Sharp degree exponents for polynomial detection of isolated negative atoms

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Abstract

We study how much polynomial degree can be required to detect an isolated negative atom of a signed measure by a negative square integral. Fix a gap $d > 0$, an exponential-moment rate $c > 0$, and $C > e^{cd}$. Among measures $\mu = -\delta_0 + \nu$ with $\operatorname{supp} \nu \subset \{|x| \geq d\}$, $\|\nu\|_{\mathrm{TV}} \leq R$, and $\int e^{c|x|}\, d|\mu| \leq 1 + CR$, let $D(R)$ be the worst least degree of a real polynomial P for which $\int P^2\, d\mu < 0$. We prove $\log D(R)/\log R \to \pi/(2cd)$, with the explicit-order upper bound $D(R) = O(R^{\pi/(2cd)} \log(2+R))$. A direct critical-weight approximation of a classical continuous Chebyshev window constructs one rational polynomial with negative square integral on the original measure, without evaluating its moments. Continuous dual Hahn laws give matching lower exponents while keeping all input budgets fixed. For finitely many isolated negative atoms we also recover a whole negative polynomial subspace with an explicit polynomial cost in inverse separation; the index is exact for a positive remainder and is a lower bound for a signed remainder. An exact infinite-support example has first negative degree $2\lfloor R \rfloor$. The theorem determines the logarithmic growth exponent.

1 The degree question and its sharp exponent

For a finite real signed Borel measure μ with all moments, write

$$I_\mu(P) = \int_{\mathbb{R}} P(x)^2\, d\mu(x), \qquad n(\mu) = \min\{\deg P : P \in \mathbb{R}[x],\ I_\mu(P) < 0\}.$$

Set $n(\mu) = +\infty$ if the set is empty. Degree always means the degree of P, rather than of its square. This distinction also applies to the reproducibility data. Fix throughout

$$c, d > 0, \qquad C > e^{cd}, \qquad R \geq 1. \tag{1}$$

Let $\mathcal{A}_R$ contain the measures

$$\mu = -\delta_0 + \nu, \quad \operatorname{supp} \nu \subset (-\infty, -d] \cup [d, \infty), \quad \|\nu\|_{\mathrm{TV}} \leq R, \quad \int e^{c|x|}\, d|\mu|(x) \leq 1 + CR. \tag{2}$$

The remainder ν may be signed. Define $D(R) = \sup_{\mu \in \mathcal{A}_R} n(\mu)$. The constants c, d, C are fixed before R grows. In particular an example whose exponential budget grows with R would not establish the lower bound below.

Degree exponent, construction and index The sharp degree statement is a worst-case exponent over fixed input budgets. The constructive upper bound retains its logarithmic factor, and the lower statement retains its subendpoint quantifiers. The rational construction rounds a finite approximation with a paid error; its bandlimited frequency is an analytic approximation resource. For several separated negative atoms, the full-index theorem concerns a common strictly negative subspace of the original form. Exact-null integer endpoints are recorded separately from strict negativity.

Theorem 1.1. *For the data in (1), a constant $K = K(c, d, C)$ satisfies*

$$D(R) \leq K R^{\pi/(2cd)} \log(2 + R). \tag{3}$$

For every $\beta < \pi/(2cd)$ there is a positive probability law τ_β, fixed before R varies, with gap d and $\int e^{c|x|}\, d\tau_\beta \leq C$, such that

$$\frac{n(-\delta_0 + R\tau_\beta)}{R^\beta} \longrightarrow +\infty. \tag{4}$$

Consequently

$$\lim_{R\to\infty} \frac{\log D(R)}{\log R} = \frac{\pi}{2cd}. \tag{5}$$

The upper bound remains true if only rational-coefficient polynomial certificates are permitted.

The upper construction specifies one rational polynomial whose original μ-integral is negative. For certified computable input data, including the known negative atom and gap, it does not require evaluation of the original moments. The hypotheses do not locate an unknown negative atom or supply uncertified parameters.

