Self-concordance and allocation for spatial power sums

Mathematical supplement to
Dimension Thresholds for Concavity of Normalized Mixed Moments

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Abstract

We study the negative logarithm of one-sided product-power smoothing of arbitrary coordinatewise decreasing weights. For continuous locally integrable inputs and decreasing lattice arrays in dimension d, a common real power $p \geq 16d$ makes this particular potential a standard self-concordant barrier, with parameter at most $2pd$. If the lattice support has at most r positive coordinates at each state, the sufficient power improves to $p \geq 32r$, independently of the magnitudes of the occupancies and of the ambient dimension. The proof uses residual Pareto tails, an exact third-cumulant identity and a sparse L^3 operator bound; it does not assume independence or log-concavity of the input. A decreasing binary mixture shows that powers must grow linearly in r for a uniform convexity theorem over this input class. For log-concave binary generating polynomials with deletion bounds, an absolute scaling retains every fixed power $p \geq 4$, independently of rank; matroid independent-set polynomials supply this premise classically. For coefficient deletion ratios up to $M \geq 1$, powers of order $(1 + \log M)^2$ give an absolute scale; a rank-one obstruction shows squared-logarithmic order necessity for uniform convexity in the stated asymptotic regime. We also give an exact quartic scalar example and an explicit budget-allocation consequence: for a listed m-state input, a balanced starting point has objective gap at most $\log m$, permitting classical damped Newton minimization. For quartic matroid utilities, a balanced KKT box and clipped sampling also give a polynomial expected-cost projected-gradient guarantee using only an independence oracle. Both optimization methods are classical.

1 Introduction and statements

Two structural regimes preserve different resources. Large powers control arbitrary decreasing inputs, with a sparsity improvement for discrete supports. For structured binary generating polynomials, a positive absolute scaling instead retains each fixed power at least four. Scaling the potential preserves its minimizer; changing the smoothing power changes the utility.

The allocation conclusions also use different access models. A listed finite-state input permits exact moment evaluation and classical damped Newton steps from a balanced start. The quartic matroid utility is handled through a balanced KKT box, clipped moment estimates and projected gradient using an independence oracle. The latter theorem has its own curvature and moment proof and does not require a self-concordance premise. Expected oracle cost and success probability are stated separately.

The companion manuscript [1] proves log-concavity of spatial product-power smoothing for decreasing inputs, using conditional residual moments and a covariance estimate. This supplement controls third derivatives of the same raw potential and studies its self-concordance. The companion's sharp leading log-concavity thresholds are distinct from the sufficient power bounds below.

An arbitrary convex negative logarithm need not have the self-concordance needed for the usual affine-invariant Newton analysis. Indeed, the elementary quadratic example in Section 6 fails every fixed positive self-concordance scaling.

For a C^3 convex function g on an open convex domain, standard self-concordance means

$$|D^3 g(A)[h,h,h]| \le 2\big(D^2 g(A)[h,h]\big)^{3/2}. \tag{1}$$

A barrier additionally tends to $+\infty$ at every finite boundary point. Its parameter is at most ν if

$$|Dg(A)[h]| \le \sqrt{\nu D^2 g(A)[h,h]}. \tag{2}$$

These definitions, the scaling and affine restriction rules, and the Newton results used below are classical [2, 3].

We consider either

$$F_p(A) = \int_{[0,\infty)^d} \mu(I) \prod_{i=1}^{d} (A_i - I_i)_+^p \, dI \tag{3}$$

or

$$F_p(A) = \sum_{n \in \mathbb{Z}^d_{\ge 0}} \mu(n) \prod_{i=1}^{d} (A_i - n_i)_+^p, \qquad A \in (0,\infty)^d. \tag{4}$$

For (3), assume that μ is nonnegative, coordinatewise nonincreasing, integrable on every finite box, and has positive integral on some box. For (4), assume that μ is finite-valued, nonzero, nonnegative and coordinatewise nonincreasing. The total continuous mass and the global array support may be infinite. All powers in this supplement are common real powers.

Theorem 1.1 (Dimension bound)**.** *For either input class, if $p \ge 16d$, then $g = -\log F_p$ is a standard self-concordant barrier on the positive orthant. Its parameter is at most*

$$\nu_d = \frac{d(p+6/5)^2}{p - 16d/5} \le 2pd.$$

Theorem 1.2 (Sparse support bound)**.** *Suppose the input in (4) is supported on states with at most r positive coordinates, where $1 \le r \le d$. If $p \ge 32r$, then $g = -\log F_p$ is a standard self-concordant barrier with parameter at most*

$$\nu_r = \frac{d(p+6/5)^2}{p - 16r/5} \le 2pd.$$

There is no bound on the sizes of the nonzero integer coordinates or on the number of states. For $r = 0$, the function is the separable barrier $-p\sum_i \log A_i - \log \mu(0)$, with parameter pd for $p \ge 1$.

For weighted independent sets of a rank-r matroid, with coordinate activities in $[0,1]$, the input satisfies Theorem 1.2. The sparse theorem itself requires no matroid hypothesis or negative association. The sparse theorem's parameter still depends on d: only the required *power* is controlled by r. Theorem 7.1 later gives an alternative for log-concave binary generating polynomials: scaling by an absolute constant retains any fixed power $p \ge 4$, including the quartic utility. A classical orthant logarithmic barrier has parameter d. They control the particular raw smoothing potential, whose optimization objective one may wish to retain.

Classical context

The entropic barrier of Bubeck and Eldan [4] is a Fenchel dual of a log-Laplace transform; its self-concordance proof uses third moments of log-concave measures. The universal barrier studied by Lee and Yue [5] is a log polar-volume potential, with sharp moment inequalities for marginals of convex bodies. Neither identification is assumed for (3)–(4): their inputs can be non-log-concave and atomic. Generalized self-concordance [6] provides broader third-order hypotheses and Newton analysis, not an implication from a Hessian sign alone. Infimal-convolution smoothing, as in [7], is a different operation from the integral or sum here. Hyperbolic-polynomial barriers [10] also have different structural hypotheses; we do not assume real stability or hyperbolicity of a power sum. Our contribution is the residual-moment verification for this specific smoothing class, and its sparse-support refinement.

2 Residual tails and exact derivatives

Every active box in (3) is bounded, and every active sum in (4) is finite. Monotonicity and nonzero input ensure $F_p(A) > 0$ throughout the orthant. In the continuous case a positive lower box lies below every positive A; in the discrete case $\mu(0) > 0$. For $p > 3$ the kernel x_+^p is C^3, and its first three derivatives are bounded on bounded intervals. Differentiation of the original fixed-input formulas on compact positive A sets is therefore justified by local integrability or local finiteness, including at activation walls.

