

Sharp regularity criteria for arithmetic cutoff functionals

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Luminary Oddities

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Abstract

We classify the coefficient penalties that control point evaluation and absolute coefficient moments of finite arithmetic cutoff certificates. In divisor coordinates, let $W_\omega = \sum_{r \geq 2} \omega(r)(J_4(r) + 10J_2(r))|v_r|^2/180$, where $\omega(r) \geq 1$ is arbitrary. Every fixed integer Bernoulli cutoff value is bounded in this norm if and only if $\sum_r \omega(r)^{-1} < \infty$. For every integer $R \geq 1$, the absolute moment $\sum_d d^R |c_d|$ is bounded if and only if $\sum_r r^{2R-2}/\omega(r) < \infty$. Both classifications persist after any fixed finite collection of bounded homogeneous constraints. A full-divisor lower bound supplies necessity and settles the critical logarithmic exponents. Applications give an exact constrained point capacity, a norm-controlled compensated-mean remainder, and an explicit native approximation floor. At the critical moment penalties, the floor gives a quantitative logarithmic lower bound on the energy required for a given native error.

1 Arithmetic setting and contribution

Coefficient control is a concrete issue in Nyman–Beurling approximation: a finite certificate must pay for the functionals used to estimate its tail. Generalized approximation constructions, including the polynomial method of Alouges, Darses and Hillion [1], make coefficient conditions explicit. The Dirichlet-polynomial asymptotics of Bettin, Conrey and Farmer [2] concern the associated approximation problem. Here we determine the exact regularity required by two arithmetic functionals. The criteria identify coefficient penalties that bound cutoff values and absolute moments. Their native approximation application also gives an explicit necessary energy cost at the critical moment penalties (Theorem 5.3).

All coefficient sequences in the statements below have finite support in $\mathbb{N}$. Write

$$A(s) = \sum_d c_d d^{-s}, \quad A(1) = 0, \quad C_c(K) = \sum_d d c_d B_2(\{K/d\}), \quad K \in \mathbb{Z}, \quad (1)$$

where $B_2(x) = x^2 - x + 1/6$. Put

$$v_r = \sum_{d|r} \frac{c_d}{d}, \quad c_d = d \sum_{k \geq 1} \mu(k) v_{dk}, \quad J_z(r) = \sum_{d|r} \mu(r/d) d^z. \quad (2)$$

Thus $v_1 = 0$, every finite sequence $(v_r)_{r \geq 2}$ gives actual anchored coefficients, and

$$A(s) = \sum_{r \geq 2} J_{1-s}(r) v_r. \quad (3)$$

For real $u > 1$, Jordan factorization gives $J_u(r) = r^u \prod_{p|r} (1 - p^{-u})$. Define

$$w_r = \frac{J_4(r) + 10J_2(r)}{180}, \quad W_\omega(c) = \sum_{r \geq 2} \omega(r) w_r |v_r|^2. \quad (4)$$

Throughout, $\omega(r)$ is finite, real, and at least 1. No monotonicity or multiplicativity is assumed.

The unpenalized covariance is predecessor material: the Bernoulli calculation in [3, Theorem 1] gives, on $A(1) = 0$,

$$\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{K=0}^{H-1} |C_c(K)|^2 = W_1(c). \quad (5)$$

Its unanchored discrete covariance kernel is $(\gcd(d, e)^4 + 10 \gcd(d, e)^2 - 6)/(180de)$. The anchor removes its constant term. The sharp tail-capacity and compensated-mean framework of [4] provides the application in Section 5.

Our contribution is the arbitrary-penalty classification in Theorems 3.2 and 3.3. The arithmetic input is a point-row estimate on every sufficiently large divisor index. It gives complete logarithmic endpoints and finite-null stability. Hilbert-space duality, divisor inversion and the Schur projection formula are used in their standard forms.

