

Sharp logarithmic sampling thresholds and lacunary interpolation in the Báez–Duarte step space

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Luminary Oddities

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Abstract

We prove a sharp distinction between discrete and continuous sampling in the harmonic-step Mellin space underlying Báez–Duarte approximation. At real part one, height-normalized derivatives of order at most R admit a uniform logarithmically weighted estimate on every separated sequence precisely for exponents $p \geq 2R + 2$. The continuous-height estimate instead holds precisely for $p > 2R + 1$. An explicit family supported at powers of two proves the discrete threshold while retaining the exact adjacent-endpoint difference of the step basis. Inside the strip $1/2 < \operatorname{Re} s < 1$, we prove count-independent sampling for Newton–Hermite packets with bounded size and arbitrary internal collisions. Complete coefficient-tail estimates then yield a full infinite Riesz Gram bound, a bounded interpolation operator and an exact projection formula for sufficiently lacunary packets. The zero-packet application controls all cross-packet terms in the native approximation norm. Finally, arbitrarily sparse interpolating nodes can admit a common-zero vector arbitrarily close to the target while excluding every nonzero Dirichlet numerator absolutely convergent at their real part.

1 The native space and the sharp threshold

Sampling, interpolation and arithmetic obstruction Three results have different endpoints: discrete sampling uses the closed threshold $p \geq 2R + 2$, continuous sampling uses $p > 2R + 1$, and the infinite lacunary interpolation theorem has its own whitened direct-sum target. The arithmetic obstruction is a third result, rather than a limitation to silently attach to the interpolation theorem. The original harmonic-step norm and the global receiving operator are kept through the transfer.

Let S be the Mellin image of the closed subspace of $L^2(0, 1)$ consisting of functions constant on

$$I_n = (1/(n+1), 1/n], \quad n \geq 1.$$

The Mellin transform is $f \mapsto \int_0^1 f(x)x^{s-1} dx$. It identifies $L^2(0, 1)$ with the half-plane Hardy space having norm $(2\pi)^{-1} \int_{\mathbb{R}} |f(1/2 + it)|^2 dt$. We use the inner product linear in its first argument. An orthonormal basis of S is

$$u_n(s) = \sqrt{n(n+1)} \frac{n^{-s} - (n+1)^{-s}}{s}, \quad f = \sum_{n \geq 1} a_n u_n, \quad \|f\|_S^2 = \sum |a_n|^2. \quad (1)$$

All evaluations and fixed derivatives in $\operatorname{Re} s > 1/2$ are continuous. In particular, the formulas below first apply to finite sums and then to S .

For $t \geq 2$ and an integer $r \geq 0$ define

$$J_r(t; f) = t^{-3/2} \partial_s^r \{t^s s f(s)\}|_{s=1+it}. \quad (2)$$

The derivative acts on the displayed function of s with t held fixed.

Theorem 1.1 (Sharp logarithmic endpoint). *Fix $R \geq 0$ and $\delta > 0$. For a real number p , there is a finite constant $C = C(R, \delta, p)$ such that*

$$\sum_a \frac{\sum_{r=0}^R |J_r(t_a; f)|^2}{[\log(2 + t_a)]^p} \leq C \|f\|_S^2 \quad (3)$$

for every δ -separated sequence $t_a \geq 2$ and every $f \in S$, if and only if $p \geq 2R + 2$. The analogous integral estimate

$$\int_2^\infty \frac{\sum_{r=0}^R |J_r(t; f)|^2}{[\log(2 + t)]^p} dt \leq C \|f\|_S^2 \quad (4)$$

holds if and only if $p > 2R + 1$.

The necessity of the discrete threshold already occurs on one arithmetic progression of heights, independently of the vector f . Thus the extra logarithmic requirement is an actual property of separated sampling in this space.

The step-space formulation comes from Báez-Duarte and Bagchi [1, 2]. Our predecessor [3] determines the fixed-packet limiting kernel and its collision-uniform positive Gram floor. The present results treat unbounded packet counts, sharp endpoint sampling and infinite lacunary interpolation. Classical ingredients include Gaussian Dirichlet mean estimates, Hardy's inequality, Abel summation, Schur bounds and Hilbert-space Gram inversion. Rational-frequency resonance in discrete zeta averages is established in Li-Radziwiłł [4]; our assertion is the exact logarithmic operator threshold for the native adjacent-endpoint basis (1). Olsen's local Dirichlet-space embeddings [5] concern a different coefficient geometry and compactly supported measures. General Gram criteria for interpolation are discussed in [6, Section 9.2].