Relation to prior work

Qualitative detection by polynomial squares is part of moment positivity and determinacy theory; exponential moments provide a standard sufficient hypothesis. Lasserre's positivity results supply an important general antecedent [7]. The Christoffel extremal principle characterizing the least square norm at a prescribed point is classical. Its interpretation as a certificate-degree barrier also appears in recent work on robust learning [9].

Critical exponential weights already have logarithmic localization scales. Fast-decreasing polynomial theory and exponential-weight Christoffel estimates are substantive antecedents [12, 8, 10]. The compact-interval estimates and full-line positive-weight estimates in those works do not directly identify the uniform exponent in (5): here the evaluation point lies in a support gap and the upper class has only variation and exponential-mass budgets. Our contribution is the matched upper and lower theorem for this precise class, together with the original-measure certificate construction. Finite real mass modifications have classical matrix-kernel descriptions, including signed weights [2]; multipoint Christoffel variation and positive mass updates are also classical [1]. Our multi-atom result supplies a uniform degree from coarse gap, separation and exponential-mass data, rather than a new finite-rank perturbation principle.

The window used below is a classical continuous Dolph–Chebyshev pattern [3]. Meixner–Pollaczek identities, continuous dual Hahn orthogonality and the Christoffel criterion are also inherited [5]. The projection method is related to the explicit exponential-weight approach of Bizeul and Klartag [4]. The exact example's median Genocchi moments are known [6]. These ingredients are used to prove a quantitative statement rather than introduced as new objects.

An earlier concrete certificate for the Vámos correction measure determines its first negative-square degree [11]. That is a fixed-measure question; (5) is a worst-case problem over a different class. Application of the theorem requires the isolated-atom structure (2). No Riemann-theta sign conclusion is drawn.

2 A compact Fourier detector for the gap

For $T > 0$ set

$$s_T(x) = \frac{\cos\Bigl(T\sqrt{x^2 - d^2}\Bigr)}{\cosh(Td)}. \tag{6}$$

The numerator means its entire power series in $x^2 - d^2$; no choice of a square-root branch is required. On $|x| \le d$ it is $\cosh(T\sqrt{d^2 - x^2})/\cosh(Td)$. Thus

$$s_T(0) = 1, \qquad |s_T(x)| \le 1 \ (x \in \mathbb{R}), \qquad |s_T(x)| \le \operatorname{sech}(Td) \ (|x| \ge d). \tag{7}$$

Lemma 2.1. *There is a positive probability measure σ_T on $[-T, T]$ such that $s_T(x) = \int e^{ixt}\, d\sigma_T(t)$. Explicitly,*

$$\sigma_T = \frac{\delta_{-T} + \delta_T}{2\cosh(Td)} + \frac{dT}{2\cosh(Td)} \frac{I_1(d\sqrt{T^2 - t^2})}{\sqrt{T^2 - t^2}} \mathbf{1}_{\{|t|<T\}}\, dt. \tag{8}$$

In particular $\|(s_T^2)''\|_\infty \le 4T^2$.

Proof. Here is a finite-state derivation that proves positivity, rather than inferring it from bandwidth. Put

$$A_x = \begin{pmatrix} ix & d \\ d & -ix \end{pmatrix}.$$

Since $A_x^2 = (d^2 - x^2)I$, its exponential has trace $2\cosh(T\sqrt{d^2 - x^2})$. Expand the exponential in the diagonal matrix $\operatorname{diag}(ix, -ix)$ and the off-diagonal matrix $d\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ by ordered time integrals. Only even numbers $2k$ of off-diagonal factors contribute to the trace. Each coefficient is positive and its phase is $\exp(ix \int_0^T \epsilon(t)\, dt)$, where ϵ alternates between 1 and -1 at the ordered times. The integrated sign lies in $[-T, T]$. For the two starting signs, the total mass at level $2k$ is $2(dT)^{2k}/(2k)!$. Division by $2\cosh(Td)$ gives total mass one. The expansion converges in total variation, proving the claimed probability representation and its Fourier transform.

