A sharp flatness transition for critical heat traces in twisted tubes

Scott Schubr
Luminary Oddities

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Abstract

We study the Dirichlet Laplacian in a three-dimensional tube with fixed annular-strip cross-section and twisting speed proportional to the square root of a logarithm. Its eigenvalue count has a leading exponential rate, which determines a critical heat-trace time but does not determine finiteness at that time. We classify the critical trace when the angular confinement potential has an infinitely flat isolated minimum of the form $a\exp(-A/|r-r_*|^q)$. The trace is finite for $q<2$ and infinite for $q>2$. At $q=2$ it is finite exactly when $A<K\pi^2/6$, where K is the critical time; equality gives divergence. Finite-order isolated width maxima also give a finite critical trace, whereas a width plateau gives divergence. The proof retains the original coupled Dirichlet form. A constant-twist heat comparison, using diamagnetic domination, preserves the helical-mode multiplicity. A family of exactly orthogonal helical trial functions supplies the matching lower bound, including the critical parameter boundary.

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1 The critical trace and the main result

Twisting can make an infinite-volume tube spectrally confining while leaving its cross-sectional area fixed. Compactness of the resolvent does not, however, imply finiteness of every heat trace. At a finite critical heat-trace time, even the leading exponential eigenvalue-counting rate is insufficient: subleading geometric confinement becomes decisive.

Fix $0<r_-<r_+<\infty$ and a positive C^1 function α on $[r_-,r_+]$. Suppose

$$\alpha(r)\le\alpha_0<2\pi,\qquad \alpha(r_*)=\alpha_0,\qquad r_*\in(r_-,r_+).$$

The fixed cross-section and physical tube are

$$\omega=\{(r,t):r_-<r<r_+,\ |t|<\alpha(r)/2\},$$
$$\Omega=\{(r\cos(t+\theta(s)),r\sin(t+\theta(s)),s):(r,t)\in\omega,\ s\in\mathbb{R}\},$$

where the angle is determined, up to a constant, by

$$\theta'(s)=b(s)=c\sqrt{\tfrac{1}{2}\log(\mathrm{e}^2+s^2)},\qquad c>0. \tag{1.1}$$

Let $H=-\Delta_D^\Omega$ be the ordinary Euclidean Dirichlet Laplacian. Put

$$\nu_0=\frac{\pi}{\alpha_0},\qquad K=\frac{1}{\nu_0^2c^2},\qquad V(r)=\left(\frac{\pi}{\alpha(r)}\right)^2-\nu_0^2. \tag{1.2}$$

All parameters and the cross-section remain fixed as the energy tends to infinity. Denote by $N_H(\Lambda)$ the number of eigenvalues strictly below Λ, counted with multiplicity.

Theorem 1.1 (Infinitely flat width maxima)**.** *Suppose, for fixed $a, A, q > 0$,*

$$V(r) = a \exp\Big(-\frac{A}{|r - r_*|^q} \Big), \quad r \neq r_*; \qquad V(r_*) = 0. \tag{1.3}$$

Then H has compact resolvent and

$$\lim_{\Lambda \to \infty} \frac{\log N_H(\Lambda)}{\Lambda} = K. \tag{1.4}$$

The heat trace is finite for every $\beta > K$ and infinite for $0 < \beta < K$. At the critical time,

$$\operatorname{Tr} e^{-KH} < \infty \quad \Longleftrightarrow \quad (q < 2) \text{ or } (q = 2 \text{ and } A < K\pi^2/6). \tag{1.5}$$

In particular $A = K\pi^2/6$ belongs to the divergent case.

The cross-sections in Theorem 1.1 are explicit: take

$$\alpha(r) = \frac{\pi}{\sqrt{\nu_0^2 + a \exp(-A/|r - r_*|^q)}}$$

with the exponential set to zero at r_*. They have the same maximal width and hence the same leading exponential rate when ν_0, c are fixed. Each has a unique C^∞, infinitely flat maximum. Nevertheless either critical-trace behavior occurs within this class.

