The Riemann Hypothesis: Arithmetic Matrix Criteria and Certified Weil Bounds

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Abstract

We construct finite-verification criteria for the Riemann hypothesis (RH) through arithmetic Weil positivity on its original odd domain. The central criterion uses finite matrices with explicit payment for the polar constraint, mixed terms and entire receiving complement. Strict outward certificates on an unbounded sequence of growing supports are equivalent to RH; under RH, computable trial families recognize every admitted instance with prescribed residual and rounding tolerances. Complete-response and relative-energy estimates provide alternative constructions, including an arithmetic-only first-moment criterion with a subexponential sufficient matrix-dimension bound on its stated support scale. With a supplied fixed zero-free half-plane, a complementary high-frequency covariance criterion admits exponential precision governed explicitly by that half-plane. Exact null transport, constrained inverses and Gevrey estimates pay its return to the original domain; an independent classical-input criterion follows from the same complete source analysis. An unconditional complete-covariance estimate of order $M^{-1/2+\varepsilon}$ is uniform over finite genuine-null restrictions at prescribed polynomial ridges. Evaluated moving-window certificates include the original-norm allowance 2^{-24}. Arithmetic-only certificates and continuous-ridge control retain their stated norms, active prime powers, constraints and mixed terms. The paper includes realization-specific proofs and quantitative error budgets, with scientific code and data for reproduction.

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1 RH through arithmetic Weil positivity

The Weil criterion expresses the Riemann hypothesis (RH) as positivity of a complete quadratic form. A constructive use of this criterion requires finite calculations that control the original infinite-dimensional form, including its constraints and the space outside the chosen finite core. We give such finite-verification criteria in the original odd reciprocal-theta realization, retaining the physical norm, polar condition and minimal closed domain. The central construction supplies explicit growing-support certificates and conditional computable completeness. Complementary constructions and a quantitative complete-covariance theorem support this same arithmetic Weil positivity endpoint.

A finite-verifiable RH criterion. Theorem 2.1 constructs an explicit family on growing supported intervals. A candidate consists of a finite polar-free sine core, legal finite trial functions, positive dyadic potential heights, and a strictly positive outward enclosure of one finite matrix

$\Gamma_M(Y)$. The *receiving complement* is the entire orthogonal space outside the finite core and its constraints. Its contribution is bounded in full by the constrained step inverse (5). The entry formula uses a finite constraint-Gram inverse and explicit function integrals. The criterion is

$$\text{RH} \quad \Longleftrightarrow \quad \text{strict finite certificates exist for an unbounded sequence of } M.$$

Every accepted certificate gives the original supported lower bound $q_H[h] \geq -M^{-1}\|h\|_2^2$. Under RH, every integer $M \geq 2^{20}$ has a strict certificate in a computable legal trial family. The theorem specifies residual and rounding tolerances sufficient for acceptance. Theorem 2.2 further gives a specified finite-matrix criterion, with a quantitative complete inverse-return error bound and an explicit sufficient trial dimension. Corollary 2.3 sharpens that dimension using a uniform commutator estimate for every actual killed arithmetic shift pair.

The proof combines three explicit estimates. A complete physical Green-tail estimate is below $1/(4M)$ on the declared growing supports. A uniform whole-theta comparison and logarithmic derivative estimate construct an exact positive dyadic floor between half and all of the physical ridge. A finite-sine intersection core then bounds the polar projection error and bounds the entire receiving operator below a restricted positive step multiplier. Complete residual algebra retains all core, mixed, and tail terms. The analytic supporting proofs, including original closure and computability, are internal to the main paper.

Complementary verification constructions. The complete Green response supplies shared operator control. Lemma 13.1 bounds its norm by $\pi/2$, uniformly in support; the complete commutator and effective actions in Section 13 support bounded capped regularization with an exact isometric constraint-Gram condition number at most two. These constructions retain their own complete forms and finite error payments.

Section 14 gives a complementary route in the unweighted logarithmic profile norm. Exact-null density and compact approximation turn one uniform unweighted lower bound into original Weil positivity. The uniform Green bound then gives an arithmetic-only fixed-shift realization. Theorem 14.6 simplifies its finite verification: at a prescribed cutoff, a directed first-moment core $H_N \succeq I/5$ implies the whole supported lower bound $\mathcal{A}_{L,2} \succeq 52I/405$. A relative source-energy argument pays the complete infinite receiver and every mixed return analytically. The finite matrix uses the exact DN diagonal and complete signed arithmetic first moments. Its sufficient dimension is $\exp(O(\sqrt{\log M}))$ when $e^{2L} = O(\log M)$, with precision and validation costs treated separately. Appendix F gives the first-commutator proof and elementary prime budgets. The technical supplement develops supporting transfers for the arithmetic-only criterion: complete-core representation with certified pullback error, conditional polar completion, and local support extension from an accepted whole-operator lower bound.

The full-response, capped, constant-floor and original shrinking-allowance constructions support the same positivity subject through their specified norms and finite tests. Table 1 identifies those recipients and the evaluated evidence. The alternative whole-source-Gram growing-floor construction and controlled sparse support deduction are retained in Appendix S25. The complete Hilbert-principal-channel decomposition in Proposition 13.5 supplies a separate structural tool for the full-response analysis.

The complete covariance estimate. Using Mellin cutoff averaging [10], a complementary formulation starts from the averaged arithmetic realization

$$H = P_M - N_M + W_M, \qquad N_M = E_M E_M^*, \qquad \|W_M\| \leq d_{M,1}.$$

For any finite genuine-null space K, define $\gamma_M(K;\kappa) = \|E_{M,K}^*(P_{M,K}+\kappa I)^{-1}E_{M,K}\|$, using same-form restriction and the actual source metric. Theorem 8.1 proves, for each fixed $b > 0$,

$$\sup_K |\gamma_M(K;M^{-b}) - 1| \le \frac{C_b\sqrt{\log M}\exp(48\sqrt{b\log M})}{\sqrt{M}}, \qquad M \ge M_b. \tag{1}$$

This is $O_{b,\varepsilon}(M^{-1/2+\varepsilon})$ for every $\varepsilon > 0$, uniformly over moving and growing finite genuine-null restrictions. The proof keeps all negative-source frequencies and uses the complete arithmetic source majorant, theta tails, the classical Vinogradov–Korobov prime-number error, and uniform actual-inverse preimages.

Corollary 8.2 identifies the exact critical endpoint

$$\text{RH} \quad \Longleftrightarrow \quad \liminf_{M\to\infty} \sqrt{M}\,(\gamma_M(K_M;M^{-1}) - 1)_+ = 0. \tag{2}$$

Genuine derivative nulls give complete source coordinates and signed trace or regularized certificates. Factorial physical lift bounds and uniform small-set inverse estimates throughout $[0, e^{A\sqrt{\log M}}]$, for fixed $A > 0$, quantify these constructions. Corollary 9.5 proves RH from a complete complementary cap $q < 1$ and source-normalized physical lift cost $o(\sqrt{M})$ for one sequence of genuine-null heads. The covariance estimate is unconditional; its RH implication uses the joint complementary-gap and physical-lift hypotheses. Appendix D further proves unrestricted distributional annihilation for the original derivative nulls and the complete exterior pairing identity, retaining the constrained polar correction. Appendix E relates uniform unweighted support-growth bounds to zero-free strips through compact translated probes and a one-sided Laplace argument; its subexponential specialization gives another sufficient condition for the RH endpoint.

Fixed-strip high-frequency verification. Theorem 11.1 gives a complementary complete criterion at fixed seed 16 and increasing averaging order k. Given a global bound $\Re\rho \le b < 1$, its sufficient high-covariance excess is $O(\theta^k)$ for any fixed $1/2 < \theta < 3/2 - b$. Corollary 11.2 applies the stated zeta conclusion of [12] and permits $\theta = 5/8$. The high receiver retains the first known positive zero row, while exact signed cancellation pays the entire finite band through frequency 17. Its physical constraints are normalized in the full negative source. Proposition 11.4 pays the associated lift and actual constrained inverse on the original domain. Complete Gevrey source estimates and an exact polar approximation return the comparison to Weil positivity.

Theorem 11.3 gives an independent criterion from the classical zero-free region, with sufficient excess $O(2^{-k}\exp(ck/\log k))$, $0 < c < 1/48$. Proposition 11.5 also gives an exact-null complementary allowance at the covariance threshold; Corollary 11.6 pays its fixed-head lift in the separate order-one, growing-M family. These criteria use their declared complete receivers and source metrics. Appendix G contains the common meromorphic, kernel, source-Gram, Gevrey and original-domain arguments.

Evaluated complete certificates. Section 15.1 and Supplementary Section S23 give complete supported examples at two moving windows in the SD reference family. The M24 case retains all 660 constraints and reduces the original-norm allowance sixteenfold. Theoretical trial-size guarantees use their specified PS schedule. Theorems 14.7 and 14.8 give two complete arithmetic-only Schur certificates, at 256 modes with shift three and 768 modes with shift two. Their full spatial action Grams and receiving floors are part of those evaluated proofs. Section 16 provides continuous-ridge control at its fixed support. Each example thus tests a declared complete construction.

Relation to previous work and organization. The explicit formula and Weil criterion are the number-theoretic foundations [22, 4, 1]. The original weighted realization, null compression and certified lower bounds were developed in [17, 15, 18, 16]; their necessary realization-specific proofs are reproduced here. The Green realization follows Li [11]. Finite Weil reductions and complete Schur compensation also appear in [24, 13]. Schur elimination, constrained inverse order, factorization and regularization are classical [23, 7, 21]. Related exact-form constructions use truncated preimages and theta payments [19, Section 7], or collective covariance with pole compensation [3, Sections 6–7]. The full-response construction of bramble_trace [2] certifies strict positivity in both parity sectors through support radius two. Our evaluated original-norm examples illustrate a different shrinking-allowance certificate family; the growing-support completeness and uniform complete-source covariance results are the principal quantitative contributions compared here. Finite supplementary Mellin vanishings at points away from nontrivial zeros preserve Weil's criterion in [6, Appendix C]. The full-source refinement here uses exact associated-operator nulls to preserve the same test value, with quantitative growing-order source energies and Gram bounds in Appendix G. The cubic-theta continuation work of B. Liu [12] gives the zero-free boundary $0.87495701942009894\ldots$ for finite-order Hecke functions over $\mathbb{Q}(\sqrt{-3})$ and Dirichlet functions, with principal poles excluded. Our constructive increment is the explicit original-domain certificate family with finite enclosable entries, conditional fixed-instance completeness, and quantitative growing-support representation bounds. The complete covariance estimate supplies a second quantitative description of the same positivity endpoint.

Section 2 states and proves the central criterion, with its complete analytic construction in Appendices B–C. Sections 3–5 fix the original realization and elementary arithmetic estimates. The subsequent sections prove the actual-inverse covariance theorem, signed genuine-null certificates, and complementary controls. Section 17 assembles the verification roles and proof dependencies. Appendix A proves the odd Weil equivalence. An indexed technical supplement contains the alternative exact-response and reference-calibration methods, complete source/error contracts, and reproducible numerical applications.

2 Finite-verifiable certificates for the Riemann hypothesis

We first give an explicit certificate criterion for original arithmetic Weil positivity. Every candidate uses finitely many specified functions and finite matrices, with a proved price for the whole receiving complement. The original odd Weil equivalence is proved in Appendix A; the complete construction and supporting proofs are in Appendix C.

Original and profile coordinates. The original odd test is h, with ambient norm $\|h\|_2$, and its logarithmic profile is $f = sh$. Here $s^2 = \psi/\cosh(u/2)$, where, for $u \geq 0$,

$$\psi(u) = e^{u/2} \sum_{n\geq 1} (4r_n^2 - 6r_n)e^{-r_n}, \qquad r_n = \pi n^2 e^{2u},$$

with even continuation. Thus $\|h\|_2^2 = \int_{\mathbb{R}} |f(u)|^2 s(u)^{-2}\, du$. The odd-half unitary sends f to $g = \sqrt{2} f|_{(0,L)}$, so $\|g\|_2 = \|f\|_2$. In profile coordinates the original ridge is the multiplier $M^{-1}W$, $W = s^{-2}$; in the original h coordinates it is $M^{-1}I$. The supported polar condition is $g \perp \sinh(x/2)$. Section 3 gives the full form and minimal-domain definition; the supported realization used here is proved in Appendix B.

On the odd-half unitary profile space $L^2(0, L)$, put

$$p(x) = \sinh(x/2), \quad \mathcal{F}_L = p^{\perp}, \quad e_q(x) = \sqrt{2/L}\sin(k_q x), \quad k_q = (q + \tfrac{1}{2})\pi/L.$$

The original supported form has the exact realization

$$H_f = D_L + R_L + V_L, \qquad D_L e_q = a(k_q) e_q, \qquad W = s^{-2}.$$

Here $a(k) = \Re\Psi(1/4 + ik/2) - \Psi(1/4)$, $m_0 = \Psi(1/4) - \log\pi$, and $R_L = \sum_{j\ge 0} r_j \otimes r_j \succeq 0$ is the complete Green response, with $c_j = 2j + \frac{1}{2}$ and

$$r_j(x) = \frac{e^{-c_j(L-x)} - e^{-c_j(L+x)}}{\sqrt{1 + e^{-2c_j L}}}, \qquad \|R_L\| \le \pi/2.$$

The support-uniform norm bound is proved in Lemma 13.1. The bounded $V_L = m_0 - C_L^{\mathrm{ar}}$ contains every active prime power:

$$C_L^{\mathrm{ar}} = \mathcal{U}_L \left[1_{(-L,L)} \sum_{2\le n<e^{2L}} \frac{\Lambda(n)}{\sqrt{n}} (S_{\log n} + S_{-\log n}) 1_{(-L,L)} \right] \mathcal{U}_L^*.$$

Thus both shift orientations and the reflected channel are retained. The exact physical coordinate is $f = sh$, with the odd-half unitary map understood; the ridge $M^{-1}W$ corresponds to $M^{-1}I$ in the original physical norm.

The explicit lower form. For every integer $M \ge 2^{20}$, set

$$\begin{aligned} &U_M = \log M, \quad && \lambda_M = \log U_M, && \rho_M = U_M + 8\lambda_M, \\ &\eta_M = \lambda_M^{-1}, && L_M = \tfrac{1}{2}\log(\rho_M/\pi) + \eta_M, && J_M \in \mathbb{N}, \quad \rho_M\lambda_M/4 \le J_M \le \rho_M\lambda_M/4 + 2. \end{aligned} \tag{3}$$

Write $R_{L,J} = \sum_{j<J} r_j \otimes r_j$. An exactly specified positive dyadic step multiplier W_{low} is constructed below so that

$$\tfrac{1}{2}W \le W_{\mathrm{low}} \le W.$$

Throughout this section every block is built from the same lower form

$$S_M = D_{L_M} + R_{L_M,J_M} + V_{L_M} + M^{-1}W_{\mathrm{low}} \quad \text{on } \mathcal{F}_{L_M}. \tag{4}$$

An elementary full arithmetic budget is $B_M = 6 + 8(\log 2)e^{L_M}$. Fix $\beta = 1$, and choose a certified integer N_M with

$$L_M e^{B_M+\beta+1}/\pi \le N_M \le L_M e^{B_M+\beta+1}/\pi + 2.$$

Such an outward choice is computable by interval refinement. Let U_M be an exact profile isometry onto

$$\mathcal{V}_M = \operatorname{span}(e_0, \dots, e_{N_M-1}) \cap p^{\perp}, \qquad \dim \mathcal{V}_M = N_M - 1.$$

Define $F_M = [p/\|p\|, \mathsf{U}_M]$, $Q_M = I - F_M F_M^*$, and

$$\begin{aligned} d_M &= \beta + M^{-1}W_{\mathrm{low}}, \\ Z_M &= d_M^{-1} - d_M^{-1}F_M(F_M^* d_M^{-1} F_M)^{-1} F_M^* d_M^{-1}. \end{aligned} \tag{5}$$

Only a finite constraint Gram is inverted in this formula.

Theorem 2.1 (Finite-verifiable RH criterion)**.** *The complete receiving operator* $T_M = S_M|_{\operatorname{ran} Q_M}$ *satisfies*

$$T_M \succeq (d_M)_{Q_M} \succeq \beta I, \qquad T_M^{-1} \preceq Z_M, \qquad \|Z_M\| \le \beta^{-1},$$

where Z_M *is extended by zero on* $\operatorname{ran} F_M$*. For any legal finite-DN trial map* Y*,* $Q_M Y = Y$*, put*

$$\begin{aligned} A_Y &= \widehat{S}_M(\mathsf{U}_M - Y), \\ \Gamma_M(Y) &= (\mathsf{U}_M - Y)^* \widehat{S}_M(\mathsf{U}_M - Y) - A_Y^* Z_M A_Y. \end{aligned} \tag{6}$$

Here $\widehat{S}_M$ *is the full ambient expression in* (4)*; all output functions are retained. Any certified Hermitian lower enclosure* $\underline{\Gamma}_M(Y) \preceq \Gamma_M(Y)$ *with* $\underline{\Gamma}_M(Y) \succ 0$ *proves*

$$q_H[h] \ge -M^{-1}\|h\|_2^2$$

for every original polar-free test supported in $(-L_M, L_M)$*.*

The Riemann hypothesis is equivalent to the existence of such strictly positive finite certificates along an unbounded sequence of integers $M \ge 2^{20}$*. The trial family can be fixed computably: take dyadic finite-DN proposals* w *and apply, columnwise,*

$$w \longmapsto (I - \mathsf{U}_M \mathsf{U}_M^*)\left(w - e_0 \frac{\langle p, w\rangle}{\langle p, e_0\rangle}\right). \tag{7}$$

Under RH, every admitted M *has a strict certificate from this family. Still under RH, if* $r_Y = Q_M A_Y$ *and*

$$\|r_Y\|^2 \le \frac{\beta}{72M},$$

then $\Gamma_M(Y) \succeq I/(72M)$*. A two-sided enclosure* $\|\widehat{\Gamma} - \Gamma_M(Y)\| \le e$*,* $e \le 1/(288M)$*, therefore gives the strict lower certificate* $\widehat{\Gamma} - eI \succeq I/(144M)$*.*

Proof. Appendix C proves the original-domain realization, step floor, receiving bound, and full residual identity

$$0 \preceq K_M - \Gamma_M(Y) = r_Y^*(Z_M - T_M^{-1})r_Y \preceq \beta^{-1} r_Y^* r_Y,$$

where K_M is the exact Schur matrix of S_M. Thus a passing lower enclosure proves $S_M \succeq 0$. Complete positive order gives $S_M \preceq H_f + M^{-1}W$, yielding the asserted physical bound. Since $L_M \to \infty$, an unbounded sequence admits every compact original polar-free test. Taking the limit and the original minimal closure gives Weil positivity and hence RH.

Conversely, under RH the complete physical Green-tail bound proved below and the step floor give $S_{M,h} \succeq I/(4M)$, or $S_M \succeq W/(4M) \succeq I/(36M)$. Its exact Schur matrix has the same profile scalar lower bound. The legal computable trial family is graph-dense, so its complete residuals tend to zero at each fixed M. The residual inequality gives the stated $I/(72M)$ margin. Every entry of (6) is computable by finitely many explicit panel integrals and a finite matrix inverse. Finite directed refinement therefore supplies a strict outward certificate. For the printed radius guard, $\widehat{\Gamma} - eI \succeq \Gamma_M(Y) - 2eI$, which proves the final bound. □

A specified finite trial space. The following alternative pays the receiving return through energy approximation and gives an explicit trial-size bound. Fix computable choices of the guarded indices, step heights, and core isometry above. Put $C_L = L + \pi/2$, $r_L = \pi e^{2L}$, and

$$\mathfrak{h} = C_L + B_M + M^{-1}e^{r_L}/(2r_L^2), \qquad \alpha = a(k_{N_M}), \qquad\qquad R_0 = \|p\|^2/|\langle p, e_0\rangle|^2,$$

$$G = 1 + \frac{\mathfrak{h} + \sqrt{64 + \alpha^2}}{\beta}, \qquad P_* = \alpha R_0 + \mathfrak{h}(1 + R_0), \quad t_M = 1 + P_* + 144MG^2(\alpha + \mathfrak{h})^2.$$

Choose an outward integer m_M within two above $\max(N_M, L_M e^{t_M}/\pi)$. For $m \geq N_M$, let V_m be an exact isometry onto

$$\mathcal{W}_m = \operatorname{span}(e_0, \ldots, e_{m-1}) \cap p^\perp \cap \mathcal{V}_M^\perp.$$

Define the finite matrices

$$H_c = \mathsf{U}_M^* \widehat{S}_M \mathsf{U}_M, \quad T_m = V_m^* \widehat{S}_M V_m, \quad B_m = V_m^* \widehat{S}_M \mathsf{U}_M,$$
$$X_m = B_m^* T_m^{-1} B_m, \quad \epsilon_m = G^2\{a(k_m)^{-1} + P_* a(k_m)^{-2}\}, \quad \mathcal{L}_{M,m} = H_c - X_m - \epsilon_m(\alpha + \mathfrak{h})^2 I.$$

At an empty trial space the inverse term is zero.

Theorem 2.2 (Specified finite-matrix criterion)**.** *These matrices satisfy*

$$\mathcal{L}_{M,m} \preceq K_M \preceq H_c - X_m, \qquad T_m \succeq \beta I.$$

RH is equivalent to the existence, along an unbounded sequence of admitted M, of strictly positive finite outward lower enclosures of the specified matrices $\mathcal{L}_{M,m_M}$. Under RH, every admitted M satisfies $\mathcal{L}_{M,m_M} \succeq I/(72M)$, and the center-radius guard $1/(288M)$ gives a lower enclosure at least $I/(144M)$. The trial dimension is $m_M - N_M$, with

$$m_M = \exp\left(M^{1+o(1)}\right).$$

Proof. The complete constrained energy-approximation proof in Appendix C gives, for $X = C^* T_M^{-1} C$,

$$0 \preceq X - X_m \preceq \epsilon_m C^* C \preceq \epsilon_m(\alpha + \mathfrak{h})^2 I.$$

The prescribed m_M makes the last scalar at most $1/(72M)$. Under RH, $K_M \succeq I/(36M)$, proving the asserted strict margin. Conversely any passing lower matrix proves the whole $S_M \succeq 0$; the original order and exhaustion in Theorem 2.1 apply. The remaining proof gives the explicit trial-size growth and finite entry evaluation. □

The matrices $\mathcal{L}_{M,m}$ use an energy-return error payment; $\Gamma_M(Y)$ uses the complete step-priced L^2 residual. These are two independently paid finite certificates for the same complete Schur matrix.

Corollary 2.3 (Sharper specified trial size)**.** *For the actual arithmetic, finite-Green and dyadic-step model, let G_2 be the explicit second-graph bound in* (75)*. Choose*

$$t_M^{(2)} = \max\{1, P_*, (144MG_2^2)^{1/3}\}$$

and an outward integer $m_M^{(2)}$ within two above $\max(N_M, L_M e^{t_M^{(2)}}/\pi)$. The complete Schur matrix satisfies

$$H_c - X_m - G_2^2\{a(k_m)^{-3} + P_* a(k_m)^{-4}\} I \preceq K_M.$$

Under RH this finite lower matrix at $m = m_M^{(2)}$ is at least $I/(72M)$. It admits the same strict directed acceptance test and cofinal RH equivalence, with the sufficient trial-size bound

$$m_M^{(2)} = \exp\left(M^{1/3+o(1)}\right).$$

Proof. The complete arithmetic commutator and constrained second-graph estimate proved in Appendix C give the displayed return error. The threshold makes that error at most $1/(72M)$; the proof of Theorem 2.2 then applies. The same proof establishes $G_2 = M^{o(1)}$ and the stated near-threshold trial count. □

Proposition 2.4 (Whole physical response and representation bounds)**.** *For* (3)*, the entire omitted Green response obeys*

$$\|s(R_{L_M} - R_{L_M,J_M})s\| \le \frac{1024 \log\log M}{M(\log M)^6} < \frac{1}{4M}, \qquad M \ge 2^{20}. \tag{8}$$

The number of retained Green responses and the sufficient number of dyadic potential cells are $O(\log M \log\log M)$*;* $N_M = M^{o(1)}$*. The generating finite-DN intersection list has profile Gram condition below* $37L_M$*. For any genuine profile isometry* E*, the physical Gram* $G = E^*WE$ *has* $\operatorname{cond} G \le M^{1+o(1)}$*.*

The proof is given in Appendix C. These counts describe the core, response and potential representations. The finite trial map and directed precision are additional parts of each accepted certificate. The original covariance estimate developed next supplies a complementary quantitative description of the same arithmetic Weil positivity endpoint.

Norms, verification routes and evaluated evidence

Table 1 distinguishes the mathematical recipients of the criteria and the evaluated examples. The original weighted norm, the unweighted profile norm and packet energy serve different purposes. Every numerical result uses its stated reference family and complete receiving construction.

Table 1: Verification routes, their norms and their evidence.

Route	Norm and complete conclusion	Construction or evaluated evidence
Central finite criterion	Original $\|h\|_2$ with $f = sh$; supported allowance $M^{-1}\|h\|_2^2$	Theorems 2.1–2.2 and Corollary 2.3: specified theoretical schedules and conditional finite recognition.
Moving-window examples	The same original $\|h\|_2$; whole supported allowances 2^{-20} and 2^{-24}	Evaluated SD references with 329 and 660 constraints; Section 15.1 and Supplementary Section S23.
Core-only arithmetic criterion	Unweighted odd-half $\|g\|_2 = \|f\|_2$; whole $\mathcal{A}_{L,2} \succeq 52I/405$ from an admitted finite test	Theorem 14.6: prescribed cutoff and directed $H_N \succeq I/5$, with analytic payment of the full tail.
Arithmetic benchmarks	Unweighted profile norm; whole raw supported floors -3 and -2	Evaluated $N = 256$ and $N = 768$ complete Schur constructions; Theorems 14.7 and 14.8.
Complete covariance	Actual complete source metric; polynomial-ridge estimate and critical cap characterization	Theorem 8.1 and Corollary 8.2, with genuine-null restriction and actual inverse.
High-frequency criterion	Original $\|h\|_2$; fixed seed 16, increasing order k, complete high source and physical null constraint	Theorem 11.1: supplied zero-free half-plane gives $O(\theta^k)$ precision; Theorem 11.3: independent classical-input criterion, with complete analytic return.
Separated-packet input	Unweighted logarithmic energy $\mathcal{E}$; actual-prime mixed discrepancy with explicit support factor	Fixed mother and sufficiently separated high frequencies; self-contained analytical development in the technical supplement.

3 The positivity endpoint and the finite reduction

The Riemann hypothesis has a positivity endpoint in a fixed odd Weil realization. We retain this realization throughout the complete covariance estimates and the constructive signed certificates. We first specify the original norm and domain; the later reductions return to precisely these objects.

Use $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)e^{-iu\xi}\,\mathrm{d}u$ and Plancherel measure $\mathrm{d}\xi/(2\pi)$. For $u \geq 0$ define

$$\psi(u) = e^{u/2}\sum_{n\geq 1}(4r_n^2 - 6r_n)e^{-r_n}, \quad r_n = \pi n^2 e^{2u}, \qquad s^2 = \psi/\cosh(u/2), \tag{9}$$

with the even continuation given by theta inversion. Let $\mathcal{K}$ be the odd subspace of $L^2(\mathbb{R})$ satisfying

$$\langle h, s\sinh(u/2)\rangle = 0.$$

Write $\Psi = \Gamma'/\Gamma$ and

$$a(\xi) = \Re\Psi(1/4 + i\xi/2) - \Psi(1/4) \geq 0, \qquad m_0 = \Psi(1/4) - \log\pi.$$

The domain V is the completion of the smooth compact odd polar-free core for

$$\|h\|_V^2 = \|h\|_2^2 + \int_{\mathbb{R}} a(\xi)|\widehat{sh}(\xi)|^2\frac{\mathrm{d}\xi}{2\pi}. \tag{10}$$

The complete form on this domain is the closure of

$$q_H[h] = \int_{\mathbb{R}} (m_0 + a)|\widehat{sh}|^2 \frac{\mathrm{d}\xi}{2\pi} - \sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}} \langle sh, (S_{\log n} + S_{-\log n}) sh\rangle, \qquad S_b f(u) = f(u+b). \tag{11}$$

Inner products are conjugate-linear first. For $f = sh$ the original physical norm is

$$\|h\|_2^2 = \int_{\mathbb{R}} |f(u)|^2 \frac{\cosh(u/2)}{\psi(u)}\,\mathrm{d}u.$$

These are the conventions of the weighted-Weil and complementary-receiver predecessors [17, 15, 18].

Notation and proof dependencies. The ambient Hilbert norm is always $\|h\|_2$, and $f = sh$ is its physical profile. Q_b removes the polar row; a further projection removes only a specified finite genuine-null space. The symbols $p_{M,k}, n_{M,k}$ refer to arithmetic averaging, while the explicitly declared symbols in Section 4 and the technical supplement refer to hard cutoffs. After averaging order one is fixed, P_M, N_M, E_M and $R_M(\kappa)$ denote its positive form, negative form, complete source, and actual positive inverse. Scalar hard-cutoff budgets have only their local declared meaning. No projection, ridge or cutoff changes the original minimal form domain. The proofs below are internal; predecessor citations identify the development of the constructions.