2 Uniform sampling in the open strip

We give the analytic estimate used both for Theorem 1.1 and for colliding packets. All constants below may depend on the fixed real-part interval, separation and derivative order, but not on input length, height or sample count.

Lemma 2.1 (Centered finite-block estimate). *Let $N \geq 1$, $0 \leq v \leq 1$ and $\lambda_n = \log((n + v)/N)$ for $N \leq n < 2N$. Suppose the weights $w_n(\sigma)$ and their first σ derivatives are bounded uniformly on a fixed real interval I of positive length. If separated points t_a belong to an interval of length $O(T)$, $T \geq 1$, then*

$$\sum_a \sup_{\sigma \in I_0} \left| \sum_{N \leq n < 2N} c_n w_n(\sigma) e^{-it_a \lambda_n} \right|^2 \leq C(T + N) \sum |c_n|^2 \quad (5)$$

for any fixed compact subinterval I_0 in the interior of I . The same estimate applies after inserting any fixed bounded powers of λ_n in the actual power weights used below.

Proof. The frequencies belong to $[0, \log 3]$ and satisfy $|\lambda_m - \lambda_n| \geq |m - n|/(2N + 1)$. Majorize the interval indicator, enlarged by one at each end, by a constant times $\exp(-(t - t_0)^2/U^2)$, where $U \asymp T$. Its Fourier Gram has entries

$$\sqrt{\pi} U e^{-U^2(\lambda_m - \lambda_n)^2/4} e^{-it_0(\lambda_m - \lambda_n)}.$$

Absolute row sums are $O(U + N)$, by comparison with $\sum_{k \in \mathbb{Z}} e^{-c(U/N)^2 k^2}$. Schur's inequality proves the continuous mean bound. It also bounds the first t , first σ , and mixed derivatives: the t derivative inserts bounded λ_n , and the other weights have the stated bounds.

Set $h = \min(\delta/3, 1/3)$. The intervals $[t_a - h, t_a + h]$ are disjoint, and the fundamental theorem of calculus with Cauchy–Schwarz gives

$$|g(t_a)|^2 \leq h^{-1} \int_{t_a-h}^{t_a+h} |g|^2 + 4h \int_{t_a-h}^{t_a+h} |g'|^2.$$

Apply this and the one-dimensional $H^1(I)$ supremum inequality successively. The four continuous bounds give (5). For the power weights, differentiation in σ inserts only bounded centered logarithms. A degenerate interval I_0 is allowed by choosing a nondegenerate enlargement I . $\square$

Fix $1/2 < b_0 \leq b_1 < 1$, $R \geq 0$, and heights $t_a \geq 2$ with gaps at least δ . Allow $\sigma_a \in [b_0, b_1]$ and reference heights h_a with $ct_a \leq h_a \leq C_0 t_a$. Write

$$\mathcal{J}_{ar} f = h_a^{-3/2} \partial_s^r \{h_a^s s f(s)\}|_{s=\sigma_a+it_a}.$$

Proposition 2.2 (Interior derivative sampling). *For the preceding fixed parameters,*

$$\sum_a \sum_{r=0}^R |\mathcal{J}_{ar} f|^2 \leq C_R \|f\|_S^2. \quad (6)$$

Proof. Take finite input. Group coefficients into $2^l \leq n < 2^{l+1}$ and reference heights into $2^k \leq h_a < 2^{k+1}$. Put $A_l = (\sum_{n \text{ in block } l} |a_n|^2)^{1/2}$. The finitely many smallest ranges are absorbed into constants. We claim the block operator bound

$$\|(\mathcal{J}_{ar} f)_l\|_2 \leq C_R (1 + |k - l|)^R d_{k-l} A_l, \quad d_j = \begin{cases} 2^{-(1-b_1)j} & j \geq 0, \\ 2^{(b_0-1/2)j} & j < 0. \end{cases} \quad (7)$$

Let $N = 2^l$, $T = 2^k$. For $N \leq T$, retain the two endpoint sums in the exact expression

$$h^s s f_l(s) = \sum_n a_n \sqrt{n(n+1)} \{(h/n)^s - (h/(n+1))^s\}.$$

Each derivative inserts $\log(h/(n+v))^r$, $v = 0, 1$. Expand this logarithm as $\log(T/N) + \log(h/T) - \log((n+v)/N)$. The first term is $O(1 + |k - l|)$; the remaining terms are bounded. Lemma 2.1 therefore gives a norm at most

$$C_R (1 + |k - l|)^R \sqrt{T + N} \sup_{b_0 \leq \sigma \leq b_1} T^{\sigma-3/2} N^{1-\sigma} A_l.$$

For $N \leq T$ this is bounded by the first branch in (7).