Theorem 1.2 (Finite-order maxima and plateaux)**.** *Under the general geometric assumptions above, suppose that, for some $a > 0$ and finite $m \geq 2$,*

$$V(r) \geq a|r - r_*|^m \qquad (r_- \leq r \leq r_+). \tag{1.6}$$

Then (1.4) *holds and $\operatorname{Tr} e^{-\beta H}$ is finite exactly for $\beta \geq K$. If instead $\alpha = \alpha_0$ on a nonempty radial open interval, then* (1.4) *still holds, but the heat trace is finite exactly for $\beta > K$.*

Predecessors and the new issue. Krejčiřík established exploding-twist confinement [1]. Barseghyan–Khrabustovskyi obtained eigenvalue-moment bounds [2]. Weyl-type accumulation in this setting remains a broader question in [3, Open Problem 8]. The author's earlier Weyl theorem treats positive-index regularly varying and certain rapidly varying speeds, with finite-band cross-sections and angular stability [4]. Its index-zero exclusion includes (1.1). The weak-twist Bose analysis [5] concerns a different limit, with the confinement parameter tending to zero and low-energy occupations becoming singular.

Non-power counting laws and heat-trace criteria on infinite-volume Euclidean domains have classical horn predecessors [6, 7]. Here the section rotates but remains *fixed*. The new issue is to combine radial confinement with the multiplicity of helical modes at the exact critical time. Characteristic slicing, IMS localization, diamagnetic domination and radial barrier comparisons are established tools.

The upper bound retains both the radial well energy and a factor of the twisting speed in the local heat trace. The lower bound constructs a matching number of orthogonal helical trial functions per axial cell. At $q = 2$, their multiplicity contributes $\Lambda^{1/2}$ to the counting lower bound, while radial confinement costs a factor $\Lambda^{-K\pi^2/(4A)}$ in the critical trace integrand. The resulting power reaches -1 at $A = K\pi^2/6$. The upper estimate proves finiteness below this boundary; the trial-space estimate proves divergence at and above it.

2 The original form and localization

The physical map defining Ω is injective: the axial coordinate is unchanged and each angular interval has width less than 2π. Pullback is unitary to $L^2(\omega \times \mathbb{R}, r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s)$ and gives the closed form

$$Q_b[v] = \int \Big(|v_r|^2 + r^{-2}|v_t|^2 + |v_s - b(s)v_t|^2 \Big)\, r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s. \tag{2.1}$$

Its domain is the closure of compactly supported smooth tests in the form norm. On bounded axial pieces it is the usual H_0^1 domain in these coordinates. The mixed square is retained throughout the argument.

Lemma 2.1 (Confinement and fixed-cost localization)**.** *The operator H has compact resolvent. There is a fixed constant C_{loc} and Dirichlet-capped operators H_j^D on the pieces $I_j = (j-1, j+1)$ such that, with extended nonnegative traces allowed,*

$$\operatorname{Tr} e^{-\beta H} \le e^{\beta C_{\mathrm{loc}}} \sum_{j\in\mathbb{Z}} \operatorname{Tr} e^{-\beta H_j^D} \qquad (\beta > 0). \tag{2.2}$$

Proof. At fixed physical polar coordinates (r, φ), where $\varphi = t + \theta(s)$, the derivative $\partial_s - b\partial_t$ becomes ordinary axial differentiation. On an exterior half-tail where $b \ge B$, every connected permitted vertical component has length at most α_0/B. The angular gap $2\pi - \alpha_0 > 0$ prevents successive passages from joining. A component meeting the artificial cap has at least one original Dirichlet endpoint. The Dirichlet–Neumann interval inequality therefore gives

$$\int_{\mathrm{tail}} |v_s - bv_t|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s \ge \frac{\pi^2 B^2}{4\alpha_0^2} \int_{\mathrm{tail}} |v|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s.$$

Integrate the one-dimensional inequalities and pass from tests by closure. Since b tends to infinity on both ends, form-bounded sets have uniformly vanishing tail mass. On every bounded piece the coordinate coefficients are bounded and elliptic, and Rellich compactness applies. A diagonal argument proves compactness of the global form embedding. Also angular Poincaré gives $H \ge \nu_0^2/r_+^2 > 0$.