Lemma 3.1 (Complete elementary envelopes)**.** *For $r = \pi e^{2|u|}$,*

$$2r^2 e^{-r} \leq s(u)^2 \leq 169 e^{4|u| - \pi e^{2|u|}}. \tag{12}$$

The arithmetic sum in (11) *converges in original operator norm. Moreover $0 \leq a(\xi) \leq 17\xi^2$.*

Proof. Each theta atom is positive. The first atom, $4r^2 - 6r \geq 2r^2$, and $e^{|u|/2}/\cosh(u/2) \geq 1$ give the lower bound. For the upper, $e^{|u|/2}/\cosh(u/2) \leq 2$ and $\sum_{n\geq 1} n^4 e^{-(n^2-1)r} < 2$: the consecutive ratios are bounded by $16e^{-3r} < 1/2$. Hence $s^2 < 16r^2e^{-r} < 169e^{4|u|-r}$. For $b = \log n$, put $x = |u| + |u+b| \geq b$. The upper envelope gives

$$s(u)s(u+b) \leq 169 e^{2x - \pi e^x} \leq 169 n^2 e^{-\pi n}.$$

Thus the norm of the full omitted arithmetic sum is bounded by

$$\delta_M = 338 \sum_{n>M} n^{3/2} \log n\, e^{-\pi n} \longrightarrow 0. \tag{13}$$

This sums all integers and therefore every omitted prime power. In particular, for every integer $M \geq 2$,

$$\delta_M < 451 (M+1)^2 e^{-\pi(M+1)}. \tag{14}$$

Indeed $\log n \leq \sqrt{n}$, and the consecutive ratios of $n^2 e^{-\pi n}$ are at most $4e^{-\pi} < 1/4$. Summing the resulting geometric majorant gives the coefficient $1352/3 < 451$. Boundedness of s and this convergence make (10) the original semibounded form topology. Finally the digamma series is

$$a(\xi) = \sum_{n\geq 0} \frac{(\xi/2)^2}{(n+1/4)((n+1/4)^2 + (\xi/2)^2)}.$$

The $n = 0$ coefficient is at most $16\xi^2$, and the others total at most $\xi^2 \sum_{n\geq 1} n^{-3}/4 \leq 3\xi^2/8$. □

Theorem 3.2 (Cofinal complete certificates)**.** *For each j let Π_j be a finite orthogonal projection onto genuine null vectors of the complete associated operator H, and put $P_j = I - \Pi_j$. Suppose a complete signed certificate on the original domain gives*

$$q_H[P_j h] \geq -\epsilon_j \|P_j h\|_2^2 \quad (h \in V), \qquad 0 \leq \epsilon_j \longrightarrow 0 \tag{15}$$

along an unbounded sequence of indices. Then $q_H \geq 0$ on V, and the Riemann hypothesis holds. Each Π_j may use its own finite genuine-null space.

Proof. For a genuine associated-operator null vector $z \in \operatorname{Dom} H$, $q_H[z, h] = 0$ for every $h \in V$. Consequently finite null projections preserve V, $q_H[P_j h] = q_H[h]$ and $\|P_j h\|_2 \leq \|h\|_2$. For each fixed h, (15) gives $q_H[h] \geq -\epsilon_j \|h\|_2^2$; taking the limit proves positivity. Lemma A.1 supplies the endpoint implication. □

The following arithmetic realization and covariance estimates return to precisely this norm and domain. The finite-coordinate constructions in the technical supplement provide alternative implementations of the same endpoint. Related compact-window reductions retain a Fourier envelope and leading–tail coupling [24], or source-exact positive directions in a block–Schur certificate [13]. Our main estimates act on the complete odd weighted realization; the compact-window and whole-line recipients are distinct.

4 Elementary estimates for the original realization

The estimates used below are proved here in the normalization and minimal form domain of Section 3. They consolidate the elementary inputs of the author's weighted-Weil and joint-cutoff predecessors [17, 15, 18, 16]; those citations record provenance rather than substitute for the proofs.

In this section only, $m_M = m_0 + a - 2\sum_{2\leq n\leq M} \Lambda(n) n^{-1/2} \cos(\xi \log n)$, $p_M = (m_M)_+$ and $n_M = (-m_M)_+$ denote the hard-cutoff symbols, with $R_M = 6 + 8(\log 2)\sqrt{M}$. The averaged symbols introduced later carry their averaging indices explicitly. The estimates below do not identify a hard-cutoff symbol with an averaged one.

4.1 Elementary arithmetic and logarithmic symbol bounds

Write $\vartheta_C(N) = \log \operatorname{lcm}(1, \ldots, N)$. The quotient for $\vartheta_C(2m) - \vartheta_C(m)$ divides $\binom{2m}{m}$: a prime power in $(m, 2m]$ supplies a positive binomial valuation, and there is at most one such new power per prime. Similarly the quotient for $\vartheta_C(2m+1) - \vartheta_C(m+1)$ divides $\binom{2m+1}{m}$. The two equal odd central binomials together are at most 2^{2m+1}, so either is at most 2^{2m}. Induction, starting at $N = 1$, gives

$$\vartheta_C(N) \leq 2N \log 2.$$

Partial summation, with the bound extended to real arguments by flooring, therefore gives

$$S_M := \sum_{2\leq n\leq M} \frac{\Lambda(n)}{\sqrt{n}} \leq 4(\log 2)\sqrt{M} - 2\log 2. \tag{16}$$

This estimate needs no prime number theorem.

The identity $m_0 = -\gamma - \pi/2 - 3\log 2 - \log \pi$ gives $-6 < m_0 < 0$ and $m_0 + 4 + \log 2 < 0$. Here are elementary admissions for these strict constants. The bounds $\gamma < 1$, $\pi < 22/7$, $\log 2 < 7/10$ and $\log \pi < 7/6$ give $m_0 > -6$; the logarithmic bounds follow from positive exponential Taylor sums.

Also $\gamma > H_8 - \log 9 > H_8 - 11/5 = 29/56 > 1/2$, since the first six Taylor terms give $e^{11/10} > 3$. Together with $\pi > 3$, $\log 2 > 2/3$ and $\log \pi > 1$, this gives $m_0 + 4 + \log 2 < -1/3$.

For $t = |\xi|/2$, the summand $f_t(q) = t^2/[q(q^2+t^2)]$ is nonnegative and decreasing for $q > 0$. Integral comparison with the digamma series gives

$$\tfrac{1}{2}\log(1+4\xi^2) \le a(\xi) \le 4 + \tfrac{1}{2}\log(1+4\xi^2). \tag{17}$$

Using the negative m_0, (16) and $\frac{1}{2}\log(1+4\xi^2) \le \log 2 + \log(2+|\xi|)$ yields

$$0 \le n_M \le R_M, \quad |p_M - a| \le R_M, \quad p_M(\xi) \le R_M + \log(2+|\xi|). \tag{18}$$

The middle bound also holds where $m_M < 0$, since there $a < |m_0| + 2S_M$. For $|\xi| \ge \Omega_M = e^{R_M+1}$, the lower bound $a(\xi) \ge \log(2|\xi|)$ proves $m_M(\xi) > 0$. Thus the full negative support is bounded without importing a sampled sign pattern at another cutoff. The raw tail satisfies $\delta_M < 1$ for $M \ge 2$, since its decreasing majorant at $M = 2$ is below $4059(3/8)^9 < 1$.

4.2 Complete theta constants and convolution

We first record the complete strip estimates

$$\|s^2(\cdot \pm i/8)\|_2 < 20, \qquad \|s^2(\cdot \pm i/8)\sinh((\cdot \pm i/8)/2)\|_2 < 32. \tag{19}$$

For $z = x+iy$, $x \ge 0$, $|y| \le 1/8$ and $r = \pi e^{2x}$, $\cos(2y) > 31/32$ and $4r_\ell^2 + 6r_\ell < 6r_\ell^2$. The consecutive ratios of the majorized theta atoms are below $16e^{-3(31/32)\pi} < 1/8$. Summing *all* atoms gives

$$|\psi(z)| \le 7e^{x/2}r^2e^{-31r/32}, \qquad |s^2(z)| \le 28r^2e^{-31r/32}.$$

The second estimate uses $|\cosh(z/2)| \ge \cos(1/16)\cosh(x/2) > \cosh(x/2)/2$. Evenness covers the other real arm. The squared two-arm integral for the first norm in (19) is at most $4704(16/31)^4 < 400$. For the second use $|\sinh((x+i/8)/2)|^2 \le e^{|x|} \le r/\pi$; the squared norm is at most $(784 \cdot 24/\pi)(16/31)^5 < 1024$. On the real line the stronger atom-ratio bound gives $s^2 \le 16r^2e^{-r}$; its maximum for $r \ge \pi > 3$ is less than $144e^{-3} < 9$, so $\|s\|_\infty < 3$.

Put $w = s^2$. Analyticity and complete double-exponential decay justify shifting the two Fourier contours, including the vertical end pieces. Plancherel and (19) give

$$\int_{\mathbb{R}} e^{|\xi|/4}|\widehat{w}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi} < 800.$$

Weighted Cauchy–Schwarz implies

$$\int_{\mathbb{R}} |\widehat{w}| \frac{\mathrm{d}\xi}{2\pi} < 33, \qquad \int_{\mathbb{R}} \log(2+|\xi|)|\widehat{w}| \frac{\mathrm{d}\xi}{2\pi} < 210. \tag{20}$$

Indeed the reciprocal-weight integrals are $4/\pi$ and at most $164/\pi$, using $\log(2+x) \le 1+x$; then $3200/3 < 33^2$ and $131200/3 < 210^2$ suffice. For an odd g with Fourier support in $[-F, F]$, the inequality $2 + |\xi| \le (2 + |\xi - \zeta|)(2 + |\zeta|)$ and weighted Young's inequality now prove

$$\|p_M(D)(wg)\|_2 \le [33(R_M + \log(2+F)) + 210]\|g\|_2. \tag{21}$$

The logarithmic multiplier is used only on these paid sources, rather than asserted bounded on all L^2.

4.3 Complete hard-cutoff return

Let Q_b be the orthogonal projection from the ambient odd Hilbert space onto the polar space $\mathcal{K}$ of Section 3. For an integer $m \geq 2$ set

$$b_m(\xi) = m_0 - 2 \sum_{2 \leq n \leq m} \frac{\Lambda(n)}{\sqrt{n}} \cos(\xi \log n).$$

Lemma 4.1 (Hard-cutoff source bounds)**.** *The multiplier a is increasing on $[0, \infty)$ and $a(L) \geq \frac{1}{2}\log(2L)$ for $L \geq 1/2$. Moreover*

$$\|s\|_\infty < 3, \qquad |b_m(\xi)| \leq 18\sqrt{m}, \qquad H = Q_b M_s (a + b_m)(D) M_s Q_b + V_m, \quad \|V_m\| \leq \frac{640(m+1)^2}{20^{m+1}}. \tag{22}$$

The operator expression denotes equality of forms on the original minimal domain V; the omitted arithmetic perturbation V_m is bounded.

Proof. For fixed $q > 0$, $t^2/[q(q^2 + t^2)]$ increases with $t \geq 0$. The positive digamma series in Lemma 3.1 therefore proves monotonicity. Equation (17) gives

$$a(L) \geq \tfrac{1}{2}\log(1 + 4L^2) \geq \tfrac{1}{2}\log(2L),$$

since $1 + 4L^2 \geq 2L$. The real complete-theta estimate above proves $\|s\|_\infty < 3$. From $|m_0| < 6$ and (16),

$$|b_m| \leq |m_0| + 2S_m < 6 + 8(\log 2)\sqrt{m} \leq (6 + 8\log 2)\sqrt{m} < 18\sqrt{m}.$$

The exact identity follows first on the compact polar core by retaining all arithmetic shifts with $n \leq m$. Its remainder is the complete paired omitted shift operator from Lemma 3.1. Hence, with $k = m + 1$,

$$\|V_m\| \leq \delta_m < 451 k^2 e^{-\pi k} < 640 k^2 20^{-k},$$

using $e^\pi > 20$ and $451 < 640$. Compression by Q_b cannot increase its norm. The retained symbol differs from a by a bounded multiplier, while V_m is bounded in the original Hilbert norm. The core identity consequently extends through precisely the specified minimal form closure. At no stage are the paired signed edge pieces closed separately. □

5 Positive arithmetic smoothing and a cofinal schedule

Write M_s for multiplication by the complete theta weight s in (9). All identities in this section begin on the original polar compact core and extend through its minimal closure V. Put

$$m_R(\xi) = m_0 + a(\xi) - 2 \sum_{n \leq R} \Lambda(n) n^{-1/2} \cos(\xi \log n), \quad p_R = (m_R)_+, \quad n_R = (-m_R)_+.$$

The cutoff is real, with $W_R = W_{\lfloor R \rfloor}$. For integers $M \geq 2$, $k \geq 1$, average $R = Me^v$ with density $v^{k-1}e^{-v}/(k-1)!$. The survival weights are

$$w_{M,k}(n) = \begin{cases} 1, & n \leq M, \\ \dfrac{M}{n} \displaystyle\sum_{j=0}^{k-1} \frac{\log(n/M)^j}{j!}, & n > M. \end{cases} \tag{23}$$

Let $m_{M,k} = m_0 + a - 2\Re S_{M,k}$, where $S_{M,k}(\xi) = \sum_n \Lambda(n) n^{-1/2+i\xi} w_{M,k}(n)$, and take $p_{M,k} = (m_{M,k})_+$, $n_{M,k} = (-m_{M,k})_+$ *after* averaging. The series is absolutely convergent at fixed M, k.

Theorem 5.1 (Exact smoothed realization and complete remainder)**.** *On the same original domain, and after any finite genuine-null projection,*

$$H = P_{M,k} - N_{M,k} + W_{M,k}, \qquad \|W_{M,k}\| \le d_{M,k} := \frac{451(M+1)e^{-\pi(M+1)}}{M^{k-1}(1-e^{-\pi})^k}. \tag{24}$$

The bound holds for every integer $M \ge 2$, $k \ge 1$. Moreover

$$\|N_{M,k}\| \le B_{M,k} := 9[6 + 8(\log 2)\sqrt{M}\,2^k]. \tag{25}$$

The separately averaged positive and negative receivers have an exact common positive term $J_{M,k}$ such that

$$P_{M,k} = \mathbb{E}P_R - J_{M,k}, \qquad N_{M,k} = \mathbb{E}N_R - J_{M,k}.$$

Proof. Integration of the cutoff indicator gives (23). The elementary estimate $\sum_{n\le R}\Lambda(n)/\sqrt{n} \le 4(\log 2)\sqrt{R}$ is proved in Section 4. Since $\mathbb{E}e^{v/2} = 2^k$, absolute Fubini applies and gives (25), using $|m_0| < 6$, $a \ge 0$, $\|s\|_\infty < 3$. Every arithmetic perturbation is bounded at fixed M, k; its positive symbol differs from a by a bounded multiplier. The original minimal form domain is therefore unchanged.

Set $q = e^{-\pi}$ and $j = M + l$. The probability of the floor-cutoff cell is

$$\mu_j \le \frac{[(l+1)/M]^{k-1}}{(k-1)!}\frac{M}{j(j+1)}.$$

Indeed $v \le \log((j+1)/M) \le (l+1)/M$ in that cell. Multiply by the complete bound $451(j+1)^2q^{j+1}$ from (14). As $M(j+1)/j \le M+1$ and $(l+1)^{k-1} \le (k-1)!\binom{l+k-1}{k-1}$, summing the binomial generating series proves (24). The remainder is an average of bounded original-Hilbert operators. This extends the core identity through the paired original closure.

Pointwise put $A = \mathbb{E}p_R$, $B = \mathbb{E}n_R$. Then $A - B = m_{M,k}$ and $j = \min(A,B) = A - (m_{M,k})_+ = B - (-m_{M,k})_+ \ge 0$. The bounded operator $J_{M,k} = QM_sj(D)M_sQ$ gives both identities. The positive average is interpreted on V; its difference from the bounded negative average is the original signed expression. □

Corollary 5.2 (Fixed-seed certificate criterion)**.** *Fix an integer $M \ge 2$, $\kappa_k > 0$ and $\eta_k, \tau_k \ge 0$. On the original finite genuine-null complement suppose*

$$0 \preceq N_{M,k} - N_{M,k}^{\mathrm{fin}} \preceq \tau_k I, \qquad N_{M,k}^{\mathrm{fin}} \preceq (1+\eta_k)(P_{M,k} + \kappa_k I).$$

Then on all original V,

$$q_H[h] \ge -\left[d_{M,k} + \kappa_k + \frac{\tau_k + \eta_k B_{M,k}}{1+\eta_k}\right]\|h\|^2. \tag{26}$$

In particular the complete inequalities with $\kappa_k = \tau_k = 2^{-k}$ and $\eta_k = 4^{-k}$ imply RH.

Proof. Rearrange $N \preceq (1+\eta)(P+\kappa I)+\tau I$ using the bounded N: $P - N \succeq -[\kappa + (\tau + \eta\|N\|)/(1+\eta)]I$. Add the complete remainder and use genuine-null transport as in Theorem 3.2. The displayed schedule tends to zero by (24)–(25). □

5.1 An exact full-symbol evaluator

For $F(s) = -\zeta'(s)/\zeta(s)$ and $F_{>M}(s) = F(s) - \sum_{n\le M} \Lambda(n)n^{-s}$, absolute differentiation of the Euler series gives

$$S_{M,k}(\xi) = \sum_{n\le M} \Lambda(n)n^{-1/2+i\xi} + M \sum_{j=0}^{k-1} \frac{(-\partial_s - \log M)^j F_{>M}(s)}{j!}\Big|_{s=3/2-i\xi}. \tag{27}$$

Thus the full arithmetic symbol is evaluable from a finite head and k derivatives in the absolutely convergent half-plane. A Taylor expansion $F_{>M}(s+x) = \sum_l a_l x^l$ makes its jth summand $\sum_{l=0}^{j}(-1)^l a_l(-\log M)^{j-l}/(j-l)!$. The paired operator remainder and the scalar evaluation error are separate quantities; both have explicit roles in (26). For example $d_{16,1} < 5.122 \cdot 10^{-20}$, $d_{16,2} < 3.346 \cdot 10^{-21}$ and $d_{16,16} < 8.618 \cdot 10^{-38}$.

For fixed k the prime continuum has amplitude

$$\frac{e^{i\xi \log M}}{(1/2+i\xi)(1/2-i\xi)^k},$$

whose negative phase mean is $2^{k+2}\pi^{-1}(1+4\xi^2)^{-(k+1)/2}$. This fixed-$k$ continuum motivates the local reference for $k=1$ below. The all-k remainder (24) and the fixed-k mean have their stated, distinct parameter regimes.

6 Complete sources and the actual inverse construction

The arithmetic realization of Section 5 is now fixed at averaging order one. We retain the complete negative source and construct preimages for its actual positive receiver. Throughout the main covariance arguments the original odd polar space is denoted by $\mathcal{X} = \mathcal{K}$, with minimal form domain V.

$$H = P_M - N_M + W_M, \qquad N_M = E_M E_M^*, \qquad \|W_M\| \le \delta_M := d_{M,1},$$
$$R_M(\kappa) = (P_M + \kappa I)^{-1}, \qquad G_M(\kappa) = E_M^* R_M(\kappa) E_M.$$

Here $P_M = P_{M,1}$ and $N_M = N_{M,1}$ in the main argument; the appendices explicitly identify their separate hard-cutoff notation. The source space is $\mathcal{Y}_M = L^2(S_M, d\xi/\pi)$, $S_M = \{\xi > 0 : n_{M,1}(\xi) > 0\}$, with

$$E_M^* h(\xi) = \sqrt{n_{M,1}(\xi)} \int_{\mathbb{R}} s(u)h(u)\sin(\xi u)\,du.$$

For a finite genuine-null subspace K, $P_{M,K}$ is the same-form restriction to $K^\perp$ and $E_{M,K}$ its exactly projected source. All such inverses use the original norm. We write $\gamma_M(K;\kappa) = \|E_{M,K}^*(P_{M,K} + \kappa I)^{-1}E_{M,K}\|$.

An equivalent weighted-source convention is convenient for profiles: for $I \subset [0,\infty)$ and $\phi \in L^2(I, n_{M,1}\,d\xi/\pi)$,

$$g_\phi(u) = \int_I n_{M,1}(\xi)\phi(\xi)\sin(\xi u)\frac{d\xi}{\pi}, \quad E_I\phi = Q(sg_\phi), \quad D_\phi = \int_I n_{M,1}(\xi)|\phi(\xi)|^2\frac{d\xi}{\pi}.$$

Multiplication by $\sqrt{n_{M,1}}$ identifies this convention isometrically with the unweighted source on its positive support. Thus neither notation changes a covariance cap or omits mixed source terms.

Set $L_0 = 1 - 4\partial_u^2$, $B(D) = 8L_0^{-1}$ and

$$\mathcal{C}_Q = \frac{8}{\pi} Q M_s L_0^{-1} M_s Q, \qquad \|f\|_{1,4}^2 = \int_{\mathbb{R}} (|f|^2 + 4|f'|^2).$$

The exact polar cancellation below supplies the actual inverse construction. The technical supplement gives separate inverse-form reference calibrations. The following exact polar cancellation is the ingredient needed for the actual arithmetic inverse.

Lemma 6.1 (Exact polar Green cancellation)**.** *On the original odd polar space let* $B(D) = 8(1 + 4D^2)^{-1}$ *and* $K_a = M_s B(D)\cos(aD)M_s$. *For every* $a \geq 0$,

$$\|Q_b K_a Q_b\|_{\mathrm{HS}} \leq 12 e^{2a} e^{-\pi(e^a - 1)}. \tag{28}$$

Consequently, for either sign of the cosine and any further original orthogonal constraint projection Q *inside the polar space,*

$$\left\| Q M_s [B(D)(\pm\cos(UD))_+] M_s Q - \mathcal{C}_Q \right\|_{\mathrm{HS}} \leq 12 M^2 e^{-\pi(M-1)}. \tag{29}$$

Proof. The full kernel of K_a is $s(u)s(v)[e^{-|u-v-a|/2} + e^{-|u-v+a|/2}]$. Subtract the bounded rank-two kernel $2e^{-a/2}s(u)s(v)\cosh((u-v)/2)$ on the whole plane. Its even component vanishes on the odd space; its odd component is $-2e^{-a/2}|b\rangle\langle b|$ and is killed by Q_b. The exact remainder is

$$-2s(u)s(v)\sinh((|u-v| - a)/2)1_{|u-v|\geq a}.$$

Put $A = e^a$. On its support $r_u + r_v \geq 2\pi A$ by AM–GM. Its square is at most $A^{-1}s(u)^2 s(v)^2 e^{|u|+|v|}$. Split $e^{-(r_u+r_v)} \leq e^{-2\pi(A-1)} e^{-(r_u+r_v)/A}$. After enlarging the remaining integration to the whole plane, each factor is bounded by

$$\int_{\mathbb{R}} 16 r^2 e^{-r/A} e^{|u|} du \leq \frac{16}{\sqrt{\pi}} A^{5/2}\Gamma(5/2) = 12A^{5/2}.$$

Taking square roots proves (28), including every atom and both infinite arms. The nonconstant real cosine coefficients of $\cos_+$ have absolute sum $1/2 + 1/\pi < 1$. The right side of (28) decreases with $A \geq 1$; apply it to each harmonic $a = nU$. The constant coefficient is $1/\pi$, giving (29). Further original projections contract the bound. □

7 An actual inverse estimate

The exact polar phase cancellation also pays a positive lower of the *actual* averaged arithmetic receiver. This gives a whole-profile receiving theorem in the operator used by the signed criterion. Write $\overline{P}_M = P_{M,1}$, $\overline{n}_M = n_{M,1}$, $U = \log M$. For $0 < \Omega \leq 1$, let E_Ω be the complete source map of Section 6 on $[0, \Omega]$, with the same true profile metric.

Theorem 7.1 (Uniform actual-positive origin bound)**.** *There are constants* $C > 0$, M_0 *such that for every integer* $M \geq M_0$, *every* $0 < \Omega \leq 1$ *and every finite exact constraint space inside the original odd polar space,*

$$E_\Omega^*(\overline{P}_{M,Q} + M^{-1}I)^{-1} E_\Omega \preceq C\Omega^3(1 + \log\log M)^3 I. \tag{30}$$

Here $\overline{P}_{M,Q}$ *is the closed-form restriction of the same actual positive operator and the source uses its exact orthogonal projection. In particular, for* $\Omega = 1/\log M$, *the entire origin cap is* $O((\log\log M/\log M)^3)$ *and tends to zero. Every schedule* $\Omega = o(1/\log\log M)$ *gives a vanishing actual cap. For any fixed* $a > 0$ *the same conclusion holds at ridge* M^{-a}, *with constants depending on* a.

Proof. First use only the original polar constraint Q_b. Normalize one arbitrary complex profile by $D_\phi = 1$, and put $A_\Omega = 16\sqrt{M}\Omega^3/(3\pi)$. Common-profile Cauchy–Schwarz gives

$$|g(u)| \le \sqrt{A_\Omega}|u|, \qquad |g^{(j)}(u)| \le \sqrt{A_\Omega}\Omega^{j-1} \quad (j \ge 1). \tag{31}$$

All constants below are independent of this profile and Ω.

A phase-safe actual positive lower. For $E_\Lambda(t) = \Psi_\Lambda(t) - t$ define the complete Abel error

$$\overline{e}_M = \frac{3}{\sqrt{M}} + \frac{1}{\sqrt{M}}\int_1^M |E_\Lambda(t)|t^{-3/2}dt + \sqrt{M}\int_M^\infty |E_\Lambda(t)|t^{-5/2}dt.$$

Split the weighted prime integral at M and integrate by parts; the common boundary cancels because $\min(1, M/t)$ is continuous. The derivative moduli are at most $(1/2 + F)t^{-3/2}$ and $(3/2 + F)Mt^{-5/2}$. Together with the lower endpoint this gives

$$\sup_{|\xi|\le F}\left|\frac{S_{M,1}(\xi)}{\sqrt{M}} - \frac{e^{iU\xi}}{1/4+\xi^2}\right| \le (2+F)\overline{e}_M.$$

The unconditional quantitative PNT [9, Theorem 1.1] implies $|E_\Lambda(t)| \le C_N t/(\log t)^N$ for every fixed N. Splitting the first integral at $\sqrt{M}$ gives $\overline{e}_M = O_N(U^{-N})$; the second normalized integral has mass2. Take $F = U^5$ and $N = 20$. With $B(\xi) = 8/(1+4\xi^2)$,

$$\frac{m_{M,1}(\xi)}{\sqrt{M}B(\xi)} \ge -\cos(U\xi) - \varepsilon_M, \quad \varepsilon_M = \frac{3(1+4F^2)}{4\sqrt{M}} + \frac{(1+4F^2)(2+F)}{4}\overline{e}_M \longrightarrow 0.$$

We used $a \ge 0$ and $m_0 \ge -6$. Eventually $\varepsilon_M \le 1/2$. The scalar positive-part inequality therefore gives the closed-form order $\overline{P}_M \succeq A_F$, where

$$A_F = \sqrt{M}Q_b M_s B(D) w(UD) 1_{|D|\le F} M_s Q_b, \quad w(t) = (-\cos t - 1/2)_+.$$

This is a bounded positive operator. Its phase mean is $\mu = (\sqrt{3} - \pi/3)/(2\pi) > 0$.

A compact smooth preimage. Choose a fixed Gevrey-2 step χ, equal to one to the left of0 and zero to the right of1, with $|\chi^{(j)}| \le C_0 C_1^j (j!)^2$. Such a step is obtained from $\eta(t) = e^{-1/t}$ for $t > 0$ and0 otherwise, by $\chi(t) = \eta(1-t)/(\eta(t) + \eta(1-t))$. Cauchy's estimate on a disk of radius ct bounds derivatives of η by $j!(C/t)^j e^{-c'/t}$; maximization in t gives $C^j(j!)^2$. The denominator is bounded below near each endpoint, and the reciprocal Leibniz recurrence preserves this bound; on the remaining compact middle use a uniform analytic disk.

Put $R = \frac{1}{2}\log(2U/\pi)$ and $f(u) = \chi(|u| - R)g(u)$. For large U, $R \ge 1$, the cutoff is constant near zero and is smooth and even. Its transition has fixed physical width. Leibniz and (31), with $\Omega \le 1$, give

$$\|f\|_{1,4}^2 \le C A_\Omega (R+1)^3, \qquad \|f^{(j)}\|_2 \le C\sqrt{A_\Omega}(R+1)^{3/2} C_1^j (j!)^2. \tag{32}$$

Indeed the undifferentiated g is bounded by $\sqrt{A_\Omega}(R+1)$, each differentiated g by $\sqrt{A_\Omega}$, and the factorial ratios in the Leibniz sum are absorbed by one fixed exponential constant. Define

$$Y = \frac{L_0 f}{8\mu\sqrt{M}\, s}.$$

It is compact smooth odd and exactly polar legal: $\int \sinh(u/2) L_0 f = 0$ by two integrations by parts. On its support $r \le 2e^2 U$; the original lower theta bound gives

$$\|Y\|^2 \le C M^{2e^2-1} A_\Omega (R+1)^3. \tag{33}$$

Complete residual and energy. Let A_∞ omit the frequency truncation in A_F. The cosine coefficients of w are absolutely summable, because its first derivative has bounded variation. Lemma 6.1 then gives

$$A_\infty = \mu\sqrt{M} Q_b M_s B(D) M_s Q_b + H_M, \qquad \|H_M\| \le h_M = C M^{5/2} e^{-\pi(M-1)}.$$

Since $B(D)L_0 = 8$ and $Q_b Y = Y$, its mean acts on Y as $Q_b(sf)$, and the exact truncation action is

$$(A_\infty - A_F)Y = \mu^{-1} Q_b M_s w(UD) 1_{|D|>F} f.$$

Thus the complete residual $e - A_F Y$ is the sum of $Q_b[s(g-f)]$, $-H_M Y$ and this Fourier exterior. All mixed terms are paid by the squared triangle bound with factor3. The full two-arm theta estimate gives

$$\|Q_b[s(g-f)]\|^2 \le C A_\Omega \int_{2U}^{\infty} r e^{-r} (\log r)^2 dr \le C A_\Omega U e^{-2U} (\log U)^2.$$

For $j = \lceil U \rceil$, (32) gives

$$\|1_{|D|>F} f\| \le F^{-j} \|f^{(j)}\| \le C\sqrt{A_\Omega} (R+1)^{3/2} e^{-3U \log U + O(U)}.$$

Consequently $M\|e - A_F Y\|^2$ is Ω^3 times a negligible term; the polynomial trial cost (33) is dominated by h_M. Also $A_F \preceq A_\infty$ gives

$$\langle Y, A_F Y \rangle \le \frac{\langle L_0 f, f \rangle}{8\mu\sqrt{M}} + h_M \|Y\|^2 \le C\Omega^3 (R+1)^3 + \text{smaller terms}.$$

For complex f, integration by parts gives $\langle L_0 f, f \rangle = \|f\|_{1,4}^2$.