For $N > T$, use the exact cancellation

$$n^{-s} - (n+1)^{-s} = s \int_0^1 (n+v)^{-s-1} dv. \quad (8)$$

Differentiating gives an integrand

$$\sum_n a_n \frac{\sqrt{n(n+1)}}{n+v} (h/(n+v))^s \{s \log(h/(n+v))^r + r \log(h/(n+v))^{r-1}\},$$

with the second term omitted when $r = 0$. Here $|s| = O(T)$ and the coefficient ratio is bounded. Apply the same lemma for each v , followed by Minkowski's inequality. The resulting bound is

$$C_R(1 + |k - l|)^R \sqrt{T + N} \sup_{b_0 \leq \sigma \leq b_1} T^{\sigma-1/2} N^{-\sigma} A_l,$$

which gives the second branch of (7).

The sequence $w_j = C_R(1 + |j|)^R d_j$ is in $\ell^1(\mathbb{Z})$. Triangle inequality in each output block and convolution on ℓ^2 give (6). These bounds are independent of finite input and output cutoffs. Coefficient approximation, continuity of each Mellin derivative and monotone convergence over finite output sets prove the full assertion. $\square$

2.1 Newton–Hermite packets

Fix $L \geq 0$, an integer $M \geq 1$, and

$$Q = [b_0, b_1] + i[-L, L].$$

Let $T_a \geq \max(4, 2L + 2)$, $|T_a - T_b| \geq 2L + \delta$. For each a choose $m_a \leq M$ nodes $q_{a1}, \dots, q_{am_a} \in Q$. Repeated nodes mean confluent, or Hermite, divided differences. Set

$$\mathcal{L}_{aj}f = T_a^{-3/2}[q_{a1}, \dots, q_{aj}]\{(iT_a + q)T_a^q f(iT_a + q)\}. \quad (9)$$

Theorem 2.3 (Global packet bound). *For any finite or countable family of these packets,*

$$\sum_a \sum_{j=1}^{m_a} |\mathcal{L}_{aj}f|^2 \leq C \|f\|_S^2, \quad (10)$$

where C depends only on b_0, b_1, L, δ, M . All off-packet Gram blocks are included; no separation is required within a packet.

Proof. Put $F_a(q) = (iT_a + q)T_a^q f(iT_a + q)$. The Hermite–Genocchi formula writes the divided difference of order $j - 1$ as

$$[q_{a1}, \dots, q_{aj}]F_a = \int_{\Delta_{j-1}} F_a^{(j-1)}\left(\sum_{\nu=1}^j \theta_\nu q_{a\nu}\right) d\theta,$$

where the simplex has mass $1/(j-1)!$. Fixing the same simplex point in every packet gives evaluation heights separated by at least δ ; the reference T_a is comparable to these heights. Derivatives of F_a differ from the derivative rows in Proposition 2.2 only by a unit scalar. Cauchy–Schwarz on the simplex, Tonelli and (6), followed by summation over $j \leq M$, prove (10). The integral formula already includes collisions. $\square$

3 Proof of the sharp endpoint theorem

3.1 Discrete sufficiency

At $\sigma = 1$ the finite-block proof still holds. Only the low-input geometric decay is lost. For $l \leq k$, (7) becomes $C_R(1 + k - l)^R A_l$. For $l > k$ exponential decay remains. Since $\log(2 + t) \asymp k + 1$ on an output height block, the low part at $p = 2R + 2$ is bounded by

$$C_R \sum_{k \geq 0} (k + 1)^{-2R-2} \left(\sum_{l=0}^k (1 + k - l)^R A_l \right)^2 \leq C_R \sum_{k \geq 0} \left(\frac{\sum_{l=0}^k A_l}{k + 1} \right)^2 \leq 4C_R \sum_l A_l^2.$$

The last step is classical discrete Hardy inequality. The high part is bounded by exponentially decaying convolution. Completion works as before. Increasing p changes only a bounded factor at the first heights, so all $p \geq 2R + 2$ follow.

