log(16M) \rceil,$$
$$n_0 = \lceil M/\sqrt{r} \rceil, \qquad n_h = \lceil M/\sqrt{r|h|} \rceil \quad (h \ne 0). \tag{6.17}$$

Then $M \geq r \geq 4$, $M \leq N < p/2$, $r \leq \log t$ and $M \geq t/(256 \log t)$. For the bound on $r$, use $\log(16M) + 1 \leq \log t$; for $M \geq r$, use $t/(256 \log t) \geq \log t$, valid throughout this range. Since $\sum_{h=1}^{M} h^{-1/2} \leq 2\sqrt{M}$,

$$\sum_h a_h/n_h \leq (1 + 4/\pi)\sqrt{r/M}.$$

Every unrounded degree is at least one, and hence

$$4^d \prod_h n_h^2 \leq \frac{16^d M^{2d}}{r^d (M!)^2}.$$

Using $\log M! \geq M \log M - M$ and discarding $-d \log r$, the logarithm of the right side is at most

$$(2M + 1) \log 16 + (2M + 2) \log M + 2M \leq 16M \log(\mathrm{e}M) \leq t/8.$$

For the first inequality, $(2M + 1) \log 16 \leq 7M$ and $(2M + 2) \log M \leq 3M \log M$ for $M \geq 2$ suffice. The final inequality follows from $\log(\mathrm{e}M) \leq \log t$. Thus the character remainder is at most $8S_M(d + 1)\mathrm{e}^{-3t/8}$. The principal terms in (6.2) are at most

$$[1024 + 32(1 + 4/\pi)]\sqrt{r/M} < 18000 \frac{\log t}{\sqrt{t}}.$$

Since $N \geq p^{1/8}/2$, $r \leq \log t$, and $M, S_M, d + 1 \leq t$, the remaining terms total at most

$$2500\mathrm{e}^{-t/32} + 384 \log t\, \mathrm{e}^{-3t/(16 \log t)} + 9t\mathrm{e}^{-t/2} + 8t^2\mathrm{e}^{-3t/8} \leq \frac{\log t}{\sqrt{t}}.$$

Each of the four terms is at most $\log t/(4\sqrt{t})$ at $t = 10^6$, as direct logarithms show, and each ratio to $\log t/\sqrt{t}$ decreases thereafter. For the second ratio its logarithmic derivative is $1/(2t) - 3(\log t - 1)/(16(\log t)^2) < 0$; the other derivatives are immediately negative. This proves (6.3) with the coarse stated constant.

### 6.5 Uniform arithmetic neighborhoods

Put $A_L = 104000(L + 2)^2$. A bump equal to one on the closed $\epsilon/2$-ball, linear on the annulus and zero outside the open $\epsilon$-ball has Lipschitz constant $2/\epsilon$. Theorem 1.1 and Theorem 6.1 yield

$$\inf_{g \in \mathcal{H}_L} \mu_p(B(g, \epsilon)) \geq \exp(-A_L \epsilon^{-2}) - \frac{2D_C(p, M, N, (n_h))}{\epsilon}, \qquad 0 < \epsilon \leq 1/4. \tag{6.18}$$

This is a finite-prime inequality, even when its right side is negative. The balls are open; using a closed inner ball needs no norm-sphere atom assumption.

**Corollary 6.3.** *Fix $L \geq 1$ and $0 < \kappa < 1/2$, and put*

$$\epsilon_p = \sqrt{\frac{A_L}{\kappa \log \log p}}.$$

*For all sufficiently large primes, with an effective threshold,*

$$\inf_{g \in \mathcal{H}_L} \#\{(A, B) \in (\mathbb{F}_p^\times)^2 : \|K_p(\cdot; A, B) - g\|_\infty < \epsilon_p\} \geq \tfrac{1}{2}(p - 1)^2 (\log p)^{-\kappa}.$$

*In particular $\kappa = 1/4$ gives $\epsilon_p = \sqrt{4A_L/\log \log p}$.*

*Proof.* The main term in (6.18) is $(\log p)^{-\kappa}$. The error is $O_L((\log p)^{-1/2}(\log\log p)^{3/2})$, eventually at most half the main term. The explicit estimate (6.2) and $\epsilon_p \le 1/4$ give an effective threshold. □

The bounds permit deterministic prime-dependent centers in one fixed class. They give very slowly shrinking radii and enormous thresholds; they are not practical small-prime estimates.

The following fixed-radius consequence is also useful independently of the quantified estimate. We include its compactness proof for later use with other fixed regularity classes.

**Corollary 6.4.** *Fix $L \ge 1$ and $0 < \epsilon \le 1/4$. Then*

$$\liminf_{\substack{q\to\infty \\ q \text{ prime}}} \inf_{g\in\mathcal{H}_L} \frac{1}{(q-1)^2}\#\{(a,b)\in(\mathbb{F}_q^\times)^2 : \|K_q(\cdot;a,b)-g\|_\infty < \epsilon\} \\ \ge \exp\{-104000(L+2)^2\epsilon^{-2}\}. \tag{6.19}$$

*In particular the same bound holds for any deterministic choice of centers $g_q \in \mathcal{H}_L$.*

*Proof.* The class $\mathcal{H}_L$ is compact in uniform norm. Indeed $|g(t)| \le L\sqrt{t} \le L$ and its common Hölder bound gives equicontinuity, hence relative compactness by Arzelà–Ascoli. It is closed: endpoints and Hölder inequalities pass to uniform limits, and for each fixed $h \ne 0$ the map

$$g \longmapsto 2\pi i h \int_0^1 [g(t) - tg(1)]\mathrm{e}^{-2\pi i h t}\,dt$$

is continuous. Its constraint to the closed real interval $[-2,2]$ is therefore preserved. The intersection of these constraints is closed.

Choose a finite $\epsilon/4$-net $g_1,\dots,g_J$ from the class. For every $g\in\mathcal{H}_L$ some $j$ satisfies $B(g_j,\epsilon/2)\subset B(g,\epsilon)$, with open uniform balls. It follows that

$$\inf_{g\in\mathcal{H}_L}\mu_q(B(g,\epsilon)) \ge \min_{1\le j\le J}\mu_q(B(g_j,\epsilon/2)).$$

Portmanteau and Theorem 1.1 give, for each $j$,

$$\liminf_q \mu_q(B(g_j,\epsilon/2)) \ge \mu(B(g_j,\epsilon/2)) \ge \exp\{-104000(L+2)^2\epsilon^{-2}\}.$$

For finitely many sequences, the liminf of their minimum is at least the minimum of their liminfs: take the largest of their finitely many eventual starting indices. This proves (6.19). The deterministic moving-center assertion follows from the infimum bound. □

# 7 The exact worst-case scale for Hölder centers

For $0<\gamma\le 1$ and $L\ge 1$, let $\mathcal{H}_{\gamma,L}$ have the same endpoint and normalized Fourier restrictions as $\mathcal{H}_L$, with its regularity condition replaced by $|g(t)-g(s)|\le L|t-s|^\gamma$. Thus $\mathcal{H}_{1/2,L}=\mathcal{H}_L$. Define

$$F_{\gamma,L}(\epsilon) = \inf_{g\in\mathcal{H}_{\gamma,L}} \mathbb{P}(\|K-g\|_\infty < \epsilon).$$

**Theorem 7.1.** *For every fixed $0<\gamma\le 1$ and $L\ge 1$, there are constants $0 < c_\gamma, C_{\gamma,L} < \infty$ and $\epsilon_\gamma > 0$ such that*

$$\exp[-C_{\gamma,L}\epsilon^{-\max(2,1/\gamma)}] \le F_{\gamma,L}(\epsilon) \le \exp[-c_\gamma\epsilon^{-\max(2,1/\gamma)}], \qquad 0<\epsilon\le\epsilon_\gamma. \tag{7.1}$$

*For $0<\gamma<1/2$, one may take $c_\gamma = 64^{-1/\gamma}/2048$.*

The rough upper bound is a worst-case statement: the center used to bound the infimum can change with $\epsilon$. Every center constructed below is piecewise linear, hence individually 1/2-Hölder with a nonuniform constant. The uniform bound on the weaker $\gamma$-Hölder norm is what makes them one admissible family. Theorem 1.1 retains its stronger each-center conclusion at exponent 1/2.