Choose smooth real functions χ_j supported in I_j with $\sum_j \chi_j^2 = 1$ and $\sum_j |\chi_j'|^2 \le C$. They can be obtained by normalizing translates of one fixed bump. Since $(\partial_s - b\partial_t)\chi_j = \chi_j'$, IMS gives

$$\sum_j Q_b[\chi_j v] \le Q_b[v] + C\|v\|^2.$$

Each localized vector has genuine Dirichlet caps. The isometric map $v \mapsto (\chi_j v)_j$ and min–max give

$$N_H(\Lambda) \le \sum_j N_{H_j^D}(\Lambda + C + 1).$$

The sum is finite at each energy: on a piece with two Dirichlet caps (DD), the same characteristic argument has *two* zero endpoints and lower bound $\nu_0^2(\inf_{I_j} b)^2$. Thus only finitely many pieces contribute. Integration against $\beta e^{-\beta\Lambda}$ proves (2.2) with $C_{\mathrm{loc}} = C+1$. All shifts are independent of j, $b(j)$ and energy. □

For a constant $B > 0$, write Q_B for (2.1) with $b = B$. If $|b - B| \le \delta$ on an interval, the norm triangle inequality for its three derivative components gives

$$(1 + r_+\delta)^{-2} Q_B \le Q_b \le (1 + r_+\delta)^2 Q_B. \tag{2.3}$$

Indeed the difference is only the third component, of norm at most $\delta\|v_t\| \le r_+\delta Q_b[v]^{1/2}$, or at most $r_+\delta Q_B[v]^{1/2}$ in the opposite direction. The common Dirichlet form domains on a bounded piece agree. For (1.1), set

$$B_j = \inf_{I_j} b, \qquad \delta_j = \sup_{I_j} |b - B_j|.$$

Differentiating (1.1) shows, as $|j| \to \infty$,

$$B_j^2 = c^2 \log|j| + O(|j|^{-1}), \qquad \delta_j = O\left(\frac{1}{|j|\sqrt{\log|j|}}\right), \qquad \delta_j(1 + B_j^2) = O(1). \tag{2.4}$$

3 A local heat estimate preserving helical multiplicity

Define the radial well energy in ordinary $L^2(\mathrm{d}r)$ by

$$e(B) = \inf_{\substack{w \in H_0^1(r_-, r_+) \\ \|w\|_2 = 1}} \int_{r_-}^{r_+} \left(|w'|^2 + B^2 V(r)|w|^2\right) \mathrm{d}r. \tag{3.1}$$

It is nonnegative, increasing, and bounded above by $C(1 + B^2)$: test one fixed smooth radial function. Put $C_r = 1/(4r_-^2)$.

Lemma 3.1 (Constant-twist heat bound)**.** *Let $H_{B,L}^{\mathrm{per}}$ be the operator of Q_B on $\omega \times (0, L)$, with Dirichlet lateral boundary and periodic axial boundary. For every $0 < \eta < 1$, $B \ge B_0 > 0$, and β in a fixed compact subinterval of $(0, \infty)$,*

$$\operatorname{Tr} e^{-\beta H_{B,L}^{\mathrm{per}}} \le C_{\eta,L}(1 + B) \exp\{-\beta\nu_0^2 B^2 - \eta\beta e(B)\}. \tag{3.2}$$

The constant is uniform in B and β on these ranges.

