Sharp Stability and Geometric Classification of a Vámos Quartic Deformation

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October 22, 2026

Abstract

We study the quartics $h_\lambda = 4e_1e_3 + \lambda a^2b^2$, where the elementary symmetric polynomials are taken in four variables. We prove that their real-stability interval is exactly $[0, 64/27]$. Throughout its positive part, the parameter is a genuine complex projective modulus, and the full linear automorphism group of the positive hyperbolicity cone consists of positive scalings and the independent interchanges of the two distinguished variable pairs. These cones are consequently pairwise linearly nonisomorphic and nonhomogeneous, including at the endpoint. The projective surfaces have eight singularities of type A_1 and one of type A_3 in the interior; one additional A_1 point appears at the endpoint. Correspondingly, the dual cone has exactly four Lorentz facets in the interior and exactly five at the endpoint. The arguments give an exact classification for this coefficient deformation of the specialized Vámos quartic, using classical hyperbolicity, singularity, and tangent-cone methods.

1 Introduction and statements

Write e_j for the degree-j elementary symmetric polynomial in a, b, c, d, and consider

$$h_\lambda(a, b, c, d) = 4(a + b + c + d)(abc + abd + acd + bcd) + \lambda a^2b^2. \tag{1}$$

The member h_1 is the specialized Vámos quartic. Its stability comes from the Vámos example of Wagner and Wei [11]; Kummer [7, Section 3] explicitly records its hyperbolicity direction $\mathbf{e} = (1, 1, 0, 0)$ and constructs a spectrahedral description of its cone. This polynomial is also associated with the classical obstruction to definite determinantal representations of powers [2, 7]. Here we examine how its stability and intrinsic geometry change when the coefficient of a^2b^2 varies.

A real polynomial is *real stable* if it does not vanish when all its arguments have positive imaginary parts. A homogeneous real polynomial p is *hyperbolic with respect to* e if $p(e) \neq 0$ and $t \mapsto p(x + te)$ has only real zeros for every real x. We use the open hyperbolicity cone containing e and its closure. The standard results of Gårding [4] identify the open cone with a component of $\{p \neq 0\}$, prove its convexity, and show that every point of that component is a hyperbolicity direction. For a homogeneous real stable polynomial with positive coefficients this cone contains the positive orthant.

Set $L = 64/27$, and, for $0 < \lambda \leq L$, let C_λ be the open hyperbolicity cone containing the positive orthant and let $K_\lambda = \overline{C_\lambda}$. The positive dual convention is

$$K_\lambda^* = \{\ell \in (\mathbb{R}^4)^* : \ell(x) \geq 0 \text{ for all } x \in K_\lambda\}.$$

The following collects the main conclusions, proved in the subsequent sections.

Theorem 1.1. *The family* (1) *has the following properties.*

1. *It is real stable exactly when* $0 \le \lambda \le L$*. For* $\lambda > 0$*, it is hyperbolic with respect to* $\mathbf{e}$ *exactly when* $\lambda \le L$*.*

2. *For* $0 < \lambda, \mu \le L$*, a complex projective linear equivalence between* h_λ *and* h_μ *exists exactly when* $\lambda = \mu$*. For every such* λ*, the projective linear automorphism group is the four-element group* $G = \langle a \leftrightarrow b,\ c \leftrightarrow d \rangle$*.*

3. *The cones* C_λ *are pairwise linearly nonisomorphic, and* $\mathrm{Aut}(C_\lambda) = \mathbb{R}_{>0}G$*. In particular all are nonhomogeneous.*

4. *The dual* K_λ^* *has exactly four facets if* $\lambda < L$ *and exactly five if* $\lambda = L$*. Every facet is linearly isomorphic to the three-dimensional Lorentz cone.*

In the interior there are eight A_1 singularities and one A_3; the endpoint adds one A_1 singularity. The stability statement also includes $\lambda = 0$. The projective and cone classifications use the positive parameter range. With the positive dual convention above, the endpoint facet is the convex-geometric consequence of the additional singular ray; the local singularity count is established in Proposition 3.1.