*Proof.* First suppose $0 < \gamma < 1/2$. Put

$$E_\gamma = \frac{2}{\gamma+1} + \frac{1}{2(1-\gamma)}, \qquad D_\gamma = 2^{2-\gamma} E_\gamma.$$

For $u = g - at$, the circular Hölder constant is at most $2^{1-\gamma}(L+2)$, by passing through the zero endpoint as in (4.1). The Fejér bounds in Section 4, with $n = M+1$, give

$$\int_0^1 F_M(t) d(t, \mathbb{Z})^\gamma \, dt \le E_\gamma n^{-\gamma}.$$

Indeed the two inner intervals contribute at most $2n \int_0^{1/n} t^\gamma \, dt = 2n^{-\gamma}/(\gamma+1)$, and the outer intervals contribute at most $(2n)^{-1} \int_{1/n}^{1/2} t^{\gamma-2} \, dt \le n^{-\gamma}/(2(1-\gamma))$. Pinning at zero therefore gives error at most $D_\gamma(L+2)(M+1)^{-\gamma}$. Choose

$$M = \left\lceil [8D_\gamma(L+2)]^{1/\gamma} \epsilon^{-1/\gamma} \right\rceil.$$

This error is at most $\epsilon/8$. The finite shifted-density proof of Lemma 4.1 uses only the endpoint and Fourier bounds, so it applies to this cutoff without change. Together with the independent tail event, it gives

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \ge c_e \epsilon^{3/2} \exp[-6M - 5120\epsilon^{-2}].$$

Since $1/\gamma > 2$, the required lower bound follows for $0 < \epsilon \le 1/4$, for example with

$$C_{\gamma,L} = 6\{[8D_\gamma(L+2)]^{1/\gamma} + 1\} + 5127. \tag{7.2}$$

We now construct centers giving the opposite estimate. For every dyadic $J = 2^m$, there are classical Rudin–Shapiro signs $\sigma_1, \ldots, \sigma_J$ with

$$\sup_{|z|=1} \left| \sum_{j=1}^J \sigma_j z^j \right| \le \sqrt{2J}. \tag{7.3}$$

The elementary construction, also used in [15], starts with $P_0 = Q_0 = 1$ and takes

$$P_{m+1} = P_m + z^{2^m} Q_m, \qquad Q_{m+1} = P_m - z^{2^m} Q_m.$$

The coefficients are signs on disjoint concatenated supports, and on the unit circle the parallelogram identity gives $|P_m|^2 + |Q_m|^2 = 2^{m+1} = 2J$. Multiplying $P_m$ by $z$ proves (7.3). These signs and their cancellation are classical.

Set

$$\lambda = \frac{1}{8\sqrt{2}}, \quad N = \lceil J^{1/(2\gamma)} \rceil, \quad A = \lambda J^{-1/2}, \quad t_j = \frac{j}{4J}, \quad \phi(x) = (1 - |x|)_+,$$

and choose $J$ large enough that $N \ge 16J$. Define

$$g_J(t) = A \sum_{j=1}^J \sigma_j \{\phi(N(t - t_j)) - \phi(N(t - (1 - t_j)))\}. \tag{7.4}$$

All supports are disjoint and interior to $[0,1]$: the left centers lie in $[1/(4J), 1/4]$, their spacing is $1/(4J)$ and their radii are at most $1/(16J)$; the remaining supports are their reflections. Thus $g_J$ is continuous, real, zero at both endpoints, and odd under $t \mapsto 1-t$. It has norm at most $A$ and global Lipschitz constant at most $AN$, because its piecewise linear slopes have that bound. It follows that

$$|g_J(t) - g_J(s)| \le \min(AN|t-s|, 2A) \le 2^{1-\gamma} AN^{\gamma} |t-s|^{\gamma} \le 2\lambda |t-s|^{\gamma}.$$

The last step uses $N \le 2J^{1/(2\gamma)}$.

The full-line Fourier transform of $\phi$ is $\widehat{\phi}(\xi) = (\sin(\pi\xi)/(\pi\xi))^2$. Since the bumps are interior, their periodic Fourier coefficients give

$$\widehat{g}_J(h) = -\frac{2iA}{N}\widehat{\phi}(h/N)\sum_{j=1}^{J}\sigma_j \sin(2\pi h t_j), \qquad h \ne 0.$$

Consequently their normalized coefficients are real and equal to

$$\beta_h = 4\pi A \frac{h}{N}\widehat{\phi}(h/N)\sum_{j=1}^{J}\sigma_j \sin(2\pi h t_j). \tag{7.5}$$

The inequality $\sin^2 x \le \min(x^2, 1) \le |x|$ gives $|\xi\widehat{\phi}(\xi)| \le 1/\pi$. The sine sum is an imaginary part of the polynomial in (7.3), so $|\beta_h| \le 4A\sqrt{2J} = 1/2$. Thus every $g_J$ lies in the single class $\mathcal{H}_{\gamma,1}$, and hence in $\mathcal{H}_{\gamma,L}$ for $L \ge 1$.

The upper probability estimate uses orthogonal increments in coefficient space, not independent process increments. Write, for $0 \le t \le 1/2$,

$$\operatorname{Re} K(t) = \sum_h c_h(t) X_h, \qquad c_0(t) = t, \quad c_h(t) = \frac{\sin(2\pi h t)}{2\pi h} \ (h \ne 0).$$

For $0 \le s \le t \le 1/2$, identity (3.4) and product-to-sum show

$$\sum_h c_h(s)c_h(t) = st + \frac{1}{2\pi^2}\sum_{h\ge 1}\frac{\sin(2\pi h s)\sin(2\pi h t)}{h^2} = \frac{s}{2}. \tag{7.6}$$

For clarity, the sine sum is half the difference of the cosine series at $t-s$ and $t+s$, and equals $\pi^2(s-2st)$. Let $s_j = t_j - 1/N$. The intervals $[s_j, t_j]$ are disjoint and inside $(0, 1/2)$. For

$$S = \sum_{j=1}^{J}\sigma_j\{\operatorname{Re} K(t_j) - \operatorname{Re} K(s_j)\} = \sum_h d_h X_h,$$

equation (7.6) gives $\sum_h d_h^2 = J/(2N)$: each diagonal contributes half the interval length, and each disjoint off-diagonal pair contributes zero. The independent bounded scalar coefficients and their exponential estimate, followed by Fatou through the convergent symmetric sums, yield $\mathbb{E}e^{uS} \le \exp(u^2 J/N)$.

If $\|K - g_J\|_\infty < \epsilon$ and $\epsilon \le A/4$, each signed target increment is $A$ and its two endpoint errors total less than $2\epsilon$. Hence $S \ge JA/2$. Exponential Markov inequality at $u = AN/4$ proves

$$\mathbb{P}(\|K - g_J\|_\infty < \epsilon) \le \exp(-JA^2N/16) = \exp(-\lambda^2 N/16). \tag{7.7}$$

To cover every sufficiently small radius, let $J_\gamma$ be a dyadic integer at least $16^{1/(1/(2\gamma)-1)}$. For $0 < \epsilon \le \lambda/(4\sqrt{2J_\gamma})$, choose the largest dyadic $J \le (\lambda/(4\epsilon))^2$. Then $J \ge J_\gamma$, $4\epsilon \le A < 4\sqrt{2}\epsilon$, and

$$N \ge J^{1/(2\gamma)} > \left(\frac{\lambda}{4\sqrt{2}}\right)^{1/\gamma} \epsilon^{-1/\gamma}.$$

Taking the infimum in (7.7) proves the rough upper bound with $c_\gamma = (\lambda^2/16)(\lambda/(4\sqrt{2}))^{1/\gamma} = 64^{-1/\gamma}/2048$.

Finally, if $1/2 \le \gamma \le 1$, then $\mathcal{H}_{\gamma,L} \subseteq \mathcal{H}_L$, since $|t-s|^\gamma \le |t-s|^{1/2}$. Theorem 1.1 supplies the uniform lower bound. The zero center belongs to every class, and its upper bound in that theorem bounds the infimum above by $\exp(-c\epsilon^{-2})$. This completes (7.1). □

For fixed $\gamma, L$, the class $\mathcal{H}_{\gamma,L}$ is compact by the same equicontinuity and closed-coefficient argument used in Corollary 6.4. Hence its lower estimate also gives

$$\liminf_{q\to\infty} \inf_{g\in\mathcal{H}_{\gamma,L}} \mu_q(B(g,\epsilon)) \ge \exp[-2^s C_{\gamma,L}\epsilon^{-s}], \qquad s = \max(2, 1/\gamma),$$

for each fixed sufficiently small $\epsilon$: repeat that corollary's finite $\epsilon/4$-net proof with radius $\epsilon/2$. This fixed-radius display follows from the qualitative finite-net argument. The stronger effective and shrinking-radius statements for the half-Hölder class are proved in Section 6; the law here remains the same two-parameter prime ensemble.

## 8 A weighted boundary condition for other coefficient laws

The same matching mechanism extends beyond Sato–Tate. Fractional Fejér tapers improve the sufficient condition on how rapidly a bounded symmetric unimodal density can vanish at its boundary. The positive-kernel representation used below is classical. This section concerns specified random Fourier series and supplies no arithmetic convergence theorem for their laws.