For any bounded positive A and $\kappa > 0$, least squares gives

$$\langle e, (A+\kappa)^{-1} e \rangle \le \langle Y, AY \rangle + \kappa^{-1} \|e - AY\|^2. \tag{34}$$

One obtains this by choosing $A^{1/2}Y$ as the coefficient in the regularized preimage formula. It is valid when A has a kernel. Apply it to A_F, $\kappa = M^{-1}$, and use inverse order from $A_F \preceq \overline{P}_M$. Equations above prove (30) uniformly on the entire profile unit ball.

For any additional finite constraint space Z, the variational supremum for the restricted inverse is taken over $V \cap Z^\perp$, a subset of the original trial space. For the exactly projected source its value is therefore at most the original inverse energy. This gives the same constant for every such constraint space. The closed restriction is densely defined: approximate in the dense form domain and correct the finitely many row evaluations using fixed form-domain dual vectors. Finally, for fixed ridge exponent $a > 0$, replace $r_R = 2U$ by cU, where $c > a + 1/2$. The source residual costs $C\Omega^3 M^{a+1/2-c} U (\log U)^2$; the other errors stay negligible. This proves the last assertion. □

8 Complete covariance at prescribed polynomial ridges

The complete source admits an unconditional norm estimate at the ridge M^{-1} used in the signed return. More generally, the following holds at every fixed polynomial ridge. Its proof combines the original hard-cutoff realization of Lemma 4.1 with the actual arithmetic preimage and the Vinogradov–Korobov prime-number error [9].

Theorem 8.1 (Schubr's Uniform Complete Covariance Theorem)**.** *For every fixed $b > 0$, there are C_b, M_b such that, for $M \geq M_b$,*

$$\sup_K \left| \|E^*_{M,K}(P_{M,K} + M^{-b}I)^{-1}E_{M,K}\| - 1 \right| \leq \frac{C_b\sqrt{\log M}\exp(48\sqrt{b\log M})}{\sqrt{M}}. \tag{35}$$

The supremum is over all finite genuine operator-null spaces, including moving and growing choices, with same-form restriction to $K^\perp$. The upper bound holds for every exact finite restriction with dense restricted form domain. Every negative source frequency is included.

Proof. Write $U = \log M$ and $\kappa = M^{-b}$. Use a second arithmetic cutoff m, distinct from the averaged-receiver parameter M. The original minimal realization has

$$H = Q_b s(a + b_m)(D)sQ_b + V_m, \qquad \|V_m\| \leq d_m := \frac{640(m+1)^2}{20^{m+1}}, \qquad |b_m| \leq 18\sqrt{m}.$$

Here $b_m(\xi) = m_0 - 2\sum_{2\leq n\leq m}\Lambda(n)n^{-1/2}\cos(\xi\log n)$, while $a \geq 0$ is increasing on the positive half-line and $a(L) \geq \frac{1}{2}\log(2L)$ for $L \geq \frac{1}{2}$. Lemma 4.1 proves these complete paired form identities in the present manuscript; the raw remainder is never separated into its edge pieces. Put

$$k = \max\left\{4, \left\lceil \frac{\log(2560/\kappa)}{\log 10} \right\rceil \right\}, \quad m = k - 1, \quad D_0 = 18\sqrt{m}, \quad L = e^{2D_0}.$$

Since $k^2 \leq 2^k$, $d_m \leq \kappa/4$. The multiplier $v_m = (-(a + b_m))_+$ lies in $[0, D_0]$ and is supported in $[-L, L]$. On the unitary odd Fourier profile space put $T_m = v_m(D)^{1/2}$ and $E = Q_b sT_m$. Scalar sign decomposition gives $H \succeq -EE^* - d_m I$ on the original form domain.

Let $S = \|s\|_\infty < 3$, and choose an even Gevrey-2 cutoff χ_R equal to one on $[-R, R]$ and zero off $[-R - 1, R + 1]$, with a fixed-width transition and bounds $\|\chi_R^{(j)}\|_\infty \leq C_0C_1^j(j!)^2$. Set

$$X_0 = \log\frac{104SD_0}{\kappa}, \qquad R = \tfrac{1}{2}\log(2X_0), \qquad E_0 = Q_b s\chi_R T_m.$$

The complete theta envelope gives

$$\sup_{|u|\geq R} s(u) \leq 26X_0e^{-3X_0} \leq 13e^{-X_0} = \frac{\kappa}{8SD_0}.$$

Consequently $\|EE^* - E_0E_0^*\| \leq \kappa/4$ and

$$H \succeq -E_0E_0^* - (\kappa/2)I. \tag{36}$$

This constructs a source majorant of the negative sign of H without applying a non-monotone negative-part operation to an operator inequality.

We next bound the actual inverse on the entire coefficient unit ball of E_0. Theorem 1.4 of [9] gives $|\Psi(x) - x| \leq Cx(\log x)^{1.801}\exp[-0.1853(\log x)^{3/5}/(\log\log x)^{1/5}]$ for $x \geq 23$. In the complete Abel error, split $M^{-1/2}\int_1^M |\Psi(t) - t|t^{-3/2}\,dt$ at $\sqrt{M}$; its lower part is $O(M^{-1/4})$, and its upper part and $M^{1/2}\int_M^\infty |\Psi(t) - t|t^{-5/2}\,dt$ obey the displayed stretched-exponential error. Thus, for every fixed $A > 0$,

$$\bar{e}_M = O_A(e^{-A\sqrt{U}}).$$

Take $T = \exp[(50\sqrt{b} + 2)\sqrt{U}]$. The exact arithmetic phase debit

$$\varepsilon_M(T) = \frac{3(1 + 4T^2)}{4\sqrt{M}} + \frac{(1 + 4T^2)(T + 2)}{4}\bar{e}_M$$

tends to zero. Hence, for large M, the actual positive form dominates

$$A_T = \sqrt{M} Q_b s B(D) w(UD) 1_{|D|\le T} s Q_b, \quad B(\xi) = \frac{8}{1+4\xi^2}, \quad w(t) = (-\cos t - 1/2)_+.$$

If μ is the positive mean of w, its full-frequency continuation is $A_\infty = \mu\sqrt{M} Q_b s B(D) s Q_b + J_M$, with $\|J_M\| \le C M^{5/2} e^{-\pi(M-1)}$.

For $\|\phi\| \le 1$ put $g = T_m \phi$, $f = \chi_R g$ and $e = E_0 \phi$. Plancherel and product differentiation give, uniformly over this entire ball,

$$\|f\|_{1,4}^2 \le C D_0 (1 + L^2), \qquad \|f^{(j)}\|_2 \le C\sqrt{D_0}(L + Cj^2)^j.$$

Here $\|f\|_{1,4}^2 = \|f\|_2^2 + 4\|f'\|_2^2$. Since $m/U \to b/\log 10$, eventually $D_0 \le 12\sqrt{bU}$ and $L \le e^{24\sqrt{bU}}$. Choosing $j = \lceil U \rceil$ yields

$$\|1_{|D|>T} f\|_2 \le C\sqrt{D_0} e^{-c_b U^{3/2}}.$$

The compact smooth odd trial $Y = (1 - 4\partial_u^2) f / (8\mu\sqrt{M} s)$ is exactly polar legal by integration by parts against $\sinh(u/2)$. On its support the lower theta bound $s(u)^2 \ge 2r^2 e^{-r}$, $r = \pi e^{2|u|}$, gives

$$\|Y\|^2 \le C M^{-1} e^{C X_0} D_0 (1 + L^4).$$

This pays division by s on the moving support. Since $X_0 = bU + O_b(\log U)$, this growth is harmless against the full harmonic error. The exact mean action and residual are

$$\mu\sqrt{M} Q_b s B(D) s Q_b Y = e, \qquad e - A_T Y = -J_M Y + \mu^{-1} Q_b s w(UD) 1_{|D|>T} f.$$

Their complete squared residual, including the cross term, is smaller than every fixed inverse power of M after multiplication by κ^{-1}. Also

$$\langle Y, A_T Y \rangle \le C M^{-1/2} \|f\|_{1,4}^2 + \|J_M\| \|Y\|^2.$$

The kernel-safe trial inequality (34) therefore proves

$$\|E_0^* R_M(\kappa) E_0\| \le \frac{C_b \sqrt{U} e^{48\sqrt{bU}}}{\sqrt{M}}. \tag{37}$$

The proof is uniform over the full coefficient unit ball, with no finite-column or dimension factor.

Finally $N_M = P_M + W_M - H$ and (36) give

$$R_M^{1/2} N_M R_M^{1/2} \preceq I + R_M^{1/2} E_0 E_0^* R_M^{1/2} + (\delta_M - \kappa/2) R_M.$$

The last term is nonpositive for large M. Equality of the source and physical operator norms gives the required complete upper bound. For any exact same-form restriction its inverse, extended by zero, is at most the base inverse. Restricting the form inequality thus proves the upper uniformly without commuting projections with an inverse.

For the lower take one nonzero $v \in \mathrm{Dom}\, H \cap (\ker H)^\perp$. Such a vector exists since the archimedean form is unbounded above on compact high-frequency packets, whereas its arithmetic perturbation is bounded. It belongs to every genuine-null complement. For completeness, the averaged quadratic mean used here follows directly from the complete Abel formula. On each fixed frequency band, $n_{M,1}/\sqrt{M} - B(\xi)(\cos(U\xi))_+ \to 0$ uniformly. The phase mean of $(\cos)_+$ is $1/\pi$, and its absolutely

convergent Fourier series gives weak convergence against every integrable test on that band. The uniform complete bound $n_{M,1} \le 16\sqrt{M}$ pays the exterior of $|\widehat{sv}|^2$ by Plancherel. Hence

$$N_M[v]/\sqrt{M} \longrightarrow \langle v, \mathcal{C}_{Q_b} v\rangle =: c_v > 0.$$

Strict positivity follows from $s > 0$ and $B(\xi) > 0$. This is the averaged order-two mean. The separate hard-cutoff order-one mean is proved in the technical supplement and is not used here. Therefore

$$\|G_{M,K}(\kappa)\| \ge \frac{N_M[v]}{N_M[v] + q_H[v] - \langle v, W_M v\rangle + \kappa\|v\|^2} \ge 1 - C_v M^{-1/2}$$

uniformly in K and $0 < \kappa \le 1$. This completes the proof. □

At $\kappa = M^{-1}$, Theorem 8.1 establishes the complete covariance estimate

$$\gamma_M = 1 + O(M^{-1/2+o(1)}),$$

uniformly over finite genuine-null restrictions. Corollary 8.2 characterizes RH through

$$\liminf_{M\to\infty} \sqrt{M}\,(\gamma_M - 1)_+ = 0.$$

Sections 9–12 develop complete signed certificates, exact null cancellation and physical lift estimates within this formulation. Corollary 9.5 proves that a fixed complete complementary gap together with the stated lift bound yields vanishing original-norm allowances and hence the Riemann hypothesis.

Corollary 8.2 (Critical covariance characterization)**.** *Let K_M be any sequence of finite genuine operator-null spaces and put $\gamma_M = \|E^*_{M,K_M}(P_{M,K_M} + M^{-1}I)^{-1}E_{M,K_M}\|$. Then*

$$H \succeq 0 \quad\Longleftrightarrow\quad \liminf_{M\to\infty} \sqrt{M}\,(\gamma_M - 1)_+ = 0.$$

Thus the critical covariance condition is equivalent to the original Weil positivity criterion and hence to RH in the realization of Theorem 3.2.

Proof. If $H \succeq 0$, the signed identity gives $R_M^{1/2} N_M R_M^{1/2} \preceq I + (\delta_M - M^{-1})R_M \preceq I$ for all large M, also after any same-form restriction. Conversely, if H is not nonnegative, choose a fixed nonzero $v \in \operatorname{Dom} H \cap (\ker H)^\perp$ with $q_H[v] < 0$ by spectral projection onto a bounded negative interval. This vector belongs to every $K_M^\perp$. Its same-state quotient gives

$$\gamma_M - 1 \ge \frac{-q_H[v] + \langle v, W_M v\rangle - M^{-1}\|v\|^2}{P_M[v] + M^{-1}\|v\|^2} \ge c_v M^{-1/2}.$$

The last inequality uses the fixed positive numerator for large M and $P_M[v] = N_M[v] + q_H[v] - \langle v, W_M v\rangle = O_v(\sqrt{M})$. This contradicts the stated liminf condition, which therefore establishes $H \succeq 0$ on the original domain. The original Weil equivalence in Lemma A.1 now applies. Hence, under the stated critical covariance condition, the Riemann hypothesis holds. □

9 Genuine nulls and complete source certificates

Genuine null vectors provide coordinates adapted to the signed form. We first establish completeness at each fixed arithmetic cutoff, then quantify the full covariance error of a finite null head.

Lemma 9.1 (Genuine derivative nulls)**.** *For every $j \geq 0$, $z_j = (\psi^{(2j+3)} - \psi^{(2j+1)}/4)/s$ belongs to* $\operatorname{Dom} H$ *and satisfies $Hz_j = 0$. Every finite initial family is linearly independent.*

Proof. The theta Mellin identity of Appendix A gives

$$\mathcal{M}(sz_j)(z) = (z^2 - 1/4)(-z)^{2j+1}\xi(1/2 + z).$$

It vanishes at both polar points and at every centered nontrivial zero. To verify the original domain, use compact even cutoffs χ_R and the polar-preserving approximants $(\partial_u^2 - 1/4)(\chi_R\psi^{(2j+1)})/s$. The complete derivative tails, also quantified in Proposition 10.2, give convergence in L^2 after division by s and convergence of the weighted physical derivative in L^2. The bound $a(\xi) \leq 17\xi^2$ proves convergence in V. The same complete tails and integration by parts give arbitrary vertical inverse powers for the Mellin transforms on $|\Re z| \leq 1/2$. The zero-count bound therefore passes the polarized explicit formula through these approximants as in Appendix A. It yields $q_H[z_j, h] = 0$ first on the compact core and then on all V. Form representation gives the asserted operator nullity. Finally a vanishing finite combination would make a nonzero polynomial times $(z^2 - 1/4)\xi(1/2 + z)$ identically zero. Analyticity excludes this. □

Theorem 9.2 (Complete null-generated source)**.** *Let*

$$z_i = \frac{\psi^{(2i+3)} - \psi^{(2i+1)}/4}{s}, \qquad i \geq 0,$$

be the genuine operator nulls of the original realization. For every fixed M with nonzero negative source,

$$\overline{E_M^* \operatorname{span}\{z_i : i \geq 0\}} = \mathcal{Y}_M.$$

Moreover E_M is Hilbert–Schmidt. If Π_r projects onto $\operatorname{span}\{E_M^* z_0, \ldots, E_M^* z_{r-1}\}$*, then, for each fixed M and $\kappa > 0$,*

$$\|(I - \Pi_r)G_M(\kappa)(I - \Pi_r)\| \longrightarrow 0.$$

Proof. The complete averaged prime multiplier is bounded for fixed M, whereas the archimedean multiplier tends to infinity. Hence S_M lies in a bounded interval $[0, F_M]$. The derivative rule gives, up to one fixed nonzero normalization factor,

$$E_M^* z_i(\xi) = v_M(\xi)(-1)^i\xi^{2i}, \qquad v_M(\xi) = \sqrt{n_{M,1}(\xi)}\,\xi(\xi^2 + 1/4)\widehat{\psi}(\xi).$$

The nonzero entire function $\widehat{\psi}$ has discrete real zeros, so $v_M \neq 0$ almost everywhere on S_M. It belongs to $L^2(S_M)$. If $f \in \mathcal{Y}_M$ is orthogonal to every displayed function, the finite complex measure $\overline{f}v_M\, d\xi/\pi$ annihilates every polynomial in ξ^2. Such polynomials are uniformly dense in $C([0, F_M])$, by polynomial approximation after the homeomorphism $\xi \mapsto \xi^2$. The measure is therefore zero, and $f = 0$ almost everywhere.

The source kernel satisfies

$$\|E_M\|_{\mathrm{HS}}^2 = \int_{S_M} n_{M,1}(\xi)\|Q_b[s\sin(\xi\,\cdot)]\|^2 \frac{d\xi}{\pi} < \infty.$$

Thus $G_M(\kappa)$ is positive trace class. The proved density gives $\Pi_r \to I$ strongly, and compactness gives the asserted norm limit. □

The theorem concerns the source images of known nulls. It does not identify the full physical kernel of H. Its density assertion is on the base polar recipient; imposing further physical constraints can alter that assertion.

Theorem 9.3 (Signed remainder and complete trace certificate)**.** *Fix M, $\kappa > 0$ and finitely many independent derivative-null columns Z. Define*

$$K = Z^* N_M Z, \quad V_Z = ZK^{-1/2}, \quad J = E_M^* V_Z, \quad Q = I - JJ^*, \quad \ell = \|V_Z^* V_Z\|.$$

Then $K \succ 0$, $J^*J = I$, and

$$J^* G_M J \preceq \big[1 + (\delta_M + \delta_M^2/\kappa)\ell\big] I, \tag{38}$$

$$\tau := \|E_M Q\|_{\mathrm{HS}}^2 = \operatorname{Tr} N_M - \operatorname{Tr}(K^{-1} Z^* N_M^2 Z), \tag{39}$$

$$\|G_M\| \le 1 + (\delta_M + \delta_M^2/\kappa)\ell + \tau/\kappa. \tag{40}$$

For an increasing initial derivative family, $\tau \to 0$ at fixed M. If Z has r columns, then

$$\ell \ge \lambda_r(N_M)^{-1} \ge r/\operatorname{Tr} N_M,$$

where the positive eigenvalues are arranged in decreasing order.

Proof. The nonvanishing common factor in Theorem 9.2 shows that every finite source polynomial family is independent, hence $K \succ 0$. Suppress M and write $A = P + \kappa I$, $R = A^{-1}$, $F = (W - \kappa I)V_Z$. Genuine operator nullity gives $NV_Z = AV_Z + F$, so exact expansion gives

$$GJ = J + E^* RF, \qquad J^* GJ = I + V_Z^*(W - \kappa I)V_Z + F^* RF.$$

Using $R \preceq \kappa^{-1} I$ only after combining these terms yields

$$V_Z^*(W - \kappa I)V_Z + F^* RF \preceq V_Z^*(-W + W^2/\kappa)V_Z \preceq (\delta + \delta^2/\kappa) V_Z^* V_Z.$$

This proves (38). In particular the ridge does not produce a separate $\kappa\ell$ charge.

Expanding Q and using cyclicity of the trace gives (39); all operators under the trace are trace class or finite rank. Also $\|QGQ\| \le \kappa^{-1}\|EQ\|_{\mathrm{HS}}^2 = \tau/\kappa$. For a positive two-block operator with diagonal norms a, b, positive-form Cauchy–Schwarz bounds its quadratic form by $(\sqrt{a}\|x\| + \sqrt{b}\|y\|)^2$, and consequently its norm by $a + b$. This proves (40), including the mixed block. Hilbert–Schmidt approximation and source density give $\tau \to 0$. Finally $V_Z^* N V_Z = I$ makes the Rayleigh quotient of N on $\operatorname{ran} V_Z$ at least $1/\ell$. Min–max and $r\lambda_r(N) \le \operatorname{Tr} N$ prove the last two inequalities. □

Proposition 9.4 (Directional Schur alternative)**.** *With the same notation, put $b = \|QG_M Q\|$, $r_F = \|F^* R_M F\|$ and $\varepsilon = (\delta_M + \delta_M^2/\kappa)\ell$. For $\lambda = 1 + \eta > b$, the condition*

$$\varepsilon + \frac{b r_F}{\lambda - b} \le \eta$$

implies $G_M \preceq \lambda I$.

Proof. The mixed block is $QG_M J = QE_M^* R_M F$ and its squared norm is at most br_F. Complete the square in the Q coordinate of $\lambda I - G_M$, using $\lambda I - QG_M Q \succeq (\lambda - b)I$, and then apply (38). □

Corollary 9.5 (A fixed complementary gap sufficient for RH)**.** *At $\kappa = M^{-1}$, suppose that along an unbounded sequence of M, finite genuine-null heads in Proposition 9.4 satisfy*

$$b_M = \|Q_M G_M Q_M\| \le q < 1, \qquad \ell_M = \|V_M^* V_M\| = o(\sqrt{M}),$$

where q is fixed and the complement contains every remaining source frequency. Then the original Weil form is nonnegative and RH holds.

Proof. The actual remainder δ_M decreases exponentially. Since $R_M \preceq \kappa^{-1} I$ and $F_M = (W_M - \kappa I)V_M$,

$$r_{F,M} \le \frac{(\kappa + \delta_M)^2}{\kappa} \ell_M.$$

Put $\varepsilon_M = (\delta_M + \delta_M^2/\kappa)\ell_M$ and $\eta_M = \varepsilon_M + q r_{F,M}/(1-q)$. The two assumed resources imply $\eta_M = o(M^{-1/2})$. With $\lambda_M = 1 + \eta_M$, one has $\lambda_M - b_M \ge 1 - q$; hence Proposition 9.4 gives $G_M \preceq (1 + \eta_M) I$, including the mixed block. The original-domain absorption bound is

$$q_H[h] \ge -\left[\delta_M + M^{-1} + \frac{\eta_M B_{M,1}}{1 + \eta_M}\right] \|h\|^2.$$

Since $B_{M,1} = O(\sqrt{M})$, its allowance tends to zero. Theorem 3.2 applies: under the stated complete complementary-gap and lift hypotheses, the Riemann hypothesis holds. □

The corollary uses a fixed gap below one, with both resources evaluated on the same head. The matrix-valued signed Schur comparison can retain additional cancellation lost in these scalar sufficient bounds.

10 Regularized null certificates and physical lift budgets

The null-source construction can avoid an unregularized source-Gram inverse. The price becomes an explicit approximation error and the physical norm of its constructed lift. In this section write $d = \|W_M\|$ and $\mathcal{D} = d + d^2/\kappa$.

Theorem 10.1 (Regularized complete certificate)**.** *Let $Z : \mathbb{C}^r \to \mathcal{X}$ be a physical isometry with range in $\operatorname{Dom} H \cap \ker H$. Put $A = E_M^* Z$, $K = A^* A$, and, for $t > 0$,*

$$J = A(K + tI)^{-1} A^*, \qquad T = Z(K + tI)^{-1} A^*.$$

Then $0 \preceq J \preceq I$, $J = E_M^* T$, and

$$\|G_M\| \le 1 + \frac{\mathcal{D}}{4t} + \frac{\tau_t}{\kappa}, \qquad \tau_t = \operatorname{Tr} N_M - \operatorname{Tr}\big((K + tI)^{-1} K (K + tI)^{-1} Z^* N_M^2 Z\big). \tag{41}$$

If a Hilbert–Schmidt map $X : \mathcal{X} \to \mathbb{C}^r$ satisfies $\|E_M^* - AX\|_{\mathrm{HS}} \le \epsilon$ and $\|X\|_{\mathrm{HS}} \le L_0$, then

$$\|G_M\| \le 1 + \frac{2\epsilon^2}{\kappa} + \sqrt{\frac{2\mathcal{D}}{\kappa}} L_0. \tag{42}$$

Every trace and approximation here uses the complete source metric.

Proof. The singular-value multipliers of T are $\sigma/(\sigma^2+t)$; hence $\|T\|^2 \le 1/(4t)$. The exact signed-null identity, applied to Tf, gives

$$\langle Jf, G_M Jf\rangle \le \|Jf\|^2 + \mathcal{D}\|Tf\|^2 \le (1+\mathcal{D}/(4t))\|f\|^2.$$

Use the positive partition $N_M = E_M J^2 E_M^* + E_M(I-J^2)E_M^*$. After inverse congruence its first term has norm $\|JG_MJ\|$, and its second is at most $\kappa^{-1}\operatorname{Tr}(E_M(I-J^2)E_M^*)$. Expanding J^2 and using cyclicity proves (41). Completing the finite-dimensional square gives

$$\operatorname{Tr}(E_M(I-J)E_M^*) = \inf_X\{\|E_M^* - AX\|_{\mathrm{HS}}^2 + t\|X\|_{\mathrm{HS}}^2\}.$$

Since $I-J^2 \preceq 2(I-J)$, the excess in (41) is at most $\mathcal{D}/(4t)+2(\epsilon^2+tL_0^2)/\kappa$. For $\mathcal{D}, L_0 > 0$ optimize at $t = \sqrt{\mathcal{D}\kappa/(8L_0^2)}$; the zero cases follow by limits. □

For independent raw columns Z_0, replace $K+tI$ by $Z_0^* N_M Z_0 + tZ_0^*Z_0$. In (42) the physical lift norm is then $L_0 = \|Z_0X_0\|_{\mathrm{HS}}$. This preserves the original physical metric without requiring explicit formation of its inverse square root.

The collar estimate of Theorem 12.1 also survives regularization. With P_I denoting source multiplication by $I \subset [0, e^{A\sqrt{\log M}}]$ for a fixed $A > 0$, suppose

$$\|(I-P_I)E_M^* - AX\|_{\mathrm{HS}} \le \epsilon_{\mathrm{out}}, \qquad \|X\|_{\mathrm{HS}} \le L_{\mathrm{out}}.$$

Completing the same square with $(I-P_I)E_M^*$ and using the triangle inequality in inverse energy gives

$$\|G_M\| \le 1 + \frac{\mathcal{D}}{4t} + \left(\sqrt{c_I} + \sqrt{\frac{2(\epsilon_{\mathrm{out}}^2 + tL_{\mathrm{out}}^2)}{\kappa}}\right)^2, \quad c_I = C_A|I|(1+\log\log M). \tag{43}$$

No projection is commuted with J. The square includes the mixed term, and the exterior includes every frequency outside I.

Proposition 10.2 (Order-uniform original-norm null budget)**.** *For* $z_j = (\psi^{(2j+3)} - \psi^{(2j+1)}/4)/s$,

$$\|z_j\|_2 \le 100\, 64^{2j+3}(2j+3)!, \qquad j \ge 0.$$

Consequently the raw map $Z_0 = (z_0, \dots, z_{r-1})$ *satisfies*

$$\|Z_0\|_{\mathrm{HS}}^2 \le B_r := 10000r\, 64^{4r+2}[(2r+1)!]^2, \qquad \|Z_0X_0\|_{\mathrm{HS}} \le \sqrt{B_r}\|X_0\|_{\mathrm{HS}}. \tag{44}$$

Proof. Set $\rho = 1/64$ and $r_u = \pi e^{2|u|}$. The complete theta series is holomorphic in $|\operatorname{Im} z| < \pi/4$ and even there. On $|z-u| = \rho$, reflect to the right half-plane if needed. Then $\operatorname{Re}(\pi e^{2z}) > 3r_u/4$ and $|\pi e^{2z}| \le e^{2\rho} r_u$. Its full series gives

$$|\psi(z)| \le 24 e^{|u|/2} r_u^2 e^{-3r_u/4} \sum_{n\ge 1} n^4 e^{-3(n^2-1)\pi/4} < 48 e^{|u|/2} r_u^2 e^{-3r_u/4}.$$

Indeed the $n=2$ summand is below $1/4$ and subsequent ratios are below $1/256$. Cauchy's estimate yields $|\psi^{(k)}(u)| \le 48k!64^k e^{|u|/2} r_u^2 e^{-3r_u/4}$. Divide by the original lower envelope $s^2 \ge 2r_u^2 e^{-r_u}$ and integrate both arms using $du = dr_u/r_u$ and $e^{|u|} = \sqrt{r_u/\pi}$. Thus

$$\|\psi^{(k)}/s\|_2^2 \le \frac{1152}{\sqrt{\pi}}(k!64^k)^2 \int_\pi^\infty r^{3/2} e^{-r/2}\, dr \le 3456\sqrt{2}(k!64^k)^2 < 80^2(k!64^k)^2.$$

The triangle inequality at $k = 2j+3$ and $k-2$ proves the first bound. Summing its squares and using the largest summand proves the second. □

Since $\log B_r = O(r\log(r+1))$, this gives a useful cofinal lift window. At $\kappa = M^{-1}$ the complete averaged remainder has $d_M \le \mathrm{poly}(M)e^{-\pi M}$. If $r_M = o(M/\log M)$ and $\log(1+\|X_{0,M}\|_{\mathrm{HS}}) = o(M)$, then

$$\sqrt{M\mathcal{D}_M}\,\|Z_{0,M}X_{0,M}\|_{\mathrm{HS}} \le e^{-\pi M/2+o(M)}.$$

Thus the lift part of (42) is paid at every fixed polynomial scale. A complete-source approximation with $\epsilon_M^2 = o(M^{-a-1})$, $1/2 < a < 1$, would then give the excess $o(M^{-a})$ and the vanishing-allowance endpoint. The approximation norm includes the entire source exterior as part of its complete-source error budget.

11 Fixed-strip high-frequency criteria for RH

The complete arithmetic form admits a high-frequency receiving criterion at a fixed averaging seed. A fixed zero-free half-plane supplies an exponential precision scale for this criterion. The construction keeps the original Hilbert norm and minimal domain; its finite null constraints are normalized in the full arithmetic source.

Fix $M = 16$ in Theorem 5.1, and abbreviate $P_k = P_{16,k}$, $N_k = N_{16,k}$ and $d_k = d_{16,k}$. All operators below act on the original odd polar space $\mathcal{K}$ of Section 3, with form domain V. Let γ be the first positive critical ordinate. The rigorous indexed zero isolation and count in [8] give $14.1 < \gamma < 14.2$, simplicity of this pair, and no other nontrivial zero with $|\Im\rho| \le 21$. Write Q_b for the original polar projection, and set

$$\begin{aligned} v_\gamma &= Q_b[s\sin(\gamma\,\cdot)], & Z_\gamma &= 2v_\gamma v_\gamma^*,\\ P_k^{>} &= Q_b M_s[1_{|\xi|>17}(m_{16,k})_+](D)M_s Q_b, & N_k^{>} &= Q_b M_s[1_{|\xi|>17}(-m_{16,k})_+](D)M_s Q_b,\\ A_k &= P_k^{>} + Z_\gamma, & \kappa_k &= (3/4)^k. \end{aligned} \tag{45}$$

Thus $0 \preceq N_k^{>} \preceq N_k$, and $Z_\gamma[h] = 2|\int s(u)h(u)\sin(\gamma u)\,du|^2$.