These statements concern a specific coefficient slice. The general hyperbolicity-cone and automorphism principles are classical; for the latter see, for example, Ito and Lourenço [6]. The contribution is the sharp parameter interval and the explicit projective, cone, and facial classifications within it. We do not propose a new general obstruction to determinantal representations or a new singularity classification. We also do not settle complete monotonicity of the full four-variable inverse half-power, including the member $h_1^{-1/2}$.

Related stable Vámos families already occur in Burton, Vinzant, and Youm [3]. Coefficient constraints and parameter reductions for the half-plane property are studied by Brändén and González D'León [1], and in more recent work of Kummer and Sawall [8]. Our claims concern the exact displayed four-variable normalization and its projective and convex geometry.

2 The sharp stability interval

The main issue is sufficiency of the upper bound. We first prove the real inequality that controls every possible collision of roots.

Lemma 2.1 (Collision inequality)**.** *Let* $t \ge 1$*,* $t \ne 2$*,* $r \ne 0, -1$*, and put*

$$D = 2t + (t+2)r \ne 0, \qquad N = (t+2)r^2 + 4r - t(t-2).$$

If $rN/D \ge 0$ *and*

$$F = -\frac{16r(1+r)^2 t}{(2-t)D} > 0,$$

then $F \ge 64/27$*, with equality only at* $t = 1, r = -1/3$*.*

Proof. The sign of r/D is the sign of $t-2$, and N has that same weak sign. Suppose first that $1 \le t < 2$. Necessarily $r = -v$, where $0 < v < 2t/(t+2) < 1$, and

$$B = (t+2)v^2 - 4v + t(2-t) \le 0.$$

Multiplication by the positive denominator shows that the desired inequality is equivalent to $H \geq 0$, where

$$\begin{aligned} H &= 27tv(1-v)^2 - 4(2-t)\big(2t-(t+2)v\big) \\ &= (8-4v)t^2 + \big(27v(1-v)^2 - 16\big)t + 16v. \end{aligned}$$

For $0 < v \leq 1/3$, use the exact decomposition

$$H = -4(2-v)B + v\big[(23v^2 - 46v + 19)(t-1) + (3v-1)(5v-3)\big].$$

The quadratic in the first bracket decreases on this interval and is $56/9$ at $v = 1/3$; both factors in the last product are nonpositive. Every summand is therefore nonnegative. Equality requires $t = 1, v = 1/3$.

For $1/3 \leq v \leq 2/5$,

$$H_{tt} = 16 - 8v > 0, \qquad H_t(1) = v\big(27(1-v)^2 - 8\big) > 0.$$

Consequently

$$H(t,v) \geq H(1,v) = (3v-1)(9v^2 - 15v + 8) \geq 0.$$

The last quadratic has discriminant -63, and equality again occurs only at $t = 1, v = 1/3$. Finally, if $2/5 \leq v < 1$, put $w = 1 - v$. The discriminant of H as a quadratic in t is

$$w^2\big[729w^2(1-w)^2 + 864w - 608\big].$$

Here $0 < w \leq 3/5$, so the bracket is at most $729/16 + 2592/5 - 608 = -3523/80 < 0$. Since its leading coefficient is positive, H is strictly positive.

Now let $t > 2$. If $r > 0$, feasibility gives $t \leq T$, where $T > 2$ is the positive root of

$$T^2 - (r^2+2)T - 2r^2 - 4r = 0.$$

For fixed r,

$$F = \frac{16r(r+1)^2 t}{(t-2)\big((r+2)t + 2r\big)}$$

decreases on $t > 2$: differentiating the quotient, its derivative has the sign of $-(r+2)t^2 - 4r$. At T, the assertion $F(T) > 4$ is equivalent to positivity of

$$\begin{aligned} J &= 4Tr(r+1)^2 - (T-2)\big((r+2)T + 2r\big) \\ &= T(3r^3 + 6r^2 + 2r) - 2r(r^2 + 4r + 2) \\ &> 4r^3 + 4r^2 > 0, \end{aligned}$$

where the quadratic relation and then $T > 2$ were used.