At a fixed A, normalize the active integral or sum to a probability law, put $z_i = A_i - I_i$ or $A_i - n_i$, and define

$$Z_i = p(A_i/z_i - 1) \geq 0. \tag{5}$$

Zero residuals carry zero tilted mass. The residual representation is used only at the fixed point; its A-dependent weight is not differentiated.

Lemma 2.1 (Residual moments)**.** *For either input class and $s \geq 0$,*

$$\Pr\{Z_i > s\} \leq (1 + s/p)^{-p}. \tag{6}$$

For $p \geq 8$ this implies

$$\mathbb{E}Z_i \leq M_1 = 6/5, \qquad \mathbb{E}Z_i^2 \leq M_2 = 16/5, \qquad \mathbb{E}Z_i^3 \leq M_3 = 15. \tag{7}$$

The coordinates may be dependent.

Proof. Condition on all other coordinates. In the continuous case the residual density is proportional to $z^p w(z)$ on $(0, A_i)$, with w nondecreasing. For $0 < u < A_i$, the integrals below and above u are bounded respectively by

$$\frac{w(u)u^{p+1}}{p+1}, \qquad \frac{w(u)(A_i^{p+1} - u^{p+1})}{p+1}.$$

Their ratio gives $\Pr\{z < u\} \leq (u/A_i)^{p+1}$; truncation or monotone approximation covers zero and unbounded endpoint weights.

In the discrete case the positive residuals are an arithmetic list $t, t+1, \ldots, A_i$ and their weights are nondecreasing. Match the k entries below u in order to the largest k entries. The weight does not decrease and the residual ratio is at least A_i/u. Thus lower-block mass is at most $(u/A_i)^p$ times upper-block mass, which is at most the total mass. The argument only compares sums and

remains valid if the blocks overlap. Setting $u = A_i/(1+s/p)$ and integrating the conditional bounds proves (6).

Tail integration gives, for $k = 1, 2, 3$ and $p > k$,

$$\mathbb{E}Z_i^k \le \frac{k!p^k}{(p-1)\cdots(p-k)}.$$

The right sides decrease with p, and at $p = 8$ are respectively 8/7, 64/21 and 512/35. These are bounded by (7). □

Fix a direction h and write $a_i = h_i/A_i$,

$$X = \sum_i a_i Z_i, \qquad Y = \sum_i a_i^2(2Z_i + Z_i^2/p). \tag{8}$$

For the logarithm ℓ of one active product kernel,

$$\ell' = p\sum_i a_i + X, \qquad \ell'' = -p\,\|a\|_2^2 - Y,$$
$$\ell''' = 2\sum_i a_i^3(p + 3Z_i + 3Z_i^2/p + Z_i^3/p^2).$$

Partition differentiation yields the exact identities

$$g'' = p\,\|a\|_2^2 + \mathbb{E}Y - \operatorname{Var} X, \tag{9}$$
$$g''' = -\mathbb{E}\ell''' + 3\operatorname{Cov}(X,Y) - \mathbb{E}(X - \mathbb{E}X)^3. \tag{10}$$

These follow by differentiating the original partition three times and collecting the first three cumulants. The moment bounds justify the identities at walls as well. In particular, the covariance and third cumulant in (10) must not be discarded.

3 Proof of the dimension bound

By Cauchy–Schwarz,

$$\operatorname{Var} X \le M_2\,\|a\|_1^2 \le \frac{16d}{5}\,\|a\|_2^2, \qquad g'' \ge (p - 16d/5)\,\|a\|_2^2. \tag{11}$$

Hölder's inequality gives $\mathbb{E}(Z_iZ_j^2) \le M_3$ and $\mathbb{E}Z_i\mathbb{E}Z_j^2 \le M_1M_2$, while $|\operatorname{Cov}(Z_i, Z_j)| \le M_2$. Therefore

$$\begin{aligned} 3|\operatorname{Cov}(X,Y)| &\le 3\left(2M_2 + \frac{M_3 + M_1M_2}{p}\right)\|a\|_1\,\|a\|_2^2 \\ &< 27\sqrt{d}\,\|a\|_2^3. \end{aligned} \tag{12}$$

Indeed at $p = 8$ the bracket with its factor 3 is $5253/200 < 27$. Since $|z - m|^3 \le z^3 + m^3$ for nonnegative z, m,

$$\mathbb{E}|Z_i - \mathbb{E}Z_i|^3 \le M_3 + M_1^3 = 2091/125 < 17.$$

Minkowski's inequality, with arbitrary dependence, gives

$$|\mathbb{E}(X - \mathbb{E}X)^3| \le 17\,\|a\|_1^3 \le 17d^{3/2}\,\|a\|_2^3. \tag{13}$$

Also

$$|\mathbb{E}\ell'''| \le \left(2p + 6M_1 + \frac{6M_2}{p} + \frac{2M_3}{p^2}\right) \|a\|_2^3 < (2p+11)\,\|a\|_2^3 . \tag{14}$$

Combining (10)–(14) and $d \ge 1$ yields

$$|g'''| \le (2p + 55d^{3/2})\,\|a\|_2^3 .$$

For $p \ge 16d$, division by the curvature lower bound gives

$$\begin{aligned}\frac{|g'''|}{(g'')^{3/2}} &\le \frac{2/\sqrt{p} + 55(d/p)^{3/2}}{(1-16d/(5p))^{3/2}} \\ &\le \frac{1/2 + 55/64}{(4/5)^{3/2}} = \frac{435\sqrt{5}}{512} < 2.\end{aligned}$$

The last comparison is the integer inequality $946125 < 1048576$. This proves (1) and strict convexity.

The first derivative obeys

$$|g'[h]| \le (p + M_1)\,\|a\|_1 \le (p + 6/5)\sqrt{d}\,\|a\|_2 ,$$

which with (11) gives ν_d. Since $p - 16d/5 \ge 4p/5$ and $(p+6/5)/p \le 43/40$, $\nu_d \le 1849pd/1280 < 2pd$. At any finite orthant boundary point one coordinate A_i tends to zero and the others stay bounded. Local integrability bounds $F_p(A)$ by a constant times A_i^p in (3); a fixed finite active box gives the same bound for (4). Thus $F_p \to 0$ and $g \to +\infty$, completing Theorem 1.1.