For a fixed integer $r \ge 1$, use the genuine derivative nulls of Lemma 9.1 and define, for all sufficiently large k,

$$Z_r = (z_0,\dots,z_{r-1}), \qquad K_{r,k} = Z_r^* N_k Z_r, \qquad \mathcal{F}_{r,k} = \ker(Z_r^* N_k), \qquad S_{r,k} = I - Z_r K_{r,k}^{-1} Z_r^* N_k. \tag{46}$$

The positive definiteness of $K_{r,k}$ and the domain properties used here are proved below. For $r = 0$ put $\mathcal{F}_{0,k} = \mathcal{K}$ and $S_{0,k} = I$. Define the *complete physical receiving norm*

$$\Gamma_{r,k} = \sup_{0\ne x\in V\cap\mathcal{F}_{r,k}} \frac{N_k^{>}[x]}{A_k[x] + \kappa_k\|x\|_2^2}. \tag{47}$$

It uses the entire high source and the inverse of the restricted closed positive form, rather than a finite compression of its outputs.

Theorem 11.1 (Fixed-strip high-frequency criterion)**.** *Suppose every nontrivial zeta zero $\rho = \beta + i\gamma_\rho$ satisfies $\beta \le b$, for some fixed $1/2 \le b < 1$. Choose fixed numbers*

$$\frac{1}{2} < \theta < R < \frac{3}{2} - b.$$

For each fixed $r \ge 0$, RH is equivalent to the existence of a constant C and an unbounded sequence of integers k on which

$$\Gamma_{r,k} \le 1 + C\theta^k. \tag{48}$$

Under RH, $\Gamma_{r,k} \le 1$ for all sufficiently large k.

Corollary 11.2 (The cited zero-free half-plane)**.** *Assume the zeta zero-free conclusion of [12, Main theorem], with $\beta \le b_{\mathrm{new}} = 0.874957019420098946\ldots$. For every fixed $r \ge 0$, a complete bound $\Gamma_{r,k} \le 1 + C(5/8)^k$ along an unbounded sequence implies RH.*

Proof. The cited statement includes zeta and allows its principal pole at 1. It specifies $b_{\mathrm{new}} = (18\xi+7)/(18\xi+9)$ with $\xi < 0.38859 < 7/18$, so $b_{\mathrm{new}} < 7/8$. Choose any fixed $5/8 < R < 3/2 - b_{\mathrm{new}}$ in Theorem 11.1. □

For the coarser supplied strip $\beta \le 7/8$, the robust rational choice $\theta = 3/5$, $R = 49/80$ has horizontal margin $1/80$ and gives the sufficient precision $C(3/5)^k$. If $b = 7/8$ exactly, $\theta = 5/8$ is an excluded endpoint of this argument.

There is also a criterion requiring only the classical zero-free region.

Theorem 11.3 (Classical high-frequency criterion)**.** *For every fixed $r \ge 0$ and $0 < c < 1/48$, RH is equivalent to a complete bound, along an unbounded integer sequence,*

$$\Gamma_{r,k} \le 1 + C2^{-k}\exp(ck/\log k). \tag{49}$$

Under RH the same eventual bound $\Gamma_{r,k} \le 1$ holds.

The following estimates provide the original-domain return and null-lift resources for both criteria. Their proofs, including every spectral collar and infinite tail, are given in Appendix G.

Proposition 11.4 (Paid finite band and full-source null resources)**.** *The positive form A_k has domain V, and*

$$\begin{aligned} &H = A_k - N_k^{>} + W_k^{>}, && \|W_k^{>}\| \le e_k := \epsilon_k + d_k, \\ &\epsilon_k = 2\cdot 10^{-17}4^{-k} + 72(3/50)^{k+1} + \frac{90}{7}(17/50)^k + 8640000000\,3^{-k}. \end{aligned} \tag{50}$$

In particular $2^k e_k/\kappa_k \to 0$. For each fixed $r \ge 1$, with $D_k = \mathrm{diag}(1, k, \ldots, k^{r-1})$,

$$(2^k k^{-3/2})^{-1} D_k K_{r,k} D_k \longrightarrow \mathsf{M}_r \succ 0, \tag{51}$$

where, for $0 \le i, j < r$ and $c_0 = -i\widehat{\psi}(0)/4 \ne 0$,

$$(\mathsf{M}_r)_{ij} = \frac{8|c_0|^2}{\pi^2}(-1)^{i+j}\int_{\mathbb{R}} t^{2i+2j+2}e^{-2t^2}\,dt.$$

Consequently

$$\lambda_{\min}(K_{r,k}) \ge c_r 2^k k^{-2r+1/2}, \qquad \|S_{r,k}\|^2 = O_r(k^{2r-1/2}), \qquad (\kappa_k + e_k)\|S_{r,k}\|^2 \longrightarrow 0. \tag{52}$$

The map $S_{r,k}$ preserves V and q_H, has range $\mathcal{F}_{r,k}$, and satisfies $N_k[S_{r,k}h] \le N_k[h]$. The head order is fixed throughout.

An exact constrained inverse retains the directional receiving terms. For $r \ge 1$, let $V_{r,k} = Z_r K_{r,k}^{-1/2}$, $B_k = N_k V_{r,k}$ and $R_k = (A_k + \kappa_k I)^{-1}$. If $R_{r,k}$ denotes the inverse of the restricted form on $\mathcal{F}_{r,k}$, extended by zero, then

$$\begin{aligned} &C_k = B_k^* R_k B_k, && X_k = V_{r,k}^*(A_k + \kappa_k I)V_{r,k}, \\ &R_{r,k} = R_k - R_k B_k C_k^{-1} B_k^* R_k, && 0 < C_k^{-1} \preceq X_k \preceq [1 + (\kappa_k + e_k)\|V_{r,k}^* V_{r,k}\|]I. \end{aligned} \tag{53}$$

Thus $\Gamma_{r,k} = \|(E_k^{>})^* R_{r,k} E_k^{>}\|$, where $N_k^{>} = E_k^{>}(E_k^{>})^*$. The inverse in (53) is an actual physical restriction; its constraint columns use N_k, not $N_k^{>}$. The one-sided bound on C_k^{-1} does not replace the complete inequality (48) or (49).

Proof of Theorems 11.1 and 11.3. Appendix G proves that compact original polar tests with Gevrey-2 physical profile form a V-dense class. For each fixed such h, under the strip hypothesis,

$$N_k^{>}[S_{r,k}h] \le C_{h,r,b,R}(1+k^{2r-1/2})\{k^8R^{-k}+2^ke^{-a_{h,r}k^2}+17^{-k}\} \tag{54}$$

when $r \ge 1$; for $r = 0$ omit the polynomial lift factor. Its product with θ^k tends to zero. Independently of a fixed strip, the classical estimate is

$$N_k^{>}[S_{r,k}h] \le C_{h,r}(1+k^{2r-1/2})2^k\{k^4\log(4k^2)e^{-k/(24\log(4k^2))}+e^{-a_{h,r}k}\}. \tag{55}$$

Its product with $2^{-k}e^{ck/\log k}$ tends to zero for $c < 1/48$. All constants concern fixed tests and a fixed head.

Write the stated positive excess bound as η_k. Apply the complete comparison to $x_k = S_{r,k}h$ and use original nullity and (50):

$$q_H[h] = q_H[x_k] \ge -(\kappa_k+e_k)\|x_k\|^2 - \frac{\eta_k}{1+\eta_k}N_k^{>}[x_k].$$

The first term tends to zero by (52); the appropriate fixed-test energy estimate pays the second. Positivity follows on the Gevrey class and then on V by density. The original odd Weil criterion, Lemma A.1, gives RH. Conversely RH implies $N_k^{>} \preceq A_k + e_kI$. Since $e_k < \kappa_k$ eventually, its restriction proves $\Gamma_{r,k} \le 1$ for every fixed r. □

11.1 Critical complementary caps and full-source precision

Proposition 11.5 (Exact-null complementary allowance)**.** *Suppose on the original domain $H = P - N + W$, where $N = EE^*$ is bounded positive, P is a closed positive form and $\|W\| \le d$. Let V_0 have finitely many genuine null columns, normalized by $V_0^*NV_0 = I$. Put*

$$J = E^*V_0, \quad Q = I - JJ^*, \quad G = E^*(P+\kappa I)^{-1}E, \quad S = I - V_0J^*E^*, \quad B_Q = \|EQE^*\|.$$

If the complete comparison $QGQ \preceq (1+\eta)Q$ holds, $\eta \ge 0$, then on all original V,

$$q_H[h] \ge -\left\{(\kappa+d)L^2 + \frac{\eta B_Q}{1+\eta}\right\}\|h\|^2, \qquad L^2 = \|S\|^2 \le 1 + \|V_0^*V_0\|\|N\|. \tag{56}$$

Proof. The source isometry $J^*J = I$ gives $E^*S = QE^*$. Nullity gives $q_H[Sh] = q_H[h]$ and $S(V) \subset V$. On $x = Sh$ the complete cap implies $N[x] \le (1+\eta)(P[x]+\kappa\|x\|^2)$, whereas $N[x] = \|QE^*h\|^2 \le B_Q\|h\|^2$. Rearrange and add W. The norm estimate is the orthogonal-range calculation proving (96). □

Corollary 11.6 (A fixed-head order-one threshold)**.** *For the actual order-one averaging, fix any initial derivative-null head of positive finite length and normalize it in $N_{M,1}$. At ridge $\kappa = M^{-1}$, a complete complementary cap $\|Q_MG_MQ_M\| \le 1+\eta_M$ with $\eta_M = o(M^{-1/2})$ on an unbounded sequence implies RH. The physical lift has $L_M^2 = O_r(1)$ without a growing-head hypothesis.*

Corollary 11.7 (Polynomial relaxation for the full source)**.** *Fix a real A. At seed* 16 *and ridge $(3/4)^k$, the complete full-source comparison*

$$N_k \preceq (1+Ck^A2^{-k})(P_k+\kappa_kI)$$

on an unbounded integer sequence implies RH.

The full-source assertion uses a fixed null-jet repair of each test. The high-frequency criteria above instead use the original or full-source-normalized physical receiving space.

12 Uniform covariance on small sets in a growing band

The null-source representation can be poorly conditioned near zeros of $\widehat{\psi}$. The actual positive inverse controls small frequency sets there without division by that transform. An exact Fourier shift extends this control beyond every fixed power of $\log M$.

Theorem 12.1 (Growing-band observation bound)**.** *Put $U = \log M$, and let $L_M \to \infty$ satisfy*

$$\frac{\sqrt{L_M}}{U} \longrightarrow \infty, \qquad L_M^3 \bar{e}_M \longrightarrow 0, \tag{57}$$

where $\bar{e}_M$ is the complete Abel error of Theorem 7.1. For all sufficiently large M and every measurable $I \subset [0, L_M]$,

$$\|E_I^*(P_M + M^{-1}I)^{-1}E_I\| \le C|I|(1 + \log\log M). \tag{58}$$

The constant and onset may depend on the admitted schedule, but not on the number or locations of the components of I. The upper bound survives additional exact finite constraints, with the same-form restricted receiver, dense restricted form domain and exactly projected source.

Proof. Take $T = 2L_M$. The complete arithmetic phase debit is

$$\varepsilon_M(T) = \frac{3(1+4T^2)}{4\sqrt{M}} + \frac{(1+4T^2)(T+2)}{4}\bar{e}_M.$$

Since $\bar{e}_M \ge 3/\sqrt{M}$, both terms tend to zero by (57). The scalar inequalities in the actual-origin proof consequently give, for all sufficiently large M,

$$P_M \succeq A_T := \sqrt{M}Q_b s B(D) w(UD) 1_{|D|\le T} s Q_b, \qquad n_{M,1}(\xi) \le \frac{12\sqrt{M}}{1+4\xi^2} \quad (0 \le \xi \le T),$$

where $B(\xi) = 8/(1+4\xi^2)$, $w(t) = (-\cos t - 1/2)_+$ and $\mu > 0$ is its mean. These are inequalities for the actual arithmetic receiver and observation, before theta sandwiching.

Put $e_\xi = Q_b[s\sin(\xi\,\cdot)]$ and $R_0 = \frac{1}{2}\log(2U/\pi)$. Let χ_{R_0} be the even fixed-width Gevrey-2 cutoff equal to one on $[-R_0, R_0]$ and zero outside $[-R_0 - 1, R_0 + 1]$. Define

$$f_\xi(u) = \chi_{R_0}(u)\sin(\xi u), \qquad Y_\xi = \frac{(1 - 4\partial_u^2) f_\xi}{8\mu\sqrt{M}\, s}.$$

This trial is compact smooth, odd and exactly polar legal: integration by parts transfers $1 - 4\partial_u^2$ to $\sinh(u/2)$ and annihilates it. Uniformly for $0 \le \xi \le L_M$, product differentiation and the complete two-arm theta envelopes give

$$\begin{aligned} \|f_\xi\|_{1,4}^2 &\le C(1+\xi^2)(1+\log U), \\ \|s(\sin(\xi\,\cdot) - f_\xi)\|^2 &\le CUe^{-2U}, \\ \|Y_\xi\|^2 &\le CM^{2e^2-1}(1+\xi^2)^2(1+\log U). \end{aligned}$$

The first bound uses a support of length $O(1 + \log U)$ and a fixed transition width. For the last bound the support satisfies $r = \pi e^{2|u|} \le 2e^2 U$, so the original lower envelope $s^2 \ge 2r^2 e^{-r}$ controls division by s.

Every positive-order derivative of χ_{R_0} is supported on two fixed-width translated intervals. Thus $\|\chi_{R_0}^{(j)}\|_1 \le C_0 C_1^j (j!)^2$ for $j \ge 1$, independently of R_0. Integration by parts with j proportional to $\sqrt{|\omega|}$ gives $|\widehat{\chi}_{R_0}(\omega)| \le Ce^{-c\sqrt{|\omega|}}$ for large $|\omega|$, uniformly in R_0. The exact shift identity

$$\widehat{f}_\xi(\omega) = \frac{\widehat{\chi}_{R_0}(\omega-\xi) - \widehat{\chi}_{R_0}(\omega+\xi)}{2i}$$

therefore proves the complete exterior bound

$$\|1_{|D|>T} f_\xi\|_2 \le Ce^{-c\sqrt{L_M}}, \qquad 0 \le \xi \le L_M.$$

Indeed both shifted frequencies have modulus at least L_M there; the polynomial factor arising on integration is absorbed by reducing c.

The full-frequency continuation satisfies

$$A_\infty = \mu\sqrt{M} Q_b s B(D) s Q_b + J_M, \qquad \|J_M\| \le CM^{5/2} e^{-\pi(M-1)}.$$

The polar Green identity gives its exact mean action $Q_b(sf_\xi)$. Consequently

$$e_\xi - A_T Y_\xi = Q_b[s(\sin(\xi\,\cdot) - f_\xi)] - J_M Y_\xi + \mu^{-1} Q_b s w(UD) 1_{|D|>T} f_\xi.$$

A squared triangle bound pays all three terms and their mixed products. At ridge M^{-1} the physical error costs $O(U/M)$; the Fourier error costs $O(Me^{-2c\sqrt{L_M}})$, smaller than every fixed inverse power of M by (57). Also that admission and $\bar{e}_M \ge 3/\sqrt{M}$ imply $L_M = o(M^{1/6})$, so the displayed trial norm is polynomial in M and is absorbed by the harmonic error. Finally

$$\langle Y_\xi, A_T Y_\xi \rangle \le CM^{-1/2} \|f_\xi\|_{1,4}^2 + \|J_M\| \|Y_\xi\|^2.$$

The kernel-safe least-squares estimate (34) and $A_T \preceq P_M$ now prove, uniformly on the whole band,

$$\langle e_\xi, (P_M + M^{-1})^{-1} e_\xi \rangle \le CM^{-1/2}(1+\xi^2)(1+\log U). \tag{59}$$

The negligible errors are absorbed uniformly; at $\xi = 0$ the source is zero. Integrating in the exact source metric gives

$$\|R_M^{1/2} E_I\|_{\mathrm{HS}}^2 = \int_I n_{M,1}(\xi) \langle e_\xi, R_M e_\xi \rangle \frac{d\xi}{\pi} \le C|I|(1+\log U).$$

This trace bounds the full operator norm, retaining every profile combination. Variational restriction gives the last assertion. □

Corollary 12.2 (Unconditional expanded bands)**.** *For every fixed $A > 0$, estimate* (58) *holds for all measurable $I \subset [0, e^{A\sqrt{\log M}}]$ and all sufficiently large M. Constants and onset may depend on A; no uniformity for a growing A is asserted.*

Proof. The complete Abel estimate proved in Theorem 8.1, using the classical Vinogradov–Korobov prime-number error [9], gives $\bar{e}_M = O_B(e^{-B\sqrt{U}})$ for every fixed $B > 0$. For $L_M = e^{A\sqrt{U}}$ choose $B > 3A$. Then $L_M^3 \bar{e}_M = O_B(e^{-(B-3A)\sqrt{U}}) \to 0$ and $\sqrt{L_M}/U = e^{A\sqrt{U}/2}/U \to \infty$. Theorem 12.1 applies in the original norm and domain. □

Corollary 12.3 (Complete collar and null-source certificate)**.** *Use the null head of Theorem 9.3, set $\kappa = M^{-1}$, and let P_I be the source multiplication projection onto $I \subset [0, L_M]$ for a schedule admitted by Theorem 12.1 (in particular $L_M = e^{A\sqrt{\log M}}$ for fixed $A > 0$). Define*

$$N_{\mathrm{out}} = E_M(I - P_I)E_M^*, \qquad \tau_{\mathrm{out}} = \operatorname{Tr} N_{\mathrm{out}} - \operatorname{Tr}(V_Z^* N_{\mathrm{out}}^2 V_Z), \qquad c_I = C|I|(1 + \log\log M).$$

If

$$(\delta_M + \delta_M^2/\kappa)\ell + \left(\sqrt{c_I} + \sqrt{\tau_{\mathrm{out}}/\kappa}\right)^2 \le \eta, \tag{60}$$

then $G_M(\kappa) \preceq (1+\eta)I$, and on the entire original domain

$$q_H[h] \ge -\left[\delta_M + \kappa + \frac{\eta B_{M,1}}{1+\eta}\right]\|h\|^2.$$

In particular, a cofinal family satisfying (60) *with $\eta = o(M^{-1/2})$ gives vanishing original-norm allowances and RH.*

Proof. Let $Q = I - JJ^*$ as above. Without commuting Q and P_I, cyclicity gives

$$\|E_M(I - P_I)Q\|_{\mathrm{HS}}^2 = \operatorname{Tr} N_{\mathrm{out}} - \operatorname{Tr}(V_Z^* N_{\mathrm{out}}^2 V_Z).$$

Here V_Z uses the full N_M normalization. The triangle inequality in actual inverse energy gives

$$\|QG_MQ\| \le \left(\|R_M^{1/2}E_M P_I Q\| + \|R_M^{1/2}E_M(I - P_I)Q\|\right)^2 \le \left(\sqrt{c_I} + \sqrt{\tau_{\mathrm{out}}/\kappa}\right)^2.$$

Combine this with (38) and the positive-block estimate in Theorem 9.3. The original-domain lower bound follows from Corollary 5.2, with complete negative source and no finite-source omission. Since $B_{M,1} = O(\sqrt{M})$ and $\delta_M \to 0$, the stated rate gives the endpoint. □

All frequencies outside I, including those above L_M, occur in τ_{out}. The mixed collar/exterior cost is the cross term in the square in (60). Thus the displayed criterion specifies a complete quantitative construction, with growing-rank conditioning and the complete trace remainder evaluated in the same source metric.

13 Complete-response verification and bounded regularization

The criterion of Theorem 2.1 concerns the original Weil positivity endpoint. We now give a complementary construction for that same endpoint, retaining the complete Green response and bounding the regularization in profile coordinates. The finite-Green construction and its evaluated examples retain their stated definitions. Each matrix in the complementary construction below is assembled from its stated full-response form.

Lemma 13.1 (A support-uniform Green norm)**.** *For every $L > 0$ and every finite Green prefix,*

$$0 \preceq R_{L,J} \preceq \frac{\pi}{2}I, \qquad 0 \preceq R_L \preceq \frac{\pi}{2}I.$$

The bounds use the complete reflected response normalization of Section 2.

Proof. Set $c_j = 2j + 1/2$ and change to the boundary coordinate $s = L - x$. The exact response profile satisfies

$$r_j(L-s) = \frac{e^{-c_j s} - e^{-c_j(2L-s)}}{\sqrt{1+e^{-2c_j L}}}, \qquad 0 \le r_j(L-s) \le e^{-c_j s}, \quad 0 < s < L.$$

The finite feature Gram $G_{ij} = \langle r_i, r_j \rangle$ therefore has nonnegative entries and obeys the entrywise bound

$$G_{ij} \le \int_0^L e^{-(c_i+c_j)s}\,\mathrm{d}s \le \frac{1}{2(i+j+1/2)} =: \mathsf{H}_{ij}.$$

Put $w_j = 4^{-j}\binom{2j}{j}$. Its generating function and Tonelli's theorem give, for every $i \ge 0$,

$$\begin{aligned}\sum_{j\ge 0} \mathsf{H}_{ij} w_j &= \frac{1}{2}\int_0^1 t^{i-1/2}(1-t)^{-1/2}\,\mathrm{d}t \\ &= \int_0^{\pi/2} \sin^{2i}\theta\,\mathrm{d}\theta = \frac{\pi}{2} w_i.\end{aligned}$$

The last identity follows from integration by parts, starting with $i = 0$. For a finite complex coefficient vector z, use symmetry and

$$2|z_i||z_j| \le |z_i|^2 \frac{w_j}{w_i} + |z_j|^2 \frac{w_i}{w_j}.$$

It follows that

$$z^* G z \le \sum_{i,j<J} \mathsf{H}_{ij}|z_i||z_j| \le \sum_{i<J} \frac{|z_i|^2}{w_i} \sum_{j<J} \mathsf{H}_{ij} w_j \le \frac{\pi}{2}\|z\|^2.$$

If $T_J z = \sum_{j<J} z_j r_j$, then $G = T_J^* T_J$ and $R_{L,J} = T_J T_J^*$ have the same operator norm. This proves the finite-prefix bound. The positive prefixes increase to the original complete response; their uniform bound gives a strong monotone limit and the same inequality for R_L. □

Lemma 13.2 (Full-response commutator and effective actions)**.** *Put $H_L = 1+8L/\pi^2$. The complete response R_L preserves* Dom D_L *and*

$$\|[D_L, R_L]\| \le 600 H_L.$$

The same bound holds uniformly for every finite Green prefix. If a column map V is supported in the first N DN modes and P_m projects onto the first $m \ge 1$ output modes, then

$$\|(I - P_m) R_L V\| \le H_L \sqrt{N/m}\,\|V\|.$$

Every finite output coefficient can be enclosed effectively. Truncating its exact reflected scalar series at index $J \ge 1$ incurs entry error at most $2/(LJ)$. An additional directed entry error ϵ gives low-output map error at most $\sqrt{mN}\{2/(LJ) + \epsilon\}\|V\|$.

Proof. Direct integration in the normalization of Section 2 gives

$$\langle r_j, e_q \rangle = \sqrt{2/L}(-1)^q \frac{c_j\sqrt{1+e^{-2c_j L}}}{c_j^2 + k_q^2}.$$

For $x, y > 0$, the function $c^2/\{(c^2+x^2)(c^2+y^2)\}$ is unimodal, with maximum $(x+y)^{-2}$ and integral $\pi/\{2(x+y)\}$. A spacing-two unimodal sum is bounded by half its integral plus twice its maximum. Consequently

$$|(R_L)_{qr}| \le \frac{\pi}{L(k_q+k_r)} + \frac{8}{L(k_q+k_r)^2} \le \frac{H_L}{q+r+1}.$$

This also bounds every partial sum, since its summands have a common sign. Writing $t = k/2$ and $b_n = n + 1/4$, the digamma series yields

$$ka'(k) = 2t^2 \sum_{n\ge 0} \frac{b_n}{(b_n^2+t^2)^2} \le 9.$$

Indeed, for $t \le 1/4$, the reciprocal-cube sum is at most 72. For $t \ge 1/4$, the unimodal sum has integral $1/(2t^2)$ and maximum at most $1/(2t^3)$, giving $1+2/t \le 9$. Thus $|a(x)-a(y)| \le 9|\log(x/y)|$. Set $x = q + 1/2$, $y = r + 1/2$ and use Schur weights $x^{-1/2}$. The bound $|\log t| \le 4(t^{1/4} + t^{-1/4})$, together with

$$\sum_{r\ge 0} \frac{x^\nu y^{-\nu}}{x+y} \le 2 + \frac{\pi}{\sin \pi\nu}, \qquad \nu = 1/4, 3/4,$$

proves a commutator norm at most $36H_L(4 + 2\pi\sqrt{2}) < 600H_L$. The displayed sum follows by bounding its decreasing summand by its first value and the positive-line integral. Each column $R_L e_r$ belongs to $\operatorname{Dom} D_L$: the coefficient envelope remains square summable after multiplication by $a(k_q) = O_L(\log(2 + q))$. On finite DN sums the bounded matrix is therefore the genuine commutator. Graph-norm approximation and closedness of D_L extend the identity and domain preservation to its full domain.

The squared Hilbert–Schmidt output tail is at most $H_L^2 \sum_{q<N}(m+q)^{-1} \le H_L^2 N/m$. For the scalar series, the omitted absolute sum is at most $(4/L)\{c_J^{-2} + (2c_J)^{-1}\} \le 2/(LJ)$. The finite-block error follows from its Frobenius norm. These low- and high-output errors concern orthogonal output spaces and may be combined in quadrature. They do not assert operator-norm convergence of $R_{L,J}$ to R_L on unrestricted inputs. □

Corollary 13.3 (Support-uniform response resources)**.** *Put* $\widehat{H}_L = \min\{1 + 8L/\pi^2, 7\}$. *The full response and every prefix have coefficient envelope* $|(R_L)_{qr}| \le \widehat{H}_L/(q+r+1)$, *and*

$$\|[D_L, R_L]\| \le 600\widehat{H}_L \le 4200, \qquad \|(I-P_m)R_L V\| \le \widehat{H}_L\sqrt{N/m}\,\|V\|.$$

For a prefix the commutator is also bounded by $\min\{14J, 600\widehat{H}_L\}$.

Proof. Keep the exact coefficient series in Lemma 13.2. Write $s = x + y$ and $f(c) = c^2/\{(c^2 + x^2)(c^2+y^2)\}$. If its maximum occurs at $\sqrt{xy} \ge 1/2$, then $s \ge 1$ and $\sup f = 1/s^2 \le 1/s$. Otherwise its maximum on $[1/2, \infty)$ is at $c = 1/2$; with $a = 2x$, $b = 2y$,

$$sf(1/2) = \frac{2(a+b)}{(1+a^2)(1+b^2)} \le 2.$$

Thus the shifted unimodal sum is at most $(\pi/4+4)/s$. The exact reflected coefficient has multiplier at most $4/L$, giving the envelope $(1 + 16/\pi)/(q+r+1) < 7/(q+r+1)$, also for every prefix. Combine this with the earlier H_L envelope and use the same weighted-Schur, domain and output-tail arguments. The prefix bound $14J$ follows independently from its finite rank-one decomposition in Appendix C. □

Proposition 13.4 (Complementary capped full-response criterion)**.** *Retain the schedule, core isometry U_M, complete constraints F_M and projection Q_M of Section 2, with $\beta = 1$. Rebuild every operator and matrix below from*

$$w_M = \min(M^{-1}W_{\mathrm{low}}, \beta), \qquad S_M^{\mathrm{cap}} = D_L + R_L + V_L + w_M, \qquad d_M^{\mathrm{cap}} = \beta + w_M.$$

Then the entire receiver obeys $T_M^{\mathrm{cap}} \succeq (d_M^{\mathrm{cap}})_{Q_M} \succeq \beta I$. For the exact isometric constraint map,

$$(2\beta)^{-1} I \preceq F_M^*(d_M^{\mathrm{cap}})^{-1} F_M \preceq \beta^{-1} I.$$

In particular this complete weighted constraint Gram has condition number at most two. Strict finite lower Schur certificates for these rebuilt forms along any unbounded subsequence of M imply RH. Conversely, under RH each admitted M has a strictly positive, effectively verifiable finite Galerkin certificate. The sufficient trial-size exponent remains $m = \exp(M^{1/3+o(1)})$ on the stated schedule.

Proof. The existing core admission proves the receiving lower for $D_L + V_L$; adding the nonnegative complete R_L and w_M gives the asserted floor. The Gram bound follows by compressing $(2\beta)^{-1} I \preceq (d_M^{\mathrm{cap}})^{-1} \preceq \beta^{-1} I$ by the exact isometry F_M. It is not a condition bound for an unnormalized raw constraint list.

In the rest of this proof all blocks are rebuilt: $H = \mathsf{U}_M^* S_M^{\mathrm{cap}} \mathsf{U}_M$, $C = Q_M S_M^{\mathrm{cap}} \mathsf{U}_M$, and $T = T_M^{\mathrm{cap}}$. Put $\alpha_D = a(k_{N_M})$, $K_F = \sqrt{64 + \alpha_D^2}$, $c_D = \alpha_D + \mathfrak{h}$, $G = 1 + (\mathfrak{h} + K_F)/\beta$ and $R_0 = \|p\|^2/|\langle p, e_0\rangle|^2 < 16L$. Define

$$P_* = \alpha_D R_0 + \mathfrak{h}(1 + R_0), \qquad F_1 = \alpha_D^2 + \mathfrak{h}\alpha_D + K_1 + K_F c_D,$$
$$G_2 = F_1 + (\mathfrak{h} + K_F) G c_D + (K_1 + K_F \mathfrak{h}) c_D/\beta.$$

Here $\mathfrak{h}$ and K_1 are the bounds specified next, not constants copied from a saved finite-Green calculation.