The other branch is $r = -v$, $v > 2t/(t+2) > 1$. In this case

$$F = \frac{16v(v-1)^2}{g(t)}, \qquad g(t) = (v-2)t + 4 - \frac{4v}{t} > 0.$$

If $1 < v < 2$, the arithmetic–geometric mean inequality gives $g(t) \leq 4 - 4\sqrt{v(2-v)}$. With $w = v-1$, the inequality $1 - \sqrt{1-w^2} \leq w^2$ implies $F \geq 4v > 4$. If $v \geq 2$, then g increases. Feasibility bounds $t \leq T$, where $T > 2$ is the positive root of

$$T^2 - (v^2+2)T - 2v^2 + 4v = 0.$$

At that root the corresponding numerator reduces to

$$\begin{aligned} J &= 4vT(v-1)^2 - (T-2)((T+2)v - 2T) \\ &= Tv(3v^2 - 6v + 2) - 2v(v^2 - 4v + 2) \\ &> 4v^3 - 4v^2 > 0. \end{aligned}$$

The coefficient of T is positive for $v \geq 2$. Thus $F > 4$ in every $t > 2$ branch, proving the lemma and its equality statement. □

Theorem 2.2. *The real-stability interval of* (1) *is exactly* $[0, L]$. *For positive parameters, hyperbolicity with respect to* $\mathbf{e}$ *holds exactly on* $(0, L]$.

Proof. Use real coordinates $s = a + b$, $\delta = a - b$, $U = c + d$, and $V = c - d$. Along an $\mathbf{e}$-line only s changes, with nonzero constant speed. The relevant polynomial is

$$p_\lambda(s) = \frac{\lambda}{16}(s^2 - \delta^2)^2 + Us^3 + (2U^2 - V^2)s^2 + (U^3 - UV^2 - U\delta^2)s - U^2\delta^2. \tag{2}$$

First restrict to the dense generic set

$$U\delta \neq 0, \quad U^2 \neq V^2, \quad \delta^2 \neq U^2, \quad U^2 - V^2 - \delta^2 \neq 0.$$

A real double root cannot be zero, because $p'_\lambda(0) = U(U^2 - V^2 - \delta^2) \neq 0$. It also cannot satisfy $s^2 = \delta^2$, since

$$p_\lambda(\delta) = \delta(U^2 - V^2)(U + \delta), \qquad p_\lambda(-\delta) = \delta(U^2 - V^2)(\delta - U).$$

At any remaining real double root define

$$r = U/s, \qquad x = \delta^2/s^2 \geq 0, \qquad z = V^2/s^2 \geq 0.$$

Eliminating λ from $p_\lambda = p'_\lambda = 0$ gives

$$z = \frac{rN}{D}, \qquad \lambda = -\frac{16r(1+r)^2(1+x)}{(1-x)D}, \tag{3}$$

where

$$D = 2 + 3r + (2+r)x, \qquad N = (1+r)(1+3r) + r^2x - x^2.$$

Here $r \neq 0$; also $r \neq -1$, since $s + U = 0$ would leave $p_\lambda = \lambda(s^2 - \delta^2)^2/16 \neq 0$. Before division by D the elimination gives $Dz = rN$. If $D = 0$, substitution of $r = -2(1+x)/(3+x)$ in N yields

$$-\frac{(x-1)^2(x+1)}{x+3}.$$

Thus $D = N = 0$ would force the already excluded $x = 1$. Applying Lemma 2.1 with $t = 1 + x$ shows that every positive collision parameter is at least L.

The known baseline h_1 is hyperbolic with respect to $\mathbf{e}$ [11, 7]. On every generic line its four roots therefore remain real as λ varies in the connected interval $(0, L)$: a nonreal conjugate pair could appear only through a real multiple root. The leading coefficient $\lambda/16$ never vanishes on a compact subinterval, so roots cannot escape to infinity there. Arbitrary real lines follow by approximation by generic lines and closedness of real-rootedness for fixed-degree polynomials. The same closedness proves the endpoint $\lambda = L$. Since $h_\lambda(\mathbf{e}) = \lambda > 0$, this is hyperbolicity in the stated direction.