For completeness the explicit density follows by differentiating the classical Bessel integral

$$\int_0^T I_0(d\sqrt{T^2 - t^2}) \cos(xt)\, dt = \frac{\sinh(T\sqrt{d^2 - x^2})}{\sqrt{d^2 - x^2}}.$$

It can be verified by convergent power series and the beta integral. The moving endpoint contributes $\cos(Tx)$; the interior derivative contributes $dTI_1(d\sqrt{T^2 - t^2})/\sqrt{T^2 - t^2}$. Even reflection gives (8). Its interior density has finite endpoint limit $d^2T/(4\cosh(Td))$, so no endpoint mass has been omitted. Finally s_T^2 is the Fourier transform of $\sigma_T * \sigma_T$, a probability measure supported on $[-2T, 2T]$. Twice differentiating proves the bound. □

Choose

$$T = \frac{\log(4\sqrt{R})}{d}. \tag{9}$$

Since $\operatorname{sech} u \le 2e^{-u}$, (7) gives, for every $\mu \in \mathcal{A}_R$,

$$\int s_T^2\, d\mu \le -1 + R \operatorname{sech}^2(Td) \le -\frac{3}{4}. \tag{10}$$

This estimate uses total variation for a possibly signed remainder.

3 Direct approximation at the critical exponential weight

Set $h = \pi/c$ and $q_n(x) = P_n^{(1/2)}(x/h; \pi/2)$. The classical generating function is [5, 18.23.7]

$$\sum_{n=0}^{\infty} q_n(x) z^n = (1 - iz)^{-1/2+ix/h}(1 + iz)^{-1/2-ix/h}, \qquad |z| < 1. \tag{11}$$

The branches are those analytic at zero. These polynomials have squared norm one for $\omega_h(x) = [h \cosh(\pi x/h)]^{-1}$, but the following bound acts directly on any measure with the prescribed exponential moment.

Lemma 3.1. *For $n \geq 0$,*

$$|q_n(x)| \leq e\sqrt{n+1}\, e^{c|x|/2}. \tag{12}$$

Suppose f is real on $\mathbb{R}$ and $f(x) = \int_{-T}^{T} e^{itx}\, d\Lambda(t)$, where $\|\Lambda\|_{\mathrm{TV}} \leq 1$. Put $r = \tanh(hT/2)$ and

$$Q_N(r) = r^{N+1}\left[\frac{N+2}{1-r} + \frac{r}{(1-r)^2}\right]. \tag{13}$$

There are explicitly given real coefficients a_n, $|a_n| \leq r^n$, such that

$$\sup_{x\in\mathbb{R}} e^{-c|x|/2}\left|f(x) - \sum_{n=0}^{N} a_n q_n(x)\right| \leq eQ_N(r). \tag{14}$$

Consequently, for any positive Borel measure ζ with $J = \int e^{c|x|}\, d\zeta < \infty$, the $L^2(\zeta)$ error is at most $e\sqrt{J}Q_N(r)$. The measure may be atomic or singular.

Proof. On $|z| = \rho < 1$, the Cayley ratio $(1-iz)/(1+iz)$ lies on the circle with center $(1+\rho^2)/(1-\rho^2)$ and radius $2\rho/(1-\rho^2)$. Its argument has absolute value at most $2\arctan\rho < \pi/2$. Since $|1+z^2| \geq 1-\rho^2$, Cauchy's estimate in (11) gives

$$|q_n(x)| \leq \rho^{-n}(1-\rho^2)^{-1/2} e^{2\arctan(\rho)|x|/h}. \tag{15}$$

For $n \geq 1$ take $\rho = n/(n+1)$. Then $\rho^{-n} \leq e$ and $(1-\rho^2)^{-1/2} \leq \sqrt{n+1}$, proving (12). The case $n = 0$ is immediate.