2 Full-divisor estimates

With $\zeta(u) = \sum_{n \geq 1} n^{-u}$ for real $u > 1$, one has

$$\frac{r^4}{180\zeta(4)} \leq w_r \leq \frac{7r^4}{360} \quad (r \geq 2). \quad (6)$$

Indeed, use Jordan factorization and $10r^2 \leq (5/2)r^4$. The point row in divisor coordinates is

$$E_K(r) = \sum_{d|r} \mu(r/d) d^2 B_2(\{K/d\}), \quad C_c(K) = \sum_{r \geq 2} E_K(r) v_r. \quad (7)$$

Lemma 2.1 (Full-divisor point row). *Put $k = |K|$ and $R_K = 24(k + k^3 + 1)$. For all $r \geq 2$,*

$$E_K(r) = J_2(r)/6 - kJ_1(r) + H_K(r), \quad |H_K(r)| \leq k^3. \quad (8)$$

Consequently

$$|E_K(r)|^2/w_r \leq 5\zeta(4)\zeta(2)^2 \quad (r \geq 2, K \in \mathbb{Z}), \quad (9)$$

$$E_K(r) \geq r^2/24, \quad |E_K(r)|^2/w_r \geq 5/56 \quad (r \geq R_K). \quad (10)$$

Proof. Bernoulli reflection [5] permits replacing K by k . For $d > k$, $d^2 B_2(\{k/d\}) = k^2 - kd + d^2/6$. For $d \leq k$, write $k/d = m + \theta$, $m \geq 1$, $0 \leq \theta < 1$. The difference from the same quadratic is $-d^2 m(m - 1 + 2\theta)$, with absolute value at most k^2 : $m(m - 1 + 2\theta) \leq (m + \theta)^2$. There are at most k such divisors. Since $\sum_{d|r} \mu(r/d) = 0$ for $r \geq 2$, this proves (8). Also $|B_2| \leq 1/6$ and $\sum_{d|r} d^2 \leq \zeta(2)r^2$, so (6) gives (9). For the lower estimate, $J_2(r) \geq r^2/\zeta(2) > r^2/2$ and $J_1(r) \leq r$. If $r \geq R_K$, then $kr + k^3 \leq r(k + k^3) \leq r^2/24$. Thus $E_K(r) \geq r^2/24$. The upper bound for w_r gives $360/(576 \cdot 7) = 5/56$. $\square$

3 Classification and stability under nulls

Let H_ω have squared norm $\|v\|_\omega^2 = \sum_{r \geq 2} p_r |v_r|^2$, $p_r = \omega(r)w_r$. A row $h = (h_r)$ is bounded if $\sum |h_r|^2/p_r < \infty$; its action is $\sum h_r v_r$. A finite list of such rows defines homogeneous constraints $Cv = 0$.

Lemma 3.1 (Finite-null stability). *For any row $a = (a_r)$, its functional on finite sequences satisfying $Cv = 0$ is bounded in H_ω if and only if $\sum |a_r|^2/p_r < \infty$.*

Proof. Sufficiency is Cauchy–Schwarz. For necessity, discard redundant constraint rows. Independence implies that some finite coordinate set F_0 has full row rank; choose a right inverse T supported there. If the dual sum of a diverges, take finite sets F outside F_0 with $L_F = \sum_{r \in F} |a_r|^2/p_r \rightarrow \infty$. Set $u_r = \overline{a_r}/(p_r \sqrt{L_F})$ on F , zero elsewhere. Then $\|u\|_\omega = 1$ and $au = \sqrt{L_F}$. Boundedness of C bounds Cu uniformly. The correction $u_0 = -TCu$ has bounded norm and bounded au_0 , because its support is fixed. The finite vector $u + u_0$ satisfies all nulls, has bounded norm and unbounded target. $\square$

Theorem 3.2 (Point regularity). *Fix $K \in \mathbb{Z}$. A constant $D < \infty$ such that $|C_c(K)| \leq D\sqrt{W_\omega(c)}$ for all finite anchored coefficients exists if and only if*

$$\sum_{r \geq 2} \frac{1}{\omega(r)} < \infty. \quad (11)$$

The equivalence persists after any fixed finite list of bounded homogeneous rows. When it holds, the same bound works for every integer K with

$$D^2 = 5\zeta(4)\zeta(2)^2 \sum_{r \geq 2} \omega(r)^{-1}. \quad (12)$$

Proof. Lemma 2.1 shows that $\sum |E_K(r)|^2/p_r$ converges exactly when (11) does. Finitely many initial indices do not affect convergence. Apply Lemma 3.1 and the exact finite inverse (2); (9) gives the uniform constant. $\square$