For necessity put $\delta = V = 0$, $U \neq 0$. After removing the root $s = 0$, the cubic has discriminant

$$\frac{\lambda(64 - 27\lambda)U^6}{256}.$$

It is negative for $\lambda > L$, excluding hyperbolicity in direction $\mathbf{e}$. For $0 < \lambda \leq L$, all coefficients are nonnegative, and the positive orthant lies in the nonvanishing component containing $\mathbf{e}$: join any positive point to $\mathbf{e}$ by a segment, on which h_λ stays positive. Gårding's theorem makes every positive direction hyperbolic. This is equivalent to real stability: writing a complex point as $x+iy$, $y > 0$, excludes the nonreal root i on that line.

Conversely, stability at $\lambda > L$ would make every positive direction hyperbolic. Taking these directions to $\mathbf{e}$, where $h_\lambda(\mathbf{e}) \neq 0$, gives hyperbolicity at $\mathbf{e}$ by closedness, a contradiction. At zero, $h_0 = 4e_1e_3$ is stable by the classical stability of elementary symmetric polynomials and closure under products. If $\lambda < 0$, $h_\lambda(R, R, 1, 1)$ is positive for sufficiently small positive R and negative for sufficiently large R. Its positive real zero gives an upper-half-plane zero after multiplying all coordinates by i, by homogeneity. Thus it is not stable. □

3 Singular points and their local types

We next determine the singular set over $\mathbb{C}$; this is essential for distinguishing genuine projective equivalence from the visible coordinate symmetries. Let $X_\lambda = \{h_\lambda = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$.

Proposition 3.1. *For $0 < \lambda < L$, the singular set of X_λ consists of the four coordinate points and*

$$q = (0, 0, 1, -1), \quad (0, 1, 0, -1), \quad (0, 1, -1, 0), \quad (1, 0, 0, -1), \quad (1, 0, -1, 0). \tag{4}$$

At $\lambda = L$, the only additional point is $v = (3, 3, -1, -1)$. The point q has type A_3, and every other listed point has type A_1.

Proof. Put $s = a + b$, $U = c + d$, $E = s + U$, and $P = ab$. Gradient differences give

$$h_a - h_b = -(a - b)(4EU + 2\lambda P), \qquad h_c - h_d = -4(c - d)Es.$$

If $E = 0$, then $h_c = 4e_3 = 0$; the other equations force $ab = 0$. Solving $E = e_3 = ab = 0$ gives exactly the five points (4). If $E \neq 0$ and $s = 0$, set $a = -b$. Then $h_c = -8a^2U$ forces $a = b = 0$, and the remaining equation $Ucd = 0$ gives the coordinate points e_c, e_d.

Otherwise $Es \neq 0$ and $c = d = t$. The restriction in coordinates (s, P, t) is

$$H = \lambda P^2 + 8Pst + 16Pt^2 + 4s^2t^2 + 8st^3.$$

If $a \neq b$, the change from (a, b) to (s, P) is nonsingular. The singularity equations are

$$H_s = 8t(P + st + t^2) = 0, \quad H_P = 2\lambda P + 8st + 16t^2 = 0,$$

$$H_t/8 = P(s + 4t) + s^2t + 3st^2 = 0.$$

When $t = 0$, they give $P = 0$, hence e_a, e_b. When $t \neq 0$, scale to $t = 1$. The first and third equations give $P = -s-1$, then $s = -2, P = 1$; the second would force $\lambda = 0$. If instead $a = b = x$, then $x \neq 0$, and the restriction is

$$J = \lambda x^4 + 16xt(x + t)^2, \qquad J_t = 16x(x + t)(x + 3t).$$

The branch $x+t=0$ forces $\lambda=0$. The branch $x+3t=0$ forces $\lambda=L$ and gives v. Direct substitution verifies the gradient vanishes at every listed point. Euler's identity supplies the equation of the hypersurface as well. This exhausts the complex projective singular set.

At a singular point the radial vector lies in the homogeneous Hessian kernel. Nonzero transverse Hessian determinants at the fixed points are

points	a nonzero transverse determinant
e_a, e_b	$32(4-\lambda)$
e_c, e_d	128
the four mixed points other than q	-32λ.

They are all nonzero in our positive interval. At v the Hessian is

$$\begin{pmatrix} 8/3 & 40/3 & 24 & 24 \\ 40/3 & 8/3 & 24 & 24 \\ 24 & 24 & 24 & 120 \\ 24 & 24 & 120 & 24 \end{pmatrix};$$

its principal minor in coordinates a, b, d is 8192. Thus these points have nondegenerate quadratic local terms in three projective coordinates, and are A_1 singularities.