3.2 An exact resonant family

Choose an integer $h \geq 1$ such that $c = 2\pi h / \log 2 \geq \delta$, and fix $t_j = cj$, $j \geq 1$. For $K \geq 1$ set

$$f_K = K^{-1/2} \sum_{k=1}^K u_{2^k}, \quad \|f_K\|_S = 1.$$

With $n_k = 2^k$, the exact expression (1) gives

$$|J_R(t_j; f_K)|^2 = t_j^{-1} |A_{KR}(j) - B_{KR}(j)|^2, \quad (11)$$

where

$$\begin{aligned} A_{KR}(j) &= K^{-1/2} \sum_{k=1}^K \sqrt{1 + 1/n_k} [\log(t_j/n_k)]^R, \\ B_{KR}(j) &= K^{-1/2} \sum_{k=1}^K \sqrt{n_k/(n_k + 1)} e^{-it_j \log(n_k + 1)} [\log(t_j/(n_k + 1))]^R. \end{aligned}$$

Indeed $n_k^{-it_j} = 1$; the second endpoint has not been replaced by the first.

Write $\vartheta_k = c \log(2^k + 1)$ modulo 2π . This is nonzero: otherwise $(2^k + 1)^h$ would be a power of two. For all sufficiently large k , its distance to $2\pi\mathbb{Z}$ is comparable to 2^{-k} , since the $k \log 2$ part is exactly resonant. Absorbing the finitely many remaining k into a constant gives

$$\left| \sum_{j=a}^b e^{-ij\vartheta_k} \right| \leq C_h 2^k. \quad (12)$$

On an integer dyadic interval $J \leq j < 2J$, $J \geq 2^{3K+C_h}$, all the logarithms are positive and $\log(cj/n_k) \geq \frac{1}{2} \log J$, after increasing the fixed constant. Thus

$$\text{av}_J A_{KR} \geq 2^{-R} \sqrt{K} (\log J)^R.$$

The supremum plus total variation of $[\log(cj/(n_k + 1))]^R$ on this interval is at most $C_{R,h} (\log J)^R$. Abel summation with (12) consequently gives

$$|\text{av}_J B_{KR}| \leq C_{R,h} \frac{2^K (\log J)^R}{J \sqrt{K}}.$$

For large K this is at most half the preceding positive lower bound, uniformly for the stated J . Jensen's inequality and (11) show that this block contributes at least

$$c_{p,R,h} K (\log J)^{2R-p} \quad (13)$$

to the sum in (3), because $t_j^{-1} [\log(2 + t_j)]^{-p} \geq cJ^{-1} (\log J)^{-p}$ on the block.

Sum (13) over $J = 2^\ell$, $\ell \geq 3K + C_h$. If $p \leq 2R + 1$ it diverges. If $2R + 1 < p < 2R + 2$, the sum is at least cK^{2R+2-p} , unbounded as $K \rightarrow \infty$. This proves necessity on the single fixed progression, including the range between the two thresholds.

3.3 Continuous sufficiency and necessity

For $t \in [T, 2T]$, group the entire low prefix $n \leq T$ before taking a mean estimate. In each endpoint sum expand $\log(t/(n+v))^r$ as powers of $\log(t/T) + \log(T/(n+v))$. The first factors are bounded. At $\sigma = 1$, the remaining coefficients have squared sum at most $C_R(1 + \log T)^{2R} \|f\|_S^2$, since $\sqrt{n(n+1)}/(n+v)$ is bounded. The full-prefix frequencies have separation at least $|m-n|/(T+1)$. The Gaussian argument in Lemma 2.1, without the point-sampling step, gives a continuous integral bound CT times this coefficient square sum. The outer factor t^{-1} cancels T . For the high input $n > T$, (8) and the exponentially decaying dyadic bounds give $C_R \|f\|_S^2$ per height block. Summing the resulting $(1+k)^{2R-p}$ proves (4) when $p > 2R+1$.