4 Sparse support and the rank bound

Lemma 4.1 (Sparse L^3 operator)**.** *Suppose $Z_i \ge 0$, $\mathbb{E}Z_i^3 \le M$, and at most r of the Z_i are nonzero on each outcome. Then, for every real vector a,*

$$\left\|\sum_i a_i Z_i\right\|_{L^3} \le M^{1/3}\sqrt{r}\,\|a\|_2 .$$

Proof. Take b with $\|b\|_{L^{3/2}} \le 1$ and put $B_i = \mathbb{E}[|b|^{3/2}\mathbf{1}_{Z_i>0}]$. Then $0 \le B_i \le 1$ and $\sum_i B_i \le r$. Hölder gives $|\mathbb{E}Z_i b| \le M^{1/3}B_i^{2/3}$, so

$$\left|\mathbb{E}b\sum_i a_i Z_i\right| \le M^{1/3}\,\|a\|_2 \left(\sum_i B_i^{4/3}\right)^{1/2} \le M^{1/3}\sqrt{r}\,\|a\|_2 .$$

The L^3 dual supremum proves the assertion. □

For the sparse lattice law, $Z_i > 0$ precisely when $n_i > 0$ on its tilted support. Pointwise Cauchy–Schwarz gives $X^2 \le r\sum_i a_i^2 Z_i^2$, hence

$$g'' \ge (p - 16r/5)\,\|a\|_2^2 . \tag{15}$$

Lemma 4.1 gives $\|X\|_{L^3} \le 15^{1/3}\sqrt{r}\,\|a\|_2$, so centering costs at most a factor two and

$$|\mathbb{E}(X - \mathbb{E}X)^3| \le 120r^{3/2}\,\|a\|_2^3 .$$

Minkowski and the monotonicity of L^q norms give

$$\|Y\|_{L^{3/2}} \le (2 \cdot 15^{1/3} + 15^{2/3}/p)\, \|a\|_2^2 .$$

Consequently

$$\begin{aligned} 3|\operatorname{Cov}(X,Y)| &\le 6\, \|X\|_{L^3}\, \|Y\|_{L^{3/2}} \\ &\le (12 \cdot 15^{2/3} + 90/p)\sqrt{r}\, \|a\|_2^3 < 85\sqrt{r}\, \|a\|_2^3 . \end{aligned}$$

Here $p \ge 8$ and $15^{2/3} < 49/8$; the latter follows by cubing, $115200 < 117649$. With (14), for $r \ge 1$,

$$|g'''| \le (2p + 216 r^{3/2})\, \|a\|_2^3 .$$

For $p \ge 32r$ its ratio to (15) to the power $3/2$ is at most

$$\frac{2/\sqrt{32} + 216/32^{3/2}}{(9/10)^{3/2}} = \frac{175\sqrt{5}}{216} < 2,$$

since $153125 < 186624$. The gradient estimate is unchanged, giving ν_r. Since the denominator is at least $9p/10$ and $p \ge 32$, $\nu_r \le pd(83/80)^2/(9/10) < 2pd$. Boundary divergence and C^3 regularity were already established. The case $r = 0$ is immediate from $\mu(0) \prod_i A_i^p$. This proves Theorem 1.2.

The sparsity hypothesis counts positive coordinates, not their magnitudes. It therefore includes unbounded occupancies along a sparse collection of axes. A nonzero ordinary Lebesgue density cannot be supported on a union of lower-dimensional coordinate subspaces.

5 A linear-order obstruction for arbitrary decreasing inputs

The companion [1] already proves dimension-linear lower obstructions, including unweighted hereditary binary supports, and a sharp leading dimension coefficient for its broader spatial classes. The following mixture is a convenient explicit smooth-cell witness for the order comparison here.

Proposition 5.1. *For every real $p \ge 1$ and every integer $d \ge 64(p+1)$, there is a finite decreasing binary input for which $-\log F_p$ is not convex at an interior point. Thus a uniform convexity, and hence self-concordance, theorem for arbitrary decreasing sparse binary inputs requires powers of order their maximal active-coordinate cardinality.*

Proof. Set $\mu(0) = 1$, $\mu(n) = q$ at every other binary state, and $\mu = 0$ outside $\{0,1\}^d$, with $0 < q < 1$. This is coordinatewise decreasing and has exact partition

$$F_p(A) = (1-q) \prod_i A_i^p + q \prod_i (A_i^p + (A_i - 1)_+^p).$$

Take $A_i = p + 1$ and put

$$b = \left(\frac{p}{p+1}\right)^p, \qquad t = \frac{b}{1+b}, \qquad q = \frac{1}{1 + (1+b)^d}.$$

The two components have equal tilted mass. The first component has zero occupancy; the other has independent Bernoulli(t) occupancies. The residual Z_i is now the occupancy indicator. In direction $h_i = A_i$, if K is total occupancy,

$$\mathbb{E}K = dt/2, \quad \operatorname{Var} K = dt(1-t)/2 + d^2t^2/4, \quad Y = (2 + 1/p)K.$$

Equation (9) gives

$$g'' = d\left[p + \frac{t}{2}(1 + 1/p + t)\right] - \frac{d^2t^2}{4}.$$

Since $\log(1+1/p) < 1/p$, we have $b > e^{-1} > 1/3$, and therefore $1/4 < t < 1/2$. Also $\frac{t}{2}(1+1/p+t) < 1$ for $p \geq 1$. Thus

$$g'' < d(p+1) - d^2/64 \leq 0.$$

The curvature is strictly negative even at the displayed endpoint. For $p \leq 3$ this is still a legitimate smooth-cell computation, since all the selected residuals are strictly positive. □

For an explicit rational witness, take $p = 4$, $A_i = 2$ and $q = 16^d/(16^d+17^d)$. The two components again have equal tilted mass, and the product component has Bernoulli(1/17) occupancies with $Z_i = 4n_i$. In direction $h_i = 2$,

$$\mathbb{E}Y = 6d/17, \qquad \operatorname{Var} X = (128d + 4d^2)/289, \qquad g'' = (1130d - 4d^2)/289.$$

At $d = 283$ the last quantity is exactly $-566/289$. The verifier checks this by differentiating the literal two-product expression, without enumerating 2^{283} states. Padding with coordinates forced to zero occupancy preserves the example in its original directions, so the obstruction is in active-coordinate cardinality r, not necessarily ambient dimension. This weight is generally not a product-activity matroid law; no such obstruction for that narrower class is inferred.