Substituting $z = i\tanh(ht/2)$ in the generating function gives

$$e^{itx} = \operatorname{sech}(ht/2) \sum_{n=0}^{\infty} q_n(x)[i\tanh(ht/2)]^n. \tag{16}$$

Replace Λ by its Hermitian symmetrization if necessary; this preserves f, its support and the variation bound. Integrating the displayed coefficients gives real a_n with $|a_n| \leq r^n$. Equation (12) supplies a summable weighted-uniform majorant for $|t| \leq T$, justifying the integration. Finally

$$\sum_{n>N} \sqrt{n+1}\, r^n \leq \sum_{n>N} (n+1) r^n = Q_N(r).$$

This proves (14), and integration of its square proves the original-measure norm bound. □

Lemma 3.2 (Rational rounding without moment evaluation)**.** *If $\int e^{c|x|}\, d\zeta \leq J_0$, then a real polynomial of degree N can be approximated by a rational polynomial of the same degree with any prescribed positive $L^2(\zeta)$ error, using only c, J_0 and its coefficients. The construction is effective when these inputs are certified computable.*

Proof. The exponential series gives

$$\int |x|^{2j}\, d\zeta \le J_0(2j)!/c^{2j}. \tag{17}$$

For monomial coefficient changes δ_j, the norm error is at most $\sqrt{J_0}\sum_{j=0}^{N}|\delta_j|\sqrt{(2j)!}/c^j$. Choose rational approximations making this finite expression smaller than the requested tolerance. No original moment value is required. □

For $\mu \in \mathcal{A}_R$ use the same T as in (9), and put

$$J_0 = 1 + CR, \qquad \delta_R = \frac{1}{16\sqrt{1+R}}.$$

Choose N such that

$$e\sqrt{J_0}Q_N(r) < \delta_R/2, \tag{18}$$

and round $\sum_{n=0}^{N} a_n q_n$ to $P \in \mathbb{Q}[x]$ with error less than $\delta_R/2$, using Lemma 3.2 for $\zeta = |\mu|$. Then $\|P - s_T\|_{L^2(|\mu|)} < \delta_R$. Since $\|s_T\|_{L^2(|\mu|)} \le \sqrt{1+R}$,

$$\left|\int (P^2 - s_T^2)\, d\mu\right| < \delta_R(2\sqrt{1+R} + \delta_R) < \frac{1}{4}. \tag{19}$$

Together with (10), this gives $\int P^2\, d\mu < -1/2$. Thus a single explicitly constructed original polynomial square is negative. The Fourier representation in (8) makes its coefficients computable on a finite interval, independently of the original measure's moments.

The uniform degree budget

Let $\alpha = \pi/(2cd)$ and $\epsilon = 1 - r$. Equation (9) gives

$$\epsilon^{-1} = (e^{hT} + 1)/2 \le e^{hT} = O_{c,d}(R^{\alpha}).$$

Take $N = \lceil \epsilon^{-1}[L\log(R+2) + L_0]\rceil$, where $L > 2 + 2\alpha$. Then

$$Q_N(r) \le \epsilon^{-2}[L\log(R+2) + L_0 + 4]e^{-L_0}(R+2)^{-L}.$$

After multiplication by $\sqrt{J_0}/\delta_R = O_C(R)$, the remaining power decays uniformly for $R \ge 1$. Increasing the fixed L_0 pays the polynomial factor in L_0 and ensures (18) for all $R \ge 1$. The resulting N is bounded by $KR^{\alpha}\log(R+2)$, proving (3). The constants depend only on the fixed data, and no strict replacement $c' < c$ is used.

4 A matching lower exponent with fixed budgets

We use classical continuous dual Hahn polynomials

$$S_n(u^2; a, b_1, b_2) = (a+b_1)_n(a+b_2)_n {}_3F_2\left(\begin{matrix}-n, a+iu, a-iu\\ a+b_1, a+b_2\end{matrix}; 1\right). \tag{20}$$

For positive parameters their probability measure on $u > 0$ is

$$\frac{1}{2\pi\Gamma(a+b_1)\Gamma(a+b_2)\Gamma(b_1+b_2)}\left|\frac{\Gamma(a+iu)\Gamma(b_1+iu)\Gamma(b_2+iu)}{\Gamma(2iu)}\right|^2 du, \tag{21}$$

and the squared norm is $n!(a+b_1)_n(a+b_2)_n(b_1+b_2)_n$ [5, 18.25.6–8, 18.26.2]. These formulas use u as the integration variable and u^2 as the polynomial variable; their distinction fixes the normalization.