**Proposition 8.1.** *Let $f$ be an even probability density supported on $[-2,2]$, nonincreasing on $[0,2)$, positive in its interior, and bounded above by $B < \infty$. Use its right-continuous monotone representative on $[0,2)$ and even extension, and put*

$$\Psi(s) = \max\{0, -\log f(2-s)\}, \qquad 0 < s \le 1.$$

*Suppose that for some fixed $1/2 < r < 1$,*

$$I_r = \int_0^1 \Psi(t^r)\,dt = \frac{1}{r}\int_0^1 \Psi(s)s^{1/r-1}\,ds < \infty. \tag{8.1}$$

*Let the independent $X_h$ have density $f$, and define $K_f = \sum_h X_h v_h$ with the same basis and symmetric summation as before. For every $L \ge 1$, $g \in \mathcal{H}_L$, and $0 < \epsilon \le 1/4$,*

$$\mathbb{P}(\|K_f - g\|_\infty < \epsilon) \ge \exp\{-C_{f,L,r}\epsilon^{-2}\}, \tag{8.2}$$

*where one possible constant is*

$$T_r = 3 + \frac{2r(1-r)}{r-1/2}, \qquad D_r = \frac{14\sqrt{2}}{3}T_r, \qquad Q_r = 64D_r^2 + 1,$$
$$C_{f,L,r} = (2r + 2I_r)Q_r(L+2)^2 + 5120 + \log 4 + 1 + 8^{1/r}I_r.$$

*For an upper estimate, put*

$$\eta = \frac{1}{4B}, \qquad p_0 = \frac{1}{2}, \qquad \delta_0 = \frac{p_0\eta^2}{32\pi^2}, \qquad c_0 = \frac{p_0\delta_0}{16}, \qquad \epsilon_f = \sqrt{\delta_0/2}.$$

*Then, for every deterministic $g \in C([0,1],\mathbb{C})$,*

$$\mathbb{P}(\|K_f - g\|_\infty < \epsilon) \le \exp(-c_0\epsilon^{-2}), \qquad 0 < \epsilon \le \epsilon_f. \tag{8.3}$$

*The class $\mathcal{H}_L$ retains its exact definition from Section 1; no separate support characterization for $f$ is needed.*

*Proof.* We first record the approximation kernel and its full estimate. For an integer $N \ge 2$ let $w_h = (1-(h/N)^r)_+$ for integers $h \ge 0$, and put

$$\alpha_m = (m+1)(w_m - 2w_{m+1} + w_{m+2}), \qquad 0 \le m \le N-1.$$

The decreasing sequence $w_h$ is convex: its second derivative on $(0,N)$ is $r(1-r)N^{-r}h^{r-2} > 0$, and its slope jumps upward to zero at $N$. Thus $\alpha_m \ge 0$. With $d_m = w_m - w_{m+1}$, telescoping gives

$$\sum_{m=0}^{N-1} \alpha_m = \sum_{m=0}^{N-1} (m+1)(d_m - d_{m+1}) = 1,$$
$$\sum_{m=h}^{N-1} \alpha_m \left(1 - \frac{h}{m+1}\right) = \sum_{m=h}^{N-1} (m+1-h)(d_m - d_{m+1}) = w_h.$$

Consequently $F_{N,r} = \sum_{m=0}^{N-1} \alpha_m F_m$ is a nonnegative integral-one kernel with Fourier coefficients $w_{|h|}$.

For $0 < \gamma < r$, these mixture coefficients satisfy

$$\sum_{m=0}^{N-1} \alpha_m (m+1)^{-\gamma} \le \left(3 + \frac{2r(1-r)}{r-\gamma}\right) N^{-\gamma}. \tag{8.4}$$

Indeed $\alpha_0 = (2-2^r)N^{-r} \le 2N^{-\gamma}$, while the last coefficient contributes at most $N^{-\gamma}$ since $\alpha_{N-1} \le 1$. For $1 \le m \le N-2$, integrating the second derivative of $x^r$ gives

$$2(m+1)^r - m^r - (m+2)^r = r(1-r)\int_0^1\int_0^1 (m+s+t)^{r-2}\,ds\,dt \le r(1-r)m^{r-2}.$$

It follows that $\alpha_m(m+1)^{-\gamma} \le 2r(1-r)N^{-r}m^{r-\gamma-1}$. Since $r-\gamma-1 \in (-1,0)$, $\sum_{m=1}^{N-2} m^{r-\gamma-1} \le N^{r-\gamma}/(r-\gamma)$, which proves (8.4).

Take $\gamma = 1/2$ and write $A_{N,r}u = F_{N,r} * u$, where $u = g - at$. The ordinary pinned Fejér error proved in Section 4 is at most $14\sqrt{2}(L+2)/(3\sqrt{m+1})$ for $m \ge 1$. For $m = 0$ its pinned mean is zero and $\|u\|_\infty \le L+2$, so that bound is still valid. Mixing the errors by (8.4) yields

$$\|A_{N,r}u - A_{N,r}u(0) - u\|_\infty \le D_r(L+2)N^{-1/2}.$$

Choose $N = \lceil 64D_r^2(L+2)^2\epsilon^{-2}\rceil$, so this error is at most $\epsilon/8$ and $N \le Q_r(L+2)^2\epsilon^{-2}$. For $0 < |h| \le N-1$ put

$$x_h = (|h|/N)^r, \qquad b_h = (1-x_h)\beta_h, \qquad r_h = \min(1/2, x_h).$$

Keep $b_0 = (1-\epsilon/8)a$ and $r_0 = \epsilon/8$. The pinned polynomial $q = b_0 t + A_{N,r}u - A_{N,r}u(0) = \sum_{|h|\le N-1} b_h v_h$ has $\|q-g\|_\infty \le 3\epsilon/8$.

Radial monotonicity makes $\Psi$ nonincreasing; it is nonnegative by definition. On the $h$th coefficient box, $|b_h + z| \le 2 - x_h$; hence $f(b_h+z) \ge \exp[-\Psi(x_h)]$. The local factor against the centered uniform density is at least $x_h \exp[-\Psi(x_h)]$, since $2r_h \ge x_h$. The decreasing function $t \mapsto \Psi(t^r)$ obeys

$$\sum_{h=1}^{N-1} \Psi((h/N)^r) \le N \int_0^{(N-1)/N} \Psi(t^r)\,dt \le N I_r.$$

Together with the factorial estimate used in (4.7), the two-sign product is at least

$$\left(\frac{(N-1)!}{N^{N-1}}\right)^{2r} \exp\left[-2\sum_{h=1}^{N-1}\Psi((h/N)^r)\right] \ge \exp[-(2r+2I_r)N].$$

The endpoint factor is at least $(\epsilon/4)\exp[-\Psi(\epsilon/8)]$. Moreover (8.1) implies

$$I_r \ge \int_0^{s^{1/r}} \Psi(t^r)\,dt \ge s^{1/r}\Psi(s), \qquad \Psi(\epsilon/8) \le 8^{1/r} I_r \epsilon^{-1/r}.$$

The bounded symmetric coefficients satisfy the same endpoint-chord exponential bound as in Lemma 2.1, so $K_f$ converges uniformly by that proof. Lemma 2.3 gives the centered finite-sum comparison. The Gaussian reference and the limiting tail argument therefore retain the bounds (3.1) and (3.3). The auxiliary uniform variables in finite density matching are themselves log-concave, so that step is unchanged. Finite density matching, followed by the centered comparison, now bounds the low-frequency event of radius $\epsilon/4$ below by

$$\frac{\epsilon}{4}\exp[-\Psi(\epsilon/8) - (2r+2I_r)N - 1024\epsilon^{-2}].$$

The independent tail event of radius $\epsilon/4$ contributes $\exp(-4096\epsilon^{-2})$. Their intersection has norm at most $7\epsilon/8 < \epsilon$. Since $1/r < 2$ and $\epsilon \le 1$, $\epsilon^{-1/r} \le \epsilon^{-2}$. Inserting the bound on $N$ and $\log(4/\epsilon) \le (\log 4 + 1)\epsilon^{-2}$ proves (8.2) with the stated constant.

For the upper bound and an arbitrary continuous complex center, define $a, u, \beta_h$ as in the proof of Theorem 1.1. The endpoint/Parseval inequality (5.1) is unchanged. Every interval of length $2\eta$ has mass at most $2B\eta = 1/2$. Thus the indicators $\mathbf{1}_{\{|X_h - \operatorname{Re}\beta_h| \ge \eta\}}$ have independent success probabilities at least $p_0$. Their sum on the small-ball event is at most $16\pi^2\eta^{-2}N^2\epsilon^2$. With $N = \lfloor \delta_0 \epsilon^{-2} \rfloor$, this is at most $p_0 N/2$. The explicit exponential-moment argument in the main proof bounds its probability by $\mathrm{e}^{-p_0 N/8}$. For $\epsilon^2 \le \delta_0/2$ this is at most $\exp[-(p_0\delta_0/16)\epsilon^{-2}]$, as asserted. □

An unweighted integrable boundary logarithm is sufficient for (8.1): $t^r \ge t$ and monotonicity give $\Psi(t^r) \le \Psi(t)$. In particular every polynomial bound $f(x) \ge d(2-|x|)^\kappa$, $d > 0$, $\kappa \ge 0$, works, since $\Psi(s) \le \max(0, -\log d) + \kappa \log(1/s)$. This includes the symmetric Jacobi densities

$$f_\kappa(x) = \frac{\Gamma(\kappa+3/2)}{2^{2\kappa+1}\sqrt{\pi}\,\Gamma(\kappa+1)}(4-x^2)^\kappa \mathbf{1}_{[-2,2]}(x), \qquad \kappa \ge 0.$$

Their normalization follows from the beta integral, and concavity of $\log(4-x^2)$ proves log-concavity. The cases $\kappa = 0$ and $\kappa = 1/2$ are uniform and Sato–Tate, respectively.