6 The scalar fourth-power repair

The quadratic function $F(x) = x^2 + (x-1)_+^2$ gives a useful failed endpoint. On $x > 1$ put $P = 2x^2 - 2x + 1$. Then

$$(-\log F)'' = \frac{8x(x-1)}{P^2}, \qquad (-\log F)''' = \frac{8(2x-1)(-2x^2+2x+1)}{P^3}.$$

As $x \downarrow 1$, curvature tends to zero while the third derivative tends to 8. Hence no fixed positive scaling is self-concordant, even inside that smooth chamber.

Proposition 6.1. *For every $0 \leq q \leq 1$, $-\log[x^4 + q(x-1)_+^4]$ is standard self-concordant on $(0, \infty)$. For every real $p \geq 4$, the function $-100\log[x^p + q(x-1)_+^p]$ is standard self-concordant, uniformly in q.*

Proof. For the quartic active chamber let $P = x^4 + q(x-1)^4$ and define

$$A = P'^2 - PP'', \qquad C = -P'''P^2 + 3PP'P'' - 2P'^3.$$

The second and third derivatives of $-\log P$ are A/P^2 and C/P^3. The substitution $x = 1 + v$, $q = w/(1+w)$, $v, w \geq 0$, clears denominators $(1+w)^2$ and $(1+w)^6$ in A and $4A^3 - C^2$. Their numerator polynomials have respectively 21 and 133 nonzero strictly positive integer coefficients, with positive constant terms. These exact polynomial identities and every coefficient are supplied by the standalone verifier. At $q = 1$ the substituted polynomials in v likewise have strictly positive coefficients. This proves the quartic inequality for all active points and both activity endpoints. On $x \leq 1$ the potential is $-4\log x$, and C^3 regularity glues the chambers.

For real $p \geq 4$, put $y = (x-1)/x$, $a = (1-y)/y$ on $x > 1$ and let $Z = pa$ with probability $w = qy^p/(1+qy^p)$ and zero otherwise. Then

$$\frac{x^2 g''}{p} = 1 + 2wa - (p-1)wa^2 + pw^2a^2.$$

Since $w \le y^p$,

$$(p-1)wa^2 \le \frac{4(p-1)}{p^2}(1-2/p)^{p-2} \le \frac{p-1}{p^2} \le \frac{3}{16}.$$

Indeed $(1-2/p)^{p-2}$ decreases for $p \ge 4$ and is $1/4$ at four; the derivative of its logarithm is $\log(1-2/p)+2/p \le 0$. Thus $g'' \ge 13p/(16x^2)$. For $k=1,2,3$,

$$\mathbb{E}Z^k \le p^k \max_{0\le y\le 1} y^{p-k}(1-y)^k \le k^k.$$

The scalar form of (10) bounds $x^3|g'''|$ by

$$2p\mathbb{E}(1+Z/p)^3 + 3|\operatorname{Cov}(Z, 2Z+Z^2/p)| + |\mathbb{E}(Z-\mathbb{E}Z)^3| \le 6p + 52 + 81/p \le 25p.$$

Here $\mathbb{E}Z \le 1$, $\mathbb{E}Z^2 \le 4$, $\mathbb{E}Z^3 \le 27$; the three terms are at most $6p$, $24+81/p$ and 28. Consequently $|g'''| \le 20(g'')^{3/2}$ for $p \ge 4$, as is checked by squaring at the worst bound $p=4$. Multiplication by 100 divides the ratio by ten. The inactive chamber and C^3 gluing are unchanged. □

7 Fixed powers with a rank-independent scale

The preceding sparse theorem changes the smoothing power. Under an additional binary log-concavity premise, one absolute scaling keeps that power fixed and preserves the original utility, independently of rank.

Theorem 7.1. *Let $Q(z) = \sum_{S\subseteq[d]} a_S \prod_{i\in S} z_i$ be multiaffine with nonnegative coefficients, $a_\varnothing > 0$, and $a_{S\cup\{i\}} \le a_S$ for $i \notin S$. Suppose $\log Q$ is concave on the whole nonnegative orthant. For every common real $p \ge 4$, set*

$$F_p(A) = \prod_i A_i^p\, Q(((1-1/A_i)_+^p)_i), \qquad g = -\log F_p.$$

Then $C_* g$, for the absolute constant $C_* = 2560000$, is a standard self-concordant barrier with parameter at most $2C_* pd$. The scale is independent of dimension, support rank and coefficients. If Q is constant, the unscaled separable barrier already works for $p \ge 1$.

Proof. Write $L = \log Q$ and $\phi_i(A_i) = (1-1/A_i)_+^p$. Decompose $Q = R_i + z_i Q_i$. Coefficient deletion gives $Q_i \le R_i$, hence $0 \le L_i \le 1/(1+z_i) \le 1$. Also $\nabla^2 L \preceq 0$. The chain rule yields

$$g''[h,h] \ge \sum_i \left(\frac{p}{A_i^2} - \phi_i'' L_i\right) h_i^2.$$

For an active coordinate let $y = 1-1/A_i$. The positive part of

$$\frac{A_i^2 \phi_i''}{p} = y^{p-2}\big((p-1)(1-y)^2 - 2y(1-y)\big)$$

is at most $3/16$ by the maximization in Proposition 6.1. Inactive coordinates contribute p/A_i^2. Consequently

$$g''[h,h] \ge \frac{13p}{16}\,\|h/A\|_2^2\,. \tag{16}$$

At the fixed point use the binary law proportional to $a_S \prod_{i\in S}\phi_i$. Write $I_i = \mathbf{1}_{i\in S}$ and, for active coordinates, $\alpha_i = p/(A_i-1)$ and $Z_i = \alpha_i I_i$; set $Z_i = 0$ on inactive coordinates. Deletion gives $\Pr(I_i=1) \le \phi_i = y_i^p$, hence $\mathbb{E}Z_i^j \le j^j$ for $j=1,2,3$. In (10) this implies

$$|\mathbb{E}\ell'''| \le 6p\,\|a\|_2^3\,, \qquad \|Y\|_{L^{3/2}} \le (6+9/p)\,\|a\|_2^2\,, \qquad a = h/A. \tag{17}$$

Indeed the first coefficient is bounded by $2p+6+24/p+54/p^2 \le 6p$ for $p \ge 4$. We next prove a rank-independent centered-score bound.

Joint inclusion from deletion. Given the complete configuration J of other indicators, a positive-probability conditional event has

$$\Pr(I_i = 1 \mid I_{-i} = \mathbf{1}_J) = \frac{a_{J\cup\{i\}}\phi_i}{a_J + a_{J\cup\{i\}}\phi_i} \le \frac{\phi_i}{1+\phi_i} \le \phi_i.$$

If $a_J = 0$, deletion makes the extended coefficient zero too, so that configuration is null. Iterated conditional expectation therefore gives

$$\Pr(T \subseteq S) \le \prod_{i\in T} \phi_i. \tag{18}$$

This is an external-field bound.