Set $b_1 = b_2 = 1/2$ and denote the law by W_a. Reflection gives its density

$$\frac{2u\tanh(\pi u)|\Gamma(a+iu)|^2}{\Gamma(a+1/2)^2}. \tag{22}$$

At the exterior evaluation point $u^2 = -a^2$, one numerator parameter in (20) is zero. Thus

$$S_n(-a^2) = (a+1/2)_n^2, \qquad \|S_n\|^2 = (n!)^2(a+1/2)_n^2.$$

The squared norm of point evaluation in degree at most k is consequently

$$K_k = \sum_{n=0}^{k}\left[\frac{(a+1/2)_n}{n!}\right]^2 \sim \frac{k^{2a}}{2a\Gamma(a+1/2)^2}. \tag{23}$$

The asymptotic follows from the gamma-ratio formula and summation of n^{2a-1}, valid for $a > 0$. Orthonormal expansion and Cauchy–Schwarz give the classical extremal identity

$$\min_{\deg Q\le k,\ Q(-a^2)=1}\int Q(u^2)^2\,dW_a = K_k^{-1}. \tag{24}$$

Equality is attained by the normalized reproducing kernel.

Fix $0 < s < \pi/c$, put $a = d/s$, and push W_a forward by $x = \pm s\sqrt{u^2+a^2}$ with equal signs. Call the resulting even law τ_s. It has support $|x| \ge d$. Stirling's formula in (22) gives a density asymptotic to a positive constant times $u^{2a}e^{-\pi u}$. Since $|x| \le su + d$, its full exponential moment

$$E_s = \int e^{c|x|}\,d\tau_s < \infty \tag{25}$$

is finite; strict $cs < \pi$ is essential.

Every real polynomial of degree at most N splits uniquely as $P(x) = Q(x^2)+xL(x^2)$. Symmetry makes the cross integral zero. The odd square is nonnegative under τ_s and vanishes at the negative atom. The even sector is precisely the polynomials in u^2 through degree $k = \lfloor N/2\rfloor$, by the affine substitution $x^2 = s^2(u^2+a^2)$. Therefore

$$n(R\tau_s - \delta_0) > N \quad\Longleftrightarrow\quad K_{\lfloor N/2\rfloor} \le R. \tag{26}$$

At equality a null square can occur; strict negativity uses $K_k > R$. Equivalently the least negative degree is asymptotic to

$$2\big[2a\Gamma(a+1/2)^2R\big]^{1/(2a)}. \tag{27}$$

The law τ_s may have $E_s > C$. To meet the *same* prescribed budget, let $\sigma_d = (\delta_d + \delta_{-d})/2$. As $E_s > e^{cd}$, choose one fixed $\eta \in (0,1]$ such that

$$(1-\eta)e^{cd} + \eta E_s \le C, \qquad \tau = (1-\eta)\sigma_d + \eta\tau_s. \tag{28}$$

For example half the minimum of 1 and $(C - e^{cd})/(E_s - e^{cd})$ works. Then $R\tau - \delta_0 \in \mathcal{A}_R$ for every $R \ge 1$. Its positive square integral dominates η times that of τ_s. Hence $K_{\lfloor N/2\rfloor} \le \eta R$ excludes a negative degree-N square. By (23), its first negative degree is bounded below by a positive fixed multiple of $R^{1/(2a)}$.

Given $\beta < \pi/(2cd)$, choose fixed $s \in (\max\{0, 2d\beta\}, \pi/c)$. Then $1/(2a) = s/(2d) > \beta$, proving (4). All s, a, η, τ are chosen before R grows. Upper and lower bounds now give (5).

5 Several negative atoms and a collision budget

For a real polynomial space, its negative index is the maximal dimension of a subspace on which the square form is strictly negative. Equivalently it is the number of strictly negative eigenvalues of its moment matrix. Zero and negative eigenvalues are distinguished.