For the remaining point q, take the chart $c=1, d=-1+z$, with $s=a+b$, $\delta=a-b$. Its exact local equation is

$$f = 4(s+z)\left[-s+sz+\frac{(s^2-\delta^2)z}{4}\right]+\frac{\lambda}{16}(s^2-\delta^2)^2.$$

Define

$$\begin{aligned} A &= -4+4z+sz+z^2+\lambda s^2/16-\lambda\delta^2/8, \\ B &= -4+4z-\delta^2, \qquad Q_0=-\delta^2. \end{aligned}$$

Completing the square gives the exact identity

$$f=\frac{\lambda\delta^4}{16}+A\left(s+\frac{Bz}{2A}\right)^2+\left(Q_0-\frac{B^2}{4A}\right)z^2.$$

At the origin $A=-4$ and $Q_0-B^2/(4A)=1$. Choose analytic square roots near these nonzero values. The coordinates

$$u=\sqrt{-A}\left(s+\frac{Bz}{2A}\right), \quad w=\sqrt{Q_0-B^2/(4A)}\,z, \quad t=(\lambda/16)^{1/4}\delta$$

have invertible Jacobian and transform the equation into $f=-u^2+w^2+t^4$. Over $\mathbb{C}$ this is the A_3 normal form. This explicit construction also proves its real analytic form and shows directly that the homogeneous Hessian rank at q is two. The terminology agrees with the standard corank-one classification [5, Section I.2.4]. □

Corollary 3.2. *Every h_λ, $0<\lambda\le L$, is absolutely irreducible.*

Proof. Two positive-degree projective surface factors in $\mathbb{P}^3_{\mathbb{C}}$ meet in a set of dimension at least one. Their intersection is singular for their product. A repeated factor likewise gives a positive-dimensional singular set. Either possibility contradicts Proposition 3.1. □

4 Projective rigidity and cone automorphisms

Theorem 4.1. *For $0 < \lambda \le L$, the projective linear automorphism group of X_λ over $\mathbb{C}$ is G. If an invertible complex linear map T and a nonzero scalar κ satisfy $h_\mu(Tx) = \kappa h_\lambda(x)$, with $0 < \lambda, \mu \le L$, then $\lambda = \mu$.*

Proof. Label the four coordinate points $0, 1, 2, 3$, the point q by 4, and the remaining four points of (4), in their displayed order, by $5, 6, 7, 8$. Their collinear triples are exactly

$$028, \quad 037, \quad 126, \quad 135, \quad 234, \quad 456, \quad 478. \tag{5}$$

This list follows by taking ranks of the indicated three point vectors; there are only $\binom{9}{3}$ triples to consider. The incidence argument below also specifies how the resulting configuration controls the group.

The point q is intrinsically distinguished as the only singular point of Hessian rank two. Within the nine-point configuration, points $2, 3, 4$ belong to three triples each and the other six to two. An incidence permutation fixing 4 therefore preserves $\{2, 3\}$. If it fixes 2 and 3 individually, the low-degree points incident with a triple through 2 and another through 3 are exactly $0, 1$. Their images can be chosen in two ways, and then the triples in (5) determine all remaining images. Interchanging $2, 3$ is realized by $c \leftrightarrow d$, so at most four permutations fixing 4 occur. All four are realized by the independent swaps of a, b and c, d.

At the endpoint the additional point v lies on no line through two of the fixed nine points. Indeed a coordinate point and any fixed point have joint support of size at most three, whereas v has four nonzero coordinates. Two noncoordinate fixed points could span v only if their two-coordinate supports were disjoint. On each such support their entries are opposite, requiring a pair of coordinates of $v = (3, 3, -1, -1)$ to sum to zero, which never occurs. Thus v is intrinsically the unique singular point lying on no collinear triple. Every endpoint automorphism fixes it and preserves the other nine points, so the preceding bound of four permutations still applies.

After composing an automorphism with its realized coordinate swap, it fixes the nine points individually. Fixing the coordinate points makes its matrix diagonal. Fixing the four mixed points forces all four diagonal entries equal. Thus it is projectively the identity. This proves the automorphism assertion, including the endpoint.