The new density hypotheses genuinely include laws that are not log-concave. For example,

$$f(x) = \begin{cases} 1/3, & |x| < 1, \\ 1/6, & 1 \le |x| < 2, \\ 0, & |x| \ge 2, \end{cases}$$

is the mixture of a uniform law on $[-1,1]$ of weight $1/3$ and a uniform law on $[-2,2]$ of weight $2/3$. It has an interior jump and therefore is not log-concave, but its boundary function satisfies $\Psi(s) = \log 6$ for $0 < s \le 1$, so the weighted condition holds for every permitted $r$.

There is also an exact threshold for the uniform lower order in a faster-vanishing family. For $A > 0$ and $a > 0$, put

$$f_{A,a}(x) = Z_{A,a}^{-1} \exp[-A(2-|x|)^{-a}]\mathbf{1}_{(-2,2)}(x). \tag{8.5}$$

This density is even, positive in the interior, bounded and log-concave: $2-|x|$ is concave, and $y \mapsto -Ay^{-a}$ is increasing and concave on positive $y$. Its boundary function is at most $|\log Z_{A,a}| + As^{-a}$. If $a < 2$, choose $1/2 < r < \min(1, 1/a)$; then $ar < 1$ and (8.1) holds. Proposition 8.1 proves the uniform $\epsilon^{-2}$ logarithmic order throughout this range.

For $a \ge 2$, Proposition 9.1 supplies a fixed-center obstruction in every prescribed $\mathcal{H}_L$, including the critical case $a = 2$. Thus $a < 2$ is the exact threshold for the existence of a uniform lower bound of order $\exp(-C\epsilon^{-2})$ in this family. The obstruction does not provide a matching lower bound or an exact critical small-deviation asymptotic.

**Corollary 8.2.** *Fix $0 < \gamma < 1/2$ and $L \ge 1$. Let $f$ satisfy the density hypotheses of Proposition 8.1, but suppose more generally that (8.1) holds for some $\gamma < r < 1$. Then*

$$\inf_{g \in \mathcal{H}_{\gamma,L}} \mathbb{P}(\|K_f - g\|_\infty < \epsilon)$$

*has negative logarithm of order $\epsilon^{-1/\gamma}$, for all sufficiently small $\epsilon$, with constants depending on the fixed parameters and law. For the family (8.5), this conclusion holds whenever $a < 1/\gamma$. Theorem 10.1 proves failure at and beyond equality in every prescribed $\mathcal{H}_{\gamma,L}$.*

*Proof.* The mixture and its moment estimate (8.4) hold for every $0 < \gamma < r < 1$, including $r \le 1/2$. With $D_\gamma$ from Section 7 and $T_{\gamma,r} = 3 + 2r(1-r)/(r-\gamma)$, the mixed pinned error is at most $D_\gamma T_{\gamma,r}(L+2)N^{-\gamma}$. For the degree-zero component the same estimate follows from $\|u\|_\infty \le L+2$, so no mixture term is omitted. Choose

$$N = \left\lceil [8D_\gamma T_{\gamma,r}(L+2)]^{1/\gamma} \epsilon^{-1/\gamma} \right\rceil.$$

The density matching proof gives the same nonzero-coefficient factor $\exp[-(2r+2I_r)N]$ and endpoint factor $(\epsilon/4)\exp[-\Psi(\epsilon/8)]$. Now $\Psi(\epsilon/8) \le 8^{1/r} I_r \epsilon^{-1/r}$ is absorbed by $\epsilon^{-1/\gamma}$ since $r > \gamma$; the Gaussian and tail cost $5120\epsilon^{-2}$ is absorbed since $\gamma < 1/2$. For example a lower-bound constant is

$$(2r+2I_r)\{[8D_\gamma T_{\gamma,r}(L+2)]^{1/\gamma} + 1\} + 5120 + \log 4 + 1 + 8^{1/r} I_r.$$

The signed-spike upper proof in Section 7 uses only independent centered coefficients in $[-2,2]$. It therefore applies to $f$ without change, with the same $c_\gamma$ and small-radius threshold. This proves matching worst-case order.

For (8.5) and $a < 1/\gamma$, choose $\gamma < r < \min(1, 1/a)$; then $ar < 1$ makes the weighted integral finite. The separate negative assertion is Theorem 10.1. □

The weighted condition is sufficient. These are same-method extensions in the stated random-series setting.

## 9 Flat-edge threshold and critical-rate bounds

The sufficient condition in Section 8 excludes the critical case $a = 2$ of (8.5). In that family the exclusion reflects an actual obstruction. Infinitely many boundary coefficients can force an additional harmonic-sum cost while classical Rudin–Shapiro cancellation keeps their center 1/2-Hölder. Rapidly vanishing origin distributions and weighted-sum small deviations belong to classical Laplace-transform theory; see, for example, Rozovsky [20]. The elementary estimate constructs a fixed admissible harmonic boundary center.

**Proposition 9.1.** *For every $L \geq 1$, there is a fixed $g_L \in \mathcal{H}_L$ such that, for the coefficient law (8.5) with $A > 0$ and $a \geq 2$, some $c_{A,a,L} > 0$ and $\epsilon_{A,a,L} > 0$ satisfy*

$$\mathbb{P}(\|K_f - g_L\|_\infty < \epsilon) \leq \exp[-c_{A,a,L}\epsilon^{-2}\log^2(1/\epsilon)], \qquad 0 < \epsilon \leq \epsilon_{A,a,L}. \tag{9.1}$$

*In particular no lower bound $\exp(-C\epsilon^{-2})$ with finite $C$ holds even at this one center for all sufficiently small radii. Consequently, within (8.5), a uniform lower bound of that order on $\mathcal{H}_L$ holds if and only if $a < 2$, for every fixed $L \geq 1$.*

*Proof.* Use the polynomials $P_m, Q_m$ in the proof of (7.3). The first $2^m$ coefficients of $P_{m+1}$ equal those of $P_m$, so they define infinite signs $s_0, s_1, \ldots$; put $r_h = s_{h-1}$. A prefix of $P_m$ or $Q_m$ decomposes recursively into signed, shifted complete $P_j$ or $Q_j$ blocks, with at most one block of each dyadic size. An entire block is kept intact. Otherwise a prefix in the first half is a prefix of $P_{m-1}$; a longer one consists of that whole first block and a signed shifted prefix of $Q_{m-1}$. Iteration follows its binary length expansion. Each complete block of length $2^j$ has norm at most $\sqrt{2 \cdot 2^j}$ by the complementary identity. Thus, uniformly on $|z| = 1$,

$$\left|\sum_{h=1}^{n} r_h z^h\right| \leq C_{\mathrm{RS}}\sqrt{n}, \qquad C_{\mathrm{RS}} = \frac{\sqrt{2}}{1 - 2^{-1/2}} < 5. \tag{9.2}$$

The shift from index zero multiplies by $z$ and preserves the norm.

Consider the deterministic center

$$g(t) = \frac{2}{\pi}\sum_{h\geq 1}\frac{r_h \sin(2\pi h t)}{h}.$$

Its uniform convergence and regularity can be checked directly. For $S_n(t) = \sum_{h=1}^{n} r_h e^{2\pi i h t}$, summation by parts gives

$$\sum_{h=M+1}^{H}\frac{r_h e^{2\pi i h t}}{h} = \frac{S_H(t)}{H} - \frac{S_M(t)}{M} + \sum_{h=M}^{H-1} S_h(t)\left(\frac{1}{h} - \frac{1}{h+1}\right).$$

The first term tends uniformly to zero as $H \to \infty$, and $\sum_{h=M}^{\infty}\sqrt{h}/(h(h+1)) \leq 3M^{-1/2}$. The complex-series tail is therefore at most $4C_{\mathrm{RS}}M^{-1/2}$, and the sine-series tail at most $8C_{\mathrm{RS}}/(\pi\sqrt{M})$. The center is consequently continuous, real, zero at both endpoints, and has normalized coefficients $\beta_h = \beta_{-h} = 2r_h$ for $h \geq 1$.