Theorem 5.1. *Fix $k \geq 1$, $B \geq 0$, $c, d, C > 0$, and $0 < w_- \leq w_+$. Let $R \geq 1$ and, if $k \geq 2$, $0 < \Delta \leq 1$. Suppose distinct $b_i \in [-B, B]$ satisfy $|b_i - b_j| \geq \Delta$, and*

$$\mu = -\sum_{i=1}^{k} w_i \delta_{b_i} + \nu, \quad w_- \leq w_i \leq w_+, \quad \operatorname{supp} \nu \subset \bigcap_i \{|x - b_i| \geq d\}, \quad \int e^{c|x|}\, d|\mu| \leq J_0 = C(1+R). \tag{29}$$

There is $A = A(k, B, c, d, C, w_-, w_+)$ such that the original degree-N form has at least k negative directions whenever

$$N \geq A R^{\alpha} \Delta^{-2(k-1)\alpha} \log \frac{R+2}{\Delta}, \qquad \alpha = \frac{\pi}{2cd}. \tag{30}$$

If ν is positive, the index is exactly k. A rational basis for a strictly negative k-dimensional polynomial subspace can be constructed without evaluating original moments, for certified computable inputs. For $k = 1$ set $\Delta = 1$ and omit the separation condition.

This is a sufficient joint contrast-and-collision bound.

Proof. Write $m = k - 1$, and use the Lagrange polynomials

$$L_j(x) = \prod_{i \neq j} \frac{x - b_i}{b_j - b_i}.$$

Their coefficient sums are at most

$$A_L = (1+B)^m \Delta^{-m}. \tag{31}$$

The main point is to pay these unbounded multipliers at the critical scale rather than sacrificing the rate c. In (15) the choice $\rho_n = n/(n+1)$ retains an exponential margin

$$\frac{\pi/2 - 2\arctan \rho_n}{h} = \frac{1}{h} \int_{\rho_n}^{1} \frac{2}{1+t^2}\, dt \geq \frac{1}{h(n+1)}.$$

For any degree-m polynomial L with coefficient sum at most A_L, put

$$U_m(h) = \max\left(1, \max_{1 \leq j \leq m} (jh/e)^j\right),$$

omitting the second maximum if $m = 0$. The inequality $\sup_{t \geq 0} t^j e^{-\delta t} = (j/(e\delta))^j$ gives

$$|L(x) q_n(x)| \leq e A_L U_m(h) (n+1)^{m+1/2} e^{c|x|/2}. \tag{32}$$

For $n = 0$ use $q_0 = 1$ and the stronger margin $c/2$; $2/c < h$ pays the same constant.

If f has Fourier variation at most one on $[-T, T]$, the polynomial $L \sum_{n=0}^{n_0} a_n q_n$ therefore approximates Lf in $L^2(|\mu|)$ with error at most

$$e A_L U_m(h) \sqrt{J_0}\, p! (n_0 + 2)^p \frac{r^{n_0+1}}{(1-r)^{p+1}}, \qquad p = m + 1. \tag{33}$$

Indeed $\sum_{n>n_0}(n+1)^p r^n$ is bounded by $(n_0+2)^p r^{n_0+1}\sum_{s\ge 0}(s+1)^p r^s$, and $(s+1)^p \le p!\binom{s+p}{p}$. Absolute weighted convergence justifies integration; the original measure need not have a density.

Put $V_0(c)=1$ and $V_q(c)=2^q[1+(q/(ce))^q]$ for $q\ge 1$. Then $\sup_{t\ge 0}(1+t)^q e^{-ct}\le V_q(c)$, so

$$\sum_{j=1}^{k}\sup_x e^{-c|x|}L_j(x)^2 \le H_0 := kA_L^2 V_{2m}(c). \tag{34}$$

Choose rational T at least

$$\max\Big(1, \frac{1}{d}\operatorname{arcosh}\max\{1, 2\sqrt{J_0H_0/w_-}\}\Big) \tag{35}$$

and at most one above this value. Define $f_j(x)=L_j(x)s_T(x-b_j)$. Its atomic evaluation matrix is exactly the identity. For every $a\in\mathbb{R}^k$, all remainder cross terms are paid by Cauchy–Schwarz and (34):

$$\int\Big(\sum_j a_j f_j\Big)^2 d\mu \le [-w_- + \operatorname{sech}^2(dT)J_0H_0]\|a\|^2 \le -\tfrac{3}{4}w_-\|a\|^2, \tag{36}$$

$$\Big\|\sum_j a_j f_j\Big\|_{L^2(|\mu|)} \le C_0\|a\|, \qquad C_0^2 = w_+ + w_-/4. \tag{37}$$

For a signed remainder these estimates use $|\nu|$, which is legitimate: the support gap makes $|\mu| = \sum_i w_i\delta_{b_i} + |\nu|$.