For equivalence between two interior parameters the same intrinsic rank and incidence argument applies to the common singular configuration. An interior parameter cannot be equivalent to the endpoint, since their singular sets have different cardinalities. Hence any equivalence in the theorem is a scalar times one of the four swaps. Comparing a coefficient such as that of a^2bc, and then the coefficient of a^2b^2, gives $\mu = \lambda$. Corollary 3.2 also makes polynomial equivalence and projective hypersurface equivalence identical here. □

Lemma 4.2. *The complex Zariski closure of ∂K_λ is the affine hypersurface $h_\lambda = 0$.*

Proof. Boundary points are zeros of h_λ. The point $x_0 = (0, 0, 1, 1)$ is approached from the positive orthant and is a boundary point, with gradient $(8, 8, 0, 0)$. By the implicit-function theorem its neighborhood in the boundary contains a real open smooth hypersurface patch. A polynomial vanishing on that patch vanishes on its complex analytic continuation: use the three real smooth coordinates as a uniqueness set for the corresponding holomorphic function. Absolute irreducibility then implies that the patch is Zariski dense in $h_\lambda = 0$. This proves the assertion. □

Theorem 4.3. *The cones C_λ, $0 < \lambda \le L$, are pairwise nonisomorphic under invertible real linear maps. For every such parameter,*

$$\mathrm{Aut}(C_\lambda) = \{t\sigma : t > 0,\ \sigma \in G\}.$$

All these cones are nonhomogeneous.

Proof. A linear cone isomorphism maps boundaries onto boundaries, hence maps their complex Zariski closures onto one another. By Lemma 4.2, irreducibility, and equal degree, it induces $h_\mu(Tx) = \kappa h_\lambda(x)$. Theorem 4.1 forces $\lambda = \mu$, and every automorphism is a real nonzero scalar times a swap. A negative scalar cannot preserve the open convex cone: it would put both a point and its negative in the cone and hence put zero there. Positive scalings and the swaps do preserve the component containing the positive orthant. The displayed group has orbits consisting of at most four rays, and therefore cannot act transitively on a four-dimensional open cone. □

The passage from a minimal defining polynomial to the cone automorphism group is a classical principle, discussed in [6, Proposition 2.6]; see also the recent formulation in [9, Proposition 1]. Lemma 4.2 supplies the concrete minimality argument needed here; no rank-one generation hypothesis is used.

5 The dual-facet transition

A facet of a full-dimensional cone in $\mathbb{R}^4$ means a face of dimension three. Denote the three-dimensional Lorentz cone by

$$\mathcal{L}_3 = \{(t, x, y) : t \ge \sqrt{x^2 + y^2}\}.$$

For a closed convex cone K and $x \in K$, use the closed tangent cone $T_K(x) = \overline{\mathrm{cone}(K - x)}$. Its positive dual satisfies

$$T_K(x)^* = K^* \cap x^\perp. \tag{6}$$

Indeed a functional nonnegative on $K - x$ vanishes at x, by testing 0 and $2x$, and is then nonnegative on K. The converse is immediate.

We use the classical localization theorem: at a point of a closed hyperbolicity cone, its tangent cone is the closed hyperbolicity cone of the lowest nonzero homogeneous term in the polynomial's expansion there. With the present tangent convention this is [10, Theorem 6].

Lemma 5.1. *The cones K_λ are solid and pointed. The four positive coordinate rays expose four distinct facets of K_λ^*, each linearly isomorphic to $\mathcal{L}_3$.*

Proof. The positive coordinate rays lie in K_λ, by closure of the positive orthant. At e_a the lowest term is

$$Q_a(b, c, d) = \lambda b^2 + 4bc + 4bd + 4cd.$$

Its coefficient matrix has determinant $4(4 - \lambda) > 0$, is indefinite, and is positive on the positive orthant. Its inertia is therefore $(1, 2)$. Its forward hyperbolicity cone is a linear image of $\mathcal{L}_3$. The same holds at e_b. At e_c, e_d, respectively, the localizations are

$$4(ab + ad + bd), \qquad 4(ab + ac + bc),$$

again nondegenerate Lorentz quadratics in the transverse coordinates. Thus each tangent cone is a Lorentz cone times its coordinate lineality. By (6) its dual is a three-dimensional Lorentz face in the corresponding coordinate hyperplane. These four faces are distinct facets.