Proof. Fourier series in the periodic variable give frequencies $\xi_k = 2\pi k/L$. The fiber form on $L^2(\omega, r\,\mathrm{d}r\,\mathrm{d}t)$ is

$$\int_\omega \left(|u_r|^2 + r^{-2}|u_t|^2 + |i\xi u - Bu_t|^2\right) r\,\mathrm{d}r\,\mathrm{d}t.$$

Completing the angular square, with $a_B(r) = B^2 + r^{-2}$, writes it as

$$\int_\omega \left(|u_r|^2 + a_B(r)\left|u_t - i\frac{\xi B}{a_B(r)}u\right|^2 + \frac{\xi^2}{1 + B^2r^2}|u|^2\right) r\,\mathrm{d}r\,\mathrm{d}t. \tag{3.3}$$

The Dirichlet diamagnetic semigroup inequality removes the real magnetic potential in this expression. It holds for the real, variable elliptic coefficients here; see [8, Proposition 6.2 and its Dirichlet-boundary observation]. One way to verify its applicability is to use $w = \sqrt{r}\,u$. The form is then a real diagonal elliptic magnetic form on ordinary $L^2(\omega)$, with additional bounded real potential $-1/(4r^2)$. The usual modulus form criterion is preserved by H_0^1, and a constant shift makes the potential nonnegative if necessary. Thus the magnetic semigroup is dominated by its nonmagnetic counterpart.

This domination also bounds heat traces: at time $\beta/2$ the absolute magnetic kernel is bounded by the nonmagnetic kernel. Squaring and integrating bounds the Hilbert–Schmidt norms, whose squares are the traces at time β. All fiber operators are elliptic on a bounded Lipschitz domain, so these kernels and traces exist. No uniform ellipticity constant in B is used for the comparison.

Let $T_B = -\partial_r^2 - B^2\partial_t^2$ on $H_0^1(\omega)$ in ordinary L^2. Dropping the extra positive angular coefficient after domination and using $r \le r_+$ bounds the fiber trace by

$$e^{\beta C_r} \exp\left(-\frac{\beta\xi^2}{1+B^2r_+^2}\right) \operatorname{Tr} e^{-\beta T_B}. \tag{3.4}$$

The potential $-1/(4r^2)$ is the only bounded radial correction.

Angular Poincaré on each fiber, followed by radial zero extension across the angular sides, gives

$$T_B \ge \nu_0^2 B^2 + e(B).$$

The zero extension is legitimate since the lateral trace is zero; almost every radial slice belongs to $H_0^1(r_-, r_+)$. Also ω lies in the rectangle of radial length $\ell = r_+ - r_-$ and angular width α_0. Domain monotonicity gives, for each ordered eigenvalue μ_j of T_B, $\mu_j \ge \lambda_j^{\mathrm{rec}}$, where the rectangle eigenvalues are

$$\lambda_{m,n}^{\mathrm{rec}} = \pi^2 m^2/\ell^2 + B^2\nu_0^2 n^2, \qquad m, n \in \mathbb{N}.$$

For each ordered eigenvalue, use η times the scalar lower bound and $1-\eta$ times the rectangle bound. The rectangle ground angular energy supplies the complementary $(1-\eta)\nu_0^2B^2$, so the coefficient of $\nu_0^2B^2$ remains one. Hence

$$\begin{aligned}\operatorname{Tr} e^{-\beta T_B} &\le e^{-\beta\nu_0^2B^2 - \eta\beta e(B)} \sum_{m\ge1} e^{-(1-\eta)\beta\pi^2m^2/\ell^2} \sum_{n\ge1} e^{-(1-\eta)\beta B^2\nu_0^2(n^2-1)} \\ &\le C_\eta e^{-\beta\nu_0^2B^2 - \eta\beta e(B)}.\end{aligned}$$

Finally

$$\sum_{k\in\mathbb{Z}} \exp\left(-\frac{\beta(2\pi k/L)^2}{1+B^2r_+^2}\right) \le C_L(1+B)$$

uniformly on the stated time range. Sum (3.4) to obtain (3.2). □

The periodic condition in this lemma is an auxiliary condition in the pulled-back coordinates, not a physical identification of the rotating caps. Dirichlet axial tests form a subspace of its periodic form domain. Therefore a DD piece has no larger heat trace, by min–max.

Proposition 3.2 (Original-operator trace bound)**.** *For every fixed $\beta > 0$ and $0 < \eta < 1$,*

$$\operatorname{Tr} e^{-\beta H} \le C_{\beta,\eta}\left(1 + \sum_{|j|\ge j_0} (1+B_j) e^{-\beta\nu_0^2B_j^2 - \eta\beta e(B_j)}\right). \tag{3.5}$$

In particular the trace is finite for every $\beta > K$.

