A sharp lower bound for adjacent-edge correlations in random subtrees

Scott Schubr
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Abstract

Give every nonempty subtree of the labeled complete graph K_n weight $\lambda^{|E(T)|}$, where $\lambda > 0$ and singleton subtrees are included. For every $n \geq 3$, the conditional probability of an edge given an adjacent edge is strictly greater than three quarters of its unconditional probability. Thus the finite lower bound proposed by Luo, Xu and Tian for the uniform law holds throughout the positive activity family. Classical fixed-forest extension counts reduce the assertion to positivity of a polynomial in λ. Convexity makes every paired contribution nonnegative except one; the coefficient of λ^2 explicitly compensates for that exception. The constant is optimal even for each fixed n, since the law converges to the uniform spanning-tree law as λ tends to infinity. We also give a direct count-based derivation of the previously known uniform-law limit as n tends to infinity.

1 The probability law and the lower bound

Let $\mathcal{T}_n$ be the nonempty connected acyclic subgraphs of K_n. Subtrees need not be induced or spanning; the n single-vertex trees are included. For $\lambda > 0$ put

$$Z_n(\lambda) = \sum_{T \in \mathcal{T}_n} \lambda^{|E(T)|}, \qquad \mathbb{P}_\lambda(T) = \frac{\lambda^{|E(T)|}}{Z_n(\lambda)}.$$

Each additional edge contributes a factor λ. The choice $\lambda = 1$ is the uniform law; large λ favors larger subtrees. An edge contained in T means an edge of its subgraph.

Luo, Xu and Tian [3] prove the adjacent-edge upper bound for the uniform law and obtain its limiting ratio $3/4$. The strict finite lower bound is proposed as Conjecture 5.1 on printed page 18 of their publicly available 2023 author manuscript [4]. We prove that bound and extend it to every positive λ. Size-weighted subtree laws are standard Boltzmann models; Fredes and Marckert [2] describe their generating functions and sampling. The weighted model and the forest-extension counts are inherited ingredients; the finite correlation inequality below is the target.

Theorem 1. *For every $n \geq 3$, $\lambda > 0$, and distinct adjacent edges e, f of K_n,*

$$\mathbb{P}_\lambda(e \in T \mid f \in T) > \frac{3}{4}\mathbb{P}_\lambda(e \in T).$$

For each fixed n, the ratio of the two probabilities tends to $3/4$ as $\lambda \to \infty$. Hence no larger constant is valid throughout the positive activity family, even for a fixed n.

2 Counts and polynomial positivity

Proof. Write $K = |V(T)|$, and let

$$w_1 = n, \qquad w_k = \binom{n}{k} k^{k-2} \quad (2 \leq k \leq n).$$

These are the classical numbers of subtrees of each order. Conditional on $K = k$, the vertex set is uniform and the tree is a uniform spanning tree on that set; weighting by λ^{k-1} does not change this conditional law.

A fixed forest on k labeled vertices, with component orders $s_1, \dots, s_c$, is contained in $k^{c-2} \prod_i s_i$ spanning trees. In particular, an edge has $2k^{k-3}$ extensions and an adjacent pair has $3k^{k-4}$ extensions. Such fixed-subtree extension counts are part of the classical framework used by Chin et al. [1, Theorem 6.5]. Including the probabilities that the required vertices are selected gives

$$\mathbb{P}_\lambda(e \in T \mid K = k) = \frac{2(k-1)}{n(n-1)}, \tag{1}$$

$$\mathbb{P}_\lambda(e, f \in T \mid K = k) = \frac{3(k-1)(k-2)}{k\, n(n-1)(n-2)}. \tag{2}$$

The first formula is zero at $k = 1$ and the second at $k = 1, 2$, as required.

Define $h(x) = (x-1)(x-2)/x = x - 3 + 2/x$ for $x > 0$, and the three polynomials

$$N = \sum_{k=1}^{n} w_k \lambda^{k-1}, \qquad B = \sum_{k=1}^{n} (k-1) w_k \lambda^{k-1}, \qquad A = \sum_{k=1}^{n} h(k) w_k \lambda^{k-1}.$$

Here $N = Z_n(\lambda)$, and $N, B > 0$. Averaging (1)–(2), the normalized joint probability is

$$R_{n,\lambda} := \frac{\mathbb{P}_\lambda(e, f \in T)}{\mathbb{P}_\lambda(e \in T)\mathbb{P}_\lambda(f \in T)} = \frac{3n(n-1)}{4(n-2)} \frac{NA}{B^2}.$$