Use the same legal constrained DN trials and second-graph derivation of Appendix C, recomputing its constants for this form. The bounded perturbation norm is at most $\mathfrak{h} = L + \pi/2 + B_M + \beta$. Its commutator budget is

$$K_1 = 600 H_L + [4(2+\pi) + 4(L + \pi/2)] \sum_{2 \le n < e^{2L}} \frac{\Lambda(n)}{\sqrt{n}} + 4(L + \pi/2) n_{\mathrm{cell}} \beta.$$

The last term uses $\mathrm{TV}(w_M) \le 2 n_{\mathrm{cell}} \beta$; no smooth derivative of the step is assumed. Every projection and polar correction in the second-graph constants G_2, P_* is retained. Thus for the actual complete return $X = C^*(T_M^{\mathrm{cap}})^{-1} C$ and its finite Galerkin return X_m,

$$0 \preceq X - X_m \preceq \epsilon_m I, \qquad \epsilon_m = G_2^2 \{a(k_m)^{-3} + P_* a(k_m)^{-4}\}.$$

All these constants are $M^{o(1)}$ on the stated schedule. Choosing $a(k_m) \ge \max\{1, P_*, (72 M G_2^2)^{1/3}\}$ pays $\epsilon_m \le 1/(36M)$ and gives the asserted near-threshold trial size.

Since $0 < w_M \le M^{-1} W$, a passing complete Schur lower implies the original supported allowance $q_H[h] \ge -M^{-1}\|h\|^2$. Support exhaustion proves the forward assertion. Under RH the original profile form is nonnegative, and $W_{\mathrm{low}} \ge W/2 \ge 1/18$ gives $S_M^{\mathrm{cap}} \succeq I/(18M)$. Its exact Schur complement has the same core lower bound. Hence $H - X_m - \epsilon_m I \succeq I/(36M)$. A two-sided matrix-center error at most $1/(144M)$, subtracted once again to form a directed lower, leaves $I/(72M)$. Lemma 13.2 encloses the exact full-response entries; all arithmetic, coordinate, constraint and inverse errors must also be paid. This proves effective strict recognition, not a total runtime bound or an evaluated new capped certificate. □

13.1 The complete Hilbert principal channel and a compact remainder

The coefficient formula also separates the infinite principal response from a remainder with a whole-operator approximation bound. Let E be the diagonal unitary $E_{qq} = (-1)^q$ on DN coefficients and let $\mathsf{H}_{qr} = (q + r + 1)^{-1}$ be the Hilbert matrix. It is bounded: the weighted Schur argument of Lemma 13.1 also bounds H, since its entries are at most twice those of that lemma's half-shift comparison matrix.

Proposition 13.5 (Complete principal-part decomposition)**.** *For every $L > 0$ there is a self-adjoint Hilbert–Schmidt operator K_L such that*

$$R_L = \tfrac{1}{2} E\mathsf{H}E + K_L.$$

Put $t_L = e^{-4L}$ and define

$$S_L = e^{-L}\left\{\frac{4t_L(1+t_L)}{(1-t_L)^3} + \frac{2t_L}{(1-t_L)^2} + \frac{1}{4(1-t_L)}\right\},$$
$$\mathfrak{a}_L = \frac{4L}{\pi^2}, \qquad \mathfrak{b}_L = \frac{2L^3 S_L}{\pi^4}.$$

Then

$$|(K_L)_{qr}| \le \frac{\mathfrak{a}_L}{(q+r+1)^2} + \frac{\mathfrak{b}_L}{(q+1/2)^2(r+1/2)^2}, \qquad \|K_L\|_{\mathrm{HS}} \le \mathfrak{a}_L\sqrt{\zeta(3)} + L^3 S_L/3.$$

For the first-m DN projection, $m \ge 1$, set

$$\eta_{L,m} = \frac{\mathfrak{a}_L}{\sqrt{2}\,m} + \frac{\mathfrak{b}_L \pi^2}{\sqrt{18}\,m^{3/2}}.$$

The entire output and finite-remainder errors satisfy

$$\|(I - P_m)K_L\| \le \eta_{L,m}, \qquad \left\| R_L - \left(\tfrac{1}{2}E\mathsf{H}E + P_m K_L P_m\right)\right\| \le 2\eta_{L,m}.$$

Proof. For $x, y > 0$ extend $f(c) = c^2/\{(c^2 + x^2)(c^2 + y^2)\}$ by zero to $c < 0$. It is continuous, nonnegative and unimodal, with maximum $(x + y)^{-2}$, total variation $2/(x + y)^2$, and integral $\pi/\{2(x + y)\}$. The length-two cells centered at $c_j = 2j + 1/2$ partition $[-1/2, \infty)$. On each cell I_j,

$$\left| f(c_j) - \tfrac{1}{2}\int_{I_j} f(c)\,\mathrm{d}c \right| \le \mathrm{Var}_{I_j}(f).$$

Summing the variations, with absolute convergence, proves the two-sided complete lattice estimate

$$\left| \sum_{j \ge 0} f(c_j) - \frac{\pi}{4(x+y)} \right| \le \frac{2}{(x+y)^2}.$$

Apply this to the exact reflected response coefficients in Lemma 13.2. The nonexponential part has main term $\pi/\{2L(k_q + k_r)\} = 1/\{2(q+r+1)\}$ and error at most $4/\{L(k_q + k_r)^2\}$. The exponential part is bounded by

$$\frac{2}{L k_q^2 k_r^2} \sum_{j \ge 0} c_j^2 e^{-2c_j L}.$$

Expansion of $c_j^2 = 4j^2 + 2j + 1/4$ and the exact first two geometric moment sums identify the last sum with S_L. Substituting the DN frequencies yields the displayed entry bound for the difference from $\frac{1}{2}E\mathsf{H}E$.

The squared Hilbert–Schmidt sum of $(q+r+1)^{-2}$ is $\sum_{n\geq 1} n/n^4 = \zeta(3)$. The second comparison is rank one, with Hilbert–Schmidt norm $\mathfrak{b}_L \sum_{q\geq 0}(q+1/2)^{-4} = \mathfrak{b}_L\pi^4/6$. Minkowski gives the asserted norm. Thus the coefficient difference defines a self-adjoint Hilbert–Schmidt operator. Equality of the coefficients proves the decomposition on finite vectors and then on the whole space by boundedness and density.

For $r \geq m \geq 1$, decreasing integral comparison gives

$$\sum_{q\geq 0}(r+q+1)^{-4} \leq \frac{1}{3r^3}, \qquad \sum_{r\geq m}\frac{1}{3r^3} \leq \frac{1}{2m^2}.$$

Convex midpoint comparison also gives $\sum_{r\geq m}(r+1/2)^{-4} \leq 1/(3m^3)$. Combine these with the full sum $\sum_{q\geq 0}(q+1/2)^{-4} = \pi^4/6$ and the two entry bounds to obtain $\|(I-P_m)K_L\| \leq \|(I-P_m)K_L\|_{\mathrm{HS}} \leq \eta_{L,m}$. Finally self-adjointness and $K_L - P_mK_LP_m = (I-P_m)K_L + P_mK_L(I-P_m)$ give the safe factor two in the whole-operator error. □

The approximation retains the exact infinite Hilbert principal channel. It is not a finite-Green approximation of R_L in operator norm. Requested finite K_L entries still need directed coefficient errors, and any solver or inverse for the Hilbert channel needs its own complete error bound. Neither the compact remainder nor its norm controls the signed arithmetic Schur matrix without those further estimates.

14 A complementary constant-shift certificate criterion

The vanishing-allowance criterion above and the criterion below retain the same original arithmetic Weil form. They use different norms and different finite receiving constructions. In particular, the fixed shift below is in the unweighted logarithmic profile norm, not in the theta-congruent norm. Its use is justified by a uniform-semiboundedness argument, which we include in full. Each criterion specifies its own operator, norm and finite verification test.

14.1 Uniform semiboundedness on the odd compact class

Let $\mathcal{Q}(f,g)$ be the original archimedean-minus-prime form on the logarithmic line, with Fourier measure $d\xi/(2\pi)$. Thus its archimedean multiplier is $m_0 + a(\xi)$ and its arithmetic term retains both translations for every prime power. Put

$$\mathcal{S}_1(f) = \sup_{u\in\mathbb{R}} e^{|u|}\max\{|f(u)|, |f'(u)|\}.$$

The argument below uses the compact and strong test classes throughout.

Lemma 14.1 (Mixed continuity and compact null approximation)**.** *There is an explicit finite constant C_* such that*

$$|\mathcal{Q}(f,g)| \leq C_*\mathcal{S}_1(f)\mathcal{S}_1(g), \qquad C_* = 17 + |m_0| + 2\sum_{n\geq 2}\frac{\log n(1+\log n)}{n^{3/2}}.$$

The complex span of the odd functions

$$r_t = \tfrac{1}{2}(\tau_t q_0 - \tau_{-t}q_0), \quad q_0 = (\partial_u^2 - \tfrac{1}{4})\psi, \quad t\in\mathbb{R},$$

is dense in odd unweighted $L^2(\mathbb{R})$. Every r_t is a strong two-moment-zero mixed null of $\mathcal{Q}$, and every finite linear combination can be approximated in $\mathcal{S}_1$ by smooth compact odd two-moment-zero functions.

Proof. Since $\int e^{-2|u|}du = 1$, the L^2 norms of f and f' are at most $\mathcal{S}_1(f)$. The digamma series gives $a(\xi) \le (\xi^2/4)\sum_{n\ge 0}(n+1/4)^{-3} \le 17\xi^2$: the sum is at most $64 + (5/4)^{-3} + 1/\{2(5/4)^2\} < 68$. This bound and Plancherel therefore bound the archimedean mixed term by $(17+|m_0|)\mathcal{S}_1(f)\mathcal{S}_1(g)$. For $t \ge 0$,

$$\int_{\mathbb{R}} e^{-|u|}e^{-|u+t|}du = (1+t)e^{-t}.$$

Apply this to the two arithmetic translates at $t = \log n$, use $\Lambda(n) \le \log n$, and sum. The displayed series converges by the integral test. This proves continuity for the complete raw form, without splitting a divergent bare prime diagonal. For completeness, each exponential moment obeys $|Mf(\pm 1/2)| \le (8/3)\mathcal{S}_1(f)$, so adding both explicit-formula pole products costs at most $(128/9)\mathcal{S}_1(f)\mathcal{S}_1(g)$.

The theta Mellin identity gives

$$Mq_0(z) = (z^2 - \tfrac{1}{4})\xi(1/2+z).$$

Translation multiplies this transform by an exponential. It still vanishes at both poles and every centered nontrivial zero. The complete polarized explicit formula on the strong test class thus proves mixed nullity, with zero multiplicities retained. Strong theta derivative decay pays all Mellin and arithmetic limits. The Fourier transform of q_0 is nonzero almost everywhere, since its entire xi factor has only isolated real-frequency zeros and its polynomial factor has none.

If an odd $h \in L^2$ is orthogonal to all r_t, parity makes its correlation with $\tau_{-t}q_0$ the negative of its correlation with $\tau_t q_0$. Both correlations are therefore zero. Fourier uniqueness, applied to the L^1 product of their two L^2 Fourier transforms, implies $\widehat{h}\,\overline{\widehat{q_0}} = 0$ almost everywhere and hence $h = 0$. This proves the asserted density, not continuity of the Weil form in the unweighted L^2 norm.

Choose even smooth compact cutoffs χ_R with uniformly bounded derivatives through order three. Then $q_R = (\partial_u^2 - 1/4)(\chi_R\psi)$ is even, compact and has both exponential moments exactly zero, by integration by parts. Complete theta derivative decay gives $\mathcal{S}_1(q_R - q_0) \to 0$. Translations satisfy $\mathcal{S}_1(\tau_t f) \le e^{|t|}\mathcal{S}_1(f)$. For each fixed finite sum, replace its finitely many translates of q_0 by translates of q_R, choosing R after the sum and the desired tolerance are fixed. Antisymmetrization preserves oddness and the two zero moments, proving the compact approximation assertion. □

Proposition 14.2 (A uniform unweighted floor)**.** *RH is equivalent to the existence of one finite $K \ge 0$ such that*

$$\mathcal{Q}[f] \ge -K\|f\|_2^2$$

for every smooth compact odd two-moment-zero test f, across all supports.

Proof. Fix such an f. By Lemma 14.1, choose finite odd mixed-null sums r_n with $\|f - r_n\|_2 \le 1/n$, and compact odd two-moment-zero approximants g_n satisfying

$$\mathcal{S}_1(g_n - r_n) \le \{n[1 + \mathcal{S}_1(f - r_n)]\}^{-1}.$$

Write $e_n = g_n - r_n$ and $v_n = f - g_n$. Mixed nullity gives $\mathcal{Q}[f - r_n] = \mathcal{Q}[f]$, whereas mixed continuity gives

$$|\mathcal{Q}[v_n] - \mathcal{Q}[f]| \le 2C_*/n + C_*/n^2, \qquad \|v_n\|_2 \le 2/n.$$

The same floor applies to every compact admissible v_n, even though their supports grow. It follows that $\mathcal{Q}[f] \ge 0$. The odd original-domain Weil criterion then gives RH. Conversely RH gives positivity and the choice $K = 0$. This proof uses neither a support-dependent floor nor a global L^2-closed extension. □

14.2 The supported constant-shift operator

For $g \in L^2(0,L)$ use the odd-half unitary identification $f(x) = g(x)/\sqrt{2}$ for $x > 0$ and $f(-x) = -g(x)/\sqrt{2}$. Then, with $p(x) = \sinh(x/2)$,

$$Mf(1/2) = \sqrt{2}\int_0^L p(x)g(x)dx, \qquad Mf(-1/2) = -Mf(1/2).$$

The sum of the two explicit-formula pole products is therefore $-4|\langle p, g\rangle|^2$. The complete raw supported odd form is

$$B_L = D_L + R_L + m_0 - C_L^{\mathrm{ar}}$$

on the full odd-half space, without a polar constraint. Its minimal supported closure and operator domain are those in Lemma B.1, with bounded arithmetic added.

Theorem 14.3 (Complementary constant-shift RH criterion)**.** *Fix $\beta > 0$ and any increasing support sequence $L_j \to \infty$. The following are equivalent:*

1. *RH;*
2. *$B_L + \beta I \succeq 0$ for every $L > 0$;*
3. *$B_{L_j} + \beta I \succeq 0$ for every member of the sequence.*

Under RH the stronger floor $B_L + \beta I \succeq \beta I$ holds.

Proof. Under RH the zero-side form is a sum of squared absolute Mellin transforms. The exact pole calculation makes the raw odd form equal to that nonnegative sum plus $4|\langle p, g\rangle|^2$. Positivity passes from compact tests to the same minimal supported closure, so $B_L \succeq 0$. Conversely every compact odd polar-free test lies in each sufficiently large support. The complete supported forms agree on this test under zero extension. Assertion 3 therefore gives the one uniform unweighted floor $\mathcal{Q}[f] \geq -\beta\|f\|_2^2$. Proposition 14.2 proves RH. No nesting of finite Schur matrices or assumption of global closability is used. □

14.3 A complete finite receiving test with fixed margin

Put $A_L = B_L + \beta I = D_L + V_L^{\mathrm{sh}}$ and choose explicit bounds $\|V_L^{\mathrm{sh}}\| \leq P$ and $\|[D_L, V_L^{\mathrm{sh}}]\| \leq K_1$, with domain preservation. The full-response commutator theorem and the complete paired-arithmetic bound supply these hypotheses; the scalar shift has zero commutator. Fix a receiving floor $\tau > 0$, independently of β. Choose a certified finite prefix $U = [e_0, \ldots, e_{N-1}]$ such that

$$a(k_N) + m_0 - \|C_L^{\mathrm{ar}}\| + \beta \geq \tau.$$

With $Q = I - UU^*$, the entire receiving operator $T = QA_LQ$ on $\operatorname{ran} Q$ satisfies $T \succeq \tau I$, because $R_L \succeq 0$. Set $H = U^*A_LU$, $C = QA_LU$, and $\alpha_N = \max_{q<N} a(k_q)$.

Proposition 14.4 (No-polar cubic return enclosure)**.** *Define*

$$G_2 = P\alpha_N + (1 + P/\tau)(K_1 + P^2).$$

Let $E_{N,m}$ be the isometry onto modes $N, \ldots, m-1$, $m > N$, and put

$$X_m = C^*E_{N,m}(E_{N,m}^*A_LE_{N,m})^{-1}E_{N,m}^*C.$$

Then

$$0 \preceq C^* T^{-1} C - X_m \preceq \epsilon_m I, \qquad \epsilon_m = G_2^2 \{a(k_m)^{-3} + P a(k_m)^{-4}\}.$$

A strictly positive directed lower enclosure of $H - X_m - \epsilon_m I$ certifies $A_L \succeq 0$ on its entire supported domain. Under RH, $\epsilon_m \le \beta/4$ and a two-sided center radius $e \le \beta/8$ leave an outward lower of at least $\beta I/2$.

Proof. Here Q commutes with D_L. The exact mixed map is $C = QV_L^{\mathrm{sh}}U$, so $\|C\| \le P$ and $\|D_L C\| \le P\alpha_N + K_1$. For $y = T^{-1}C$, coercivity and the operator equation give

$$\|y\| \le P/\tau, \qquad \|D_L y\| \le P + P^2/\tau.$$

Domain preservation and the bounded commutator imply that $D_L y = C - QV_L^{\mathrm{sh}} Qy$ belongs to $\operatorname{Dom} D_L$. Applying D_L gives

$$D_L^2 y = D_L C - QV_L^{\mathrm{sh}} Q D_L y - Q[D_L, V_L^{\mathrm{sh}}]Qy, \qquad \|D_L^2 y\| \le G_2.$$

This supplies the required second-graph regularity rather than assuming it for a generic bounded perturbation.

Galerkin energy minimization uses the legal receiving trial obtained by retaining modes below m in y. The omitted vector h obeys

$$\langle h, D_L h\rangle \le a(k_m)^{-3}\|D_L^2 y\|^2, \qquad \|h\|^2 \le a(k_m)^{-4}\|D_L^2 y\|^2.$$

Adding the bounded perturbation fee proves the complete return estimate in finite coefficient order. Schur square completion proves the certificate implication. Under RH, $A_L \succeq \beta I$, so its exact Schur complement is at least βI. Subtracting $\epsilon_m I$ leaves at least $3\beta I/4$. A center enclosure of radius e followed by subtraction of eI can lose $2e$, which leaves at least $\beta I/2$ under the stated guard. □

Every finite coefficient in this test is effectively enclosable: the complete Green coefficient series has its paid tail, every active arithmetic pair is retained, and the receiving finite matrix is coercive. All inverse, coefficient and final matrix errors must be included in the directed lower. This gives the complete finite-recognition test for the stated supported form.

For a sufficient explicit threshold take

$$a(k_m) \ge \max\{1, P, (8G_2^2/\beta)^{1/3}\}.$$

It gives $\epsilon_m \le \beta/4$. For fixed β, choose $\tau = 1$ and a near-threshold outward prefix. The complete bounds yield $P = O(e^L + L)$, $\alpha_N = O(e^L)$ and $K_1 = O(Le^L)$, hence $G_2 = O(e^{3L})$ and a sufficient near-threshold choice has $\log m = O(e^{2L})$ for $L \ge 1$. On $L = \frac{1}{2}\log(10 \log M/\pi) + 2$, this is $m = M^{O(1)}$. Here M labels support growth in the fixed-shift realization. This trial-dimension bound has the fixed conditional margin and unweighted norm of Theorem 14.3. Arithmetic precision and validation costs enter separately in the directed finite test.

14.4 An arithmetic-only fixed-shift construction

The support-uniform response norm gives a further finite target for the same original positivity question. Keep the unweighted odd-half coordinates and minimal supported domain of Section 14. For a fixed $\beta > \pi/2$, define

$$\mathcal{A}_{L,\beta} = D_L + m_0 - C_L^{\mathrm{ar}} + \beta I, \qquad \delta_\beta = \beta - \pi/2 > 0.$$

The complete arithmetic operator retains both killed translation orientations and the reflected odd channel for every active prime power.

Corollary 14.5 (Arithmetic-only RH criterion)**.** *Fix $\beta > \pi/2$ and any increasing sequence $L_j \to \infty$. The following are equivalent:*

1. *RH;*
2. *$\mathcal{A}_{L,\beta} \succeq 0$ for every $L > 0$;*
3. *$\mathcal{A}_{L_j,\beta} \succeq 0$ for every j.*

Under RH,

$$\mathcal{A}_{L,\beta} \succeq \delta_\beta I \quad \text{uniformly in } L.$$

In particular $\beta = 2$ has conditional margin greater than $2/5$, and $\beta = 3$ has conditional margin greater than $7/5$.

Proof. Write the complete raw odd form as $B_L = D_L + R_L + m_0 - C_L^{\mathrm{ar}}$. In the half-unitary convention, the two explicit-formula pole products are $-4|\langle \sinh(x/2), g\rangle|^2$. Thus under RH the raw form B_L is the nonnegative zero-side form plus the positive term $4|\langle \sinh(x/2), g\rangle|^2$. This proves $B_L \succeq 0$ on compact odd tests and on its same minimal supported closure. Lemma 13.1 now gives

$$\mathcal{A}_{L,\beta} = B_L + \beta I - R_L \succeq (\beta - \pi/2) I.$$

Conversely, cofinal positivity of $\mathcal{A}_{L,\beta}$ gives $B_L = \mathcal{A}_{L,\beta} + R_L - \beta I \succeq -\beta I$ on each accepted support. Every compact odd polar-free test belongs to all sufficiently large supports, and the complete forms agree on it under zero extension. Hence the original raw form has one uniform lower bound $-\beta\|f\|_2^2$ on that compact class. The exact-null density and strong compact-approximation argument of Proposition 14.2 converts this bound to original Weil positivity and RH. The stated rational margins follow from $\pi < 16/5$. □

Complete arithmetic and commutator bounds. Let

$$\Sigma_L = \sum_{2 \le n < e^{2L}} \frac{\Lambda(n)}{\sqrt{n}}, \qquad P_\beta = |m_0| + \beta + 2\Sigma_L, \qquad K_\beta = (8 + 6\pi)\Sigma_L.$$

Then $V_{L,\beta} = m_0 - C_L^{\mathrm{ar}} + \beta I$ preserves $\operatorname{Dom} D_L$ and

$$\|V_{L,\beta}\| \le P_\beta, \qquad \|[D_L, V_{L,\beta}]\| \le K_\beta.$$

Indeed each killed symmetric pair has norm at most two. The crosscut domain argument in Appendix C gives, before odd restriction, a commutator bound $4(2+\pi)$ for the supported archimedean operator. Since that operator equals $D_L + R_L$ on the odd sector, bounded subtraction and Lemma 13.1 give for the same pair

$$\|[D_L, (T_t + T_{-t})|_{\mathrm{odd}}]\| \le 4(2 + \pi) + 4\|R_L\| \le 8 + 6\pi.$$

Bounded subtraction preserves the supported domain, and the killed-pair domain invariance is supplied by the crosscut argument. Summation with the complete arithmetic coefficients proves the two bounds above.

Complementary complete-return implementations. The same domain and commutator bounds allow Proposition 14.4 to be applied directly with $V = V_{L,\beta}$, $P = P_\beta$ and $K_1 = K_\beta$. Choose a DN prefix U and $\tau > 0$ with $a(k_N) + m_0 - 2\Sigma_L + \beta \geq \tau$, put $Q = I - UU^*$, and set $H = U^*\mathcal{A}_{L,\beta}U$, $C = -QC_L^{\mathrm{ar}}U$. The proposition gives a complete finite Galerkin return enclosure; its constants and induced error bounds are recomputed for these data.

A scalar receiving payment gives the complementary finite lower

$$\Gamma = H - \tau^{-1}C^*C, \qquad C^*C = U^*(C_L^{\mathrm{ar}})^2U - (U^*C_L^{\mathrm{ar}}U)^2.$$

The first term integrates complete spatial output functions. The identity retains every prime-pair and reflected mixed contribution. A positive directed lower for Γ proves positivity on the entire supported domain by Schur square completion. This is the full-output construction used by the evaluated arithmetic examples below. The following theorem gives an additional first-moment-only test by paying the receiving return analytically at its prescribed cutoff.

14.5 First-moment certificates with analytic tail payment

The arithmetic-only realization admits a finite test involving its first moments alone. A prescribed spectral cutoff and the complete first commutator bound pay the entire receiving return analytically. This provides a simpler verification object alongside the complete-return constructions above.

Fix $\mathcal{A}_{L,2} = D_L + V_L$, where $V_L = m_0 - C_L^{\mathrm{ar}} + 2I$, and set $\delta_0 = 1/5$. Choose rational budgets $P \geq 1$ and $K \geq 0$ satisfying

$$\|V_L\| \leq P, \qquad \|[D_L, V_L]\| \leq K.$$

The arithmetic domain invariance established above is retained. Explicit outward budgets are available from

$$P \geq 8 + 12e^L, \qquad K \geq 192e^L;$$

Lemma F.1 below proves their elementary prime estimate. The dimension estimate below uses rational budgets within one unit above these two elementary lower bounds. Sharper complete budgets may also be used. Let $n \geq 1$ be the least integer such that $4^n \geq 1 + P$, and choose a rational A between the following maximum and that maximum plus one:

$$A \geq \max\left\{8, 16P, \left[\frac{1024(n+1)^2(K^2+P^2)}{\delta_0}\right]^{1/3}\right\}. \tag{61}$$

Choose an outward integer N with $Le^A/\pi \leq N \leq Le^A/\pi + 2$. All subsequent matrices use these same choices. Rational enclosures of width less than one give these choices without deciding equality of computable real numbers.

Theorem 14.6 (Core-only arithmetic certificate)**.** *Let $U = [e_0, \ldots, e_{N-1}]$ be the ordinary DN mode isometry and put*

$$H_N = U^*\mathcal{A}_{L,2}U = \mathrm{diag}_{q<N}\, a(k_q) + (m_0 + 2)I - U^*C_L^{\mathrm{ar}}U. \tag{62}$$

If a complete directed lower enclosure certifies $H_N \succeq I/5$, then, on the entire original supported domain,

$$\mathcal{A}_{L,2} \succeq \frac{52}{405}I > \frac{1}{8}I.$$

For any prescribed increasing unbounded support sequence L_j, RH is equivalent to the existence of these directed certificates at every L_j. Under RH, the exact cores satisfy $H_N \succ 2I/5$. For the near-threshold choices above, the sufficient matrix dimension is

$$\log N = O(e^L + \log L), \qquad N = \exp(O(\sqrt{\log M})) \quad \text{when } e^{2L} = O(\log M),\ L \geq 1.$$

The dimension bound concerns this finite verification object; precision and validation costs enter separately.

Proof. Write $Q = I - UU^*$, $C = QV_LU$ and $T = Q\mathcal{A}_{L,2}Q|_{\operatorname{ran} Q}$. The omitted diagonal satisfies $a(k_N) > A$. For any rational $\tau \in [A/2, A/2+1]$, $T \succeq (A-P)I \succeq \tau I$. Once the finite premise $H_N \succeq \delta_0 I$ is verified, Lemma F.2 gives the complete covariance payment

$$\tau^{-1}C^*C \preceq 3H_N/16.$$

Thus the true Schur complement satisfies $H_N - C^*T^{-1}C \succeq 13H_N/16 \succeq 13I/80$. Moreover $\|T^{-1}C\| \leq P/\tau \leq 1/8$ and $\tau \geq 8$. Schur square completion, with triangular norm fee at most $(9/8)^2$, therefore yields the whole bound $\min(13/80, 8)(9/8)^{-2}I = 52I/405$.

Under RH, Corollary 14.5 gives $\mathcal{A}_{L,2} \succeq (2-\pi/2)I \succ 2I/5$, hence the same floor for each compression. Directed recognition has a strict margin: a Hermitian center with induced radius $e \leq 1/20$ has outward lower strictly above $3I/10$. The possible center displacement and the subtraction together cost at most $2e$. Entrywise radii are first converted to an induced matrix bound.

Conversely, successful tests on the prescribed cofinal supports give $B_{L_j} = \mathcal{A}_{L_j,2} + R_{L_j} - 2I \succeq -2I$. Every compact odd polar-free test embeds in a sufficiently large support, with its original unweighted norm preserved. The uniform-floor implication in Proposition 14.2 proves RH.

Finally $P, K = O(e^L)$ and $n = O(L)$ permit $A = O(e^L)$ in (61). The outward choice of N gives the displayed dimension bounds. The full proof of the relative payment and its domain requirements is in Appendix F. □

Complete finite entries. For $t = \log n \in (0, 2L)$ and $w_n = \Lambda(n)/\sqrt{n}$, the arithmetic entry in (62) is

$$(U^*C_L^{\mathrm{ar}}U)_{ij} = \frac{2}{L}\sum_{n<e^{2L}} w_n \Bigg[\int_0^{\max(0,L-t)} \sin(k_i x)\sin(k_j(x+t))\,dx + \int_{\max(0,t-L)}^{L} \sin(k_i x)\sin(k_j(x-t))\,dx \Bigg].$$

The second sine itself implements odd reflection when its argument is negative. Both orientations and every active prime power are retained. The identity $2\sin(k_ix)\sin(k_j(x+\sigma t)) = \cos((k_i-k_j)x - \sigma k_j t) - \cos((k_i+k_j)x + \sigma k_j t)$ gives exact finite primitives. When $i = j$, the zero-frequency primitive is evaluated as a constant integral, with no division by an interval containing zero. The diagonal is also directly enclosable through $a(k_q) + m_0 = \operatorname{Re}\psi(1/4 + ik_q/2) - \log\pi$. Thus the test uses only signed arithmetic first moments and the known diagonal. Its cutoff supplies the analytic whole-tail guarantee. The two evaluated certificates below retain their original complete Schur proofs and specified cutoffs.