Proof. By (2.3), the local trace at time β is bounded by the constant-B_j DD trace at time

$$\beta_j = \beta(1 + r_+\delta_j)^{-2}.$$

For large $|j|$ these times lie in $[\beta/2, \beta]$. Apply Lemma 3.1 with $L = 2$. Replacing β_j by β in its exponent changes the bound by a uniformly bounded factor, since

$$(\beta - \beta_j)(\nu_0^2B_j^2 + \eta e(B_j)) \le C\delta_j(1 + B_j^2) = O(1).$$

The finitely many central DD pieces have finite traces: each physical piece is a bounded domain contained in a fixed cylinder, so Dirichlet domain monotonicity suffices. Now use (2.2). If $\beta > K$, discard $e(B_j) \ge 0$ and use (2.4); the resulting series is bounded by a constant times

$$\sum_{j\ge j_0} j^{-\beta\nu_0^2c^2} (\log j)^{1/2},$$

which converges because $\beta\nu_0^2c^2 > 1$. □

4 Radial flat wells and critical finiteness

The radial energy isolates the geometric scale needed in (3.5). Its flat-well asymptotic is the Dirichlet-interval version of the semiclassical result of Bony and Popoff [9, Theorem 3.4 and Example 3.8], under the scaling $h = B^{-1}$. We give the short barrier proof on the radial interval for completeness.

Lemma 4.1 (Radial flat-well energy)**.** *For* (1.3)*,*

$$e(B) \sim \frac{\pi^2}{4}\left(\frac{2\log B}{A}\right)^{2/q} \qquad (B \to \infty). \tag{4.1}$$

Proof. Fix $0 < \varepsilon < 2$. The normalized Dirichlet cosine on the centered radial interval of half-width

$$h_- = \left(\frac{A}{(2+\varepsilon)\log B}\right)^{1/q}$$

has derivative energy $\pi^2/(4h_-^2)$. Its support is interior for large B, and there $B^2V \le aB^{-\varepsilon}$. This gives the required upper limit as $\varepsilon \downarrow 0$.

For the lower limit put $h_+ = (A/((2-\varepsilon)\log B))^{1/q}$. Outside this interval $B^2V \ge aB^{\varepsilon}$. Zero extension to $\mathbb{R}$ and scaling give

$$e(B) \ge h_+^{-2}\lambda(aB^{\varepsilon}h_+^2),$$

where $\lambda(M)$ is the bottom of $-\,\mathrm{d}^2/\,\mathrm{d}x^2 + M\mathbf{1}_{|x|>1}$ on the line. Here $\lambda(M) \to \pi^2/4$ as $M \to \infty$. Indeed the Dirichlet cosine gives the upper bound. Normalized approximate minimizers have bounded derivative energy and outside mass $O(M^{-1})$. Weak H^1 and local strong L^2 convergence yield a norm-one limit supported in $[-1,1]$, hence in $H_0^1(-1,1)$. Lower semicontinuity and DD Poincaré give the matching lower limit. There is no escaping mass. Since $aB^{\varepsilon}h_+^2 \to \infty$, let $B \to \infty$ and then $\varepsilon \downarrow 0$. □

At $\beta = K$, the series in (3.5) is, up to fixed factors,

$$\sum_{j \ge j_0} \frac{(\log j)^{1/2}}{j} e^{-\eta K e(B_j)}. \tag{4.2}$$

If $q < 2$, (4.1) makes $e(B_j)$ a positive multiple of $(\log\log j)^{2/q}$ to leading order. This dominates every multiple of $\log\log j$, so (4.2) converges.

If $q = 2$, for every $d < \pi^2/(2A)$ one has $e(B) \ge d\log B$ eventually. Thus (4.2) is bounded by

$$C\sum_{j \ge j_0} \frac{1}{j}(\log j)^{(1-\eta K d)/2}.$$

It converges when $\eta K d > 3$. There are fixed $\eta < 1$ and $d < \pi^2/(2A)$ satisfying this inequality exactly when $A < K\pi^2/6$. This proves the finite part of (1.5).