Thus it suffices to show

$$P(\lambda) := n(n-1)NA - (n-2)B^2 > 0.$$

For $n \geq 4$, symmetrize the expansion as

$$P(\lambda) = \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j Q_{ij} \lambda^{i+j-2}, \qquad Q_{ij} = n(n-1)[h(i) + h(j)] - 2(n-2)(i-1)(j-1).$$

We first check that every Q_{ij} except Q_{22} is nonnegative. If $i = 1$ or $j = 1$, this follows from $h(1) = h(2) = 0$ and $h(k) \geq 0$ for integer $k \geq 1$. If $i = 2$ and $j \geq 3$, then

$$Q_{2j} = (j-1)\left[\frac{n(n-1)(j-2)}{j} - 2(n-2)\right] \geq 0,$$

because $(j-2)/j$ increases for $j \geq 3$, so the bracket is minimized at $j = 3$ and is at least $(n-3)(n-4)/3$. Symmetry treats $j = 2$.

If $i, j \geq 3$, put $m = (i+j)/2$. Since $h''(x) = 4/x^3 > 0$ and $(i-1)(j-1) \leq (m-1)^2$,

$$\begin{aligned} Q_{ij} &\geq 2\left[n(n-1)h(m) - (n-2)(m-1)^2\right] \\ &= \frac{2(m-1)}{m}(n-2)(n-m)\left(m - 1 - \frac{n}{n-2}\right) \geq 0. \end{aligned}$$

The last sign uses $3 \leq m \leq n$ and $n/(n-2) \leq 2$.

The sole negative contribution, $Q_{22} = -2(n-2)$, occurs at degree two. The other pairs at that degree are $(1,3)$ and $(3,1)$. Their combined coefficient is

$$\begin{aligned} [\lambda^2]P &= n(n-1)w_1 w_3 h(3) - (n-2)w_2^2 \\ &= \frac{n^2(n-1)^2(n-2)(4n-3)}{12} > 0. \end{aligned}$$

All other coefficients are nonnegative. Therefore $P(\lambda) > 0$ for every $\lambda > 0$. When $n = 3$, direct substitution gives $N = 3 + 3\lambda + 3\lambda^2$, $B = 3\lambda + 6\lambda^2$, $A = 2\lambda^2$, and $P(\lambda) = 27\lambda^2 > 0$. Thus, after combining the degree-two contributions,

every full coefficient is nonnegative and that coefficient is strictly positive. This proves the strict inequality in every case. It also gives the explicit finite margin

$$R_{n,\lambda} - \frac{3}{4} = \frac{3P(\lambda)}{4(n-2)B(\lambda)^2} \geq \frac{n^2(n-1)^2(4n-3)\lambda^2}{16B(\lambda)^2}.$$

For $n = 3$ this is equality; equivalently, $R_{3,\lambda} = 3(1+\lambda+\lambda^2)/(1+2\lambda)^2$.

For fixed n, the highest-degree terms in N come from the spanning trees. Thus $K \to n$ in probability as $\lambda \to \infty$, and the limiting conditional law is uniform on the spanning trees of K_n. Formulas (1)–(2) give marginal probability $2/n$ and joint probability $3/n^2$. Their ratio is $3/4$, proving optimality. □

3 The uniform law as the number of vertices grows

At $\lambda = 1$, the theorem proves the author-manuscript conjecture. Its sharpness within the uniform laws also follows from the limit already obtained by Luo, Xu and Tian [3, 4]. A short derivation from the same counts is included here.

For $3 \leq k \leq n$,

$$\frac{w_{k-1}}{w_k} = \frac{(1-1/k)^{k-3}}{n-k+1} \leq \frac{1}{n-k+1}, \qquad \frac{w_1}{w_2} = \frac{2}{n-1}.$$

Consequently $w_{n-j}/w_n \leq 1/j!$ for $0 \leq j \leq n-2$, and $w_1/w_n \leq 2/(n-1)!$. As $N \geq w_n$, this gives

$$\mathbb{E}_1(n-K) \leq \sum_{j=1}^{n-2} \frac{j}{j!} + \frac{2}{(n-2)!} \leq e+2.$$

Thus $\mu = \mathbb{E}_1(K-1) = n + O(1)$ and $\mathbb{E}_1 h(K) = n + O(1)$. The exact ratio formula becomes

$$R_{n,1} = \frac{3n(n-1)}{4(n-2)\mu^2}\mathbb{E}_1 h(K) \longrightarrow \frac{3}{4}.$$

Both limits have distinct roles: $\lambda \to \infty$ proves optimality in the weighted family for each fixed n, while $n \to \infty$ proves optimality already among the uniform subtree laws.

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