Low scores. The zero direction is trivial. Normalize $\|a\|_2 = 1$, ignore inactive coordinates, and put $\gamma_i = a_i\alpha_i$. Split the indices into $\mathcal{L} = \{i : |\gamma_i| \le 4\}$ and $\mathcal{H} = \{i : |\gamma_i| > 4\}$, at this fixed point and direction. Let $X_{\mathcal{L}} = \sum_{i\in\mathcal{L}} \gamma_i I_i$ and $m_i = \Pr(I_i = 1)$. Exponentially changing just these fields and using the tangent inequality for the concave function $\log Q$ gives, for either sign of t,

$$\log \mathbb{E}e^{t(X_{\mathcal{L}} - \mathbb{E}X_{\mathcal{L}})} \le \sum_{i\in\mathcal{L}} m_i(e^{t\gamma_i} - 1 - t\gamma_i).$$

The varied fields can exceed one; this is why whole-orthant concavity is assumed. Zero fields remain fixed. Since $\sum_i m_i\gamma_i^2 \le 4\sum_i a_i^2 = 4$ and $e^u - 1 - u \le 3u^2/2$ for $|u| \le 1$, taking $t = \pm 1/4$ bounds each logarithmic moment by $3/8$. Both exponential moments are below two, because $e^{3/8} \le (1 - 3/8)^{-1} < 2$. Now $|x|^3 \le 6 \cdot 4^3 e^{|x|/4}$ gives

$$\mathbb{E}|X_{\mathcal{L}} - \mathbb{E}X_{\mathcal{L}}|^3 < 1536 < 12^3. \tag{19}$$

Rare high scores. For $p \ge 4$,

$$\phi_i = (1 + \alpha_i/p)^{-p} \le (1 + \alpha_i/4)^{-4} \le 256\alpha_i^{-4},$$

since $p\log(1 + \alpha/p)$ increases with p. On $\mathcal{H}$, $\alpha_i|a_i| > 4$. Thus, for $j = 1, 2, 3$,

$$A_j := \sum_{i\in\mathcal{H}} \phi_i|\gamma_i|^j \le 256 \cdot 4^{j-4} \sum_i |a_i|^4 \le 4^j.$$

For $H = \sum_{i\in\mathcal{H}} |\gamma_i| I_i$, expansion of its cube and (18) give

$$\mathbb{E}H^3 \le A_3 + 3A_2A_1 + A_1^3 \le 64 + 3 \cdot 16 \cdot 4 + 4^3 = 320 < 7^3.$$

Writing $X_{\mathcal{H}} = \sum_{i\in\mathcal{H}} \gamma_i I_i$, centering therefore costs at most a factor two and $\|X_{\mathcal{H}} - \mathbb{E}X_{\mathcal{H}}\|_{L^3} < 14$. No fourth tilted moment or independence is used. Minkowski with (19), and then homogeneity, proves

$$\|X - \mathbb{E}X\|_{L^3} < 26\,\|a\|_2 \quad (a \ne 0). \tag{20}$$

The score split is not differentiated; it only bounds moments of the already evaluated directional derivatives.

Standard self-concordance. The centered cubic in (10) is at most $26^3 \|a\|_2^3 = 17576 \|a\|_2^3$. By Hölder and (17),

$$3|\operatorname{Cov}(X,Y)| = 3|\mathbb{E}[(X-\mathbb{E}X)Y]| \le 3 \cdot 26(6+9/p) \|a\|_2^3 < 644 \|a\|_2^3 .$$

Consequently $|g'''| \le (6p+18220) \|a\|_2^3$. Dividing by (16) to the power 3/2 yields

$$\frac{|g'''|}{(g'')^{3/2}} \le \frac{6p+18220}{(13p/16)^{3/2}} \le \frac{18244}{(13/4)^{3/2}} < 3200.$$

The bound decreases with $p \ge 4$, and the last squared comparison is $18244^2 \cdot 64 < 3200^2 \cdot 13^3$. Multiplication by $C_* = 1600^2$ divides this ratio by 1600, proving (1). The gradient bound $(p+1)\sqrt{d} \|a\|_2$ and (16) give parameter $16d(p+1)^2/(13p) < 2pd$ before scaling. Scaling multiplies it by C_*. Positivity, C^3 gluing and boundary divergence follow from the finite binary partition, including constant Q. □

For matroid independent sets with activities $\lambda_i \in [0,1]$, $a_S = \mathbf{1}_{S \text{ independent}} \prod_{i \in S} \lambda_i$ has the deletion property. Ordinary Q log-concavity is classical: Anari–Liu–Oveis Gharan–Vinzant [8, Theorem 4.1] prove complete log-concavity of

$$\sum_{S \text{ independent}} y^{d-|S|} \prod_{i \in S} z_i .$$

The homogenizing degree is the *ground-set size* d, not rank. Restricting to $y = 1$ and inserting nonnegative activities gives exactly the premise used here.

For positive support rank r, the earlier sparse estimate also gives $\|X\|_{L^3} \le 3\sqrt{r} \|a\|_2$, hence $|g'''| \le (6p + 365r^{3/2}) \|a\|_2^3$. With (16) its ratio is at most $70r^{3/2}$, so $1225r^3$ is another sufficient scale. Either sufficient scale may be chosen. It is numerically smaller than C_* for $r \le 12$; the new gain is uniformity in rank.

In Corollary 9.1, scaling by C_* preserves the fixed-p utility maximizer. Its starting gap is at most $C_* \log m$, so the classical Newton count is $O(1 + C_* \log m + \log(1 + \log(1/\epsilon)))$ at every $p \ge 4$ for this structured binary class. Since C_* is absolute, its hidden constant is independent of rank. The guarantee is still for an explicit state list, not an implicit matroid counting oracle.

8 Large activities and squared-logarithmic powers

The deletion constant one cannot be removed while retaining a fixed power. Increasing the common power gives a different sufficient theorem for the same smoothing construction.