Under (1.6), comparison with the line oscillator gives

$$e(B) \ge \lambda_m a^{2/(m+2)} B^{4/(m+2)}, \qquad \lambda_m = \inf \mathrm{spec}(-\,\mathrm{d}^2/\,\mathrm{d}x^2 + |x|^m) > 0. \tag{4.3}$$

The positivity follows, for example, by controlling mass outside $[-1,1]$ with the potential, and the mass inside with the derivative and mass on $[1,2]$. Zero extension and dilation prove the stated scale. This reserve also makes (4.2) converge. Hence the finite critical assertion of Theorem 1.2 follows.

5 Orthogonal helical packets and critical divergence

To resolve the boundary, the lower count must retain order B helical modes per unit axial cell. The following construction controls the Rayleigh quotient on their whole span, as required by min–max.

Lemma 5.1 (A helical trial family)**.** *Fix a constant-twist unit piece and a contained constant-opening radial box. Let ϕ be real, belong to H_0^1 of its radial interval, and be normalized in $L^2(r\,\mathrm{d}r)$, and let ψ be the normalized angular Dirichlet cosine with frequency $\nu = \pi/\alpha$. There is a trial space of dimension at least $c_1 B$ for B large whose Q_B quotient is at most*

$$B^2\nu^2 + \int |\phi'|^2 r\,\mathrm{d}r + C(1+\nu^2). \tag{5.1}$$

Here C depends only on the fixed radial bounds, not on the box width, its radial half-width or B.

Proof. On $0 < s < 1$ set $\chi(s) = \sqrt{2}\sin(\pi s)$ and use

$$v_k(r,t,s) = \phi(r)\psi(t)\chi(s)e^{i\xi_k(s+t/B)}, \qquad \xi_k = 4\pi k, \quad 0 \le k \le \lfloor B/(4\pi)\rfloor.$$

These are Dirichlet form vectors. Their norms are one. For $k \neq l$, the axial integral against χ^2 vanishes because its Fourier support is $\{0, \pm 1\}$, whereas $(\xi_k - \xi_l)/(2\pi)$ is a nonzero even integer. Thus the vectors are orthogonal, even though their angular phases differ.

The phase is annihilated by $\partial_s - B\partial_t$, so

$$(\partial_s - B\partial_t)v_k = \phi e^{i\xi_k(s+t/B)}(\chi'\psi - B\chi\psi').$$

All off-diagonal form terms vanish as well: their axial factors are χ^2, $|\chi'|^2$ or $\chi\chi'$, each supported in frequencies $\{0, \pm 1\}$. This verifies orthogonality in the form, not only in the norm. For a diagonal term the real mixed product integrates to zero. Its axial-square energy is $\pi^2 + B^2\nu^2$; the angular physical energy is at most

$$r_-^{-2}(\nu^2 + \xi_k^2/B^2) \le r_-^{-2}(\nu^2 + 1).$$

Adding the radial energy proves (5.1) on the entire span. Zero extension from the contained box is legitimate, and translation gives the same construction on any unit cell. □

Proposition 5.2 (An all-large-energy lower count)**.** *For (1.3), set $p = 2/q$ and $\gamma = \pi^2/(4A^{2/q})$. There are $c_2 > 0$ and Λ_0 such that*

$$N_H(\Lambda) \ge c_2\sqrt{\Lambda}\,\exp\{K\Lambda - K\gamma(\log\Lambda)^p\} \qquad (\Lambda \ge \Lambda_0). \tag{5.2}$$

Proof. Put $h = (A/\log\Lambda)^{1/q}$. On $|r - r_*| < h$, the actual width well satisfies $V \le a/\Lambda$, so the original section contains the constant-opening box with

$$\nu_\Lambda^2 = \nu_0^2 + a/\Lambda.$$

Use the normalized radial cosine $w = h^{-1/2}\cos(\pi(r - r_*)/(2h))$, divided by $\sqrt{r}$ to give ϕ. Its physical radial energy satisfies

$$\int |\phi'|^2 r\,\mathrm{d}r = \frac{\pi^2}{4h^2} - \int \frac{|w|^2}{4r^2}\,\mathrm{d}r \le \gamma(\log\Lambda)^p.$$