For necessity, $u_1(s) = \sqrt{2}(1-2^{-s})/s$ alone suffices. Since $0 \leq \log(t/2) \leq \log t$ for $t \geq 2$, the reverse triangle inequality yields

$$|J_R(t; u_1)| \geq \frac{1}{\sqrt{2}} t^{-1/2} (\log t)^R.$$

The corresponding integral diverges for $p \leq 2R+1$. This finishes Theorem 1.1.

4 Coefficient localization and lacunary interpolation

Return to the fixed compact rectangle $Q \subset \{1/2 < \operatorname{Re} q < 1\}$. Identify S with its coefficient space by (1). Let λ_{Tj} be the coefficient row of (9).

Lemma 4.1 (Complete row tails). *Uniformly over all packets of size at most M and all sufficiently large T ,*

$$\|\mathbf{1}_{n \leq U} \lambda_{Tj}\|_2^2 \leq C(U/T)^{3-2b_1} [1 + \log(T/U)]^{2M-2}, \quad 1 \leq U \leq T, \quad (14)$$

$$\|\mathbf{1}_{n \geq U} \lambda_{Tj}\|_2^2 \leq C(U/T)^{1-2b_0} [1 + \log(U/T)]^{2M-2}, \quad U \geq T. \quad (15)$$

Proof. For derivative order at most $M-1$, the separate endpoint expression gives the low-index squared coefficient majorant

$$CT^{2\sigma-3} n^{2-2\sigma} [1 + \log(T/n)]^{2M-2}.$$

Write its power factor as $T^{-1}(n/T)^{2-2\sigma}$ and use $\sigma \leq b_1$ for $n \leq T$. Summing, or comparing with the integral after splitting the first cells, proves (14); the positive power $3-2b_1$ absorbs all fixed logarithmic moments. The $n+1$ endpoint changes only the constant.

At the high end, (8) and its differentiated form bound the coefficient by

$$CT^{\sigma-1/2} n^{-\sigma} [1 + \log(n/T)]^{M-1}.$$

Square, use $\sigma \geq b_0$, and sum the integrable power n^{-2b_0} to obtain (15). The Hermite–Genocchi formula passes both bounds to Newton rows; all convex combinations stay in Q , including at repeated nodes. $\square$

Let D_a be the exact diagonal Gram of the normalized rows in packet a . The predecessor’s collision-uniform kernel convergence and positive floor [3, Sections 3–4] imply that for all $T_a \geq T_0$,

$$cI \leq D_a \leq C_0 I \quad (16)$$

uniformly on Q^{m_a} , $m_a \leq M$. The constants and T_0 depend only on the fixed packet parameters. Apply the positive matrix $D_a^{-1/2}$ to the output rows in packet a , and denote the resulting analysis operator by J . This is exact within-packet whitening, rather than replacement by a limiting Gram.

Theorem 4.2 (Infinite lacunary interpolation). *There exist $B_0 > 1$, T_0 and $C_2 < \infty$, depending only on the fixed packet parameters, such that the following holds. If $T_1 \geq T_0$ and $T_{a+1}/T_a \geq B \geq B_0$, then the full whitened row Gram $\Gamma = JJ^*$ on $\bigoplus_a \mathbb{C}^{m_a}$ satisfies*

$$(1 - \theta_B)I \leq \Gamma \leq (1 + \theta_B)I, \quad \theta_B = \frac{2C_2B^{-\eta}}{1 - B^{-\eta}} < 1, \quad \eta = (2b_0 - 1)/8. \quad (17)$$

For every $b \in \bigoplus_a \mathbb{C}^{m_a}$, the unique minimum-norm solution of $Jf = b$ is

$$f_b = J^*\Gamma^{-1}b, \quad \|f_b\|_S^2 = \langle \Gamma^{-1}b, b \rangle, \quad \frac{\|b\|_2^2}{1 + \theta_B} \leq \|f_b\|_S^2 \leq \frac{\|b\|_2^2}{1 - \theta_B}. \quad (18)$$

All statements allow countably many packets and arbitrary internal collisions.