Let

$$\eta = \min\{\sqrt{w_-}/(4\sqrt{k}),\ w_-/(16C_0\sqrt{k})\}.$$

Apply (33) to each shifted wave, using a common n_0 so that the displayed error is below $\eta/2$. Translation preserves Fourier variation one. After multiplying by L_j, round the real monomial coefficients to rationals with error below $\eta/2$ using Lemma 3.2. This step also permits irrational atom positions; the rounded polynomials need not interpolate exactly. Their full errors are included in the following estimate. For the resulting P_j, the norm error of $\sum a_jP_j$ is less than $\sqrt{k}\eta\|a\|$, so its quadratic-form error is less than

$$\sqrt{k}\eta(2C_0+\sqrt{k}\eta)\|a\|^2 \le \tfrac{3}{16}w_-\|a\|^2.$$

Equation (36) implies strict negativity on the entire span. In particular the P_j are independent. If $\nu\ge 0$, any $(k+1)$-dimensional polynomial subspace contains a nonzero polynomial vanishing at all b_i, whose square integral is nonnegative. The negative index is then at most k, proving exactness. This last upper bound is not available for a signed remainder.

It remains to bound the degree uniformly in collisions. Equations (31)–(35) give

$$e^{hT} \le K_1[1+J_0H_0/w_-]^\alpha \le K_2R^\alpha\Delta^{-2m\alpha}, \qquad (1-r)^{-1}\le e^{hT}.$$

Put $S=(R+2)/\Delta\ge 3$, $\epsilon=1-r$, and choose $n_0=\lceil\epsilon^{-1}[L\log S+L_0]\rceil$. Then $r^{n_0+1}\le e^{-L_0}S^{-L}$, whereas $A_L\le K_3S^m$, $\sqrt{J_0}\le K_4S^{1/2}$ and $\epsilon^{-1}\le K_5S^{\alpha(1+2m)}$. Also $(n_0+2)^p\le K_6(L,L_0)\epsilon^{-p}(\log S)^p$. Choosing

$$L > m+\tfrac{1}{2}+\alpha(1+2m)(2p+1)+p+1$$

makes (33) at most a fixed polynomial in L_0 times $e^{-L_0}S^{-1}$. A sufficiently large fixed L_0 makes it less than $\eta/2$ for all $R\ge 1$ and $0<\Delta\le 1$. Thus n_0+m is bounded by the right side of (30), after increasing A. Nestedness of polynomial spaces gives the claim for every larger N. □

The direct norm control is important for an index statement. On degree at most N, $-\delta_0$ has negative index one, while convolution with any nondegenerate positive Gaussian gives a negative Gaussian measure whose index is $N+1$. The valid finite identity

$$e^{\rho D^2/2}(P^2) = \sum_{j=0}^{\deg P} \frac{\rho^j}{j!}\big[(e^{\rho D^2/2}P)^{(j)}\big]^2$$

returns at least one negative square from a negative smoothed integral, but does not preserve a whole negative subspace. Direct simultaneous approximation avoids that inference. The collision bound is stated for $\Delta > 0$, with distinct atoms.

6 An exact infinite-support example

The half-parameter case makes the threshold elementary. Let $U > 0$ have density

$$f(u) = \frac{2\pi u \sinh(\pi u)}{\cosh(\pi u)^2}, \tag{38}$$

and choose an independent uniform sign. For fixed $s > 0$ let $X = \pm s\sqrt{U^2 + 1/4}$ and let τ be its law. Integration by parts gives $\int f = 2\int_0^\infty \operatorname{sech}(\pi u)\,du = 1$. It has gap $s/2$ and finite exponential moments at every rate $c < \pi/s$. For instance

$$\mathbb{E}e^{c|X|} \le \frac{4\pi e^{cs/2}}{(\pi - cs)^2}. \tag{39}$$

This follows from $\sqrt{u^2+1/4} \le u + 1/2$ and $\sinh(\pi u)/\cosh(\pi u)^2 \le 2e^{-\pi u}$.