Apply Lemma 5.1 on a unit axial cell with constant $B \le \sqrt{\Lambda}/\nu_0$. Its whole trial-space quotient is at most

$$\nu_0^2 B^2 + \gamma(\log\Lambda)^p + C_0, \tag{5.3}$$

where C_0 is uniform in cell and Λ. In particular the added twist cost is $(a/\Lambda)B^2 \le a/\nu_0^2$, and ν_Λ stays bounded.

Let

$$T_\Lambda = \frac{\Lambda - \gamma(\log\Lambda)^p - C_0 - 2}{\nu_0^2}, \qquad R_\Lambda = \sqrt{\exp(2T_\Lambda/c^2) - \mathrm{e}^2}.$$

For large Λ, $T_\Lambda > 0$ and $b(R_\Lambda)^2 = T_\Lambda$. Use all integer unit cells contained in $[R_\Lambda/2, R_\Lambda]$, with $B = b$ at the left endpoint. There are at least cR_Λ such cells, and $B \asymp \sqrt{\Lambda}$ uniformly over them. Their speed oscillation is $O((R_\Lambda\sqrt{\Lambda})^{-1})$. By (2.3), replacing the constant speed by the actual speed changes every quotient in (5.3) by at most $C\sqrt{\Lambda}/R_\Lambda = o(1)$. Thus the actual quotients are strictly below Λ for all sufficiently large Λ.

Each cell supplies at least $c\sqrt{\Lambda}$ orthogonal vectors, and different cells have disjoint form supports. Their direct-sum span is therefore a legal min–max space of dimension at least $c\sqrt{\Lambda}R_\Lambda$. Finally

$$R_\Lambda = \exp(T_\Lambda/c^2)(1 + o(1)) \asymp \exp\{K\Lambda - K\gamma(\log\Lambda)^p\}.$$

All constants are independent of Λ. This proves (5.2) on the entire sufficiently large energy half-line. □

The positive compact-resolvent operator satisfies, by Tonelli,

$$\operatorname{Tr} e^{-\beta H} = \beta \int_0^\infty e^{-\beta\Lambda} N_H(\Lambda)\,\mathrm{d}\Lambda, \tag{5.4}$$

with infinity allowed. If $q > 2$, then $p < 1$ and (5.2) makes the critical integrand bounded below by

$$c\Lambda^{1/2}\exp\{-K\gamma(\log\Lambda)^p\};$$

this is not integrable, since the exponential factor is $\Lambda^{-o(1)}$. If $q = 2$, the lower integrand is $c\Lambda^{1/2 - K\pi^2/(4A)}$. This lower bound has divergent integral when $A \ge K\pi^2/6$; at equality its exponent is exactly -1. The finite-side estimate proves convergence when $A < K\pi^2/6$. Together these bounds establish the critical classification.

6 The leading rate, the plateau comparison, and Gibbs states

We finish the assertions common to both theorems. For any continuous width attaining α_0 at an interior point, every small $\varepsilon > 0$ gives a fixed radial interval on which $\alpha \ge \alpha_0 - \varepsilon$. Choose one fixed real radial test and the angular Dirichlet cosine of that opening. Real unit-cell axial sine tests have quotient at most

$$\nu_\varepsilon^2 \sup_{\text{cell}} b^2 + C_\varepsilon, \qquad \nu_\varepsilon = \pi/(\alpha_0 - \varepsilon).$$

The mixed term vanishes by $\int \psi\psi' = 0$. Disjoint unit cells below the exact inverse-speed threshold give

$$N_H(\Lambda) \ge c_\varepsilon \exp\!\left(\frac{\Lambda - C_\varepsilon}{\nu_\varepsilon^2 c^2}\right).$$

Take $\Lambda \to \infty$ first and then $\varepsilon \downarrow 0$ to obtain $\liminf \log N_H(\Lambda)/\Lambda \ge K$. Proposition 3.2 gives a finite trace for every $\beta > K$, hence $N_H(\Lambda) \le e^{\beta\Lambda}\operatorname{Tr} e^{-\beta H}$ and the opposite upper limit after $\beta \downarrow K$. This proves (1.4). The same lower count and (5.4) imply divergence for every $0 < \beta < K$.