Theorem 8.1. *Let $Q(z) = \sum_S a_S z^S$ be multiaffine with nonnegative coefficients, $a_\varnothing > 0$, and $\log Q$ concave on the whole nonnegative orthant. Suppose $a_{S+i} \le M a_S$ for $i \notin S$, where $1 \le M < \infty$. Put $\ell = \log M$ and $H = 1 + \ell$. For every common real power $p \ge 72H^2$, the function*

$$g(A) = -\log \left[\prod_i A_i^p Q(((1 - 1/A_i)_+^p)_i) \right]$$

has Cg a standard self-concordant barrier, with $C = 360000$ and parameter at most $3Cpd$. Weighted matroid independent-set polynomials with activities in $[0, M]$ satisfy these hypotheses by classical ordinary log-concavity [8] and heredity.

Proof. At fixed A, put $\phi_i = (1 - 1/A_i)_+^p$ and tilt S with weights $a_S \phi^S$. On active coordinates let $\alpha_i = p/(A_i - 1)$ and $Z_i = \alpha_i \mathbf{1}_{i \in S}$; elsewhere set $Z_i = 0$. Conditioning on all other indicators gives

$$\Pr(i \in S \mid S \setminus \{i\} = J) \le \frac{M\phi_i}{1 + M\phi_i} \le \beta_i := \min(1, M\phi_i).$$

If $a_J = 0$, the numerator also vanishes; null conditioning events cause no exception. Iteration gives $\Pr(T \subset S) \le \prod_{i \in T} \beta_i$. This is an external-field bound. Both Z_i and the artificial variable $\alpha_i B_i$ with $B_i \sim \text{Bernoulli}(\beta_i)$ have tail bounded by $\min(1, M(1 + t/p)^{-p})$, because $\phi_i = (1 + \alpha_i/p)^{-p}$. This bound is exactly the tail of $b + cY$, where $\Pr(Y > t) = (1 + t/p)^{-p}$, $c = M^{1/p}$ and $b = p(c - 1)$. Here $c < 3$, $b \le 3\ell$, since $\ell/p \le 1$. For $1 \le k \le 4$,

$$\mathbb{E}Y^k = \frac{k! p^k}{(p-1)\cdots(p-k)}.$$

These moments decrease with p; at $p = 8$ they are $8/7, 64/21, 512/35, 4096/35$, whose roots are bounded respectively by $4/3, 2, 8/3, 10/3$. Minkowski therefore gives, for both variables,

$$\|Z_i\|_1 \le 4H, \quad \|Z_i\|_2 \le 6H, \quad \|Z_i\|_3 \le 8H, \quad \|Z_i\|_4 \le 10H. \tag{21}$$

The artificial Bernoulli bound is needed for the joint estimates below.

For $u_i = h_i/A_i$, set $X = \sum_i u_i Z_i$ and $Y_0 = \sum_i u_i^2 (2Z_i + Z_i^2/p)$. The original partition identities of Section 2 apply. Write $B = \log Q$ and $\pi_i = \phi_i B_i = \Pr(i \in S)$. On an active coordinate,

$$A_i^2 \phi_i'' B_i = \pi_i[(p-1)\alpha_i^2/p - 2\alpha_i] \le \mathbb{E}Z_i^2.$$

The compensated chain rule and $\nabla^2 B \preceq 0$ imply

$$g''[h,h] \ge (p - 36H^2)\,\|u\|^2 \ge (p/2)\,\|u\|^2\,. \tag{22}$$

This uses the actual $\phi_i B_i$, not the generally false large-activity bound $B_i \le 1$. Inactive coordinates have separable curvature. Also

$$\begin{aligned} \|Y_0\|_{3/2} &\le (16H + 64H^2/p)\,\|u\|^2 \le 17H\,\|u\|^2\,, \\ |\mathbb{E}\ell'''_{\text{kernel}}| &\le [2p + 24H + 216H^2/p + 1024H^3/p^2]\,\|u\|^3 \le 3p\,\|u\|^3\,. \end{aligned}$$

For the last comparison the extra terms divided by p are at most $1/3 + 1/24 + 1024/72^3 < 1$.

Normalize $\|u\| = 1$ and put $c_i = u_i \alpha_i$. For the low set $|c_i| \le 12H$, concavity of $\log Q$ applied to fields $\phi_i e^{tc_i}$ yields, for either sign of t,

$$\log \mathbb{E}e^{t(X_{\text{lo}} - \mathbb{E}X_{\text{lo}})} \le \sum_{\text{lo}} \pi_i (e^{tc_i} - 1 - tc_i).$$

All fields remain in the whole-orthant concavity domain. Since $\sum_i \pi_i c_i^2 \le 36H^2$, at $t = \pm(12H)^{-1}$ this is at most $3/8$, using $e^v - 1 - v \le 3v^2/2$ for $|v| \le 1$. Both exponential moments are less than two. Hence $|z|^3 \le 6(12H)^3 e^{|z|/(12H)}$ gives

$$\|X_{\text{lo}} - \mathbb{E}X_{\text{lo}}\|_3^3 < 24(12H)^3 < 35^3 H^3.$$

For the high set $|c_i| > 12H$, the artificial fourth moment gives $\beta_i \alpha_i^4 \le (10H)^4$. Thus

$$A_k := \sum_{\text{hi}} \beta_i |c_i|^k \le (10H)^4 (12H)^{k-4} \quad (k = 1, 2, 3), \qquad A_1 \le 6H,\ A_2 \le 70H^2,\ A_3 \le 834H^3,$$

since $\sum_i |u_i|^4 \le 1$. Expanding the cube of $V = \sum_{\mathrm{hi}} |c_i| \mathbf{1}_{i \in S}$ and using joint inclusion gives $\mathbb{E}V^3 \le A_3 + 3A_2A_1 + A_1^3 \le 2310H^3 < 14^3H^3$. Centering costs at most two in L^3, so homogeneity yields $\|X - \mathbb{E}X\|_3 < 63H\,\|u\|$. The exact third-cumulant identity now implies

$$|g'''[h,h,h]| \le [3p + 253260H^3]\,\|u\|^3, \qquad 253260 = 63^3 + 3 \cdot 63 \cdot 17,$$

where $H \ge 1$ absorbs the covariance term's H^2. Consequently

$$\frac{|g'''|}{(g'')^{3/2}} \le \frac{3p + 253260H^3}{(p/2)^{3/2}} \le \frac{253476}{216} < 1200.$$

Each summand decreases with p; the last comparison uses $p = 72H^2$ and $H \ge 1$. Scaling by 600^2 gives the standard constant two. Finally $|g'[h]| \le (p+4H)\sqrt{d}\,\|u\|$, so the gradient parameter before scaling is at most $2d(p+4H)^2/p \le 2pd(19/18)^2 < 3pd$. The finite binary sum is positive, is C^3 across walls, and tends to zero at every finite orthant boundary. This proves the barrier claim. □