Theorem 3.3 (Absolute moment regularity). *For each integer $R \geq 1$ set $S_R(c) = \sum_{d \geq 1} d^R |c_d|$. A support-independent bound $S_R(c) \leq D_R \sqrt{W_\omega(c)}$ on finite anchored coefficients exists if and only if*

$$\sum_{r \geq 2} \frac{r^{2R-2}}{\omega(r)} < \infty. \quad (13)$$

The equivalence persists after any fixed finite list of bounded homogeneous rows. A sufficient explicit constant is

$$D_R^2 = 180\zeta(4)\zeta(R+1)^2 \sum_{r \geq 2} \frac{r^{2R-2}}{\omega(r)}. \quad (14)$$

Proof. Including c_1 in divisor inversion gives

$$S_R(c) \leq \sum_{r \geq 2} |v_r| \sum_{d|r} d^{R+1} \leq \zeta(R+1) \sum_{r \geq 2} r^{R+1} |v_r|.$$

Cauchy–Schwarz and (6) give (14). Conversely, $S_R(c) \geq |\sum_d d^R c_d| = |\sum_{r \geq 2} J_{R+1}(r) v_r|$, and

$$\frac{|J_{R+1}(r)|^2}{p_r} \geq \frac{360}{7\zeta(R+1)^2} \frac{r^{2R-2}}{\omega(r)}.$$

If (13) fails, Lemma 3.1 makes this signed moment, hence also its absolute moment majorant, unbounded after the prescribed nulls. $\square$

Corollary 3.4 (Sharp logarithmic endpoints). *For $t \geq 0$, $b \in \mathbb{R}$, put $\omega_{t,b}(r) = \max\{1, r^{2t}[\log(er)]^b\}$. The point bound holds precisely for $t > 1/2$, or $t = 1/2$ and $b > 1$. The S_R bound holds precisely for $t > R - 1/2$, or $t = R - 1/2$ and $b > 1$. Both retain the stated null stability.*

Proof. At a positive critical power the maximum with 1 is eventually inactive. The relevant series has a power exponent away from the critical power and becomes $\sum 1/[r(\log(er))^b]$ at it. The integral test proves the alternatives. For $t = 0$ the weight is constant or logarithmic, and both series diverge. $\square$

Thus $r[\log(er)]^2$ controls point evaluation and $r^{2R-1}[\log(er)]^2$ controls S_R . Each is asymptotically smaller than the corresponding critical power with any extra fixed positive exponent. Necessity uses the full integer series: restricting to primes would lose part of the logarithmic endpoint.

4 Exact Dirichlet nulls and point capacities

Put

$$0 < a < \frac{1}{2}, \quad \sigma = \frac{1}{2} - a, \quad \alpha = \frac{1}{2} + a, \quad q = \sigma + iT, \quad \lambda = \alpha + iT, \quad T \in \mathbb{R}. \quad (15)$$

The nulls $A(q) = A(\lambda) = 0$ have rows $h_q(r) = J_{1-q}(r)$ and $h_\lambda(r) = J_{1-\lambda}(r)$. For $0 < \Re s < 1$, the divisor count $\tau(r) \leq 2\sqrt{r}$ gives

$$|J_{1-s}(r)| \leq 2r^{3/2-\Re s}, \quad \sum_{r \geq 2} \frac{|J_{1-s}(r)|^2}{p_r} \leq 720\zeta(4) \sum_{r \geq 2} r^{-1-2\Re s} < \infty. \quad (16)$$

Thus both classifications hold on these actual nulls. Their independence has a finite witness: $J_z(4) = 2^z J_z(2)$. The determinant on indices 2, 4 has modulus at least

$$(2^\alpha - 1)(2^\sigma - 1)(2^\alpha - 2^\sigma) > 0.$$

Indeed its exact value is $J_{1-q}(2)J_{1-\lambda}(2)(2^{1-\lambda} - 2^{1-q})$. Null corrections can always use these two indices.

Proposition 4.1 (Constrained capacity). *Assume (11). For $i, j \in \{q, \lambda\}$ put*

$$S = \sum_{r \geq 2} \frac{|E_K(r)|^2}{p_r}, \quad \Gamma_{ij} = \sum_{r \geq 2} \frac{h_i(r)\overline{h_j(r)}}{p_r}, \quad g_j = \sum_{r \geq 2} \frac{E_K(r)\overline{h_j(r)}}{p_r}.$$

The sharp squared point constant under both nulls is

$$\Psi_{K,\omega} = S - g\Gamma^{-1}g^* > 0. \quad (17)$$

The capacities on supports $\{2, \dots, N\}$ increase to this value.