Proof. The full row norms are uniformly bounded, and whitening preserves Lemma 4.1 by (16). For $T_b/T_a = r \geq 1$, split a coefficient inner product at $U = \sqrt{T_a T_b}$. Above U use the high tail of the earlier row; below U use the low tail of the later row. Cauchy–Schwarz gives the complete off-block norm bound

$$\|\Gamma_{ab}\| \leq C_1(1 + \log r)^{M-1} \{r^{-(2b_0-1)/4} + r^{-(3-2b_1)/4}\}.$$

These are complete coefficient sums. Since $b_0 \leq b_1 < 1$, both exponents exceed η . Absorbing the logarithm yields

$$\|\Gamma_{ab}\| \leq C_2 r^{-\eta} \leq C_2 B^{-\eta|a-b|}, \quad a \neq b.$$

The block Schur row sum is at most θ_B , tending to zero as $B \rightarrow \infty$. Since $\Gamma_{aa} = I$, this proves (17) first on every finite set and then on the full Hilbert direct sum. Its inverse exists, for example by the norm-convergent Neumann series for $I - \Gamma$; no finite-section inverse limit is needed.

The bounded analysis operator is provided by Theorem 2.3, after increasing T_0 to ensure its interpacket gap, or directly by the just-proved finite Gram upper bound. Hence $JJ^* = \Gamma$. Formula (18) solves $Jf_b = b$. Every other solution differs by a vector in $\ker J$, orthogonal to $\text{ran } J^*$, proving minimality, uniqueness and the norm identity. The norm bounds follow from (17). $\square$

5 The arithmetic approximation interface

The native Báez–Duarte target and generators are

$$E(s) = 1/s, \quad G_l(s) = (l^{-s} - l^{-1})\zeta(s)/s, \quad l \geq 2, \quad V = \overline{\text{span}\{G_l : l \geq 2\}} \subset S.$$

Write $D_\infty = \text{dist}(E, V)$. The step-space approximation criterion [1, 2] identifies vanishing of this distance with RH.

Corollary 5.1 (A complete separated zero-packet receiver). *Suppose the nodes $iT_a + q_{aj}$ in Theorem 2.3 are zeros of ζ , with repetitions not exceeding their multiplicities. Set*

$$z_{aj} = [q_{a1}, \dots, q_{aj}]T_a^q.$$

Then

$$D_\infty^2 \geq C^{-1} \sum_a T_a^{-3} \|z_a\|_2^2. \quad (19)$$

If the packets satisfy Theorem 4.2, put $Z = \ker J$ and $b_a = D_a^{-1/2} T_a^{-3/2} z_a$. Then

$$\text{dist}(E, Z)^2 = \langle \Gamma^{-1}b, b \rangle, \quad D_\infty^2 = \langle \Gamma^{-1}b, b \rangle + \text{dist}(P_Z E, V)^2. \quad (20)$$

Proof. The zero jets annihilate every generator and hence V . Their gauged target values are exactly $\mathcal{L}_{aj}E = T_a^{-3/2}z_{aj}$. Apply (10) to $E-v$ and take the infimum over $v \in V$ to obtain (19). For lacunary packets, the target $b = JE$ belongs to the output Hilbert sum. More explicitly Hermite–Genocchi and (16) give

$$\|b_a\|^2 \leq CT_a^{2b_1-3}(1 + \log T_a)^{2M-2},$$

which is summable along the lacunary heights. By Theorem 4.2, $P_{Z^\perp}E = J^*\Gamma^{-1}b$. As $V \subset Z$, orthogonal Pythagoras gives (20). $\square$

The first term in (20) is an exact full-Gram interpolation cost. An arithmetic approximation upper estimate must also control the second term in the same norm. Finite off-line spectral synthesis and its projected-target rates are studied in [7, 8]; the latter’s rate theorem assumes RH or finitely many off-line zeros. The formulas here separate the count-independent geometric estimate from that arithmetic synthesis problem. The endpoint theorem also shows why replacing arbitrary separated samples by a continuous-height mean can lose a necessary logarithmic factor. The following construction gives a second, coefficient-sensitive distinction.

5.1 Sparse interpolating nodes and arithmetic regularity

Sparse uniqueness sets for almost-periodic function classes are classical; see Ferri, Galindo and Gómez [9, Section 2] for finite-character matching and its uniqueness consequences. We use only the countable character family with frequencies $\log n$, and combine its elementary recurrence with the native interpolation estimate. For $\beta \in (1/2, 1)$ let $\mathcal{A}_\beta$ denote the class of $f \in S$ whose numerator $sf(s)$ is represented, for $\operatorname{Re} s > \beta$, by an ordinary Dirichlet series $U(s) = \sum c_n n^{-s}$ with $\sum |c_n| n^{-\beta} < \infty$.