Its degree-$M$ partial sum has derivative $4\sum_{h=1}^{M} r_h \cos(2\pi h t)$, of norm at most $4C_{\mathrm{RS}}\sqrt{M}$. For $\delta = |t - s| \in (0, 1]$, choose $M = \lfloor \delta^{-1} \rfloor$. The partial sum and its two tails give

$$|g(t) - g(s)| \leq 4C_{\mathrm{RS}}\sqrt{M}\,\delta + \frac{16C_{\mathrm{RS}}}{\pi\sqrt{M}} \leq \left(4 + \frac{16\sqrt{2}}{\pi}\right)C_{\mathrm{RS}}\sqrt{\delta} < 80\sqrt{\delta}.$$

Thus we may fix $L_0 = 80$.

Extend $g$ periodically. Its zero endpoints give circular 1/2-Hölder constant at most $\sqrt{2}L_0$, by the same wrapped-pair argument as in (4.1). For a positive integer $q$, put $g_q(t) = q^{-1}g(qt)$. This has Hölder constant at most $\sqrt{2}L_0/\sqrt{q}$, zero endpoints, and normalized coefficients

$$\beta_{qk} = \beta_{-qk} = 2r_k \quad (k \geq 1), \qquad \beta_h = 0 \quad (q \nmid h).$$

The formula follows directly from its sine series, or by partitioning the Fourier integral into $q$ periods. Choose the fixed integer $q = \max\{1, \lceil 2L_0^2/L^2 \rceil\}$, and let $g_L = g_q$. It belongs to $\mathcal{H}_L$ and is independent of the radius.

On $\|K_f - g_q\|_\infty < \epsilon$, the endpoint identity gives $|X_0| < \epsilon$. Thus $D = K_f - tX_0 - g_q$ has norm less than $2\epsilon$. Parseval, retaining positive frequencies $qk$ for $1 \leq k \leq M$, gives

$$\sum_{k=1}^{M} \frac{(X_{qk} - 2r_k)^2}{(qk)^2} \leq 16\pi^2\epsilon^2.$$

Write $d_k = 2 - r_k X_{qk} = |X_{qk} - 2r_k| \in (0, 4)$ almost surely, and $H_M = \sum_{k=1}^{M} 1/k$. Cauchy's inequality implies

$$\left(\frac{H_M}{q}\right)^2 \leq \left(\sum_{k=1}^{M} \frac{d_k^2}{(qk)^2}\right)\left(\sum_{k=1}^{M} d_k^{-2}\right), \qquad \sum_{k=1}^{M} d_k^{-2} \geq \frac{H_M^2}{16\pi^2 q^2 \epsilon^2}.$$

For $a \geq 2$ and $\delta_x = 2 - |x| \in (0, 2]$, $\delta_x^{-a} \geq 2^{2-a}\delta_x^{-2}$. Also $\delta_{X_{qk}} \leq d_k$ for either sign $r_k$. With $A_* = A2^{2-a}$ and $Z = Z_{A,a}$, the marginal density is at most $Z^{-1}\exp(-A_*/d_k^2)$. Independence of these $M$ coefficients, integration over their finite necessary set, and the full box volume $4^M$ therefore give

$$\mathbb{P}(\|K_f - g_q\|_\infty < \epsilon) \leq (4/Z)^M \exp\left[-\frac{A_* H_M^2}{16\pi^2 q^2 \epsilon^2}\right]. \tag{9.3}$$

All unselected coefficients were discarded through a necessary-event upper bound; no infinite density product is used.

Take $M = \lfloor \epsilon^{-2} \rfloor$ for $\epsilon \leq 1/4$. Then $H_M \geq \log(M + 1) \geq 2\log(1/\epsilon)$ and $M \leq \epsilon^{-2}$. Put $b = \log(4/Z) > 0$. The right side of (9.3) is at most

$$\exp\left\{\epsilon^{-2}\left[b - \frac{A_*}{4\pi^2 q^2}\log^2(1/\epsilon)\right]\right\}.$$

For $\epsilon \leq \min\{1/4, \exp[-\sqrt{8\pi^2 q^2 b/A_*}]\}$, this proves (9.1) with $c_{A,a,L} = A_*/(8\pi^2 q^2)$. The extra unbounded factor $\log^2(1/\epsilon)$ excludes every finite-$C$ lower bound of order $\exp(-C\epsilon^{-2})$ at $g_L$. For $a < 2$, Proposition 8.1 already proves the positive assertion, completing the stated threshold. □

## 9.1 A lower bound at the critical law

For $a = 2$, a borderline positive taper supplies a complementary uniform lower bound. Write $K_A$ for the process with density $f_{A,2}$, and $Z_A = Z_{A,2}$.

**Theorem 9.2.** *For every fixed $A > 0$ and $L \geq 1$, there is an explicitly finite $C_{A,L}$ such that*

$$\mathbb{P}(\|K_A - g\|_\infty < \epsilon) \geq \exp[-C_{A,L}\epsilon^{-2}\log^3(\mathrm{e}/\epsilon)], \qquad g \in \mathcal{H}_L, \quad 0 < \epsilon \leq 1/4. \tag{9.4}$$

*For the single fixed center $g_L$ of Proposition 9.1, therefore,*

$$c_{A,L}\epsilon^{-2}\log^2(1/\epsilon) \leq -\log \mathbb{P}(\|K_A - g_L\|_\infty < \epsilon) \leq C_{A,L}\epsilon^{-2}\log^3(\mathrm{e}/\epsilon) \tag{9.5}$$

*at every sufficiently small radius. The upper and lower estimates have the displayed logarithmic powers.*

*Proof.* Use the mixture from Section 8 with $w_h = (1-\sqrt{h/N})_+$, $N \ge 2$, and $\alpha_m = (m+1)(w_m - 2w_{m+1} + w_{m+2})$. Its positivity, unit sum and exact Fejér representation hold also at this borderline exponent. For $1 \le m \le N-2$, the second-difference integral gives

$$\alpha_m (m+1)^{-1/2} \le \tfrac{1}{4} N^{-1/2} (m+1)^{1/2} m^{-3/2} \le \tfrac{1}{2} N^{-1/2}/m.$$

The $m=0$ and $m=N-1$ contributions are at most $2N^{-1/2}$ and $N^{-1/2}$, respectively. Hence, with $H_0 = 0$,

$$\sum_{m=0}^{N-1} \alpha_m (m+1)^{-1/2} \le N^{-1/2}[3 + \tfrac{1}{2} H_{N-2}] \le N^{-1/2}[4 + \tfrac{1}{2}\log N]. \tag{9.6}$$

Put

$$D = \frac{14\sqrt{2}}{3}, \quad S = 8D(L+2), \quad B = 256S^2, \quad \ell = \log(\mathrm{e}/\epsilon), \quad N = \lceil B^2 \epsilon^{-2} \ell^2 \rceil.$$

These constants satisfy

$$\sqrt{N} \ge B\epsilon^{-1}\ell, \quad N \le 2B^2\epsilon^{-2}\ell^2, \quad \log N \le \log 2 + 2\log B + 4\ell.$$

In particular $4 + \frac{1}{2}\log N \le (7 + \log B)\ell$ and $B \ge S(7 + \log B)$: use $S \ge 1$, $7 + \log 256 + 2\log S \le 13S$ and $B/S = 256S$. For $u = g - g(1)t$, the ordinary pinned Fejér error is at most $D(L+2)/\sqrt{m+1}$, including $m = 0$. Its mixture and (9.6) consequently give error at most $\epsilon/8$.

For $0 < |h| \le N-1$, put

$$x_h = \sqrt{|h|/N}, \quad b_h = (1 - x_h)\beta_h, \quad r_h = \min(1/2, x_h),$$

and put $b_0 = (1-\epsilon/8)g(1)$, $r_0 = \epsilon/8$. The pinned polynomial $q = \sum_{|h| \le N-1} b_h v_h$ has $\|q - g\|_\infty \le 3\epsilon/8$. For $|z| \le r_h$, $h \ne 0$, $|b_h + z| \le 2 - x_h$, so comparison with the centered uniform law on $[-r_h, r_h]$ pays at least

$$x_h \exp[-AN/|h| - z_A], \qquad z_A = \max(0, \log Z_A).$$

The factorial estimate, over both frequency signs, gives the nonzero-frequency product at least

$$\exp[-(1 + 2z_A)N - 2ANH_{N-1}].$$

For $h = 0$ the factor is at least $(\epsilon/4)\exp[-64A\epsilon^{-2} - z_A]$.