Proposition 8.2 (A rank-one obstruction)**.** *For each fixed $0 < \gamma < 1/4$, powers $p = \gamma(\log M)^2$ do not give a uniform convexity theorem as $M \to \infty$, even for $Q(z) = 1 + Mz$. At $p = 4, M = 625, A = 5/4$, one has $[-\log(A^4 + 625(A-1)^4)]'' = -192/25$.*

Proof. Set $\ell = \log M$, $y = e^{-\ell/p}$, $A = (1-y)^{-1}$ and $\alpha = p(e^{\ell/p} - 1)$. The two product terms have equal tilted mass, so occupation is Bernoulli 1/2. Direct differentiation gives, for $p > 2$,

$$A^2 g''(A) = p + \alpha - \frac{p-2}{4p}\alpha^2.$$

For $p = \gamma\ell^2$, $\alpha/\ell \to 1$; division by ℓ^2 gives limit $\gamma - 1/4 < 0$. In the finite example $y = 1/5$, $\alpha = 16$, $A^2 g'' = -12$, and therefore $g'' = -192/25$. □

This establishes squared-logarithmic order necessity in the stated asymptotic regime, not a sharp finite-activity threshold. Changing the power changes the objective. In particular it does not extend the quartic sampled allocation theorem to activities above one.

9 An explicit budget-allocation consequence

Corollary 9.1. *Let a finite decreasing lattice input be explicitly listed with m nonzero states and maximal active-coordinate cardinality r. Assume $p \ge 32r$ for $r \ge 1$, or $p \ge 1$ for $r = 0$. For $c_i > 0$ and $B > 0$, maximize $F_p(A)$ subject to $A_i \ge 0$ and $\sum_i c_i A_i = B$. There is a unique maximizer $A^* > 0$. Starting at*

$$A_i^0 = \frac{B}{dc_i},$$

classical damped Newton minimization of $-\log F_p$ on the affine simplex attains $F_p(A) \ge e^{-\epsilon} F_p(A^*)$ in

$$O\big(1 + \log m + \log(1 + \log(1/\epsilon))\big)$$

iterations for $0 < \epsilon \le 1/2$, with an absolute hidden constant. Each iteration uses $O(md^2 + d^3)$ arithmetic operations and $O(md)$ positive-real-power oracle calls.

Proof. Since $\mu(n) \le \mu(0)$ and every residual is bounded by its endpoint, weighted AM–GM gives

$$F_p(A) \le m\mu(0) \prod_i A_i^p \le m\mu(0) \frac{(B/d)^{pd}}{\prod_i c_i^p} \le mF_p(A^0).$$

Thus the initial objective gap is at most $\log m$. The closed simplex is compact, the partition is positive at A^0 and zero whenever a coordinate vanishes. Its maximizer is therefore interior. Strict convexity of the negative logarithm on the affine slice proves uniqueness. For $d = 1$ or $m = 1$ the displayed point is already optimal.

In affine coordinates on the tangent space, the restriction f is a nondegenerate standard self-concordant barrier. Its Newton decrement is $\delta^2 = \nabla f^\mathsf{T} (\nabla^2 f)^{-1} \nabla f$. The classical damped step [3, Lecture 2, IX–X]

$$x^+ = x - \frac{(\nabla^2 f)^{-1} \nabla f}{1 + \delta}$$

remains feasible and decreases the objective by at least $\delta - \log(1+\delta)$. While $\delta \ge 1/4$ the decrease is at least $1/4 - \log(5/4) > 0$, so there are $O(1 + \log m)$ such steps. Once $\delta \le 1/4$, the same classical theorem gives

$$\delta_+ \le 2\delta^2, \qquad f(x) - f^* \le \frac{\delta^2}{2(1-\delta)}.$$

The remaining count is $O(1 + \log(1 + \log(1/\epsilon)))$. Additive error in $-\log F_p$ is precisely the claimed multiplicative utility accuracy. Summing each listed state's gradient and Hessian costs $O(md^2)$, and a dense tangent or bordered Newton solve costs $O(d^3)$. Product rules or prefix/suffix products avoid division by a zero residual at a wall. □

This is an application of classical Newton theory to the stated explicit state-list model. The guarantee concerns this specific utility and retains its objective, rather than replacing it by the standard orthant logarithmic barrier.

10 Sampled allocation for an implicit matroid

The preceding Newton consequence uses an explicit state list. For the specific quartic matroid utility, sampling instead gives an implicit independence-oracle guarantee. Projected gradient descent and matroid sampling are classical; the point here is the uniform balanced box for this utility.

Theorem 10.1. *Let $\mathcal{I}$ be the independent sets of a matroid on $[d]$, accessible by an independence oracle of cost t_I. Let $0 \le \lambda_i \le 1$, $c_i > 0$ and $B > 0$, and put*

$$F(A) = \sum_{S \in \mathcal{I}} \prod_{i \in S} \lambda_i \prod_{i=1}^{d} (A_i - \mathbf{1}_{i \in S})_+^4.$$

For $0 < \epsilon, \delta \le 1/2$, there is a randomized algorithm producing $A > 0$, $c \cdot A = B$, with $-\log F(A) \le \min_{c \cdot A' = B} [-\log F(A')] + \epsilon$ with probability at least $1 - \delta$. Its expected number of arithmetic operations and independence-oracle calls is polynomial in $d, \epsilon^{-1}, \log(1/\delta)$, independently of the smallest positive activity, c and B, in the exact real-arithmetic sampling model. Every history produces a feasible output. No partition-value, counting, or Hessian oracle, bit-complexity bound, or self-concordance premise is needed for this algorithm.

Proof. Write $g = -\log F$. At a positive A, the tilted independent-set law has fields $w_i = \lambda_i(1 - 1/A_i)_+^4$. Set $Z_i = 4/(A_i - 1)$ on occupation of coordinate i, and zero otherwise (in particular when $A_i \le 1$). The deletion estimates and the compensated log-concavity calculation of Section 7 give, with $m_i = \mathbb{E}Z_i$ and $D_A = \operatorname{diag}(A_i)$,

$$0 \le m_i \le 1, \quad \mathbb{E}Z_i^2 \le 4, \quad \mathbb{E}Z_i^4 \le 256, \qquad \partial_i g = -(4 + m_i)/A_i,$$
$$\frac{13}{4} I \preceq D_A \nabla^2 g D_A = 4I + \operatorname{diag} \mathbb{E}(2Z_i + Z_i^2/4) - \operatorname{Cov}(Z) \preceq 7I.$$

The fourth moment here is a moment bound, not a fourth derivative at a wall. The upper matrix bound uses $\operatorname{Cov}(Z) \succeq 0$.