Proof. The minor above makes Γ positive definite. Projecting the point row off the span of the two Dirichlet rows in the dual metric $\sum a_r \overline{b_r}/p_r$ gives (17). To recover its supremum with finite vectors, truncate a null vector and correct the constraint residual on 2, 4. Boundedness of the rows and the fixed inverse minor make the correction tend to zero in H_ω . This proves density and monotone convergence. A zero capacity would put E_K in the span of the two rows. Its quadratic lower growth on a full ray contradicts the strictly smaller growth of both rows in (16). $\square$

5 Applications to certificate costs

5.1 A norm-controlled compensated-mean remainder

For (15) impose $A(1) = A(q) = 0$. Let $K \geq 1$ be an integer and $F_K(x) = x^{-\lambda} - K^{-2a}x^{-q}$. The compensated mean tail from [4] is

$$\mathcal{M}_K(c) = \sum_d c_d \int_K^\infty B_1(\{x/d\}) F'_K(x) dx, \quad (18)$$

where $B_1(x) = x - 1/2$ in the integral. Set $(s)_j = s(s+1) \cdots (s+j-1)$, $\Delta_j = (\lambda)_j - (q)_j$, and

$$t_{R,K}c = \sum_{j=1}^R \frac{\Delta_j K^{-\lambda-j}}{(j+1)!} \sum_d c_d d^j B_{j+1}(\{K/d\}).$$

With

$$C_R = \frac{\max_{0 \leq x \leq 1} |B_{R+1}(x)|}{(R+1)!}, \quad \mathcal{A}_R = \frac{|\Delta_{R+1}|}{\alpha + R} + \frac{2a|(q)_{R+1}|}{(\sigma + R)(\alpha + R)},$$

one has

$$|\mathcal{M}_K(c) - t_{R,K}c| \leq C_R \mathcal{A}_R K^{-\alpha-R} S_R(c). \quad (19)$$

On each smooth interval, $dB_{j+1}(\{x/d\})/(j+1)$ is a primitive of $B_j(\{x/d\})$; these primitives are continuous for $j+1 \geq 2$. Repeated integration by parts gives the stated boundary terms and a remainder bounded by $C_R S_R \int_K^\infty |F_K^{(R+1)}(x)| dx$. At the boundary, $F_K^{(j)}(K) = (-1)^j \Delta_j K^{-\lambda-j}$. After setting $x = Ky$, bound the derivative difference by

$$|(\lambda)_{R+1} y^{-2a} - (q)_{R+1}| \leq |\Delta_{R+1}| y^{-2a} + |(q)_{R+1}| (1 - y^{-2a}).$$

The two resulting integrals are $1/(\alpha + R)$ and $2a/[(\sigma + R)(\alpha + R)]$, proving (19).

Corollary 5.1. *If (13) holds, then*

$$|\mathcal{M}_K(c) - t_{R,K}c| \leq C_R \mathcal{A}_R D_R K^{-\alpha-R} \sqrt{W_\omega(c)}.$$

The critical penalty $r^{2R-1}[\log(er)]^b$ is sufficient exactly when $b > 1$ for this support-independent S_R budget.

The classification is sharp for the absolute coefficient moment that controls (19).

5.2 An explicit native approximation floor

For finite coefficients define

$$\mathcal{F}(c) = \sum_{n \geq 1} \frac{|1 + \sum_{d \geq 2} c_d \{n/d\}|^2}{n(n+1)}. \quad (20)$$

This converges because the inner finite sum is bounded.

Proposition 5.2 (Least-common-multiple floor). *For $L \geq 1$ put*

$$Q_\omega(L)^2 = \sum_{\substack{r \geq L \\ r \geq 2}} \frac{\sigma_1(r)^2}{\omega(r)w_r}, \quad D_L = \text{lcm}(1, \dots, L), \quad \sigma_1(r) = \sum_{d|r} d.$$

Then $Q_\omega(L) \rightarrow 0$. If $W_\omega(c) \leq B^2$ and $BQ_\omega(L) \leq 1/2$, then

$$\mathcal{F}(c) \geq \frac{1}{4D_L(D_L + 1)}. \quad (21)$$

For each fixed $\omega \geq 1$, any finite-trial sequence with $\mathcal{F}(c) \rightarrow 0$ therefore has $W_\omega(c) \rightarrow \infty$.