Proposition 6.1. *For $\mu_R = R\tau - \delta_0$ and every $R > 0$,*

$$n(\mu_R) = 2\lfloor R \rfloor. \tag{40}$$

At integer $R = m \ge 1$ there is a null polynomial of degree $2(m-1)$, before the first negative degree $2m$.

Proof. In (23), $a = 1/2$ makes every summand equal to one, so $K_k = k+1$. The parity argument above shows that all degree-N squares are nonnegative exactly when $\lfloor N/2 \rfloor + 1 \le R$. This proves (40), including $0 < R < 1$.

More explicitly, put $v = x^2/s^2$ and define

$$e_n(v) = \sum_{j=0}^{n} \frac{(-n)_j}{(j!)^3} \prod_{\ell=0}^{j-1} [v + \ell(\ell+1)]. \tag{41}$$

These are the orthonormal continuous dual Hahn polynomials after division by $(n!)^2$, and $e_n(0) = 1$. The normalized kernel

$$P_k(x) = \frac{1}{k+1} \sum_{j=0}^{k} e_j(x^2/s^2) \tag{42}$$

satisfies $P_k(0) = 1$ and

$$\int P_k^2\, d\mu_R = \frac{R}{k+1} - 1. \tag{43}$$

For integer $R = m$, choose $k = m$ for value $-1/(m+1)$ and $k = m-1$ for the null value. The degree convention and strict inequality matter at these integer thresholds. □

The even moments for $s = 1$ begin $1, 1, 2, 8, 56, 608$. They are the classical median Genocchi moments [6]. They can be reconstructed without quadrature from the monic polynomials $Q_n(v) = (-1)^n (n!)^2 e_n(v)$:

$$vQ_n = Q_{n+1} + (2n^2 + 2n + 1)Q_n + n^4 Q_{n-1}, \qquad Q_0 = 1. \tag{44}$$

Their norms are $(n!)^4$. Orthogonality to Q_0 is a triangular recurrence for successive moments. This is the alternative reconstruction used in the independent checker.

The same example is detected by the bandlimited square s_T^2 once $R\,\mathrm{sech}^2(Ts/2) < 1$, using $T = O_s(\log R)$. Thus logarithmic bandwidth and linear ordinary polynomial degree coexist on the same fixed infinite-support background. These are distinct test resources, not a comparison of algorithmic runtime.

7 Reproduction, scope, and remaining questions

The supplementary package includes the editable source, exact example data, two standard-library Python checkers, reader instructions, citation metadata and a checksum manifest. An additional finite cardinal checker verifies whole mixed Gram and finite-support index controls. The first example checker reconstructs (41); the second reconstructs (44). They verify exact finite Gram identities, witness square integrals and null thresholds against the original moments. Both can be run normally and with Python's `-O` option. Their finite controls corroborate the example; the analytic proofs above establish the all-degree example and general minimax theorem. A finite control does not verify the continuum asymptotic theorem.

The single-atom upper proof constructs one rational polynomial directly from certified computable data and the known compact Fourier representation. It requires no original-moment evaluation. The multi-atom theorem likewise constructs a whole rational negative subspace; its exact-index conclusion requires a positive remainder. The lower theorem is worst-case over (2), not a lower bound for every measure with those data. The condition $C > e^{cd}$ and the order of choosing parameters are retained in the lower proof.

Different support assumptions, stronger tail bounds, or additional structural information can change degree costs. Applying the theorem to a spectral problem requires an actual signed deficit, a certified gap and the full exponential variation bound. Those hypotheses cannot be inferred from positivity of a related trace or a finite list of moments. The present result settles the declared certificate-degree problem and leaves such application-specific sign and representation questions separate.

Assistance disclosure

AI tools assisted mathematical development, drafting, literature comparison and independent internal checks. The author is responsible for the statements and presentation.

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