The centered uniform acceptance event of radius $\epsilon/4$ has probability at least $\exp(-1024\epsilon^{-2})$ by the peakedness and Gaussian lemmas. The independent tail of $K_A$ has acceptance probability at least $\exp(-4096\epsilon^{-2})$. Indeed the even log-concave density $f_{A,2}$ dominates $N(0,4)$ in central peakedness, exactly as in Section 8. The deterministic and two random errors total $7\epsilon/8$. Thus the strict-ball probability is at least

$$\frac{\epsilon}{4}\exp[-z_A - (1 + 2z_A)N - 2ANH_{N-1} - (5120 + 64A)\epsilon^{-2}].$$

Since $H_{N-1} \le (6 + 2\log B)\ell$, one admissible constant is

$$C_{A,L} = 2B^2[1 + 2z_A + 2A(6 + 2\log B)] + 5120 + 64A + \log 4 + 1 + z_A.$$

This proves (9.4); combining it with (9.1) gives (9.5). □

The extra harmonic sum in (9.6) explains the loss in this lower-bound proof. Determining the exact logarithmic order at the fixed boundary center remains a separate problem.

## 10 The exact joint roughness–density threshold

Corollary 8.2 gives a sufficient range for rough centers. The equality case in the flat-edge family can also be decided. The following construction uses classical Rudin–Shapiro cancellation and positive Fejér tests to force a separate density penalty at every geometric frequency scale.

**Theorem 10.1.** *Fix $0 < \gamma < 1/2$, $L \geq 1$, $A > 0$, and put $a = 1/\gamma$. There is one fixed real zero-endpoint $g_{\gamma,L} \in \mathcal{H}_{\gamma,L}$ such that, for the law $f_{A,a}$ in (8.5),*

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \leq \exp[-c_{A,\gamma,L}\epsilon^{-1/\gamma}\log(1/\epsilon)] \tag{10.1}$$

*for every sufficiently small $\epsilon$, with $c_{A,\gamma,L} > 0$. The same center excludes a bound of the form*

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \geq \exp(-C\epsilon^{-1/\gamma})$$

*with finite $C$ for all sufficiently small radii, for every law $f_{A,a'}$ with $a' \geq 1/\gamma$. Consequently, within this family a uniform bound*

$$\inf_{g\in\mathcal{H}_{\gamma,L}} \mathbb{P}(\|K_f - g\|_\infty < \epsilon) \geq \exp(-C\epsilon^{-1/\gamma})$$

*with a finite constant and a sufficiently small radius exists exactly when $a' < 1/\gamma$.*

*Proof.* For $j \geq 1$, put

$$N_j = 4^j, \quad J_j = 2^{\lfloor 2\gamma\log_2 N_j\rfloor}, \quad m_j = N_j/J_j, \quad B_j = 4N_j.$$

Here $J_j, m_j$ are positive integers, $\frac{1}{2}N_j^{2\gamma} < J_j \leq N_j^{2\gamma}$, and $J_j m_j = N_j$. Choose independently for each band a deterministic Rudin–Shapiro polynomial $R_{J_j}(z) = \sum_{k=0}^{J_j-1}\sigma_k z^k$ of norm at most $\sqrt{2J_j}$, using the complete recursion in (7.3). No consistency between different finite sign blocks is needed.

In band $j$, assign $r_h = \sigma_k$ at

$$h = B_j + 2m_j k + \ell, \qquad 0 \leq k < J_j, \quad |\ell| \leq m_j - 1.$$

Subblocks leave a one-frequency gap; all band frequencies lie in $[3N_j, 6N_j]$, while the next band lies in $[12N_j, 24N_j]$. Thus every assignment is unambiguous. With $D_{m-1}(t) = \sum_{\ell=-m+1}^{m-1} e^{2\pi i\ell t}$,

$$T_j(t) := \sum_{h \text{ in band } j} r_h e^{2\pi i h t} = e^{2\pi i B_j t} D_{m_j-1}(t) R_{J_j}(e^{2\pi i 2m_j t}), \qquad \|T_j\|_\infty \leq 2\sqrt{2}m_j\sqrt{J_j}.$$

Define

$$g_j(t) = \frac{2}{\pi}\sum_{h \text{ in band } j}\frac{r_h\sin(2\pi h t)}{h}, \qquad g(t) = \sum_{j\geq 1} g_j(t).$$

If $b_j \geq 3N_j$ is the smallest band frequency, write $T_j(z) = z^{b_j}Q_j(z)$. Maximum modulus gives $|Q_j(z)| \leq \|T_j\|_\infty$ for $|z| \leq 1$, and

$$\sum_{h \text{ in band } j}\frac{r_h z^h}{h} = z^{b_j}\int_0^1 t^{b_j-1}Q_j(tz)\,dt.$$

Therefore

$$\|g_j\|_\infty \leq \frac{4\sqrt{2}}{3\pi\sqrt{J_j}} \leq \frac{8}{3\pi}N_j^{-\gamma} < N_j^{-\gamma}, \qquad \left\|g_j'\right\|_\infty \leq 16N_j^{1-\gamma}.$$

This proves uniform convergence, zero endpoints, reality, and exact normalized coefficients $\beta_h = \beta_{-h} = 2r_h$ on the bands, zero elsewhere. No prefix estimate for a truncated band is assumed.

Splitting bands at $N_k \le |t-s|^{-1} < N_{k+1}$ proves a Hölder-$\gamma$ bound with constant

$$L_\gamma^* = \frac{16}{1-4^{-(1-\gamma)}} + \frac{2}{1-4^{-\gamma}}.$$

Indeed low-band increments are bounded by $16|t-s|\sum_{j\le k} N_j^{1-\gamma}$, and the two tails by $2\sum_{j>k} N_j^{-\gamma}$; their geometric sums give the two displayed terms. For $|t-s| \ge 1/4$, the full tail bound gives the second term directly. Extend $g$ periodically. Its circular constant is at most $2^{1-\gamma}L_\gamma^*$ by splitting wrapped pairs at the common zero endpoint. Fix an integer

$$q \ge \max\left\{1, (2^{1-\gamma}L_\gamma^*/L)^{1/(1-\gamma)}\right\}$$

and set $g_{\gamma,L}(t) = q^{-1}g(qt)$. Its Hölder constant is at most $2^{1-\gamma}L_\gamma^* q^{\gamma-1} \le L$, and its normalized boundary coefficients at $qh$ are still $2r_h$. Thus the center is fixed in the required class.

For a band, suppress indices and put $N = Jm$, $B = 4N$. The analytic test

$$P_j(t) = \mathrm{e}^{2\pi iBt}F_{m-1}(t)R_J(\mathrm{e}^{2\pi i2mt})$$

has coefficient $r_h w_h$ on the band, with $w_h = 1 - |\ell|/m \in (0,1]$, and

$$W_j := \sum_h w_h = N, \qquad \|P_j\|_1 \le \sqrt{2J}.$$

The derivative satisfies

$$\|P_j'\|_1 \le DN\sqrt{J}, \qquad D = 4\pi(3+\mathrm{e})\sqrt{2}. \tag{10.2}$$

For completeness, $F_{m-1} = |D_m^+|^2/m$ with $D_m^+ = \sum_{k=0}^{m-1}\mathrm{e}^{2\pi ikt}$; Cauchy–Schwarz gives $\|F_{m-1}'\|_1 \le 4\pi m$. An analytic polynomial $R$ of degree $d \ge 1$ obeys $\|R'\|_{|z|=1} \le \mathrm{e}d\|R\|_{|z|=1}$: reverse the polynomial to bound its modulus outside the unit disk by $|z|^d\|R\|$, and use Cauchy's formula on radius $1/d$. For degree zero the derivative vanishes. Differentiating the three factors now bounds $\|P_j'\|_1$ by

$$2\pi B\sqrt{2J} + 4\pi m\sqrt{2J} + 4\pi \mathrm{e}m(J-1)\sqrt{2J},$$

which proves (10.2), including $m = 1$ or $J = 1$.

On the small-ball event, $|X_0| < \epsilon$ and $U = K_f - tX_0 - g_{\gamma,L}$ has norm less than $2\epsilon$. For a selected positive frequency define $d_h = 2 - r_h X_{qh} > 0$. Fourier multiplication gives the exact nonnegative identity

$$S_j := \sum_{h \text{ in band } j} w_h d_h = \int_0^1 U(t)\overline{\frac{d}{dt}P_j(qt)}\,dt \le 2qD\epsilon N_j\sqrt{J_j}.$$

In fact $\widehat{U}(qh) = (X_{qh} - 2r_h)/(2\pi iqh)$, and the conjugate derivative coefficient is $-2\pi iqh\, r_h w_h$, giving precisely $w_h d_h$. The $L^1$ derivative norm after integer dilation is $q\|P_j'\|_1$, so no further factor appears.

For $a = 1/\gamma$, weighted Jensen gives

$$\sum_{h \text{ in band } j} d_h^{-a} \ge \sum_h w_h d_h^{-a} \ge \frac{W_j^{a+1}}{S_j^a} \ge \frac{N_j}{(2qD\epsilon\sqrt{J_j})^a} \ge c_*\epsilon^{-a}, \qquad c_* = (2qD)^{-a}.$$

The last inequality uses $J_j^{a/2} \le N_j$. The actual distance to the coefficient boundary is at most $d_h$, so its density is at most $Z_{A,a}^{-1}\exp(-Ad_h^{-a})$.