14.6 A complete arithmetic-only supported certificate

The arithmetic-only criterion of Corollary 14.5 admits a complete finite calculation at the same support used by the second SD example, now in the unweighted odd-half profile norm. Set

$$L = \frac{1}{2}\log\Big(\frac{96\log 2}{\pi}\Big) + \frac{1}{10}, \qquad N = 256, \qquad \beta = 3,$$

and use the original minimal supported realization of

$$\mathcal{A}_{L,3} = D_L + m_0 - C_L^{\mathrm{ar}} + 3I, \qquad B_L = \mathcal{A}_{L,3} + R_L - 3I.$$

Here $U = [e_0, \dots, e_{255}]$ is the orthonormal DN core and $Q = I - UU^*$ is its entire receiving complement.

Theorem 14.7 (Evaluated arithmetic-only certificate)**.** *At the stated support, $\mathcal{A}_{L,3}$ is nonnegative on its complete odd supported domain. Consequently the complete raw odd form satisfies*

$$B_L \succeq -3I.$$

Proof. The complete active prime powers are

$$\{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25\}.$$

The arithmetic row bound can be verified directly in the unweighted logarithmic norm. On $(-L, L)$ use the positive even Schur weight $h(x) = \cosh(x/2)$. For $x \geq 0$, define

$$\begin{aligned} p_a(x) &= \sum_{n<e^{L+x}} \Lambda(n), \quad h_a(x) = \sum_{n<e^{L+x}} \Lambda(n)/n, \\ p_b(x) &= \sum_{n<e^{L-x}} \Lambda(n), \quad h_b(x) = \sum_{n<e^{L-x}} \Lambda(n)/n, \end{aligned}$$

where all sums run over the displayed active prime powers. The two killed translation orientations give the weighted row price

$$r(x) = \frac{A(x) + e^x B(x)}{1 + e^x}, \qquad A = p_a + h_b, \quad B = p_b + h_a.$$

Indeed their respective weight ratios are $\Lambda(n)(1 + e^x/n)/(1 + e^x)$ and $\Lambda(n)(1/n + e^x)/(1 + e^x)$, with the corresponding support indicators. Symmetry makes r even, and the complete paired operator is symmetric on the full supported interval. Applying

$$2|v(x)v(y)| \leq \frac{h(y)}{h(x)}|v(x)|^2 + \frac{h(x)}{h(y)}|v(y)|^2$$

to every paired shift proves its absolute quadratic-form bound by $\int r(x)|v(x)|^2\,\mathrm{d}x$. Restriction to the invariant odd sector, followed by the unitary odd-half identification, retains the reflected channel and gives the same bound for C_L^{ar}.

The coefficients A, B are constant on each complete reference cell. The derivative of its row price has the sign of $B - A$; both one-sided endpoint bounds therefore control the whole cell. The verifier checks all 7,828 cells and their exact consecutive endpoints. The reference uses the identical expression $(\sqrt{K}A + zB)/(\sqrt{K} + z)$, where $K = e^{2L}$ and $z = e^{L+x}$. Consequently the exact dyadic

upper r_{up} gives $\|C_L^{\mathrm{ar}}\| \le r_{\max} := \max r_{\mathrm{up}}$. Monotonicity of the digamma diagonal gives, on the whole suffix,

$$T = Q\mathcal{A}_{L,3}Q \succeq [\Re\Psi(1/4 + ik_N/2) - \log\pi - r_{\max} + 3]\, I \succ \tau I, \qquad \tau = \frac{71111}{65536}.$$

The directed bracket before I exceeds 1.0850825.

Define the complete finite matrices

$$\mathsf{B} = U^* C_L^{\mathrm{ar}} U, \qquad \mathsf{S} = (C_L^{\mathrm{ar}} U)^*(C_L^{\mathrm{ar}} U), \qquad H = U^* \mathcal{A}_{L,3} U.$$

The diagonal and scalar terms preserve the DN core, so the exact mixed map is $C = Q\mathcal{A}_{L,3}U = -QC_L^{\mathrm{ar}}U$ and

$$C^*C = \mathsf{S} - \mathsf{B}^*\mathsf{B}.$$

The full arithmetic-output Gram S is integrated on $(0, L)$. The 19 arithmetic integration intervals are formed from $0, L$ and the interior points $\log n$, $L - \log n$, $\log n - L$. On each interval the action columns are finite sine/cosine combinations. Their exact elementary antiderivatives, evaluated by directed arithmetic, enclose every entry of S and B. The separate closed-form first-moment formulas check all $65,536$ entries of B.

Inverse order therefore gives the complete Schur lower

$$H - C^*T^{-1}C \succeq \Gamma_{\mathrm{ar}} := H - \tau^{-1}(\mathsf{S} - \mathsf{B}^*\mathsf{B}).$$

The scientific enclosures give 256 strictly positive directed pivots for $\Gamma_{\mathrm{ar}} - I/10$. The portable verifier reconstructs the digamma diagonal, first moments and complete Schur matrix, reverses the DN coordinate order, and checks the factorization independently. Thus $\Gamma_{\mathrm{ar}} \succ I/10$ in the orthonormal core coordinates. Completion of squares against $T \succeq \tau I$ proves $\mathcal{A}_{L,3} \succeq 0$ on its entire supported domain. Finally $B_L = \mathcal{A}_{L,3} + R_L - 3I$ and $R_L \succeq 0$ give the raw-form bound. □

Scientific data and error bounds. The arithmetic-only companion contains the original 768-bit directed enclosures for H, S, B, $\mathsf{S} - \mathsf{B}^*\mathsf{B}$ and Γ_{ar}, together with the exact reference-cell data and rational τ. The source Gram includes the entire arithmetic output before the finite-core subtraction. The manifest identifies each scientific file and the prescribed support. The generator recomputes the continuous elementary integrals; the independent verifier checks the complete reference, recomputes every first moment from its separate closed-form identity and validates all 256 reversed-order pivots at 2,048 bits. All coefficient, integration, subtraction, inverse-reference and factorization errors are included in the interval operations. The margin $1/10$ is used for the finite Schur inequality; whole-domain positivity follows from its stated receiving bound and square completion.

14.7 A stronger arithmetic-only supported certificate

Increasing the DN core at the same support strengthens the supported lower bound in Theorem 14.7. Set

$$L = \tfrac{1}{2}\log(96\log 2/\pi) + \tfrac{1}{10}, \qquad N = 768, \qquad \beta = 2.$$

Use $U = [e_0, \dots, e_{767}]$ and its entire orthogonal complement $Q = I - UU^*$ in the unweighted odd-half realization.

Theorem 14.8 (Evaluated shift-two arithmetic certificate)**.** *At this support, $\mathcal{A}_{L,2} = D_L + m_0 - C_L^{\mathrm{ar}} + 2I$ is nonnegative on its complete odd supported domain. The complete raw odd form therefore satisfies $B_L \succeq -2I$.*

Proof. The active arithmetic shifts and the weighted-Schur majorant are those of Theorem 14.7: all fourteen prime powers are retained. On each of the 7,828 reference cells, the price $(A + e^x B)/(1 + e^x)$ has constant-sign derivative, so checking both endpoints bounds the complete cell. The even weight $\cosh(x/2)$ then gives $\|C_L^{\mathrm{ar}}\| \le r_{\max}$ on the full supported space and its invariant odd sector. Monotonicity of the diagonal symbol on the entire DN suffix gives

$$Q\mathcal{A}_{L,2}Q \succeq t_* Q, \qquad t_* = \Re\Psi\left(\tfrac{1}{4} + \frac{i(768 + 1/2)\pi}{2L}\right) - \log\pi - r_{\max} + 2 > 1.18239457729109417.$$

The rational receiving bound $\tau = 77489/65536$ satisfies $0 < \tau < t_*$.

Define the complete continuous-output moments

$$H = U^* \mathcal{A}_{L,2} U, \quad J = U^* C_L^{\mathrm{ar}} U, \quad S = (C_L^{\mathrm{ar}} U)^* (C_L^{\mathrm{ar}} U).$$

Integration over all nineteen arithmetic panels encloses every entry of J and S, including all reflected and killed-shift terms. The full mixed return is $S - J^*J$. The scientific interval matrices enclose

$$\Gamma = H - \tau^{-1}(S - J^*J).$$

Directed factorization of $\Gamma - \frac{1}{10}I$ gives all 768 strictly positive pivots. Thus the actual Schur complement dominates $\Gamma \succeq \frac{1}{10}I$ on the finite core. Completing the square with the positive entire receiving suffix proves $\mathcal{A}_{L,2} \succeq 0$. Finally $B_L = \mathcal{A}_{L,2} + R_L - 2I$ and $R_L \succeq 0$ give the stated raw-form bound. □

The companion supplies all $768^2 = 589,824$ entries of each continuous moment matrix, the exact receiving bound, and the interval Schur matrix. The independent verifier reconstructs every arithmetic first moment from a separate closed formula, reconstructs all three Schur ingredients, checks every reference cell, and reverses the core order before factorization. The coefficient $1/10$ is the finite Schur-core margin used in the completion-of-squares argument. The result strengthens the supported raw lower bound from -3 to -2 at this fixed support.

Supporting certificate transfers. Section S22 of the technical supplement gives three complementary transfers for this arithmetic-only realization: complete-core representation with certified pullback error, conditional completion of a polar-free raw bound by a smooth edge direction, and local support extension from a positive whole-operator margin. Each retains its stated assumptions, original profile coordinates and complete mixed-error payment.

15 Complete supported certificates

The principal evaluated moving-window instance below uses the original norm and the complete receiving space. We first state its common construction, so its proof is independent of the earlier numerical instance retained in the supplement, Section S23.

Fix an admitted support $L > 0$, integer Green rank $J \ge 0$, ridge $\kappa > 0$ and integer DN prefix length $N \ge 2$. Let $W/2 \le W_{\mathrm{low}} \le W$ be the certified theta floor and put

$$S = D_L + R_{L,J} + m_0 - C_L^{\mathrm{ar}} + \kappa W_{\mathrm{low}}.$$

The arithmetic operator retains every active prime power and both killed translation orientations. Write $p(x) = \sinh(x/2)$ and $p_q = \langle p, e_q \rangle$. The columns

$$C = (e_j - (p_j/p_0)e_0)_{1 \le j < N}, \qquad F = [p, C], \qquad Q = \mathrm{proj}_{\ker F^*}$$

give the full low-band polar-free intersection and its infinite receiver; the raw core Gram is $G_C = C^*C$.

For the reflection-symmetric weight $\phi(u) = e^{-u/2} + e^{-(2L-u)/2}$, applying $2|fg| \le |f|^2\phi(y)/\phi(x) + |g|^2\phi(x)/\phi(y)$ to every arithmetic edge gives $m_0 - C_L^{\mathrm{ar}} \succeq m_0 - \mathfrak{r}$, where $\mathfrak{r}$ is the complete finite row sum. A receiving vector's first N DN coefficients are parallel to the low-band pole projection. The polar equation bounds their squared norm fraction by $\delta_N \le 9L^3/[\pi^4(N+1/2)^3]$. Hence

$$D_L|_Q \succeq \ell_N I, \qquad \ell_N = a(k_N) - \delta_N\{a(k_N) - a(k_0)\}.$$

Certified cell bounds $r_{\mathrm{up}} \ge \mathfrak{r}$ and $\alpha \le \ell_N + m_0$, together with $R_{L,J} \succeq 0$, prove the whole receiving inequality

$$T = S_Q \succeq d_Q, \qquad d(x) = \kappa W_{\mathrm{low}}(x) + \alpha - r_{\mathrm{up}}(x) \ge d_* > 0.$$

Here the admitted cell bounds include the uniform constant $d_* > 0$.

Set $G = F^*d^{-1}F$ and

$$Z = d^{-1} - d^{-1}FG^{-1}F^*d^{-1}.$$

The complete constrained inverse satisfies $T^{-1} \preceq Z$ on $\operatorname{ran} Q$. If an invertible upper-triangular dyadic conditioner R obeys $\|R^*GR - I\| < \delta$, then $G^{-1} \succeq RR^*/(1+\delta)$. Here $\widehat{S}$ denotes the full ambient expression defining S, before restriction to $p^\perp$. With $E = \widehat{S} - d$, retain all output functions before forming

$$B_E = C^*EC, \qquad K_E = (EC)^*d^{-1}(EC), \qquad H = F^*C + F^*d^{-1}EC.$$

The identical ridge cancels symbolically in E. The complete zero-trial Schur lower is

$$K_{\mathrm{core}} \succeq H^*G^{-1}H - B_E - K_E \succeq A := \frac{H^*RR^*H}{1+\delta} - B_E - K_E. \tag{63}$$

This is the constrained receiving comparison of Section C; all mixed terms and the polar row are included. Positive directed pivots of an invertible congruence of A therefore prove $S \succeq 0$ by completion of squares against the full positive receiver. Finally $R_L - R_{L,J} \succeq 0$, $W - W_{\mathrm{low}} \ge 0$ and $f = sh$, $sWs = I$, give $q_H[h] \ge -\kappa\|h\|_2^2$ on the stated original supported domain. Instance-specific cell and integral verification is indexed in the technical supplement.

15.1 The M24 complete moving-window certificate

The common construction gives the following evaluated moving-window instance in the original norm. Its reference and constraint metric are computed at this support; its constants are specific to M24. Put

$$M = 2^{24}, \qquad \kappa = 2^{-24}, \qquad L = \tfrac{1}{2}\log(4\log M/\pi) + \tfrac{1}{10}, \qquad J = 88, \qquad N = 660.$$

The exact full-theta floor and common partition are specified in the companion M24 scientific data and verified in the technical supplement. Let

$$S_{24} = D_L + R_{L,88} + m_0 - C_L^{\mathrm{ar}} + \kappa W_{\mathrm{low}} \quad \text{on } p^\perp, \qquad p(x) = \sinh(x/2).$$

Theorem 15.1 (Evaluated M24 complete supported certificate)**.** *The form S_{24} is nonnegative on the original minimal closed odd supported polar-free profile domain. Consequently every original odd polar-free test supported in $(-L, L)$ satisfies*

$$q_H[h] \ge -2^{-24}\|h\|_2^2.$$

Proof. Let C have the 659 columns $e_j - (p_j/p_0)e_0$, $1 \le j \le 659$, set $F = [p, C]$, and let Q project onto $\ker F^*$. The common pole-leakage and complete arithmetic row bounds, with the M24 cell data, give

$$T = (S_{24})_Q \succeq d_Q, \qquad d = \kappa W_{\mathrm{low}} + \alpha - r_{\mathrm{up}} > 1/200.$$

This is an inequality on the whole infinite receiving space. The 7,828 common cells retain all 14 active prime powers and both translated channels; the 88 retained Green features are positive. The supplement gives their complete reference and weighted-Gram checks.

Write $G = F^* d^{-1} F$. The exact upper-triangular dyadic conditioner R, with positive diagonal, satisfies

$$\|R^* G R - I\| < 4.747 \times 10^{-74} < \delta, \qquad \delta = 10^{-60}.$$

Hence $G^{-1} \succeq RR^*/(1+\delta)$. With the complete output integrals B_E, K_E, H of (63), the same constrained-inverse comparison gives the zero-trial Schur lower

$$K_{24} \succeq A_{24} := \frac{H^* R R^* H}{1+\delta} - B_E - K_E.$$

The data and replay in the supplementary section *Verification of the second moving-window certificate* give all 659 positive directed pivots after the invertible raw-core congruence $P_C = G_C^{-1} R_{uu}^{-*}$, where $G_C = C^*C$ and R_{uu} is the lower-right 659×659 core block of the M24 conditioner R. Thus $A_{24} \succ 0$. Completion of squares against the positive whole receiver proves $S_{24} \succeq 0$. Finally $R_L - R_{L,88} \succeq 0$ and $W - W_{\mathrm{low}} \ge 0$ imply $H_f + \kappa W \succeq S_{24}$; the exact physical congruence $f = sh$, $sWs = I$, proves the stated bound. □

The complete evaluated cases are summarized in Table 2.

Table 2: Complete supported zero-trial certificates in the SD family.

Case	Support half-width	Constraints	Core	Allowance
M20	1.535391914…	329	328	2^{-20}
M24	1.626552692…	660	659	2^{-24}

The M24 case extends the evaluated support and reduces this family's supported allowance by a factor of sixteen. The M20 construction and its full certificate are retained in Supplementary Section S23. Both cases use the support/reference family of Section 15; the quantified trial-size bound for the distinct schedule (3) is not transferred to these examples. The two evaluated instances are complete supported certificates; the cofinal assertion of Theorem 2.1 retains its unbounded-sequence quantifier.

16 Complete fixed-support ridge continuation

The M24 certificate also supplies a continuous family at the same fixed support. With $\kappa_0 = 2^{-24}$, the complete derivative-weight integrals and all 659 directions certify the interval

$$\kappa_0 - h \le \kappa \le \kappa_0, \qquad h = \frac{1125889145388605}{75557863725914323419136}.$$

Its lower endpoint is approximately $4.47036260082 \times 10^{-8}$. The full constrained reference-inverse Schur lower obeys $\kappa\Gamma'_\kappa \preceq 3\Gamma_\kappa$ throughout this band. The complete derivative identity, downward-loss estimate, capped-return comparison and numerical proof are in Supplementary Section S24.

This extension has separate derivative primitives and exact congruences, identified in the scientific data index. Support, constraints, Green model and legal trials stay fixed throughout its comparison.

17 Consequences and verification structure

The central finite criterion retains the original norm and converts each accepted complete matrix certificate into a supported lower bound. Its unbounded-sequence condition reaches the original odd Weil positivity endpoint. Table 1 records the distinct norms, references and evidence for the complementary constructions.

The unconditional complete-covariance theorem, full-source null resources and high-frequency criteria provide supporting analytic control. The fixed-strip criterion makes the sufficient precision depend explicitly on the supplied zero-free width; its classical-input companion is independent of that additional premise. Their actual constrained inverse and complete original-domain return are proved in Appendix G.

The M24 example and its fixed-support ridge continuation supply evaluated instances with their own complete error budgets. The arithmetic-only first-moment criterion gives a complementary admitted finite test in its stated unweighted norm. Earlier numerical details, the alternative growing-floor construction and packet-family estimates are indexed in the technical supplement. Each retained conclusion uses its own full receiving comparison, constraints and norm.

A The odd original-domain criterion

We give the needed realization-specific reduction to the classical Weil criterion [22, 1]. For $F(z) = \int f(u)e^{zu}\,\mathrm{d}u$, the complete explicit formula has zero-side form

$$W[f,g] = \sum_{\rho} F(\rho - 1/2)\overline{G(1/2-\overline{\rho})}. \tag{64}$$

The sum counts multiplicities. On odd real tests $W[f] = -\sum_{\rho} F(\rho-1/2)^2$. The physical form in (11) equals $W[f] + 2|F(1/2)|^2$ on odd tests; the polar term vanishes on $\mathcal{K}$.

Lemma A.1 (Odd original-domain criterion)**.** *Nonnegativity of q_H on V is equivalent to RH.*

Proof. The theta Mellin identity, in the normalization (9), is

$$M\psi(z) = \xi(1/2+z), \qquad M\psi'(z) = -z\xi(1/2+z), \qquad M\psi'(1/2) = -1/4.$$

Indeed, with $t = e^{2u}$ and $A(t) = \sum_{n\geq 1} e^{-\pi n^2 t}$, one has $\psi = e^{u/2}(4t^2A'' + 6tA')$. Two integrations by parts on $0 < t < \infty$, initially for $\Re s > 1$, give

$$M\psi(s-1/2) = \frac{s(s-1)}{2}\int_0^\infty A(t)t^{s/2-1}\,\mathrm{d}t = \xi(s).$$

Theta inversion gives the entire continuation. For any compact smooth real odd f, put $g = f + 4F(1/2)\psi'$. Its polar moment is zero and its transform agrees with F at every nontrivial zero, so $W[g] = W[f]$.

Here g/s belongs to the original minimal domain. To see this directly, take even smooth cutoffs χ_R and a fixed compact real odd b with $Mb(1/2) = 1$. Set $g_R = \chi_R g - M(\chi_R g)(1/2)b$. Then g_R is in the exact polar compact core. Complete theta derivative envelopes give

$$\|(g_R - g)/s\|_2 \to 0, \qquad \|(g_R - g)'\|_2 \to 0.$$

For example $|\psi'| \le 40e^{|u|/2}r^3e^{-r}$ and the lower bound (12) give an integrable upper $800e^{|u|}r^4e^{-r}$ for $|\psi'/s|^2$. The derivative atom polynomials are $-8r^3+30r^2-15r$ and $16r^4-112r^3+165r^2-75r/2$. Their absolute coefficient bounds for $r \ge \pi > 3$, and consecutive atom ratios, give $|\psi''| \le 160e^{|u|/2}r^4e^{-r}$. These complete tails pay both limits. Bounded cutoff derivatives and the vanishing compact moment correction preserve them. The bound $a \le 17\xi^2$ now gives convergence in (10).

The same tails give exponential-weighted L^1 convergence of $g_R - g$ and its first derivative. Integration by parts therefore gives

$$\sup_{|\Re z|\le 1/2} (1+|\Im z|)|M(g_R-g)(z)| \longrightarrow 0.$$

The zero count implies $\sum_\rho (1+|\Im\rho|)^{-2} < \infty$, so the complete quadratic zero sum converges as well. Passing the compact explicit formula to the limit yields $q_H[g/s] = W[g] = W[f]$. Thus positivity on V implies $W[f] \ge 0$ for every real odd compact smooth f.

Suppose an off-line centered zero $\sigma_0 = \delta + i\gamma$ exists, with $\delta, \gamma > 0$ by symmetry. Nontrivial real zeros are absent because the alternating eta series gives $\zeta(s) < 0$ for real $0 < s < 1$. Fix $\Gamma > \gamma + 1$. An even real polynomial P annihilates every zero orbit in $|\Im\sigma| \le \Gamma$ except the quartet of σ_0, and $P(\sigma_0) \ne 0$. Choose real a, b with $iP(\sigma_0)(a + b\sigma_0^2) = 1$, possible because $\Im\sigma_0^2 \ne 0$. Put

$$F_T(z) = iP(z)(a+bz^2)[e^{T(z-i\gamma)^2} - e^{T(z+i\gamma)^2}], \quad T \ge 1.$$

This entire function is odd and real type. It is the bilateral Laplace transform of a real odd Gaussian multiplied by polynomial and trigonometric factors. At each fixed T, even compact cutoffs of that physical function converge in exponential Schwartz seminorms. Repeated integration by parts gives transform convergence with every polynomial vertical decay uniformly on $|\Re z| \le 1/2$. The zero count $N_\zeta(Y) = O(Y\log(Y+2))$ therefore permits dominated convergence in (64), so $W[f_T] \ge 0$.

At σ_0,

$$e^{-T\delta^2}F_T(\sigma_0) = 1 - e^{-4T\gamma^2+4iT\delta\gamma} \longrightarrow 1.$$

The target quartet, of multiplicity m, contributes $-4m\Re F_T(\sigma_0)^2 \sim -4me^{2T\delta^2}$. All other zeros in the finite window are annihilated. For exterior zeros put $d_\rho = ||\Im\sigma_\rho| - \gamma| > 1$. A fixed polynomial envelope bounds their squared transform by

$$C(1+|\Im\sigma_\rho|)^{2D}e^{2T(1/4-d_\rho^2)} \le Ce^{-3T/2}(1+|\Im\sigma_\rho|)^{2D}e^{-(d_\rho^2-1)}.$$

The full remaining zero sum is $O(e^{-3T/2})$ by the same zero-count bound. The negative quartet dominates, a contradiction. This proves RH. Conversely, RH makes (64) a sum of squared absolute transforms on the polar core; original form closure extends positivity to V. □

B The supported Green realization

The supported Green realization used in Theorem 2.1 follows the positive exponential construction of Li [11]. Its domain identification is included here. The complete physical-tail estimate and finite certificate construction are proved in Appendix C. All arithmetic sums below include every active prime power. Optional critical carriers are empty, and the original polar restriction is retained.

B.1 Exact realization and domain

On the odd-half unitary space $L^2(0,L)$ put

$$c_j = 2j + \tfrac{1}{2}, \quad k_q = (q+\tfrac{1}{2})\pi/L, \quad e_q(x) = \sqrt{2/L}\sin(k_qx), \quad D_Le_q = a(k_q)e_q.$$

Lemma B.1 (Complete Green realization)**.** *The minimal zero-extended supported archimedean form equals $D_L + R_L$, where*

$$R_L = \sum_{j\ge 0} r_j \otimes r_j, \qquad r_j(x) = \frac{2\sinh(c_j x)}{\sqrt{e^{2c_j L}+1}}, \tag{65}$$

$$0 \preceq R_L, \qquad \|R_L\| \le C_L := L + \pi/2. \tag{66}$$

The associated operator domain is $\operatorname{Dom} D_L$*; finite DN spans are an operator graph core.*

Proof. The positive digamma difference expansion is

$$a(\xi) = \sum_{j\ge 0} \left(\frac{2}{c_j} - \frac{2c_j}{c_j^2 + \xi^2} \right).$$

For the Dirichlet half-line and Dirichlet–Neumann interval Green functions,

$$G_{\infty,c}(x,y) = \frac{e^{-c|x-y|} - e^{-c(x+y)}}{2c},$$
$$G_{\mathrm{DN},c}(x,y) = \frac{\sinh(c\min(x,y))\cosh(c(L-\max(x,y)))}{c\cosh(cL)},$$

direct subtraction gives $2c(G_{\mathrm{DN},c} - G_{\infty,c}) = r_c \otimes r_c$. For $z = 2L - x - y > 0$, the sum kernel is bounded by

$$\sum_{j\ge 0} e^{-c_j z} = \frac{e^{-z/2}}{1 - e^{-2z}} \le 1 + \frac{1}{2z}.$$

The constant kernel has norm L; the Carleman kernel $[2(u+v)]^{-1}$, $u = L - x$, has Schur bound $\pi/2$ with weight $u^{-1/2}$. All positive partial sums therefore converge strongly to the bounded R_L in (66).

The identity first holds on smooth compact odd tests. To identify its closure, extend a finite DN combination oddly across zero and evenly across L, with period $4L$. Cut it off in endpoint layers of width ε and mollify preserving parity. The reflected error has H^t norm $O(\varepsilon^{1/2-t})$ for every fixed $0 < t < 1/2$: rescale each bounded smooth boundary bump in the fractional Sobolev norm. The bound $a(k) \le C_t(1+k^2)^t$, from the logarithmic symbol estimate, shows convergence in the diagonal form norm. Conversely smooth compact tests have rapidly decreasing DN coefficients. Finite sequence truncation is a form core of D_L; hence the original minimal closure is precisely $D_L + R_L$. Boundedness of R_L then gives its operator domain and graph core. This argument retains the supported realization, rather than silently imposing a different endpoint domain. □

At fixed L, the original positive smooth s has positive minimum. Thus $s^{\pm 1}$ are bounded on this interval. The original physical congruence is exactly $S_h = sS_f s$, with physical norm $\|h\|_2$. The polar condition becomes $f \perp p$, $p(x) = \sinh(x/2)$, in profile coordinates. Restrict the closed forms to this subspace, retaining their minimal supported closure; one can enforce the constraint on the smooth approximants by subtracting a fixed compact test with nonzero polar pairing. Finite bounded arithmetic and ridge terms preserve these domain statements. No condition number is assumed uniform as L grows.

C Proofs for the finite-verifiable criterion

All forms use the original supported minimal closure and its exact polar restriction. Lemma B.1 proves $H_f = D_L + R_L + V_L$, the domain $\operatorname{Dom} D_L$, and the finite-DN graph core. The arithmetic bound follows from the complete weighted Chebyshev estimate in Section 4. The construction below combines that realization with positive form order, constrained inversion, and a complete residual identity.

C.1 Uniform theta brackets and the dyadic floor

For $x \geq 0$, put $r = \pi e^{2x}$, $t = (1 + e^{-x})/2$, and

$$\Theta(x) = \sum_{n\geq 1} (4r^2n^4 - 6rn^2)e^{-rn^2}, \qquad W = t/\Theta.$$

Every summand is positive. Write $q_1 = (4r^2 - 6r)e^{-r}$ and

$$W_1 = t/q_1 = \frac{(1 + e^{-x})e^r}{4r(2r - 3)}.$$

Lemma C.1 (Whole-theta brackets and variation)**.** *On $x \geq 0$,*

$$q_1 \leq \Theta \leq (129/128)q_1, \qquad (128/129)W_1 \leq W \leq W_1, \qquad |(\log W)'| < 21\pi e^{2x}. \tag{67}$$

Proof. Since $r > 3$, $q_1 \geq 2r^2e^{-r}$. For $n = 2$, $4\log 2 < 3$; for $n \geq 3$, $4\log n \leq 4(n - 1) \leq n^2 - 1$. Thus

$$\frac{\Theta - q_1}{q_1} \leq 2\sum_{n\geq 2} n^4 e^{-r(n^2-1)} < 2\sum_{n\geq 2} e^{-2(n^2-1)} \leq \frac{2e^{-6}}{1 - e^{-10}} < \frac{5971968}{1073682775} < \frac{1}{128}.$$

The penultimate inequality uses $e > 8/3$. This proves the first two bounds for the entire series.