Theorem 5.2 (A sparse arithmetic-regularity obstruction). *Fix $\beta \in (1/2, 1)$ and $0 < \varepsilon < 1$. There are arbitrarily high, sufficiently lacunary simple nodes $s_a = \beta + iT_a$ whose whitened native evaluation Gram satisfies (17), but whose common-zero space*

$$Z = \{f \in S : f(s_a) = 0 \text{ for every } a\}$$

satisfies

$$0 \neq g = P_Z E, \quad \|E - g\|_S \leq \varepsilon, \quad Z \cap \mathcal{A}_\beta = \{0\}.$$

Thus even a full native interpolation inverse and a small target projection cost do not imply density of absolutely convergent arithmetic numerators in the common-zero space.

Proof. For finitely many real frequencies $\lambda_1, \dots, \lambda_N$ and $\eta > 0$, there are arbitrarily large positive integers q such that $\max_{n \leq N} |e^{-iq\lambda_n} - 1| < \eta$. Indeed, cover the compact finite torus by finitely many sets of diameter less than η . One contains infinitely many points $(e^{-ik\lambda_n})_{n \leq N}$. Fix one such index k_0 ; the positive differences $k - k_0$ are arbitrarily large and their phase quotients have the desired property. This is elementary simultaneous almost-periodic recurrence.

Enumerate the rational heights as $\tau_1, \tau_2, \dots$. At stage N append one height for every $m \leq N$, of the form $T_{N,m} = \tau_m + q$, using the preceding recurrence with $\lambda_n = \log n$, $n \leq N$, and tolerance $1/N$. Choose q large enough that each new height exceeds B times its predecessor, and that the first height exceeds any prescribed threshold. For every fixed m, n this gives

$$n^{-iT_{N,m}} \longrightarrow n^{-i\tau_m} \quad (N \rightarrow \infty).$$

If U is absolutely convergent at β and vanishes at every constructed node, dominated convergence in its coefficient series implies $U(\beta + i\tau_m) = 0$ for every m . Uniform absolute convergence makes it continuous on the vertical line, so it vanishes there identically. Finally

$$c_k k^{-\beta} = \lim_{H \rightarrow \infty} \frac{1}{2H} \int_{-H}^H U(\beta + it) e^{it \log k} dt = 0.$$

Termwise integration and the limit are justified by $\sum |c_n| n^{-\beta} < \infty$; averaged frequency differences tend to zero except at $n = k$. Hence every coefficient is zero. This proves the required uniqueness of absolutely convergent numerators using classical Dirichlet Fourier uniqueness.

Apply Theorem 4.2 with singleton packets $q_a = \beta$. Choose B so $\theta_B < 1$ and all heights beyond the diagonal threshold. The exact whitened target is $b_a = D_a^{-1/2} T_a^{\beta-3/2}$. Therefore

$$\|E - P_Z E\|_S^2 = \langle \Gamma^{-1} b, b \rangle \leq \frac{T_1^{2\beta-3}}{c(1 - \theta_B)(1 - B^{2\beta-3})}.$$

As $2\beta - 3 < 0$, the first height can make this at most ε^2 . Since $\|E\|_S = 1$, the resulting g is nonzero. If $f \in Z \cap \mathcal{A}_\beta$, continuity from the right identifies its analytic numerator with the absolute series also on $\text{Re } s = \beta$. The uniqueness just proved forces $sf(s) = 0$, and thus $f = 0$. $\square$

The constructed nodes are arbitrary interpolation nodes, rather than specified zeta zeros. Their exponentially large gaps exclude a local zero-density explanation for the obstruction. The conclusion concerns exact arithmetic regularity at the node real part; it does not concern the existence of a far-right Dirichlet representation. In fact, if v_n are the physical step values of any $f \in S$, telescoping gives

$$sf(s) = \sum_{n \geq 1} (v_n - v_{n-1}) n^{-s}, \quad v_0 = 0,$$

absolutely for $\text{Re } s > 3/2$, by Cauchy-Schwarz and $\sum |v_n|^2 / [n(n+1)] < \infty$. The loss between this coefficient class and $\mathcal{A}_\beta$ is substantive. The finite-off-line hypotheses in [7, 8] provide arithmetic information beyond the geometric sampling assumptions used here.

References

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