Select the first $K$ bands, containing

$$V_K = \sum_{j=1}^{K} J_j(2m_j - 1) \le 2\sum_{j=1}^{K} N_j \le \tfrac{8}{3} N_K$$

independent positive-frequency coefficients. Integrate just these finite marginals over their necessary set; the enclosing box has volume at most $4^{V_K}$. It follows that

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \le (4/Z_{A,a})^{V_K} \exp[-Ac_* K\epsilon^{-a}].$$

This uses independence of scalar coefficients, not of process increments.

Put

$$b = \log(4/Z_{A,a}) > 0, \qquad K = \lfloor \log_4 \epsilon^{-a} \rfloor, \qquad K_0 = \max(2, \lceil 16b/(3Ac_*) \rceil).$$

For $0 < \epsilon \le \epsilon_0$, where $\epsilon_0 = \min(1/4, 4^{-\gamma(K_0+1)})$, the conditions are

$$N_K \le \epsilon^{-a}, \qquad K \ge K_0, \qquad K \ge \frac{a\log(1/\epsilon)}{2\log 4}.$$

Absorbing the finite volume term therefore gives

$$\mathbb{P}(\|K_f - g_{\gamma,L}\|_\infty < \epsilon) \le \exp\left[-\frac{Aac_*}{4\log 4}\epsilon^{-a}\log(1/\epsilon)\right].$$

For $a' \ge a$, the inequality $\delta^{-a'} \ge 2^{a-a'}\delta^{-a}$ on $(0, 2]$ applies the same proof with $A$ replaced by $A2^{a-a'}$ and with the actual normalization constant. The center depends only on $\gamma, L$, not on the law parameters or the radius. The positive assertion for $a' < 1/\gamma$ is Corollary 8.2; the extra logarithm excludes every finite-constant lower bound at and beyond equality. □

Together with Proposition 9.1, this determines the flat-edge lower-order property for all $0 < \gamma \le 1/2$: its threshold is $a < 1/\gamma$. The theorem supplies a one-sided critical obstruction below $\gamma = 1/2$; it does not identify the exact critical logarithmic order for those rough centers.

## 11 Supercritical laws and smoother centers

The same positive taper also covers the range beyond the joint threshold. Its approximation order becomes slower when its exponent is below the Hölder exponent, but the resulting density cost has exactly the required power. This also distinguishes the critical half-Hölder case from smoother centers.

**Theorem 11.1.** *Fix $0 < \gamma < 1$, $L \ge 1$, $A > 0$, and $a > 1/\gamma$. For the law $f_{A,a}$ of (8.5), there are explicitly finite constants $C_a, C_2$ such that*

$$\mathbb{P}(\|K_f - g\|_\infty < \epsilon) \ge \exp[-C_a\epsilon^{-a} - C_2\epsilon^{-2}], \qquad g \in \mathcal{H}_{\gamma,L}, \quad 0 < \epsilon \le 1/4. \tag{11.1}$$

*If $a > \max(2, 1/\gamma)$, a single fixed smooth center in $\mathcal{H}_{\gamma,L}$ gives a matching upper bound $\exp(-c\epsilon^{-a})$ at all sufficiently small radii. Consequently the worst-case negative logarithm has exact order $\epsilon^{-a}$ in this range. For $1/2 < \gamma < 1$ and $a = 2$, it instead has exact order $\epsilon^{-2}$.*

*Proof.* Choose $1/a < r < \gamma$. Use $w_h$ and $\alpha_m$ from Section 8. The positive-kernel representation is unchanged, but the mixture moment is now

$$\sum_{m=0}^{N-1} \alpha_m (m+1)^{-\gamma} \le T N^{-r}, \qquad T = 3 + 2r(1-r)\left(1 + \frac{1}{\gamma - r}\right). \tag{11.2}$$

Indeed the initial term is at most $2N^{-r}$, and the last term is at most $N^{-\gamma} \le N^{-r}$. For $1 \le m \le N-2$, the same second-difference integral gives

$$\alpha_m (m+1)^{-\gamma} \le 2r(1-r) N^{-r} m^{r-\gamma-1}.$$

The sum of $m^{-1-(\gamma-r)}$ is at most $1 + 1/(\gamma - r)$, which proves (11.2).

Put

$$E_\gamma = \frac{2}{\gamma+1} + \frac{1}{2(1-\gamma)}, \qquad D_\gamma = 2^{2-\gamma} E_\gamma, \qquad U = 8 D_\gamma T (L+2), \qquad N = \left\lceil (U/\epsilon)^{1/r} \right\rceil.$$

The elementary Fejér moment calculation in Section 7 is valid for every $0 < \gamma < 1$. Thus the ordinary pinned approximation to $u = g - g(1)t$ has error at most $D_\gamma (L+2)(m+1)^{-\gamma}$, including $m = 0$. Mixing by (11.2) gives error at most $\epsilon/8$.

As before set, for $0 < |h| < N$,

$$x_h = (|h|/N)^r, \qquad b_h = (1 - x_h)\beta_h, \qquad \rho_h = \min(1/2, x_h),$$

and set $b_0 = (1 - \epsilon/8) g(1)$, $\rho_0 = \epsilon/8$. The pinned polynomial $q = \sum_{|h|<N} b_h v_h$ has $\|q - g\|_\infty \le 3\epsilon/8$. Writing $z = \max(0, \log Z_{A,a})$, the local density factor against the centered uniform comparison is at least

$$x_h \exp[-A N^{ar} |h|^{-ar} - z] \quad (h \ne 0).$$

Now $ar > 1$, so

$$\sum_{h=1}^{N-1} h^{-ar} \le S := 1 + \frac{1}{ar - 1}.$$

The factorial estimate and both frequency signs give a joint factor at least

$$\exp[-2rN - 2zN - 2ASN^{ar}] \ge \exp[-(2r + 2z + 2AS)N^{ar}].$$

The endpoint factor is at least $(\epsilon/4)\exp[-A 8^a \epsilon^{-a} - z]$. The unchanged centered acceptance and independent tail cost at most $5120\epsilon^{-2}$ in the negative logarithm. Their two errors are $\epsilon/4$ each, so the total error is $7\epsilon/8 < \epsilon$. Since $N \le 2(U/\epsilon)^{1/r}$, one possible choice in (11.1) is

$$C_a = (2r + 2z + 2AS) 2^{ar} U^a + A 8^a, \qquad C_2 = 5120 + z + \log 4 + 1.$$

For the opposite estimate, choose a fixed positive integer $q$ and put

$$g_q(t) = \frac{2}{\pi q} \sin(2\pi q t).$$

This center is real and has zero endpoints, with normalized coefficients $\beta_q = \beta_{-q} = 2$ and all others zero. Its increments satisfy

$$|g_q(t) - g_q(s)| \le \min(4|t-s|, 4/(\pi q)) \le 4\pi^{\gamma-1} q^{\gamma-1} |t-s|^\gamma.$$

Taking $q \geq \max\{1, (4\pi^{\gamma-1}/L)^{1/(1-\gamma)}\}$ places it in $\mathcal{H}_{\gamma,L}$. On the small-ball event, $|X_0| < \epsilon$, and the $q$th Fourier coefficient of $K_f - tX_0 - g_q$ gives $|X_q - 2| \leq 4\pi q\epsilon = \delta$. For $0 < \delta \leq 1$, this necessary event has probability at most

$$\frac{\delta}{Z_{A,a}} \exp(-A\delta^{-a}) \leq \exp\left[\max(0, -\log Z_{A,a}) - A(4\pi q)^{-a}\epsilon^{-a}\right].$$

At sufficiently small radii this is bounded by $\exp[-\frac{A}{2}(4\pi q)^{-a}\epsilon^{-a}]$. Combining with (11.1) proves the supercritical assertion. For $\gamma > 1/2, a = 2$, both powers in (11.1) equal two, and the same witness supplies the reverse bound. □

For clarity, set

$$F^{A,a}_{\gamma,L}(\epsilon) = \inf_{g \in \mathcal{H}_{\gamma,L}} \mathbb{P}(\|K_f - g\|_\infty < \epsilon).$$

Theorem 11.1, the earlier subcritical results, and the inclusion $\mathcal{H}_{\gamma,L} \subseteq \mathcal{H}_{1/2,L}$ for $\gamma > 1/2$ give the following complete noncritical power orders. All parameters in the table are fixed, with $A > 0, L \geq 1$.

| Hölder exponent | Density exponent | Order of $-\log F^{A,a}_{\gamma,L}$ |
|---|---|---|
| $0 < \gamma \leq 1/2$ | $a < 1/\gamma$ | $\epsilon^{-1/\gamma}$ |
| $0 < \gamma \leq 1/2$ | $a > 1/\gamma$ | $\epsilon^{-a}$ |
| $1/2 < \gamma < 1$ | every $a > 0$ | $\epsilon^{-\max(2,a)}$ |

The excluded equality line $a = 1/\gamma$ for $0 < \gamma \leq 1/2$ retains its proved obstruction and unresolved logarithmic order. In particular, the fixed half-Hölder RS center still has the bracket (9.5) at $a = 2$. For that same fixed center, however, the new theorem and any one of its saturated coefficients give exact order $\epsilon^{-a}$ for every $a > 2$.