Termwise differentiation, justified by uniform Gaussian tails on finite intervals, gives

$$\Theta'(x) = \sum_{n\geq 1} (-8y^3 + 28y^2 - 12y)e^{-y}, \qquad y = rn^2.$$

For $y \geq 3$ the absolute polynomial is at most $20y^3$. Also $\sum_{n\geq 1} n^6 e^{-r(n^2-1)} < 2$: for $n \geq 2$, $6\log n \leq 2n^2$, and the remaining tail is bounded by

$$\sum_{n\geq 2} e^{3-n^2} \leq \frac{e^{-1}}{1 - e^{-5}} < 16/31 < 1.$$

Consequently $|\Theta'/\Theta| < 20r$. Finally $|t'/t| \leq 1/2$, proving the logarithmic derivative bound. □

Let $K_L = 21\pi e^{2L}$. Partition $[0, L]$ into cells of length at most $h \leq \log(4/3)/(2K_L)$. At each left node use directed evaluation $0 < a \leq W_1 \leq b$, $b/a \leq 17/16$. The complete-theta node enclosure

$$a_j = (128/129)a \leq W \leq b = b_j, \qquad b_j/a_j \leq 2193/2048 < 4/3$$

then needs only the elementary expression W_1. Choose an exactly represented positive dyadic d_j with

$$(9/10)a_j e^{-K_L h} \leq d_j \leq a_j e^{-K_L h},$$

and put $W_{\mathrm{low}} = d_j$ on that cell. The nonempty strict interior of this interval permits a finite certified dyadic choice. Logarithmic variation gives

$$d_j \le W(x), \qquad \frac{d_j}{W(x)} \ge (9/10)(a_j/b_j)e^{-2K_L h} \ge 81/160 > 1/2.$$

Thus $W/2 \le W_{\mathrm{low}} \le W$ as multiplication operators. One sufficient cell count is $\lceil 2K_L L/\log(4/3)\rceil$. At each fixed L, both potentials are bounded. Their jumps change the operator by a bounded multiplier and preserve the supported form and operator domains.

C.2 The whole physical Green response

Lemma C.2 (Complete physical sandwich)**.** *For $E_J = R_L - R_{L,J}$, $0 < \eta < L$, and $\sigma_\eta^2 = \operatorname{ess\,sup}_{L-\eta<x<L} s(x)^2$,*

$$0 \preceq sE_J s \preceq \left\{ \sqrt{(L+\pi/2)\sigma_\eta^2} + 3\sqrt{\frac{e^{-2c_J\eta}}{2c_J(1-e^{-4\eta})}} \right\}^2 I. \tag{68}$$

Proof. Use the ambient L^2 edge and interior projections $P_e = 1_{(L-\eta,L)}$, $P_i = I - P_e$. Since $0 \preceq E_J \preceq R_L$,

$$\|E_J^{1/2} P_e s\| \le \sqrt{L+\pi/2}\,\sigma_\eta.$$

On the interior $0 \le r_j(x) \le e^{-c_j(L-x)}$, whence Tonelli and the complete positive feature sum give

$$\operatorname{Tr}(P_i E_J P_i) \le \sum_{j\ge J} \frac{e^{-2c_j\eta}}{2c_j} \le \frac{e^{-2c_J\eta}}{2c_J(1-e^{-4\eta})}.$$

Use $\|s\|_\infty < 3$ for the interior factor, and apply the operator triangle inequality before squaring $\|E_J^{1/2}s\|^2 = \|sE_J s\|$. All cross terms are included. Positive partial sums pass monotonically to the bounded response. The characteristic functions act only in L^2, so no multiplication of a form-domain vector by a discontinuous cutoff is required. Restriction then preserves every polar constraint. □

All response indices are chosen outward as in (3); finite interval refinement supplies such an integer without deciding an exact-ceiling equality. Here are the explicit admissions. Write $U = \log M$, $\lambda = \log U$, $\rho = U + 8\lambda$. For $M \ge 2^{20}$, $13 < U$, $2 < \lambda$, and $\log U \le U/4$. The last inequality follows at 13 from $\log 13 < 3 < 13/4$, and then by differentiation. Thus $\rho \le 3U$, $\eta < 1/2$, and

$$C_L = L + \pi/2 < \tfrac{1}{2}\lambda + \tfrac{5}{2} < 2\lambda.$$

The width satisfies $L > 1$. Indeed its derivative has the sign of $U(\lambda^2 - 2) + 8\lambda(\lambda - 2) > 0$. At $U_0 = 20\log 2$, $\rho/\pi > (40/3 + 16)/4 = 22/3 > 4$, $\lambda_0 < 3$, and hence $L(U_0) > \log 2 + 1/3 > 1$.

At the edge start $\pi e^{2(L-\eta)} = \rho$. The complete theta envelope $s(x)^2 \le 169 e^{4x - \pi e^{2x}}$, decreasing for $x \ge 0$, therefore gives

$$\sigma_\eta^2 \le 169(\rho/\pi)^2 e^{-\rho} \le \frac{169}{MU^6}, \qquad C_L\sigma_\eta^2 \le \frac{338\lambda}{MU^6}.$$

Moreover $4J\eta \ge \rho$, $2c_J \ge \rho/\eta$, and $1 - e^{-t} \ge t/(1+t)$ imply

$$2c_J(1-e^{-4\eta}) \ge \frac{4\rho}{1+4\eta} > \frac{4}{3}U, \qquad \tau_J := \frac{e^{-2c_J\eta}}{2c_J(1-e^{-4\eta})} \le \frac{3}{4MU^9}.$$

Applying $(a+b)^2 \le 2a^2+2b^2$ in (68) proves

$$\Delta_J \le \frac{676\lambda}{MU^6} + \frac{27}{2MU^9} \le \frac{1024\lambda}{MU^6}, \qquad M\Delta_J \le \frac{1024}{U^5} < \frac{1024}{13^5} < \frac{1}{4}.$$

These elementary inequalities prove (8) for every admitted integer, not just asymptotically.

The complete lower form satisfies $S_M \preceq H_f + M^{-1}W$. Under original Weil positivity, its exact physical congruence is

$$S_{M,h} = H_h - sE_J s + M^{-1} sW_{\mathrm{low}} s \succeq \left(\frac{1}{2M} - \Delta_J\right) I \succeq I/(4M).$$

Equivalently $S_M \succeq W/(4M) \succeq I/(36M)$. The bounded response and potential changes preserve the same original minimal supported closure. Compact polar-free exhaustion uses the exact physical row $s(x)\sinh(x/2)$, as in the proof of the odd criterion.

C.3 The small-support intersection core and receiving floor

The scalar symbol bounds needed below follow directly from

$$a(k) = \sum_{j\ge 0} \frac{2k^2}{c_j(c_j^2+k^2)}.$$

For $k \ge 1$, comparison with the decreasing integral gives $a(k) \ge \log k$; separating $j=0$ and bounding the remaining sum by its integral gives

$$a(k) \le 5 + \log(1+k), \qquad k \ge 0. \tag{69}$$

For the lower bound, the integral over $c \in [1/2, \infty)$, divided by the spacing 2, is $\frac{1}{2}\log(1+4k^2) \ge \log k$. For the upper, the first summand is at most 4; the remaining integral is at most $\frac{1}{2}\log(1+4k^2) \le \log 2 + \log(1+k)$.

Let $\bar{p} = p/\|p\|$, P_n be the first n DN projection, and $\delta_n = \|(I-P_n)\bar{p}\|^2$. Integration by parts gives the exact coefficients

$$|\langle \bar{p}, e_q\rangle|^2 = \frac{\cosh(L/2)^2}{L(\sinh L - L)(1/4+k_q^2)^2}. \tag{70}$$

For $L \ge 1$, the ratio $\cosh(L/2)^2/(\sinh L - L)$ decreases: its derivative is $-L\sinh L/[2(\sinh L - L)^2]$. Its value at 1 is below 9, since $\cosh 1 < 2$ and $\sinh 1 - 1 > 1/6$. The first-term-plus-integral estimate of the remaining fourth-power series therefore yields, for $n \ge 1$,

$$\delta_n \le \frac{9L^3}{\pi^4(n+1/2)^3} = \frac{9}{\pi k_n^3}.$$

If $f \perp \mathcal{V}_M$ and $f \perp p$, its low-frequency component is proportional to $P_{N_M}p$. The exact polar row and Cauchy–Schwarz give $\|P_{N_M}f\|^2 \le \delta_{N_M}\|f\|^2$, hence

$$D_L[f] \ge \{a(k_{N_M}) - \delta_{N_M}[a(k_{N_M}) - a(k_0)]\}\|f\|^2.$$

Put $X = B_M + \beta + 1 > 2$. Since $k_{N_M} \ge e^X$, the decreasing function $[5+\log(1+k)]/k^3$ and (69) give

$$\delta_{N_M}[a(k_{N_M}) - a(k_0)] \le \frac{9}{\pi}(X+6)e^{-3X} < 3(X+6)e^{-3X} \le 24e^{-6} < 3/8.$$

The deliberately generous $X+6$ bound follows from $\log(1+e^X) < X+1$. Thus D_L on the whole receiving complement exceeds $B_M + \beta + 5/8$. Dropping only the positive retained responses and using $V_L \succeq -B_M I$ proves

$$T_M \succeq \beta I + M^{-1} Q_M W_{\mathrm{low}} Q_M. \tag{71}$$

This proof is safe for every larger outward N_M.

For completeness write $v_q = \langle e_q, p\rangle$, $v_0 > 0$. The finite generating list $w_j = v_0 e_j - v_j e_0$, $1 \le j < N_M$, has Gram $v_0^2 I + v_{\mathrm{tail}} v_{\mathrm{tail}}^*$. By (70) and $\cosh(L/2)^2/(\sinh L - L) > 1/2$,

$$\operatorname{cond} G_w \le \frac{\|p\|^2}{v_0^2} < 2L(1/4 + \pi^2/(4L^2))^2 < 37L, \qquad L \ge 1.$$

Its inverse square root defines a computable exact profile isometry. All inputs remain finite DN combinations.

C.4 Constrained step inversion and the full residual

Let $F = F_M$, $Q = Q_M$, $d = d_M$, and $G_d = F^* d^{-1} F$. Since $F^*F = I$ and d is a finite positive step multiplier,

$$d_{\max}^{-1} I \preceq G_d \preceq d_{\min}^{-1} I.$$

Thus the finite inverse is both well-defined and susceptible to directed verification with an explicit lower bound. For $g \in \operatorname{ran} Q$, the function

$$z = d^{-1} g - d^{-1} F G_d^{-1} F^* d^{-1} g$$

satisfies $F^* z = 0$ and $Q d z = g$. It is the solution for the restricted multiplier form d_Q. The same formula annihilates F, so it defines $Z = (d_Q)^{-1}$ extended by zero. From (71), inverse form order gives

$$0 \preceq T_M^{-1} \preceq Z, \qquad \|Z\| \le \beta^{-1}.$$

Write $H_c = \mathsf{U}_M^* S_M \mathsf{U}_M$, $C = Q \widehat{S}_M \mathsf{U}_M$, and $K = H_c - C^* T_M^{-1} C$. For a legal operator-domain Y, $QY = Y$, the complete residual identity is

$$K = (\mathsf{U}_M - Y)^* \widehat{S}_M (\mathsf{U}_M - Y) - (C - T_M Y)^* T_M^{-1} (C - T_M Y).$$

This follows by expanding both squares and cancelling all linear and quadratic Y terms. As $r_Y = C - T_M Y = Q \widehat{S}_M (\mathsf{U}_M - Y)$ and $ZF = 0$, replacing T_M^{-1} by Z yields exactly

$$K - \Gamma_M(Y) = r_Y^* (Z - T_M^{-1}) r_Y, \qquad 0 \preceq K - \Gamma_M(Y) \preceq \beta^{-1} r_Y^* r_Y.$$

The positive receiving block makes this Schur identity valid even when the core block is indefinite. Acceptance retains the complete ambient action covariance, not a square of a finite output compression.

C.5 Computable trials, complete entries, and strict acceptance

Finite DN sums form a graph core for D_L. Formula (70) also shows $p \in \operatorname{Dom} D_L$, because $a(k_q) = O(\log(2+q))$ and the coefficient squares are $O(q^{-4})$. Restriction to $p^\perp$, and then to the finite-DN core complement, has the expected operator domain: the off-diagonal rows against these finitely many domain vectors are bounded. More explicitly, $\langle p, D_L f\rangle = \langle D_L p, f\rangle$, and the same identity holds for the finite core columns. Thus their removed row components are bounded in $\|f\|_2$, making

the ambient and restricted graph norms equivalent on the constrained domain. All other terms in $\widehat{S}_M - D_L$ are bounded at fixed M.

For a true receiving-domain f, DN truncation converges in this graph norm. Exact pole correction using e_0, followed by projection off the finite core, preserves graph convergence and finite-DN structure. The corrections in (7) are graph-continuous and act identically on true receiving vectors. Dyadic finite-DN proposals therefore give a computable graph-dense trial family. Each column of $T_M^{-1}C$ belongs to the receiving operator domain. Approximating these finitely many columns by that family makes $\|C - T_M Y\| \to 0$.

We spell out finite evaluability. The function d^{-1} is a finite step function. For finite-DN inputs, diagonal actions are finite sine sums; each retained response is an explicit exponential; each killed prime translation is a finite sine expression on its overlap panels. The odd-half identification supplies the reflected pieces as well. The potential action is another finite piecewise sine sum. Hence every entry in $A_Y^* d^{-1} A_Y$, $F^* d^{-1} A_Y$, G_d, and the form matrix uses finitely many sine, cosine and exponential integrals over a common finite partition. For example, the entire expression

$$\int_a^b e^{zx}\,dx = e^{za}(b-a)\sum_{m\ge 0}\frac{[z(b-a)]^m}{(m+1)!}$$

has an explicit exponential-series remainder and includes $z = 0$. It avoids an equality decision at a vanishing denominator. Outward panel endpoints and continuous zero-overlap limits similarly handle coincident shift endpoints.

All scalar coefficients are computable with directed errors. The displayed positive series for $a(k)$ has tail at $j \ge R$ at most $2k^2[c_R^{-3} + c_R^{-2}/4]$, by its first term and integral. Logarithms, exponentials, trigonometric functions and square roots have their usual convergent rational enclosures. The Euler constant entering m_0 may be enclosed between $H_n - \log n - 1/n$ and $H_n - \log n$. Finite-dimensional positive square roots and inverses can be enclosed using the explicit Gram lower bounds. The outward arithmetic index may be chosen within two of e^{2L}; any additional killed shift has zero overlap on the actual support. Consequently a finite certificate contains directed bounds for every scalar, node, height, action, constraint inverse, and final matrix.

Under RH the complete lower form is at least $I/(36M)$ in profile norm. Minimization over the entire receiving space with fixed core coefficient therefore gives $K \succeq I/(36M)$. The residual tolerance $\|r_Y\|^2 \le \beta/(72M)$ proves $\Gamma_M(Y) \succeq I/(72M)$. The computable graph core attains this tolerance at each fixed M. Directed refinement of the finitely many entries can attain any positive error radius. With $\|\widehat{\Gamma} - \Gamma_M(Y)\| \le e \le 1/(288M)$, the lower matrix obeys $\widehat{\Gamma} - eI \succeq I/(144M)$. This proves conditional fixed-instance completeness and finite recognition by a strict positive matrix test. Conversely, every accepted certificate gives a complete original-norm allowance M^{-1}; cofinal exhaustion proves Theorem 2.1.

C.6 A quantitative complete receiving-return enclosure

We prove Theorem 2.2. Write $\widehat{S}_M = D_L + \mathcal{B}$, where

$$\|\mathcal{B}\| \le \mathfrak{h} = C_L + B_M + M^{-1}e^{r_L}/(2r_L^2).$$

The potential bound follows from the first theta term, as detailed below. For the normalized polar row, cosine Parseval and integration by parts give

$$\sum_{q\ge 0} k_q^2 |\langle p, e_q\rangle|^2 = \|p'\|^2, \qquad \frac{\|p'\|^2}{\|p\|^2} = \frac{\sinh L + L}{4(\sinh L - L)} < 4, \quad L \ge 1.$$

The derivative ratio is decreasing, and the last inequality follows at 1 from $\sinh 1 > 7/6 > 17/15$. The preceding symbol integral gives $a(k) \le 4 + \frac{1}{2}\log(1+4k^2) \le 4 + 2k$. Hence $a(k)^2 \le 32 + 8k^2$ and $\|D_L\bar{p}\| \le 8$. For $F = [\bar{p}, \mathsf{U}_M]$ this proves the dimension-independent estimate

$$\|D_L F\| \le \sqrt{64 + \alpha^2}.$$

The graph-domain argument above identifies $\operatorname{Dom} T_M = \operatorname{Dom} D_L \cap \operatorname{ran} Q_M$. For $y = T_M^{-1} f$, $f \in \operatorname{ran} Q_M$,

$$D_L y = f - Q_M \mathcal{B} y + F(D_L F)^* y, \qquad \|D_L y\| \le G\|f\|.$$

Also $C = Q_M \widehat{S}_M \mathsf{U}_M$ obeys $\|C\| \le \alpha + \mathfrak{h}$.

Let P_m be the first m DN projection, $m \ge N_M$, and set $z_0 = (I - \mathsf{U}_M \mathsf{U}_M^*)e_0$. Then z_0 lies in the first N_M modes, $\|z_0\| \le 1$, $D_L[z_0] \le \alpha$, and $\langle p, z_0\rangle = \langle p, e_0\rangle \ne 0$. For $y = T_M^{-1} f$, the legal approximation

$$w_m = P_m y - z_0 \frac{\langle p, P_m y\rangle}{\langle p, e_0\rangle} \in \mathcal{W}_m$$

retains both the polar and core constraints. Put $h = (I - P_m)y$ and $\zeta = \langle p, P_m y\rangle / \langle p, e_0\rangle = -\langle p, h\rangle / \langle p, e_0\rangle$. The high and low pieces of $y - w_m = h + z_0\zeta$ are orthogonal in both L^2 and the diagonal form. Thus

$$\|y - w_m\|^2 \le (1 + R_0)\|h\|^2, \qquad D_L[y - w_m] \le D_L[h] + \alpha R_0 \|h\|^2.$$

The exact pole formula gives

$$R_0 = \frac{L(\sinh L - L)}{\cosh(L/2)^2}\left(\tfrac{1}{4} + \frac{\pi^2}{4L^2}\right)^2 < 16L.$$

Indeed the hyperbolic ratio is below 2, $\pi^2 < 10$, and $2(1/4 + 10/4)^2 = 121/8 < 16$. Since a is increasing,

$$\|h\|^2 \le a(k_m)^{-2}\|D_L y\|^2, \qquad D_L[h] \le a(k_m)^{-1}\|D_L y\|^2.$$

On the same receiving domain these estimates imply

$$T_M[y - w_m] \le \{a(k_m)^{-1} + P_* a(k_m)^{-2}\}\|D_L y\|^2.$$

Let $Y = T_M^{-1} C$ and $Y_m = V_m T_m^{-1} B_m$. The classical Galerkin energy projection [5] gives orthogonality in the complete T_M energy:

$$0 \preceq X - X_m = (Y - Y_m)^* T_M (Y - Y_m).$$

Apply its best-approximation property to w_m, for each common core coefficient vector. The graph bound for T_M^{-1} then proves

$$0 \preceq X - X_m \preceq G^2\{a(k_m)^{-1} + P_* a(k_m)^{-2}\} C^* C \preceq \epsilon_m (\alpha + \mathfrak{h})^2 I.$$

Every omitted receiving direction is priced by this error. The full C^*C is an optional sharper payment; the printed scalar alternative uses the complete bound $\|C\| \le \alpha + \mathfrak{h}$.

For the specified outward m_M, $k_{m_M} \ge e^{t_M}$ and $a(k_{m_M}) \ge t_M \ge \max(1, P_*)$. Consequently

$$\epsilon_{m_M}(\alpha + \mathfrak{h})^2 \le \frac{2G^2(\alpha + \mathfrak{h})^2}{a(k_{m_M})} \le \frac{1}{72M}.$$

This proves the conditional margin and the strict finite enclosure in Theorem 2.2. The entries of H_c, T_m, B_m are finite-DN form integrals with the same step and shift panels already described. Their receiving matrix has the explicit floor β. They therefore admit finite directed inversion and error bounds.

C.7 Actual arithmetic commutators and the sharper return bound

We prove Corollary 2.3, including the support-domain identification. On the full line let $A = a(D)$ and use the bounded nonnegative partial multipliers

$$A_n = a_n(D), \qquad a_n(\xi) = \sum_{j<n} \{2/c_j - 2c_j/(c_j^2 + \xi^2)\}.$$

Their off-diagonal kernel is $-K_n(|x-y|)$, with

$$0 \le K_n(d) \le \frac{e^{-d/2}}{1-e^{-2d}} \le e^{-d/2} + \frac{1}{2d}.$$

Across a half-line cut, write the distances as $u, v > 0$. The exponential cross kernel has rank one and norm 1; the Carleman kernel $1/[2(u+v)]$ has norm at most $\pi/2$. The commutator consists of a cross block and its negative adjoint, so its norm is at most $1 + \pi/2$. For the interval projection $P = 1_{(-L,L)}$, subtraction of the two half-line projections gives uniformly in n, L

$$\|[A_n, P]\| \le c_* := 2 + \pi. \tag{72}$$

The diagonal constants cancel before this estimate.

For $f \in \operatorname{Dom} A$, $A_n f \to Af$, and (72) bounds $\|A_n P f\|$ uniformly. Monotone convergence of $a_n(\xi)^2 |\widehat{Pf}(\xi)|^2$ proves $Pf \in \operatorname{Dom} A$. Strong limits then give the bounded commutator $[A, P]$. The off-diagonal blocks $PA(I-P)$ and $(I-P)AP$ are bounded by c_*. Removing their self-adjoint sum from A yields a self-adjoint block-diagonal operator on $\operatorname{Dom} A$. Its interval part A_I has domain $\operatorname{Dom} A \cap \operatorname{ran} P$, and represents the restricted full-line closed form.

This is the original minimal supported realization. Indeed the block argument also bounds every interval projection uniformly in the norm $\|f\|^2 + \|A^{1/2} f\|^2$. Interior interval cutoffs converge in that norm: prove it first on the global Schwartz core, where a boundary strip of width ε has H^t norm $O(\varepsilon^{1/2-t})$, $0 < t < 1/2$, and then use the uniform projection bound and density. The logarithmic multiplier is bounded by every positive Sobolev power. An interior-supported vector can then be mollified, with convergence by Fourier dominated convergence and with support retained. Even mollifiers preserve odd parity. Thus compact smooth odd tests are a form core of the odd restricted form. Lemma B.1 identifies this same closure with $D_L + R_L$.

Let τ_a be full-line translation and $T_a = P\tau_a P$ the actual killed translation on $\operatorname{ran} P$. Both P and τ_a preserve $\operatorname{Dom} A$; translation commutes with A. On the compressed domain the exact identity is

$$[A_I, T_a] = -PA(I-P)\tau_a P + P\tau_a (I-P)AP.$$

It proves $\|[A_I, T_a]\| \le 2c_*$ for every real a. Only the pair $T_a + T_{-a}$ is restricted to the odd sector. It has norm at most 2, preserves the odd supported domain, and the half-unitary identity $D_L = A_I|_{\mathrm{odd}} - R_L$ gives

$$\|[D_L, (T_a + T_{-a})|_{\mathrm{odd}}]\| \le 4(2+\pi) + 4C_L. \tag{73}$$

There is no shift-size singularity. Triangle inequality in this commutator estimate retains every active prime power, while the arithmetic operator in the form keeps its actual signed coefficients.

We also pay the other components. For a step b on $(0, L)$, each jump at t has

$$\|[D_L, 1_{(t,L)}]\| \le 2C_L, \qquad \|[D_L, b]\| \le 2C_L \ \mathrm{TV}(b).$$

To prove the first estimate, use the finite digamma identity $D_n = A_{I,n}|_{\mathrm{odd}} - R_{L,n}$. In half-interval coordinates the opposite-side kernel of $A_{I,n}$ is bounded by $1 + 1/[2(u+v)]$, giving norm at most C_L.

The $R_{L,n}$ commutator costs at most C_L. Uniformity in n, followed by the same diagonal monotone convergence argument, proves both preservation of $\operatorname{Dom} D_L$ and the stated bound. Sum the jumps to obtain the variation estimate.

For the finite Green terms,

$$\|r_j\|^2 \le (2c_j)^{-1}, \qquad \|r_j'\|^2 \le 2c_j, \qquad \|D_L r_j\|^2 \le 16/c_j + 16c_j.$$

These follow from the explicit exponential profiles and the same DN cosine Parseval estimate used for p. Hence $\|D_L r_j\|\|r_j\| \le \sqrt{8/c_j^2 + 8} < 7$. The rank-one commutator identity then gives $\|[D_L, R_{L,J}]\| \le 14J$, with domain preservation. If n_{cell} is the potential cell count, an explicit bound for the whole perturbation $\mathcal{B}$ is therefore

$$\begin{aligned} K_1 = 14J + [4(2+\pi) + 4C_L] \sum_{2 \le n < e^{2L}} \frac{\Lambda(n)}{\sqrt{n}} \\ + 4C_L M^{-1} n_{\mathrm{cell}} \|W_{\mathrm{low}}\|_\infty \;\ge\; \|[D_L, \mathcal{B}]\|. \end{aligned} \tag{74}$$

Here $\mathrm{TV}(W_{\mathrm{low}}) \le 2n_{\mathrm{cell}}\|W_{\mathrm{low}}\|_\infty$; monotonicity of the heights is unnecessary. Every component preserves $\operatorname{Dom} D_L$.

Put $P_F = F_M F_M^*$, $K_F = \sqrt{64 + \alpha^2}$, $c = \alpha + \mathfrak{h}$, and define

$$F_1 = \alpha^2 + \mathfrak{h}\alpha + K_1 + K_F c, \qquad G_2 = F_1 + (\mathfrak{h} + K_F)Gc + (K_1 + K_F\mathfrak{h})c/\beta. \tag{75}$$

The complete forcing $C = Q_M \widehat{S}_M \mathsf{U}_M$ belongs to $\operatorname{Dom} D_L$, with $\|D_L C\| \le F_1$: expand $D_L Q_M (D_L + \mathcal{B})\mathsf{U}_M$, using $\|D_L P_F\| \le K_F$ and $D_L\mathcal{B} = \mathcal{B}D_L + [D_L, \mathcal{B}]$. For $y = T_M^{-1} f$ with $f \in \operatorname{Dom} D_L$, the previously proved identity $D_L y = f - Q_M \mathcal{B} y + P_F D_L y$ has its right side in $\operatorname{Dom} D_L$. Thus $y \in \operatorname{Dom} D_L^2$, and applying D_L yields

$$\|D_L^2 y\| \le \|D_L f\| + (\mathfrak{h} + K_F)\|D_L y\| + (K_1 + K_F\mathfrak{h})\|y\|.$$

With $f = Cv$, this proves $\|D_L^2 T_M^{-1} C\| \le G_2$, including every finite polar/core projection term.

For the same legal approximation w_m as above,

$$D_L[h] \le a(k_m)^{-3}\|D_L^2 y\|^2, \qquad \|h\|^2 \le a(k_m)^{-4}\|D_L^2 y\|^2.$$

The complete Galerkin energy identity now proves

$$0 \preceq X - X_m \preceq G_2^2\{a(k_m)^{-3} + P_* a(k_m)^{-4}\} I.$$

The threshold in Corollary 2.3 pays this by $1/(72M)$. On the actual support schedule, the elementary weighted-prime bound, the response and cell counts, and $M^{-1}\|W_{\mathrm{low}}\|_\infty = M^{o(1)}$ give $K_1 = M^{o(1)}$. Consequently $G_2 = M^{o(1)}$ and $\log m_M^{(2)} = M^{1/3 + o(1)}$, completing its proof.

C.8 Representation growth

For (3),

$$e^{2L} = \rho e^{2/\lambda}/\pi = (1/\pi + o(1)) \log M, \quad J = O(\log M \log\log M), \quad L = O(\log\log M).$$

The elementary arithmetic budget gives $B = O(\sqrt{\log M})$, so $\log N = O(\sqrt{\log M})$ and $N = M^{o(1)}$. Also $K_L = 21\pi e^{2L} = O(\log M)$, proving the cell count in Proposition 2.4. The first theta term gives

$s(x)^2 = \Theta/t \geq 2r^2 e^{-r}$; this last function decreases for $r \geq \pi$. With $r_L = \pi e^{2L} = (1+o(1))\log M$, any profile isometry E therefore has

$$I/9 \preceq E^* W E \preceq \frac{e^{r_L}}{2r_L^2} I, \qquad \operatorname{cond}(E^* W E) \leq \frac{9e^{r_L}}{2r_L^2} = M^{1+o(1)}.$$

These are representation and metric bounds for the stated core and profile isometries. The explicit residual tolerance governs trial enrichment and finite precision in each certificate.

For the quantitative energy-return construction, the same potential bound gives $\mathfrak{h} = M^{o(1)}$ on the actual schedule. The certified near-threshold core index has $\alpha = O(B_M + 1)$, while $R_0 = O(L_M)$. Thus $G, P_*, (\alpha + \mathfrak{h})^2 = M^{o(1)}$, and the explicit t_M is $M^{1+o(1)}$. Choosing m_M within two above its displayed threshold gives $\log m_M = M^{1+o(1)}$, as asserted. This trial-size bound quantifies conditional completeness; all scalar, matrix inversion, and final enclosure precisions are paid separately in the finite certificate.

D Unrestricted derivative nulls and exterior response pairing

The derivative nulls in Lemma 9.1 also satisfy an uncompressed distributional equation. This extension concerns the same original null family. We retain the normalization $\mathcal{M}f(z) = \int_{\mathbb{R}} f(x) e^{zx}\,\mathrm{d}x$ and Fourier measure $\mathrm{d}\xi/(2\pi)$, with inner products conjugate-linear in their first argument. Write $\eta_j = s z_j = (\partial_x^2 - 1/4)\psi^{(2j+1)}$. The transform identity and complete derivative tails are those already proved in Lemma 9.1 and Appendix A.

Define the full signed geometric expression by

$$\begin{aligned} \mathcal{A}_{\mathrm{geo}} f(x) = (a(D) + m_0) f(x) - \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \{ f(x - \log n) + f(x + \log n) \} \\ + \int_{\mathbb{R}} 2\cosh((x-y)/2) f(y)\,\mathrm{d}y. \end{aligned} \tag{76}$$

For compact input the prime sum is locally finite. For the theta-tail inputs η_j it converges locally absolutely with every derivative. The expression is interpreted distributionally; no global self-adjoint realization of its infinite prime convolution is assumed. The pole term in (76) is part of the definition. On odd tests its quadratic form is $-2|\mathcal{M}f(1/2)|^2$, so the expression agrees with the complete Weil zero form in (64), rather than with the pole-free geometric form on an unrestricted test class.