If $\alpha = \alpha_0$ on an open radial interval, take the fixed test there with no ε loss. Then $N_H(\Lambda) \ge ce^{K\Lambda}$ for all large Λ, which proves critical divergence. This completes Theorem 1.2 and the remaining assertions of Theorem 1.1.

Corollary 6.1 (Fixed-chemical-potential ideal Bose gas)**.** *Fix $\mu < \lambda_1(H)$. The ideal Bose grand partition function and the sum of mean occupations are finite exactly at those inverse temperatures for which the heat trace in Theorems 1.1–1.2 is finite.*

Proof. On the high-energy tail both $-\log(1 - e^{-\beta(\lambda_j-\mu)})$ and $(e^{\beta(\lambda_j-\mu)} - 1)^{-1}$ are bounded above and below by positive fixed multiples of $e^{-\beta\lambda_j}$. The finitely many lower modes cause no problem because $\mu < \lambda_1$. Summing proves the assertion. □

This is a trace-existence statement at fixed geometry and chemical potential. The exact speed (1.1) matters at the critical time: a leading equivalence alone need not preserve logarithmic corrections to an exponential count. For other flat wells, Proposition 3.2 supplies a useful sufficient criterion through $e(B)$, while a matching necessity argument requires trial spaces adapted to that well. Multiple maxima and sharp subleading counting equivalents provide further questions beyond the explicit classification proved here.

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References

[1] D. Krejčiřík, *Waveguides with asymptotically diverging twisting*, Applied Mathematics Letters **46** (2015), 7–10. doi:10.1016/j.aml.2015.01.016.

[2] D. Barseghyan and A. Khrabustovskyi, *Spectral estimates for Dirichlet Laplacian on tubes with exploding twisting velocity*, Operators and Matrices **13** (2019), 311–322. doi:10.7153/oam-2019-13-21.

[3] D. Krejčiřík, *Spectral geometry: old questions and new answers*, preprint (2026). arXiv:2609.28602v1.

[4] S. Schubr, *Weyl Laws and Angular Stability for Rapidly Twisted Tubes*, Luminary Research Archive, edition 2 (2026). `https://research.luminaryoddities.com/weyl-laws-angular-stability-rapidly-twisted-tubes/v/2`.

[5] S. Schubr, *Bose gases in weakly twisted annular-sector waveguides*, Luminary Research Archive, edition 2 (2026). `https://research.luminaryoddities.com/bose-gases-in-weakly-twisted-annular-sector-waveguides/v/2`.

[6] M. van den Berg, *On the spectrum of the Dirichlet Laplacian for horn-shaped regions in $\mathbb{R}^n$ with infinite volume*, Journal of Functional Analysis **58** (1984), 150–156. doi:10.1016/0022-1236(84)90036-3.

[7] R. Bañuelos and M. van den Berg, *Dirichlet eigenfunctions for horn-shaped regions and Laplacians on cross sections*, Journal of the London Mathematical Society (2) **53** (1996), 503–511. doi:10.1112/jlms/53.3.503.

[8] A. F. M. ter Elst and E. M. Ouhabaz, *The diamagnetic inequality for the Dirichlet-to-Neumann operator*, Bulletin of the London Mathematical Society **54** (2022), 1978–1997. doi:10.1112/blms.12674; arXiv:2004.09782, Proposition 6.2.

[9] J.-F. Bony and N. Popoff, *Low-lying eigenvalues of semiclassical Schrödinger operator with degenerate wells*, Asymptotic Analysis **112** (2019), 23–36. doi:10.3233/ASY-181493; arXiv:1802.02882v1.