These are extensions of the same kernel and density-matching argument. They assert neither a new general summability principle nor an arithmetic convergence result for the flat-edge laws.

# A Why continuity alone does not give the lower order

**Proposition A.1.** *There is a continuous real $g$ with $g(0) = g(1) = 0$ and every normalized Fourier coefficient real in $[-1, 1]$ for which no finite $C$ satisfies $\mathbb{P}(\|K - g\|_\infty < \epsilon) \geq \exp(-C\epsilon^{-2})$ for all sufficiently small $\epsilon$. The center is not 1/2-Hölder.*

*Proof.* Put $a_j = 16^{-j}$, $M_j = a_j^{-8} = 16^{8j}$, and

$$g_j(t) = a_j \sum_{h=M_j}^{2M_j} \frac{\sin(2\pi ht)}{\pi h}, \qquad g(t) = \sum_{j \geq 1} g_j(t).$$

Because $\|g_j\|_\infty \leq 2a_j/\pi$, this is a uniformly convergent continuous real series, zero at both endpoints. The frequency blocks are disjoint. Its normalized Fourier coefficients at both signs of a frequency in the $j$th block are $a_j$, and all others are zero, hence are strictly inside $[-1, 1]$.

At $t_j = 1/(8M_j)$, all sines of the current block are at least $\sqrt{2}/2$, and those of earlier blocks are nonnegative. The later blocks have total norm at most $2a_j/(15\pi)$. Therefore

$$g(t_j) \geq \left(\frac{\sqrt{2}}{4\pi} - \frac{2}{15\pi}\right) a_j \geq a_j/16. \tag{A.1}$$

For the real part of $K(t)$, the real scalar coefficients are $c_0 = t$ and $c_h = \sin(2\pi ht)/(2\pi h)$ for $h \neq 0$. If $0 < t \leq 1/2$, identity (3.4) gives

$$\sum_h c_h^2 = t^2 + \frac{1}{2\pi^2} \sum_{h \geq 1} \frac{\sin^2(2\pi ht)}{h^2} = t/2.$$

Hoeffding's estimate, independence, and Fatou through the convergent symmetric sums yield $\mathbb{E}\exp(\lambda \operatorname{Re} K(t)) \leq \exp(\lambda^2 t)$. Optimizing exponential Markov inequality gives

$$\mathbb{P}(\operatorname{Re} K(t) \geq A) \leq \exp(-A^2/(4t)), \quad A > 0.$$

For $\epsilon_j = a_j/64$, equations (A.1) and the small-ball event imply $\operatorname{Re} K(t_j) > 3a_j/64$. Consequently

$$\mathbb{P}(\|K - g\|_\infty < \epsilon_j) \leq \exp\{-\tfrac{9}{2048} a_j^2 M_j\} = \exp\{-\tfrac{9}{2048} a_j^{-6}\}.$$

Since $\epsilon_j^{-2} = 4096 a_j^{-2}$, this contradicts the proposed lower bound for every fixed $C$ when $j$ is sufficiently large. Finally, $g(t_j)/\sqrt{t_j} \geq (\sqrt{8}/16) a_j^{-3} \to \infty$, so this center is not 1/2-Hölder. □

The proposition shows that the regularity hypothesis cannot be replaced by continuity alone, even strictly inside the permitted Fourier interval. This proposition alone does not identify a regularity threshold; Theorem 7.1 separately determines the uniform worst-case scale within the Hölder hierarchy. Neither result characterizes which individual centers have the $\epsilon^{-2}$ rate. Its admissibility also places it in the previously known support by [14]; no new support theorem is needed for the displayed probability obstruction.

# B Center-dependent concentration obstructions

The next consequences expose the bounded-coefficient concentration mechanism behind the spike and continuity examples. They use classical exponential estimates and are not separate comparison principles.

**Proposition B.1** (An increment-energy obstruction)**.** *Let $g \in C([0,1], \mathbb{C})$, let $\epsilon > 0$, and let $[s_j, t_j] \subset [0, 1/2]$ be finitely many intervals with disjoint interiors and positive lengths $\ell_j = t_j - s_j$. Then*

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \leq \exp\left\{ -\frac{1}{4} \sum_j \frac{(|g(t_j) - g(s_j)| - 2\epsilon)_+^2}{\ell_j} \right\}.$$

*In particular, the same bound holds with the supremum of the displayed sum over all such finite collections; an infinite supremum gives probability zero.*

*Proof.* Write $v_h = c_h + i d_h$, so

$$c_0(t) = t, \quad d_0(t) = 0, \qquad c_h(t) = \frac{\sin(2\pi ht)}{2\pi h}, \quad d_h(t) = \frac{1 - \cos(2\pi ht)}{2\pi h} \quad (h \neq 0).$$

For $0 \leq s \leq t \leq 1/2$, the cosine-series identity (3.4) gives

$$\sum_h c_h(s) c_h(t) = \sum_h d_h(s) d_h(t) = s/2, \qquad \sum_h c_h(s) d_h(t) = 0.$$

The first identity for $c_h$ was used in (7.6). For $d_h$, write the product of $1-\cos(2\pi hs)$ and $1-\cos(2\pi ht)$ using the cosine series at $0, s, t, t-s, t+s$; its value is $s/2$. The cross sum vanishes by pairing $h$ and $-h$. Consequently, for arbitrary complex weights $z_j$, the real scalar variable

$$S = \sum_j \mathrm{Re}\{\overline{z_j}(K(t_j) - K(s_j))\}$$

has a coefficient vector of squared norm $\frac{1}{2}\sum_j |z_j|^2 \ell_j$. Independence and the bounded-variable exponential estimate from Section 2 therefore give

$$\mathbb{E}\mathrm{e}^{uS} \le \exp\{u^2 \sum_j |z_j|^2 \ell_j\}.$$

For the infinite series this follows by Fatou from the symmetric partial sums, which converge almost surely at all the finitely many endpoints.

Put $\Delta_j = g(t_j) - g(s_j)$ and $\eta_j = (|\Delta_j| - 2\epsilon)_+$. Take $z_j = 0$ if $\eta_j = 0$, and otherwise take $z_j = \eta_j \Delta_j / (\ell_j |\Delta_j|)$. If $E = \sum_j \eta_j^2/\ell_j > 0$, the strict small-ball event implies $S > E$, whereas $\sum_j |z_j|^2 \ell_j = E$. Exponential Markov inequality at $u = 1/2$ yields $\mathbb{P}(S > E) \le \mathrm{e}^{-E/4}$. If $E = 0$ the conclusion is trivial. Taking the infimum over these bounds proves the final assertion. □

**Corollary B.2** (A necessary regularity condition at a fixed center)**.** *Suppose $g \in C([0,1], \mathbb{C})$ and there exist $C, \epsilon_* > 0$ such that*

$$\mathbb{P}(\|K - g\|_\infty < \epsilon) \ge \exp(-C\epsilon^{-2}), \qquad 0 < \epsilon \le \epsilon_*.$$

*Then $g$ is 1/4-Hölder on $[0,1]$. This is a necessary condition, not a sufficient condition or a characterization of centers with the displayed lower order.*

*Proof.* For $s < t$ in $[0, 1/2]$, put $d = |g(t) - g(s)|$. When $0 < d \le 4\epsilon_*$, apply the preceding proposition to the single interval and take $\epsilon = d/4$. The assumed lower bound and the resulting upper bound imply

$$\exp(-16C/d^2) \le \mathbb{P}(\|K - g\|_\infty < d/4) \le \exp\{-d^2/[16(t-s)]\}, \qquad d^4 \le 256C(t-s).$$

By uniform continuity, the restriction $d \le 4\epsilon_*$ holds for all sufficiently short intervals. The longer intervals are controlled using the boundedness of $g$. Thus $g$ is 1/4-Hölder on the first half interval.

Every sample path satisfies $K(t) + \overline{K(1-t)} = K(1)$: the identity holds termwise for the symmetric partial sums and passes to their uniform limit. The assumed positive small-ball probabilities for arbitrarily small radii force the same identity for $g$. Indeed on such an event the uniform defect in that identity is less than $3\epsilon$. Reflection therefore gives the same Hölder bound on $[1/2, 1]$. Splitting an interval at 1/2 and using $x^{1/4} + y^{1/4} \le 2^{3/4}(x+y)^{1/4}$ completes the proof. □

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