A sharp lower bound for adjacent-edge correlations in random subtrees

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Abstract

Give every nonempty subtree of the labeled complete graph K_n weight $\lambda^{|E(T)|}$, where $\lambda > 0$ and singleton subtrees are included. For every $n \geq 3$, the conditional probability of an edge given an adjacent edge is strictly greater than three quarters of its unconditional probability. Thus the finite lower bound proposed by Luo, Xu and Tian for the uniform law holds throughout the positive activity family. Classical fixed-forest extension counts reduce the assertion to positivity of a polynomial in λ. Convexity makes every paired contribution nonnegative except one; the coefficient of λ^2 explicitly compensates for that exception. The constant is optimal even for each fixed n, since the law converges to the uniform spanning-tree law as λ tends to infinity. For each fixed positive activity, we quantify the approach to this bound: the correlation ratio is $3/4 + 3/(4e\lambda n) + o(n^{-1})$ as n tends to infinity. The proof uses the classical Poisson law for the number of missing vertices.

1 The probability law and the lower bound

Let $\mathcal{T}_n$ be the nonempty connected acyclic subgraphs of K_n. Subtrees need not be induced or spanning; the n single-vertex trees are included. For $\lambda > 0$ put

$$Z_n(\lambda) = \sum_{T \in \mathcal{T}_n} \lambda^{|E(T)|}, \qquad \mathbb{P}_\lambda(T) = \frac{\lambda^{|E(T)|}}{Z_n(\lambda)}.$$

Each additional edge contributes a factor λ. The choice $\lambda = 1$ is the uniform law; large λ favors larger subtrees. An edge contained in T means an edge of its subgraph.

Luo, Xu and Tian [3] prove the adjacent-edge upper bound for the uniform law and obtain its limiting ratio $3/4$. The strict finite lower bound is proposed as Conjecture 5.1 on printed page 18 of their publicly available 2023 author manuscript [4]. We prove that bound and extend it to every positive λ. Size-weighted subtree laws are standard Boltzmann models; Fredes and Marckert [2] describe their generating functions and sampling. The weighted model and the forest-extension counts are inherited ingredients; the finite correlation inequality below is the target.

Theorem 1. *For every $n \geq 3$, $\lambda > 0$, and distinct adjacent edges e, f of K_n,*

$$\mathbb{P}_\lambda(e \in T \mid f \in T) > \frac{3}{4}\mathbb{P}_\lambda(e \in T).$$

For each fixed n, the ratio of the two probabilities tends to $3/4$ as $\lambda \to \infty$. Hence no larger constant is valid throughout the positive activity family, even for a fixed n.

2 Counts and polynomial positivity

Proof. Write $K = |V(T)|$, and let

$$w_1 = n, \qquad w_k = \binom{n}{k} k^{k-2} \quad (2 \leq k \leq n).$$

These are the classical numbers of subtrees of each order. Conditional on $K = k$, the vertex set is uniform and the tree is a uniform spanning tree on that set; weighting by λ^{k-1} does not change this conditional law.

A fixed forest on k labeled vertices, with component orders $s_1, \dots, s_c$, is contained in $k^{c-2} \prod_i s_i$ spanning trees. In particular, an edge has $2k^{k-3}$ extensions and an adjacent pair has $3k^{k-4}$ extensions. Such fixed-subtree extension counts are part of the classical framework used by Chin et al. [1, Theorem 6.5]. Including the probabilities that the required vertices are selected gives

$$\mathbb{P}_\lambda(e \in T \mid K = k) = \frac{2(k-1)}{n(n-1)}, \tag{1}$$

$$\mathbb{P}_\lambda(e, f \in T \mid K = k) = \frac{3(k-1)(k-2)}{k\, n(n-1)(n-2)}. \tag{2}$$

The first formula is zero at $k = 1$ and the second at $k = 1, 2$, as required.

Define $h(x) = (x-1)(x-2)/x = x - 3 + 2/x$ for $x > 0$, and the three polynomials

$$N = \sum_{k=1}^{n} w_k \lambda^{k-1}, \qquad B = \sum_{k=1}^{n} (k-1) w_k \lambda^{k-1}, \qquad A = \sum_{k=1}^{n} h(k) w_k \lambda^{k-1}.$$

Here $N = Z_n(\lambda)$, and $N, B > 0$. Averaging (1)–(2), the normalized joint probability is

$$R_{n,\lambda} := \frac{\mathbb{P}_\lambda(e, f \in T)}{\mathbb{P}_\lambda(e \in T)\mathbb{P}_\lambda(f \in T)} = \frac{3n(n-1)}{4(n-2)} \frac{NA}{B^2}.$$

Thus it suffices to show

$$P(\lambda) := n(n-1)NA - (n-2)B^2 > 0.$$

For $n \geq 4$, symmetrize the expansion as

$$P(\lambda) = \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j Q_{ij} \lambda^{i+j-2}, \qquad Q_{ij} = n(n-1)[h(i) + h(j)] - 2(n-2)(i-1)(j-1).$$

We first check that every Q_{ij} except Q_{22} is nonnegative. If $i = 1$ or $j = 1$, this follows from $h(1) = h(2) = 0$ and $h(k) \geq 0$ for integer $k \geq 1$. If $i = 2$ and $j \geq 3$, then

$$Q_{2j} = (j-1)\left[\frac{n(n-1)(j-2)}{j} - 2(n-2)\right] \geq 0,$$

because $(j-2)/j$ increases for $j \geq 3$, so the bracket is minimized at $j = 3$ and is at least $(n-3)(n-4)/3$. Symmetry treats $j = 2$.

