Sharp limitations of two permanental cofactor comparisons

Scott Schubr
Luminary Oddities

22 October 2026

Abstract

We study two auxiliary cofactor comparisons motivated by permanent Hadamard inequalities. For a complex positive definite correlation matrix B, let R be its transposed permanental cofactor matrix and L that of its entrywise squared modulus. We compare $z^T Lz$ and $z^T|R|^{\circ 2}z$, where $z_i = |x_i|^2$, with $(\operatorname{per} B)(x^*B^{-1}x)(x^*Rx)$. Both comparisons have sharp universal constant one in orders one and two. At every fixed order at least three, neither admits any finite universal constant, even among dense irreducible positive definite correlation matrices with nonreal triangle phases. An explicit roots-of-unity family gives quantitative divergence.

1 Definitions and statement

A correlation matrix is a Hermitian positive semidefinite matrix with diagonal entries one. Throughout the theorem B is positive definite of order m. Write $B(j \mid i)$ for deletion of row j and column i, and put

$$a = \operatorname{per} B, \quad R_{ij} = \operatorname{per} B(j \mid i), \quad L_{ij} = \operatorname{per}\big(|B|^{\circ 2}(j \mid i)\big).$$

Here $|B|^{\circ 2}_{ij} = |B_{ij}|^2$, and the permanent of the empty matrix is one. For $0 \neq x \in \mathbb{C}^m$, define

$$s = x^*B^{-1}x, \quad r = x^*Rx, \quad z_i = |x_i|^2, \qquad \ell = z^T Lz, \quad \ell_R = z^T|R|^{\circ 2}z.$$

The cofactor convention ensures $\frac{d}{dh}\operatorname{per}(B + hxx^*)\big|_{h=0} = x^*Rx$. In particular R is Hermitian. Deleted symmetric products give its Gram representation (up to conjugation), so $R > 0$ when $B > 0$. The quantities a, s, r are therefore positive. Also

$$\ell_R = \operatorname{Tr}\Big[\big(\operatorname{diag}(\overline{x})R\operatorname{diag}(x)\big)^2\Big].$$

Theorem 1. *For each of the inequalities*

$$\ell \leq C\,asr, \qquad \ell_R \leq C\,asr,$$

the following classification holds.

1. *If $m = 1$ or $m = 2$, $C = 1$ works for every B, x above and is sharp.*
2. *For every fixed $m \geq 3$, no finite constant C works. Both ratios are unbounded on dense irreducible positive definite correlation matrices having a nonreal gauge-invariant triangle product.*

Reading the ratios. Both numerators are compared with the same strictly positive quantity asr. Sharpness in small orders is attained at $B = I$, $x = e_1$; within the irreducible class it is approached by a limit. For each fixed $m \geq 3$, the construction below normalizes $s = 1$ and gives a common lower bound $t^{2-m}/(4(m!)^2)$ for the two ratios. This is the quantitative family proving unboundedness. The full compensated border expression discussed in the final section retains additional terms, so the auxiliary comparison and the complete border budget have separate mathematical roles.

The incoherent sum in ℓ_R need not be controlled by the coherent quadratic form r. The distinction persists after multiplying by any order-dependent finite constant. Roots-of-unity cancellation in symmetric products is established, including its use in bosonic NOON-state constructions [2]. The proof below applies that mechanism to the particular comparisons in Theorem 1. Factorial-weighted polynomial product norms and their contraction identities are classical as well [1].

2 The sharp small orders

For $m = 1$ the ratios equal one. For $m = 2$, a diagonal unitary gauge reduces B to

$$B = \begin{pmatrix} 1 & p \\ p & 1 \end{pmatrix}, \qquad 0 \leq p < 1,$$

without changing either ratio. Take an invertible Gram realization $B = VV^*$, put $x = Vw$, and let v_i be row i of V. Define $P_i = v_i^* v_i$ and $S = P_1 + P_2$. Direct cofactor evaluation gives

$$a = 1 + p^2, \quad R = B, \quad T := V^* R V = S^2, \quad L = |R|^{\circ 2}.$$

Since $s = \|w\|^2$ and $r = w^* T w$, it suffices to bound

$$W = P_1 \otimes P_1 + P_2 \otimes P_2 + p^2 (P_1 \otimes P_2 + P_2 \otimes P_1).$$

Each tensor product of the P_i is positive semidefinite, so $W \preceq S \otimes S$. For any eigenvalues $h, k \geq 0$ of S,

$$\frac{a}{2}(h^2 + k^2) - hk = \frac{a}{2}(h - k)^2 + (a - 1)hk \geq 0.$$

Consequently

$$W \preceq \frac{a}{2}(T \otimes I + I \otimes T).$$

Evaluation on $w \otimes w$ yields $\ell \leq asr$. Since $L = |R|^{\circ 2}$, the second comparison is identical. Equality at $B = I$, $x = e_1$, proves sharpness. Restricting to irreducible matrices preserves the sharp supremum by taking $p \downarrow 0$.

3 Explicit positive definite counterfamilies

Fix $m \geq 3$, put $\theta = 2\pi/m$, and let $u_j = e^{\mathrm{i}j\theta}$ for $0 \leq j < m$. For $0 < t \leq 1/(2m)$, set

$$\eta = t^{m+2}, \qquad B_0 = (1 - t)J_m + tuu^*, \qquad B = (1 - \eta)B_0 + \eta I_m.$$

The vectors $\mathbf{1}, u$ are orthogonal and have squared norm m. The eigenvalues of B are

$$\lambda_0 = m(1 - \eta)(1 - t) + \eta, \qquad \lambda_1 = m(1 - \eta)t + \eta,$$

and $m-2$ copies of η. Thus $B>0$, with diagonal one. Set $q=\lambda_1/m$ and $x=\sqrt{q}\,u$. Then

$$s=1, \qquad t\le q\le 2t.$$

All off-diagonal entries have modulus at least $(1-\eta)(1-2t)>0$, so B is dense and irreducible. The triangle product on indices $0,1,2$ satisfies

$$\Im(B_{01}B_{12}B_{20}) = (1-\eta)^3 t(1-t)(1-2t)\big(\sin(2\theta)-2\sin\theta\big)\neq 0.$$

Indeed $\sin(2\theta)-2\sin\theta = 2\sin\theta(\cos\theta-1)<0$ for $m\ge 3$. No diagonal unitary gauge makes this family real.