Proof. The bound $\sigma_1(r) \leq r(1 + \log r)$ and (6) make the series for $Q_\omega(1)^2$ convergent. Divisor inversion and Cauchy–Schwarz give

$$\sum_{d>L} |c_d| \leq \sum_{r>L} \sigma_1(r) |v_r| \leq \sqrt{W_\omega(c)} Q_\omega(L).$$

At $n = D_L$, all $2 \leq d \leq L$ divide n . Their fractional parts vanish; the remaining contribution has modulus at most $1/2$. The summand at D_L proves (21). For every fixed B this is a positive uniform floor on $W_\omega \leq B^2$, proving the last assertion. $\square$

For $\omega(r) = r^2$, $B = 1$, one may take $L = 32$: using $\zeta(4) < 10/9$, $\log 32 < 4$ and integration,

$$Q_\omega(32)^2 \leq \frac{200}{32^3} \left(\frac{25}{3} + \frac{10}{9} + \frac{2}{27} \right) < \frac{1}{4}.$$

Here $D_{32} = 144403552893600$. The least common multiple in place of $32!$ gives a larger explicit floor with the same coefficient-tail payment.

Theorem 5.3 (A quantitative energy cost). *Fix an integer $R \geq 1$ and put $\omega_R(r) = r^{2R-1} \log^2(er)$ and $W_R = W_{\omega_R}$. Every finite anchored trial satisfies*

$$W_R(c) \geq \frac{R}{400} \left[\frac{\log_+(1/(2048\mathcal{F}(c)))}{\log 256} \right]^{2R}, \quad \log_+ x = \max\{0, \log x\}. \quad (22)$$

The inequality remains valid after imposing additional constraints on the trials.

Proof. The bounds $\sigma_1(r) \leq r \log(er)$ and $w_r \geq r^4/(180\zeta(4))$ give

$$Q_{\omega_R}(L)^2 \leq 180\zeta(4) \sum_{r>L} r^{-2R-1} \leq \frac{90\zeta(4)}{RL^{2R}} < \frac{100}{RL^{2R}}.$$

For $B = \sqrt{W_R(c)} > 0$ choose $L = \lceil (20B/\sqrt{R})^{1/R} \rceil$. Then $BQ_{\omega_R}(L) \leq 1/2$. We use the elementary bound $D_L \leq 16^L$. To see it, D_{2m}/D_m divides $\binom{2m}{m}$: each new prime power lies in $(m, 2m]$, contributes one to this quotient, and contributes at least one to the binomial valuation. Iterating over powers of two gives $D_{2^k} < 4^{2^k}$. The next power of two above L is less than $2L$, proving the bound (with $L = 1$ immediate). Proposition 5.2 now yields

$$\mathcal{F}(c) \geq \frac{1}{8 \cdot 256^L} \geq \frac{1}{2048} \exp \left[-\log 256 (20B/\sqrt{R})^{1/R} \right].$$

Taking logarithms and rearranging proves (22). If $B = 0$, divisor inversion gives $c = 0$ and $\mathcal{F}(c) = 1$. Additional constraints restrict the same set of finite trials, so preserve the inequality. $\square$

For $R = 1$, native error tending to zero requires at least logarithm-squared energy growth. This gives an explicit necessary lower bound on the energy of finite native approximants.

6 Exact finite controls and use of the criteria

The accompanying `verify.py` uses rational arithmetic only. It checks 1,001 full-divisor row cases, the covariance against actual finite coefficients, a complete-period mean in a small case, exact anchor and Dirichlet nulls at 1, 2, 3, finite Schur optimizers, and the native floor. The rational nulls are convenient finite controls; the general complex q, λ case is proved in Section 4. The output is in `verification.json`; its decimals are display aids.

At $K = 0$, finite vectors from the unpenalized point row, corrected on 2, 4, give squared point-to-energy ratios about 1.674 at support 8 and 13.519 at support 16. The infinite-limit classifications are proved analytically above. To use them, choose the target functional, check its summability series, and apply its W_ω bound to the same trial. The logarithmic weights yield these bounds at the critical power.

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