The utility is continuous on the compact budget simplex, positive in its interior and zero on its coordinate boundary. Strict convexity of g therefore gives a unique interior minimizer A^*. Its original budget KKT equations imply

$$\frac{4 + m_i(A^*)}{A_i^*} = \tau c_i, \qquad \tau B = 4d + \sum_i m_i(A^*).$$

Set $A_i^0 = B/(dc_i)$, $x_i = A_i/A_i^0$ and $h(x) = g(A^0 x)$. Then

$$x^* \in K := \{x \in [4/5, 5/4]^d : \sum_i x_i = d\}, \qquad \mu I \preceq \nabla^2 h(x) \preceq LI \quad (x \in K), \quad \mu = \frac{52}{25}, \quad L = \frac{175}{16}.$$

Moreover $\nabla h(x^*)$ is parallel to $(1, \dots, 1)$, even if x^* is on a face of this artificial box. Thus for every $x \in K$,

$$h(x) - h(x^*) \le (L/2) \|x - x^*\|^2 . \tag{23}$$

The Euclidean projection $P_K(u)$ is exactly $[P_K(u)]_i = \min(5/4, \max(4/5, u_i - \theta))$, with θ making the sum d. Sort the $2d$ breakpoints $u_i - 5/4, u_i - 4/5$ and solve the piecewise affine sum on its appropriate interval; a flat interval may use either endpoint. This costs $O(d \log(2d))$ arithmetic operations. The projection variational inequalities give $\|P_K(u) - P_K(v)\|^2 \le (P_K(u) - P_K(v)) \cdot (u - v)$, hence nonexpansiveness. Integrating the Hessian on a segment in K shows that $x \mapsto x - \nabla h(x)/L$ has Lipschitz constant $q = 1 - \mu/L = 3543/4375$. Also $P_K(x^* - \nabla h(x^*)/L) = x^*$. Consequently the iteration

$$x_{j+1} = P_K(x_j - \widehat{v}_j/L), \quad x_0 = \mathbf{1}, \qquad \|\widehat{v}_j - \nabla h(x_j)\| \le \eta$$

satisfies $\|x_T - x^*\| \le q^T \sqrt{d}/4 + \eta/\mu$.

Here are explicit finite sampling budgets. Choose the largest power of two $\eta \le 1$ with $\eta^2 \le \mu^2 \epsilon/(2L)$, and the largest power of two $\zeta \le 1$ with $\zeta^2 \le 16\eta^2/(25d)$. Choose the smallest integer $T \ge 1$ with $q^{2T} d \le 8\epsilon/L$ by successive rational multiplication. These choices give distance at most $\sqrt{2\epsilon/L}$ when all T errors are accurate, and $T = O(1 + \log(d/\epsilon))$, $\zeta^{-1} = O(\sqrt{d/\epsilon})$. Let R be the smallest power of two at least one with $R^3 \ge 1024/\zeta$, and choose the smallest integer $b \ge 1$ such that $2^b \ge 2dT/\delta$. Take

$$M = \left\lceil 8R^2 \zeta^{-2} b \right\rceil, \qquad \xi = \zeta/(8R).$$

All choices use rational comparisons and integer rounding; no square root, cube root or logarithm evaluation is required.

At each iterate take M fresh independent draws of the tilted law, each within total variation ξ, and average $Z_i' = \min(Z_i, R)$ to obtain $\widehat{m}_i$; use $\widehat{v}_i = -(4 + \widehat{m}_i)/x_i$. The clipping bias is at most $256/R^3 \le \zeta/4$. The sampling-law bias is at most $2R\xi \le \zeta/4$. Hoeffding's inequality and a union bound give empirical error at most $\zeta/2$ in every coordinate except with probability at most

$2d\exp(-M\zeta^2/(2R^2)) \le \delta/T$. Thus $\|\widehat{v} - \nabla h\| \le (5/4)\sqrt{d}\zeta \le \eta$. Conditional on every previous history the same statement holds, because the iterate is always in K. A union bound over the deterministic T steps, followed by (23), proves the objective assertion.

For completeness the required sampler is precisely the weighted independent-set sampler of Chen–Guo–Zhang–Zou [9, Theorem 1]: positive fields w admit a total-variation ξ sample in expected $O((1 + \max_i w_i)d\log(d/\xi)(\log d + t_I))$ time. At each call delete all zero-field elements, restricting the matroid; their occupation and score are zero but their separable derivative $-4/A_i$ remains. An empty restriction has the deterministic empty-set sample. Positive remaining fields may be arbitrarily small. Since $w_i \le 1$, each draw costs at most $O(d\log(2d/\xi)(\log(2d) + t_I))$, plus $O(d)$ score arithmetic. The bound is uniform conditional on every history. There are exactly TM draws, $R = O(\zeta^{-1/3})$ and $M = O(\zeta^{-8/3}\log(2dT/\delta))$; their unconditional expected total cost is therefore polynomial as stated. If $d = 1$, or all positive-activity elements are loops, the balanced point is already optimal. An empty restriction at just one iterate is not a reason to stop. No function value or decrement is used as a stopping criterion. □

11 Reproducibility and remaining scope

The accompanying Python 3 verifier uses only the standard library and explicit exceptions, including under optimized execution. It constructs the scalar quartic polynomials by integer coefficient arithmetic, performs the positive-variable substitution, and checks every sign and both activity endpoints. It also differentiates the literal two-product curvature witness with exact rational arithmetic and checks the arithmetic comparisons used in the constant bounds. The resulting JSON records the full coefficient payload. It also checks the fixed-power binary scaling constants. The verifier also checks the exact balanced-box gradient constants, the large-activity constants and its literal scalar curvature witness. These computations support the finite polynomial identities; the arbitrary-dimensional inequalities are proved analytically above.

The constants 16 and 32, the scalar multiplier 100, and the barrier parameters are not sharp. The lower construction establishes the linear order in active support for the unrestricted decreasing class, not a sharp coefficient. The fixed-power structured result has an absolute sufficient scale. The large-activity coefficient 72 and scale 360000 are sufficient; the rank-one obstruction establishes the stated squared-logarithmic asymptotic order. Arbitrary-input counterexamples do not settle this narrower matroid question. The explicit-state Newton result does not furnish an implicit partition oracle; the separate matroid sampling result instead uses an independence oracle and retains the quartic objective, without a counting or Hessian oracle.

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