Solidity follows because $\mathbf{e}$ is an interior hyperbolicity direction. For pointedness, $K_\lambda \subseteq T_{K_\lambda}(x)$ for every $x \in K_\lambda$: if $k \in K_\lambda$, then $x + tk \in K_\lambda$ for $t \geq 0$. Hence the lineality of K_λ lies in the lineality of both coordinate tangent cones at e_a and e_b. Their linealities are the distinct coordinate lines, whose intersection is zero. The dual is consequently also solid and pointed. □

Lemma 5.2. *For $0 < \lambda < L$, the nonzero singular points in K_λ are exactly its four positive coordinate rays. At $\lambda = L$, there is exactly one further singular ray, generated by $v = (3, 3, -1, -1)$.*

Proof. Membership in the closed cone is equivalent to every zero of $t \mapsto h_\lambda(x+t\mathbf{e})$ being nonpositive. For $m = (1, 0, 0, -1)$, direct expansion gives

$$h_\lambda(m + t\mathbf{e}) = t^2(t + 1)\big(\lambda(t + 1) - 8\big).$$

It has a positive root $8/\lambda - 1$. Replacing m by $-m$ reflects the roots and supplies the positive root 1. The swaps exclude both orientations of all four mixed points. At q,

$$h_\lambda(q + t\mathbf{e}) = t^2(\lambda t^2 - 16),$$

which has roots of both signs, and the same is true at $-q$. Negative coordinate rays are excluded by pointedness.

At the endpoint,

$$h_L(v + t\mathbf{e}) = \frac{4}{27}(6 + 2t)(2t)^2(9/2 + 2t).$$

Its roots are zero, twice, -3, and $-9/4$; hence $v \in K_L$. Pointedness excludes $-v$. Proposition 3.1 now exhausts the possibilities. □

Theorem 5.3. *For $0 < \lambda < L$, the dual K_λ^* has exactly the four coordinate-exposed facets of Lemma 5.1. At $\lambda = L$ it has exactly one additional facet, namely*

$$F_v = \{(\alpha, \beta, \gamma, \eta) : \gamma + \eta = 3(\alpha + \beta), \quad \alpha + \beta \geq 0, \quad (\alpha + \beta)^2 \geq \tfrac{3}{2}(\alpha - \beta)^2 + \tfrac{1}{6}(\gamma - \eta)^2\}. \tag{7}$$

This fifth facet is also linearly isomorphic to $\mathcal{L}_3$.

Proof. At v, for a displacement y, put

$$S = y_a + y_b, \quad \delta = y_a - y_b, \quad U = y_c + y_d, \quad V = y_c - y_d, \quad T = S + 3U.$$

The localization is

$$Q_v = 4T^2 - \frac{8}{3}\delta^2 - 24V^2.$$

The forward tangent cone containing $\mathbf{e}$ is $T \geq \sqrt{(2/3)\delta^2 + 6V^2}$, with U free; its lineality is precisely $\mathbb{R}v$. In these coordinates the dual pairing with $(\alpha, \beta, \gamma, \eta)$ is

$$\frac{\alpha + \beta}{2}T + \frac{\alpha - \beta}{2}\delta + \frac{\gamma - \eta}{2}V + \frac{\gamma + \eta - 3(\alpha + \beta)}{2}U.$$

Nonnegativity for all U forces the linear equation in (7). Duality of the weighted Lorentz cone, or direct Cauchy–Schwarz, gives its remaining inequalities. By localization and (6) this is exactly $K_L^* \cap v^\perp$, a full three-dimensional Lorentz cone in that hyperplane. It is distinct from the four coordinate facets.