3.1 The cancellation and derivative

For $A,D\ge 0$, the standard symmetric-product Gram identity gives

$$\operatorname{per}(AJ_m + Duu^*) = m!(A^m+D^m). \tag{1}$$

For clarity, if a polynomial $f(z)=\sum_\alpha f_\alpha z^\alpha$ has factorial norm $\|f\|_F^2=\sum_\alpha \alpha!|f_\alpha|^2$, expanding the product of m Gram linear forms and pairing their coefficients gives $\|\prod_j f_j\|_F^2=\operatorname{per}(\langle f_i,f_j\rangle)$. Here

$$\prod_{j=0}^{m-1}(\sqrt{A}\,z_0+\sqrt{D}\,e^{\mathrm{i}j\theta}z_1) = A^{m/2}z_0^m-(-1)^m D^{m/2}z_1^m.$$

Its two distinct monomials are orthogonal, each with squared norm $m!$, which proves (1). The sign has no effect on that norm.

Let R_0 be the transposed cofactor matrix of B_0. Applying (1) to B_0+htuu^* and differentiating at $h=0$ gives

$$u^*R_0u = m\,m!\,t^{m-1}. \tag{2}$$

This derivative calculation uses singular B_0 only in a polynomial identity; no inverse of B_0 is used.

3.2 Uniform perturbation estimates

Every entry of B,B_0 has modulus at most one, and corresponding entries differ by at most η. A cofactor is a sum of $(m-1)!$ products of $m-1$ entries. Telescoping each product gives

$$|R_{ij}-(R_0)_{ij}|\le (m-1)(m-1)!\eta.$$

Using $|u_j|=1$ and (2),

$$u^*Ru\le m\,m!\,t^{m-1}+m(m-1)m!\eta.$$

As $q\le 2t$ and $(m-1)t^3\le 1$,

$$r=q\,u^*Ru\le 2m\,m!\,t^m\big(1+(m-1)t^3\big)\le 4m\,m!\,t^m. \tag{3}$$

Also $a\le m!$ by the triangle inequality. For each i, $L_{ii}\ge 1$ because its entrywise nonnegative correlation minor contains the identity-permutation term one. The PSD permanent diagonal-product inequality gives $R_{ii}\ge 1$. The entries of L and of $|R|^{\circ 2}$ are nonnegative. Since $|x_i|^2=q\ge t$, their diagonal contributions imply

$$\ell\ge mt^2, \qquad \ell_R\ge mt^2. \tag{4}$$

Combining $s = 1$, (3), and (4) proves

$$\frac{\ell}{asr}, \frac{\ell_R}{asr} \geq \frac{t^{2-m}}{4(m!)^2} \longrightarrow +\infty \quad (t \downarrow 0).$$

Here m is fixed. For any prescribed $C > 0$, it is enough to choose

$$0 < t < \min\left\{\frac{1}{2m}, \left[\frac{1}{4C(m!)^2}\right]^{1/(m-2)}\right\}.$$

Nonpositive constants fail because the left sides are positive. This proves Theorem 1.

For completeness, the PSD permanent diagonal-product bound used above can be recovered from multiplication in the factorial norm. Multiplication by a unit linear form is unitarily equivalent to multiplication by z_1; on each monomial this increases its squared factorial weight by $\alpha_1 + 1 \geq 1$. Iterating over unit Gram forms gives $\operatorname{per} B \geq 1$, including the principal minors. For $B > 0$, the Gram linear forms are independent, so their m deleted products are linearly independent: after an invertible change of variables they are the distinct monomials $\prod_{j \neq i} z_j$. Their Gram matrix, and hence R, is positive definite.

4 What the obstruction does not imply

Let $c = \operatorname{per} |B|^{\circ 2}$. The actual compensated border expression is

$$(a^2 - c)s^2 + 2asr + r^2 - \ell.$$

Theorem 1 rejects comparisons that discard its baseline and other compensating terms. Nor does it decide the separate matrix comparison $|R|^{\circ 2} \succeq L$. Related permanental spectral comparisons involve different cofactor matrices and different normalization [3]; they must not be identified with the ratios here. The present result is a sharp limitation of two auxiliary estimates, with the underlying roots-of-unity mechanism credited to prior work.

Methods disclosure. AI-assisted derivation, exact symbolic checks and separate computational checks were used in developing this note. The theorem is analytic and its proof above does not depend on a numerical search or a finite-sample inference.

References

[1] B. Reznick, An inequality for products of polynomials, *Proceedings of the American Mathematical Society* **117** (1993), 1063–1073. `https://reznick.web.illinois.edu/Paper28.pdf`.

[2] B. Seron, L. Novo and N. J. Cerf, Boson bunching is not maximized by indistinguishable particles, *Nature Photonics* **17** (2023), 702–709. `https://arxiv.org/abs/2203.01306`.

[3] L. Pioge, K. K. Pietrasz, B. Seron, L. Novo and N. J. Cerf, A logical implication between two conjectures on matrix permanents, *Linear Algebra and its Applications* **725** (2025), 309–318. `https://arxiv.org/abs/2508.00111`.