If $i, j \geq 3$, put $m = (i+j)/2$. Since $h''(x) = 4/x^3 > 0$ and $(i-1)(j-1) \leq (m-1)^2$,

$$\begin{aligned} Q_{ij} &\geq 2\big[n(n-1)h(m) - (n-2)(m-1)^2\big] \\ &= \frac{2(m-1)}{m}(n-2)(n-m)\left(m - 1 - \frac{n}{n-2}\right) \geq 0. \end{aligned}$$

The last sign uses $3 \leq m \leq n$ and $n/(n-2) \leq 2$.

The sole negative contribution, $Q_{22} = -2(n-2)$, occurs at degree two. The other pairs at that degree are $(1, 3)$ and $(3, 1)$. Their combined coefficient is

$$\begin{aligned} [\lambda^2]P &= n(n-1)w_1 w_3 h(3) - (n-2)w_2^2 \\ &= \frac{n^2(n-1)^2(n-2)(4n-3)}{12} > 0. \end{aligned}$$

All other coefficients are nonnegative. Therefore $P(\lambda) > 0$ for every $\lambda > 0$. When $n = 3$, direct substitution gives $N = 3 + 3\lambda + 3\lambda^2$, $B = 3\lambda + 6\lambda^2$, $A = 2\lambda^2$, and $P(\lambda) = 27\lambda^2 > 0$. This proves the strict inequality in every case and gives the explicit finite margin

$$R_{n,\lambda} - \frac{3}{4} = \frac{3P(\lambda)}{4(n-2)B(\lambda)^2} \geq \frac{n^2(n-1)^2(4n-3)\lambda^2}{16B(\lambda)^2}.$$

For $n = 3$ this is equality; equivalently, $R_{3,\lambda} = 3(1 + \lambda + \lambda^2)/(1 + 2\lambda)^2$.

For fixed n, the highest-degree terms in N come from the spanning trees. Thus $K \to n$ in probability as $\lambda \to \infty$, and the limiting conditional law is uniform on the spanning trees of K_n. Formulas (1)–(2) give marginal probability $2/n$ and joint probability $3/n^2$. Their ratio is $3/4$, proving optimality. □

3 First-order approach at fixed activity

The uniform-law limit was obtained by Luo, Xu and Tian [3, 4]; their Theorem 2.1 also gives its Poisson missing-vertex law. We use the same counting mechanism to retain the first-order correlation margin at every fixed activity. Broader dense-graph subtree-size asymptotics are developed by Wagner and Wang [5]. This is distinct from taking $\lambda \to \infty$ at fixed n.

Theorem 2. *For each fixed $\lambda > 0$, as $n \to \infty$,*

$$R_{n,\lambda} = \frac{3}{4} + \frac{3}{4e\lambda n} + o(n^{-1}).$$

Proof. Put $D_n = n - K$ and $t = 1/(e\lambda)$. Relative to the spanning-tree weight, define

$$a_{n,j} = \frac{w_{n-j}\lambda^{n-j-1}}{w_n\lambda^{n-1}}, \qquad 0 \le j \le n-1,$$

and set it to zero at larger integers. For fixed j, the exact count gives

$$a_{n,j} = \frac{\lambda^{-j}}{j!}\left[\prod_{r=0}^{j-1}(1 - r/n)\right](1 - j/n)^{n-j-2} \longrightarrow \frac{t^j}{j!}.$$

For $3 \le k \le n$,

$$\frac{w_{k-1}}{w_k} = \frac{(1 - 1/k)^{k-3}}{n-k+1} \le \frac{1}{n-k+1}, \qquad \frac{w_1}{w_2} = \frac{2}{n-1}.$$

Thus $a_{n,j} \le \lambda^{-j}/j!$ for $j \le n-2$, and the singleton term satisfies $a_{n,n-1} \le 2\lambda^{-(n-1)}/(n-1)!$. These estimates give the common summable bound $2\lambda^{-j}/j!$. Dominated summation, also after multiplication by any fixed power of j, proves

$$\sum_j a_{n,j} \to e^t, \qquad D_n \Rightarrow \mathrm{Poisson}(t), \qquad d_n := \mathbb{E}_\lambda D_n \to t.$$

In particular $K \to \infty$ in probability, so $\mathbb{E}_\lambda(1/K) = o(1)$ by bounded convergence in probability. The exact ratio from the proof of Theorem 1 is

$$R_{n,\lambda} = \frac{3}{4}\frac{n(n-1)}{n-2}\frac{n - 3 - d_n + 2\mathbb{E}_\lambda(1/K)}{(n - 1 - d_n)^2}.$$

Since d_n is bounded and $n(n-1)/(n-2) = n + 1 + O(n^{-1})$, its numerator before division by $(n - 1 - d_n)^2$ equals $n^2 - (2 + d_n)n + o(n)$, while that denominator equals $n^2 - 2(1 + d_n)n + O(1)$. Their quotient is $1 + d_n/n + o(n^{-1})$. Taking $d_n \to t$ proves the assertion. □

At $\lambda = 1$ this yields $R_{n,1} - 3/4 \sim 3/(4en)$ and, in particular, the known uniform-law limit. Here the activity remains fixed as n grows.

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