It remains to exclude additional facets. Every facet F of a closed full-dimensional convex cone is exposed. To see this here, choose a nonzero point in $\mathrm{ri}(F)$ and a supporting hyperplane there. Its normal can be taken nonzero in $(K_\lambda^*)^* = K_\lambda$. The functional vanishes on F by relative interior. Its zero face contains F and has the same dimension three, so equals F: a proper face cannot have the full relative dimension of its containing face. Thus $F = K_\lambda^* \cap x^\perp$ for a nonzero boundary point $x \in K_\lambda$. If this were a smooth point of $h_\lambda = 0$, its tangent cone would be a halfspace, and (6) would make F a ray rather than a facet. Hence x is singular. Lemma 5.2 gives exactly the already constructed four or five rays, proving exhaustion. □

For example, intersecting the endpoint facet with $\alpha = \beta = 1$ gives $\gamma + \eta = 6$, $|\gamma - \eta| \leq \sqrt{24}$. Thus the extra algebraic singular point creates an actual nonpolyhedral dual facet, not merely a new point on the surface.

6 Scope and reproducibility

The classification separates three issues. Stability changes sharply at 64/27; inside the positive interval the parameter cannot be removed by complex projective coordinates; and at its endpoint a new singular ray produces a fifth Lorentz facet of the dual. The exact automorphism group stays unchanged at that transition. Classical localization and minimal-boundary arguments explain how the algebraic calculations control the cone geometry.

The proofs are algebraic and analytic; finite computation is not used to infer an infinite parameter statement. Exact rational checks can independently verify the gradient and Hessian values, the finite collinearity list, its incidence permutations, and the local completed-square identity. All analytic steps, including singular-locus exhaustion, root continuation, the local coordinate change, and dual facet exhaustion, have been stated separately in the proofs.

The projective classification and cone automorphism groups concern the positive parameter range and linear changes of coordinates.

Acknowledgment of assistance

AI-assisted research tools were used for literature discovery, proof development, independent adversarial checks, exact symbolic and rational-arithmetic verification, and manuscript preparation. Responsibility for the statements, proofs, and citations rests with the author.

References

[1] P. Brändén and R. S. González D'León, *On the half-plane property and the Tutte group of a matroid*, Journal of Combinatorial Theory, Series B **100** (2010), 485–492. doi:10.1016/j.jctb.2010.04.001.

[2] P. Brändén, *Obstructions to determinantal representability*, Advances in Mathematics **226** (2011), 1202–1212. doi:10.1016/j.aim.2010.08.003.

[3] S. Burton, C. Vinzant, and Y. Youm, *A real stable extension of the Vámos matroid polynomial*, preprint, 2014. arXiv:1411.2038.

[4] L. Gårding, *An inequality for hyperbolic polynomials*, Journal of Mathematics and Mechanics **8** (1959), 957–965. doi:10.1512/IUMJ.1959.8.58061.

[5] G.-M. Greuel, C. Lossen, and E. Shustin, *Introduction to Singularities and Deformations*, Springer Monographs in Mathematics, Springer, Berlin, 2007, Section I.2.4, Theorems 2.47–2.48.

[6] M. Ito and B. F. Lourenço, *Automorphisms of rank-one generated hyperbolicity cones and their derivative relaxations*, SIAM Journal on Applied Algebra and Geometry **7** (2023), 236–263. doi:10.1137/22M1513964.

[7] M. Kummer, *A note on the hyperbolicity cone of the specialized Vámos polynomial*, Acta Applicandae Mathematicae **144** (2016), 11–15. doi:10.1007/s10440-015-0036-z.

[8] M. Kummer and D. Sawall, *Three results related to the half-plane property of matroids*, Algebraic Combinatorics **8** (2025), no. 1, 1–15. doi:10.5802/alco.399.

[9] M. Orlitzky, *Automorphisms of hyperbolic polynomials*, preprint, June 11, 2026. https://optimization-online.org/wp-content/uploads/2026/06/automorphisms_of_hyperbolic_polynomials.pdf.

[10] J. Saunderson and V. Chandrasekaran, *Terracini convexity*, Mathematical Programming **198** (2023), 399–441. doi:10.1007/s10107-022-01774-y.

[11] D. G. Wagner and Y. Wei, *A criterion for the half-plane property*, Discrete Mathematics **309** (2009), 1385–1390. doi:10.1016/j.disc.2008.02.005; arXiv:0709.1269. Corrigendum: Discrete Mathematics **313** (2013), no. 6, 830.