Proposition D.1 (Unrestricted distributional annihilation)**.** *For every fixed $j \geq 0$ and every $g \in C_c^\infty(\mathbb{R})$,*

$$\langle \eta_j, \mathcal{A}_{\mathrm{geo}} g \rangle = 0.$$

Equivalently $\mathcal{A}_{\mathrm{geo}} \eta_j = 0$ in distributions on $\mathbb{R}$.

Proof. Use the polar-preserving approximants already employed in Lemma 9.1, now before division by s:

$$\eta_{j,R} = (\partial_x^2 - 1/4)(\chi_R \psi^{(2j+1)}),$$

where χ_R is even with fixed-width transition and uniformly bounded derivatives of each fixed order. These functions are compact, smooth, odd, and have both polar moments exactly zero. The complete theta derivative bounds give convergence to η_j in every fixed exponentially weighted

derivative L^1 seminorm and in every fixed H^k. In particular repeated integration by parts yields, uniformly in R and $|\Re z| \le 1/2$,

$$|\mathcal{M}\eta_{j,R}(z)| \le C_{j,N}(1+|\Im z|)^{-N} \quad \text{for each fixed } N.$$

For odd compact g, polarize the complete compact explicit formula with $\eta_{j,R}$ in the first inner-product slot. Its zero side is

$$\sum_{\rho} \mathcal{M}g(\rho-1/2)\overline{\mathcal{M}\eta_{j,R}(1/2-\overline{\rho})}.$$

The zero count and the uniform strip estimates give an absolutely summable majorant counting every multiplicity. The limit of each term is zero, since

$$\mathcal{M}\eta_j(z) = (z^2-1/4)(-z)^{2j+1}\xi(1/2+z)$$

vanishes at both polar points and at every centered nontrivial zero; the reflected conjugate argument is another such zero. Multiplicity repeats the same vanishing value and introduces no derivative premise.

The geometric limit is also justified directly. The H^2 convergence and $0 \le a(\xi) \le 17\xi^2$ control the archimedean pairing. For x in a fixed compact interval $[-b,b]$, the complete theta tails and the cutoff derivative bounds give, uniformly in R and large n,

$$|\eta_{j,R}(x \pm \log n)| + |\eta_j(x \pm \log n)| \le Cn^C \exp(-c_b n^2).$$

Thus the entire prime pairing is dominated by a convergent series $\sum_n \Lambda(n) n^{C-1/2}\exp(-c_b n^2)$. Scalar terms converge, and polar cross terms vanish exactly. Dominated convergence proves the assertion for odd g without a polar restriction on g. Even g gives zero by parity of every term in (76); decomposition proves the general case. Symmetry of the distributional expression identifies the equivalent annihilation statement. No finite zero list or prime truncation is used. □

Proposition D.2 (Complete exterior identity)**.** *Let $I = (-b,b)$ and let u, extended by zero, be odd and belong to the actual minimal supported local geometric operator domain. Suppose that its constrained equation inside I is*

$$\mathcal{A}_{\mathrm{geo}}u = \lambda u + \beta_u p, \qquad p(x) = \sinh(x/2), \quad \lambda \in \mathbb{R}.$$

Then

$$\lambda\langle \eta_j, u\rangle_I = -\langle \eta_j, \mathcal{A}_{\mathrm{geo}}u - \beta_u p\rangle_{\mathbb{R}\setminus I}. \tag{77}$$

Every prime and polar term in the exterior pairing is retained and absolutely integrable against η_j. The entire exterior response need not belong to ordinary L^2.

Proof. The domain and supported-closure identification in Appendix C gives an L^2 archimedean action inside I. Outside I, the off-diagonal kernel of $a(D)$ is exactly

$$-J(x-y), \qquad J(t) = \frac{e^{-|t|/2}}{1-e^{-2|t|}}, \quad t \ne 0.$$

Indeed the positive digamma series has summands $2/c_j - 2c_j/(c_j^2+\xi^2)$, and the inverse Fourier transform of its second term is $-e^{-c_j|t|}$ away from the diagonal. Near either endpoint the cross kernel is bounded by a bounded remainder plus $1/[2(s+t)]$, for positive distances s,t to that

endpoint. The Carleman part has L^2 norm at most $\pi/2$, by Schur weights $t^{-1/2}$; the remaining local kernel is bounded and the far kernel decays exponentially. Hence the cross map from $L^2(I)$ to $L^2(\mathbb{R} \setminus I)$ is bounded.

These interior and exterior pieces represent the full archimedean distribution, including the endpoints. To see the last detail without assuming a global operator domain, logarithmic symbol growth gives $a(D)u \in H^{-\epsilon}(\mathbb{R})$ for every $\epsilon > 0$. The assembled interior/exterior action is in L^2. Their difference is supported at the two endpoints. A point-supported distribution is a finite sum of delta derivatives, none of which belongs to $H^{-\epsilon}(\mathbb{R})$ when $0 < \epsilon < 1/2$. The difference therefore vanishes. Fourier symmetry now justifies the archimedean pairing with the Schwartz function η_j.

For arithmetic, change variables to the compact input interval and use the complete theta bound to obtain

$$\sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}} \int_I |u(y)|\,|\eta_j(y \pm \log n)|\,\mathrm{d}y < \infty.$$

This follows from $\|u\|_{L^1(I)} < \infty$ and the same $n^C \exp(-c_b n^2)$ majorant. It pays every far-prime translate and permits reversal of the full sum. The pole grows at most like $e^{|x|/2}$, whereas η_j has every exponential moment, so its pairing is absolute and can also be reversed. Together with the scalar and archimedean parts, Proposition D.1 gives

$$\langle \eta_j, \mathcal{A}_{\mathrm{geo}} u\rangle = \langle \mathcal{A}_{\mathrm{geo}} \eta_j, u\rangle = 0.$$

Split inside and outside I, substitute the stated local equation, and use the global identity $\langle \eta_j, p\rangle_{\mathbb{R}} = 0$. The interior polar moment need not vanish separately. Moving it to the exterior gives (77), with the displayed $\beta_u p$ term and sign. □

These propositions extend the derivative nulls to unrestricted tests and complete exterior responses. For packets null for an operator with a subtracted critical-value correction, that correction remains in the equation. An application to quantitative boundary observation uses the exact exterior identities together with an observation norm and remainder estimates for the actual eigenresponses.

E Supported unweighted growth and zero-free half-planes

The fixed-shift criterion of Theorem 14.3 uses one uniform unweighted lower bound across supports. This appendix quantifies the corresponding implication when the allowed negative lower bound depends on the support radius. The argument combines the complete Weil explicit formula [22, 4] with the classical singularity principle for nonnegative Laplace transforms. Related eventual-sign criteria for zero-free half-planes appear in Suzuki's shifted screw-function formulation [20, Section 11, Theorem 11.1].

For a smooth compact profile f write

$$\mathcal{M}f(z) = \int_{\mathbb{R}} f(u) e^{zu}\,\mathrm{d}u, \qquad \|f\|_2^2 = \int_{\mathbb{R}} |f(u)|^2\,\mathrm{d}u.$$

On profiles satisfying $\mathcal{M}f(1/2) = \mathcal{M}f(-1/2) = 0$, put $q_{\mathrm{geo}}[f] = \mathcal{Q}[f]$, with the complete archimedean-minus-prime form $\mathcal{Q}$ of Section 14. The compact explicit formula (64) identifies this form with its complete zero sum. Let $\mathcal{Z}$ be the set of distinct centered nontrivial zeros $\sigma = \rho - 1/2$, and write m_σ for their positive integer multiplicities. The Fourier measure remains $\mathrm{d}\xi/(2\pi)$.

Proposition E.1 (Supported growth controls zero locations)**.** *Fix $a \geq 0$. Suppose that, for every $\varepsilon > 0$, there are constants $C_\varepsilon \geq 0$ and L_ε such that*

$$q_{\mathrm{geo}}[f] \geq -C_\varepsilon e^{(a+\varepsilon)L} \|f\|_2^2 \tag{78}$$

for every $L \geq L_\varepsilon$ and every real odd $f \in C_c^\infty(\mathbb{R})$ supported in $[-L, L]$ with $\mathcal{M}f(\pm 1/2) = 0$. Then every nontrivial zeta zero satisfies

$$\Re\rho \leq \frac{1+a}{2}.$$

In particular, the hypothesis with $a = 0$ implies RH.

Proof. Choose a real even $g \in C_c^\infty(\mathbb{R})$, supported in $[-r, r]$, with both polar moments zero, and put $G = \mathcal{M}g$. For $R > r$ define

$$f_R = \tau_R g - \tau_{-R} g, \qquad \tau_R g(u) = g(u - R).$$

This test is real, odd and polar-free, has support in $[-(R+r), R+r]$, and its disjoint translated supports give

$$\|f_R\|_2^2 = 2\|g\|_2^2, \qquad \mathcal{M}f_R(z) = 2\sinh(Rz)G(z). \tag{79}$$

The complete zero form on a real even test has summands $G(\sigma)^2$; on a real odd test it has summands $-\mathcal{M}f(\sigma)^2$. Using the symmetry $\sigma \mapsto -\sigma$, define

$$C_g(R) = \sum_{\sigma \in \mathcal{Z}} m_\sigma e^{2R\sigma} G(\sigma)^2 = \sum_{\sigma \in \mathcal{Z}} m_\sigma \cosh(2R\sigma) G(\sigma)^2.$$

Conjugation symmetry makes C_g real for real R. Expansion of $\sinh^2$ in the complete explicit formula yields

$$q_{\mathrm{geo}}[f_R] = 2q_{\mathrm{geo}}[g] - 2C_g(R). \tag{80}$$

For every fixed integer $n \geq 0$, integration by parts gives

$$|G(z)| \leq A_{g,n}(1 + |\Im z|)^{-n}, \qquad |\Re z| \leq 1/2.$$

Together with the multiplicity-counted zero bound $N_\zeta(T) = O(T\log(T+2))$, this proves $\sum_\sigma m_\sigma |G(\sigma)|^2 < \infty$. Thus the sum defining C_g converges absolutely, locally uniformly for real R, and satisfies $|C_g(R)| \leq A_g e^R$ for $R \geq 0$. In particular it is continuous. Applying (78) at $L = R + r$ and using (79)–(80) gives

$$C_g(R) \leq q_{\mathrm{geo}}[g] + C_\varepsilon \|g\|_2^2 e^{(a+\varepsilon)(R+r)}.$$

After enlargement of a constant on a compact initial interval,

$$C_g(R) \leq K_{g,\varepsilon} e^{(a+\varepsilon)R}, \qquad R \geq 0. \tag{81}$$

For $\Re s > 1$, absolute convergence permits termwise Laplace integration:

$$F_g(s) = \int_0^\infty e^{-sR} C_g(R)\, \mathrm{d}R = \sum_{\sigma \in \mathcal{Z}} \frac{m_\sigma G(\sigma)^2}{s - 2\sigma}. \tag{82}$$

The right side converges normally on compact subsets avoiding its discrete pole set. Indeed the high-zero denominators are bounded below by a constant times their heights, whereas G has arbitrary inverse-power vertical decay. It therefore defines a meromorphic continuation with residue

$m_{\sigma_0} G(\sigma_0)^2$ at $s = 2\sigma_0$. This continuation is holomorphic on the positive real axis: nontrivial real zeta zeros are absent, as recalled in Appendix A.

We recall the positive-Laplace singularity principle with its elementary proof. If $h \geq 0$ is locally integrable and its Laplace integral has a finite real convergence abscissa α, the transform has a singularity at the real point α. Otherwise, let it continue holomorphically to a disk of radius r about α and set $s_0 = \alpha + r/4$. Its Taylor series at s_0 converges at $s_0 - u$, where $u = r/3$. Differentiation in the convergence half-plane and Tonelli's theorem give

$$\sum_{n\geq 0} \frac{u^n}{n!} \int_0^\infty R^n e^{-s_0 R} h(R)\, \mathrm{d}R = \int_0^\infty e^{-(s_0-u)R} h(R)\, \mathrm{d}R < \infty.$$

All Taylor terms in this real direction are nonnegative, while $s_0 - u = \alpha - r/12 < \alpha$, a contradiction. The derivatives under the integral are justified by a smaller positive convergence margin.

Fix $b > a$ and choose $\varepsilon > 0$ with $a + \varepsilon < b$. Equation (81) and continuity allow a constant $A > 0$ such that

$$h_b(R) = Ae^{bR} - C_g(R) \geq 0 \quad (R \geq 0), \qquad h_b(R) \geq (A/2)e^{bR} \quad \text{for large } R.$$

Its Laplace abscissa α is finite and lies between b and $\max\{b, 1\}$. The transform has meromorphic continuation

$$H_b(s) = \frac{A}{s-b} - F_g(s).$$

If $\alpha > b$, this expression is holomorphic near the real point α, contradicting the singularity principle. Hence $\alpha = b$, and the genuine Laplace transform is holomorphic throughout $\Re s > b$. By meromorphic continuation the residue in (82) must therefore vanish at every pole in that half-plane.

Suppose now that a centered nontrivial zero σ_0 has $2\Re\sigma_0 > a$. Choose b strictly between these two numbers. Let ϕ_t be an even real smooth mass-one mollifier with compact support shrinking to zero, and set $g_t = (\partial_u^2 - 1/4)\phi_t$. Integration by parts gives both polar moments zero and

$$\mathcal{M}g_t(\sigma_0) = (\sigma_0^2 - 1/4)\mathcal{M}\phi_t(\sigma_0) \longrightarrow \sigma_0^2 - 1/4 \neq 0.$$

Fix a sufficiently small t. This one fixed admissible probe has a nonzero residue at $2\sigma_0$, contradicting holomorphy for $\Re s > b$. Multiplicity contributes the positive integer m_{σ_0} to this same residue. We conclude $2\Re\sigma \leq a$ for all centered zeros, which is the asserted closed boundary. For $a = 0$, functional symmetry places every nontrivial zero on the critical line. □

Corollary E.2 (Fixed-probe growth characterization)**.** *For $a \geq 0$, the following are equivalent:*

1. *Every nontrivial zeta zero has $\Re\rho \leq (1 + a)/2$.*
2. *For each fixed real even smooth compact polar-free g and each $\varepsilon > 0$, there is a finite $K_{g,\varepsilon} \geq 0$ such that $C_g(R) \leq K_{g,\varepsilon} e^{(a+\varepsilon)R}$ for every $R \geq 0$.*

Proof. Under the first assertion, absolute summation gives

$$|C_g(R)| \leq e^{aR} \sum_{\sigma\in\mathcal{Z}} m_\sigma |G(\sigma)|^2, \qquad R \geq 0,$$

which proves the second assertion. Conversely, apply the positive-Laplace argument in the preceding proof to a fixed detecting probe for each proposed zero with $2\Re\sigma > a$. The second assertion supplies exactly the one-sided bound (81) used in that argument, and the nonzero residue excludes that proposed zero. □

The constants in the fixed-probe characterization may depend on the chosen probe and its support. Proposition E.1 uses a single C_ε for all admitted profiles at every sufficiently large support. These two quantifier patterns specify the respective supported-form and translated-probe verification targets.

F Relative source energy and complete tail payment

This appendix proves the whole-tail payment used by Theorem 14.6. The spectral bands are proof devices; the finite test in that theorem requires only the first-moment core.

Lemma F.1 (Elementary prime budget)**.** *For $X \geq 1$,*

$$\sum_{n \leq X} \frac{\Lambda(n)}{\sqrt{n}} < 6\sqrt{X}.$$

Consequently $\Sigma_L < 6e^L$.

Proof. For an integer $h \geq 1$, Legendre's valuation formula gives

$$v_p\binom{2h}{h} = \sum_{k \geq 1} \left(\left\lfloor \frac{2h}{p^k} \right\rfloor - 2 \left\lfloor \frac{h}{p^k} \right\rfloor \right).$$

Each summand is nonnegative and equals one when $h < p^k \leq 2h$. Writing $\Psi(X) = \sum_{n \leq X} \Lambda(n)$, this proves $\Psi(2h) - \Psi(h) \leq 2h \log 2$. Dyadic telescoping and monotonicity give $\Psi(X) < 4X \log 2$ for $X > 1$. Partial summation yields

$$\sum_{n \leq X} \frac{\Lambda(n)}{\sqrt{n}} = \frac{\Psi(X)}{\sqrt{X}} + \frac{1}{2} \int_1^X \Psi(t) t^{-3/2}\, dt \leq 8(\log 2)\sqrt{X} - 4 \log 2 < 6\sqrt{X}.$$

The case $X = 1$ is immediate, and restriction to the strict active cutoff proves the last assertion. This is the elementary Chebyshev argument. □

Lemma F.2 (Nikki's Relative Tail-Payment Lemma)**.** *Let $D \succeq 0$ be self-adjoint with an increasing discrete eigenbasis. Let V be bounded self-adjoint, preserve $\operatorname{Dom} D$, and satisfy $\|V\| \leq P$, $\|[D, V]\| \leq K$, where $P \geq 1$ is rational. Let U contain its first N modes, $Q = I - UU^*$, and assume that the first omitted diagonal is at least A. Put*

$$H = U^*(D + V)U \succeq \delta I, \quad \delta > 0, \qquad C = QVU.$$

Let $n \geq 1$ be minimal with $4^n \geq 1 + P$, and suppose

$$A \geq \max\{8, 16P, [1024(n+1)^2(K^2 + P^2)/\delta]^{1/3}\}.$$

For $\tau \in [A/2, A/2 + 1]$, the entire receiver satisfies $Q(D + V)Q \succeq \tau I$ and

$$\tau^{-1} C^* C \preceq \frac{3}{16} H, \qquad H - \tau^{-1} C^* C \succeq \frac{13}{16} H. \tag{83}$$

Proof. Write $V_N = U^* V U$ and $D_Q = QDQ|_{\operatorname{ran} Q}$. Since U reduces D, domain invariance gives the actual operator identities

$$[H, V_N] = U^*[D, V]U, \qquad D_Q C - CH = Q[D, V]U - CV_N.$$

In particular C maps the finite source space into $\operatorname{Dom} D_Q$. Let $s_j = 1/4 + j/(4n)$ and $E_j = \mathbf{1}_{(-\infty, s_j A]}(H)$, $j = 0, \dots, n$. Adjacent thresholds have gap $A/(4n)$, so the finite Sylvester integral for the first identity gives

$$\|E_{j+1}^{\perp} V_N E_j\| \le 4nK/A.$$

The second identity has receiving-source gap at least $A/2$ on every E_j. To justify its Sylvester integral, differentiate $e^{-tD_Q} C E_j e^{tH|_{E_j}}$. The domain assertion above justifies the derivative and the terminal term decays at least as $e^{-At/2}$.

Let $c_j = \|CE_j\|$ and $r = 2P/A \le 1/4$. Splitting $V_N E_j$ at E_{j+1} and using the two gap bounds gives

$$c_j \le q + r c_{j+1}, \quad q = 2K/A + 8nPK/A^2 \quad (j < n), \qquad c_n \le 2(K + P^2)/A.$$

Iterating this finite recurrence proves

$$c_0 \le \frac{q}{1-r} + r^n \frac{2(K+P^2)}{A} \le \frac{4(n+1)K}{A} + \frac{2P}{A}.$$

Indeed $q \le (n+2)K/A$, $(1-r)^{-1} \le 4/3$, $(4/3)(n+2) + 2 \le 4(n+1)$, and $P 4^{-n} \le 1$. All growing-band constants are explicit.

On E_0, $H \succeq \delta I$, while on $E_0^{\perp}$ it exceeds $AI/4$. Thus $\|CH^{-1/2}\| \le c_0/\sqrt{\delta} + 2P/\sqrt{A}$. Two applications of Young's inequality and $\tau \ge A/2$ yield

$$\frac{\|CH^{-1/2}\|^2}{\tau} \le \frac{128(n+1)^2 K^2}{\delta A^3} + \frac{32P^2}{\delta A^3} + \frac{16P^2}{A^2} \le \frac{1}{8} + \frac{1}{16} = \frac{3}{16}.$$

This bounds the complete mixed covariance, including low/high cross terms. Finally $Q(D+V)Q \succeq (A-P)I \succeq \tau I$, since $A \ge 16P$, $P \ge 1$. Congruence by $H^{1/2}$ proves (83). □

The proof uses only the first commutator. Its narrow bands all lie between $A/4$ and $A/2$, so their number can grow while the full-source Sylvester gap remains at least $A/2$. When $P, K = O(e^L)$, the prescribed threshold has $A = O(e^L)$. This is the resource improvement used by the fixed-shift core-only criterion. The complete-Galerkin and whole-source Gram tests remain complementary constructions with their own admissions.

G Proofs for the high-frequency criteria

We use the Fourier convention and original paired closure of Section 3. In this appendix $N_k = N_{16,k}$ is always the full gamma negative source and $N_k^{>}$ its part on $|\xi| > 17$.

G.1 A complete meromorphic bound and the finite-band return

Let $F = -\zeta'/\zeta$, and let ξ denote the completed entire xi function. Its Hadamard product and functional equation give

$$\frac{\xi'}{\xi}(w) = B + \sum_{\rho} \frac{w}{\rho(w-\rho)}, \qquad B = \log(2\sqrt{\pi}) - 1 - \gamma_E/2 \in (-1/40, 0), \qquad \sum_{\rho} |\rho|^{-2} < 1/10. \tag{84}$$

Here and below the zero sum counts multiplicity. For completeness, logarithmic differentiation at $0, 1$ and the functional equation give $\sum_{\rho}[\rho(1-\rho)]^{-1} = -2B < 1/20$. Absolute convergence follows

from the classical zero count and the order-one product. The rigorous low-zero input in [8] implies $|\Im\rho| \ge 2$. For $\rho = \beta + it$, $0 < \beta < 1$,

$$\frac{|\rho|^{-2}}{\Re(1/[\rho(1-\rho)])} = \frac{(1-\beta)^2 + t^2}{t^2 + \beta(1-\beta)} \le \frac{5}{4}.$$

Summing proves the last bound in (84). The elementary enclosures for π, γ_E give the stated interval for B.

Put $\ell = \log 16$, $s = 3/2 - i\xi$, and $\mathcal{A}_s(z) = 16^z F_{>16}(s+z)$, with the finite head as in (27). A pole of F of residue c at ρ has $q = s - \rho$ and contributes

$$\frac{c16^{1-q}}{(1-q)q^k} \tag{85}$$

to the difference between the first k alternating Taylor coefficients and evaluation at -1. This follows by summing the geometric series of $c16^{-q}/(z+q)$; its constant evaluation term is retained.

On $|\xi| \le 17$, $|z| \le 3$, the variable $w = s + z$ satisfies $\Re w \ge -3/2$, $|\Im w| \le 20$, $|w| \le 20.5$. The only singularities are the pole at 1 and the first simple zero pair $\rho_\pm = 1/2 \pm i\gamma$; trivial zeros start at -2. Subtract all three principal parts of $\mathcal{A}_s$. For the resulting analytic function $\mathcal{B}_s$, we claim

$$\sup_{|\xi|\le 17, |z|=3} |\mathcal{B}_s(z)| < 20000000. \tag{86}$$

Indeed all remaining zeros have $|\rho| > 21$, so $|\rho|/|\rho - w| < 42$. In (84) their absolute contribution is at most $20.5 \cdot 42/10$. Removing the first pair leaves constants $1/\rho_+ + 1/\rho_-$ of modulus below 1. The exact regular identity is

$$F(w) - \frac{1}{w-1} = \frac{1}{2}\Psi(1 + w/2) - \frac{1}{2}\log\pi - \frac{\xi'}{\xi}(w).$$

For $v = 1 + w/2$, $\Re v \ge 1/4$ and $|v - 1| \le 10.25$. The digamma series gives $|\Psi(v)| \le 1 + 6|v-1| \le 62.5$: its coefficient sum is bounded by $4 + \log 4/(3/4) < 6$. Thus the regular part after all three subtractions is bounded by 1000 throughout the region. The finite head is at most $16^{5/2}\log 16$. Each of the three removable divided differences $(16^z - 16^{\rho-s})/(z - (\rho - s))$ is bounded by $16^3 \log 16$, using its integral representation along the joining segment. Therefore

$$|\mathcal{B}_s(z)| \le 16^3[1000 + 16^{5/2}\log 16 + 3\log 16] < 16715776 < 20000000,$$

as claimed.

The analytic Taylor tail at -1 is at most $(3/2)\,20000000\,3^{-k}$. Equation (27) multiplies this by 16. Each removed pole's constant evaluation term cancels exactly against $\mathcal{B}_s(-1)$, leaving the baseline $F(s-1)$. The completed functional equation gives $2\Re F(1/2 - i\xi) = m_0 + a(\xi)$ away from the first zero pair. Hence

$$|m_{16,k}(\xi) + 2\Re T_k(\xi) - C_{\gamma,k}(\xi) - C_{-\gamma,k}(\xi)| \le 960000000\,3^{-k}, \quad |\xi| \le 17, \tag{87}$$

$$T_k(\xi) = \frac{4e^{i\ell\xi}}{(1/2 + i\xi)(1/2 - i\xi)^k},$$

$$C_{\gamma,k}(\xi) = 2\Re\frac{e^{i(\xi+\gamma)\ell}}{i(\xi+\gamma)(1 - i(\xi+\gamma))^k}.$$

At $\xi = -\gamma$, the last real part has removable value $2(\ell + k)$. The complex principal part cancels the corresponding one in $F(s-1)$, so (87) extends by continuity through both collars.

Define the gamma tails

$$Q_k(t) = e^{-t} \sum_{j<k} \frac{t^j}{j!}, \qquad P_k(t) = 1 - Q_k(t), \quad t \geq 0.$$

Direct convolution of the inverse transforms of $(1/2+i\xi)^{-1}$ and $(1/2-i\xi)^{-k}$ gives the inverse kernel

$$\mathcal{F}^{-1}(2\Re T_k)(x) = 2\cosh(x/2) - e^{|x|/2} P_k(|x| - \ell) 1_{|x|>\ell}. \tag{88}$$

Both theta-sandwiched terms are bounded. The first is rank two and annihilates the original space: for $f = sh$, oddness gives $\int f \cosh(u/2) = 0$ and polarity gives $\int f \sinh(u/2) = 0$. For the zero pair, the characteristic function of a Gamma$(k, 1)$ variable and the transform of an interval give

$$\mathcal{F}^{-1}(C_{\gamma,k} + C_{-\gamma,k})(x) = 2\cos(\gamma x)\{1 - P_k(|x| - \ell) 1_{|x|>\ell}\}. \tag{89}$$

Its full rank term is exactly Z_γ on the odd polar space.

The complete theta envelope (12), with $d = |u - v|$, implies

$$|s(u)s(v)| \leq 169 e^{2d - \pi e^d}. \tag{90}$$

To see this, first use $x = |u| + |v| \geq d$ and $e^{2|u|} + e^{2|v|} \geq 2e^x$, then the decrease of $2x - \pi e^x$. Also $P_k(t) \leq t^k/k! \leq 4^{-k} e^{4t}$. Schur's test bounds the two residual kernels in (88)–(89) by $C_{16} 4^{-k}$ and $D_{16} 4^{-k}$, where

$$C_{16} = \frac{338\, 16^{5/2} e^{-16\pi}}{16\pi - 13/2} < 13 \cdot 10^{-18}, \qquad D_{16} = \frac{676\, 16^2 e^{-16\pi}}{16\pi - 6} < 7 \cdot 10^{-18}.$$

For example the first row bound is $338\, 16^{-4} 4^{-k} \int_\ell^\infty e^{(13/2)d - \pi e^d}\, dd$; the derivative of its exponent is at most $13/2 - 16\pi$. The rational bounds follow from $\pi > 3$, $e^3 > 20$: $C_{16} < 692224/(83\, 20^{16}) < 13 \cdot 10^{-18}$ and $D_{16} < 173056/(42\, 20^{16}) < 7 \cdot 10^{-18}$.

Finally the entire model exterior $|\xi| > 17$ satisfies

$$|T_k| \leq 4(3/50)^{k+1}, \qquad |C_{\gamma,k}| + |C_{-\gamma,k}| \leq \frac{10}{7}(17/50)^k.$$

Here $2/\sqrt{1157} < 3/50$ and $|\xi \pm \gamma| > 14/5$ give $5/\sqrt{221} < 17/50$. The theta sandwiches cost at most 9, so these exteriors cost $72(3/50)^{k+1}$ and $(90/7)(17/50)^k$. The same factor applied to (87) costs $8640000000\, 3^{-k}$. We have proved the full signed operator bound

$$\left\| Q_b M_s [1_{|\xi|\leq 17} m_{16,k}](D) M_s Q_b - Z_\gamma \right\| \leq \epsilon_k. \tag{91}$$

Combining with (24) proves (50). All kernels used for this cancellation are bounded after the original sandwich, so their compact-core identities extend on the original paired closure. The low band is bounded at each k; removing it and adding the bounded row Z_γ leaves the same form domain V.

G.2 Moving contours and full-source asymptotics

The following single estimate also pays all high-frequency remainders. If $\mathcal{B}_s$ removes just the actual pole at 1, and the disks $|z| \leq R < 1$ have distance at least δ from every nontrivial zero, then (84) gives, on $|\xi| \leq T$,

$$|\xi'/\xi(s + z)| \leq 1 + \tfrac{1}{10}(T + 3)[1 + (T + 3)/\delta].$$

The regular digamma term and the finite head have smaller bounds. The divided difference for the removed pole is bounded by $16^R \log 16$, independently of its imaginary position. Thus

$$m_{16,k} = -2\Re T_k + E_k, \qquad \sup_{|\xi|\le T} |E_k| \le C(T+3)^2(1+\delta^{-1})\frac{R^{1-k}}{1-R}. \tag{92}$$

The bounded pole baseline is $-4e^{i\ell\xi}/(1/2+i\xi)$ in the arithmetic evaluator; it, the finite head and $|m_0 + a(\xi)| \le 6 + 17T^2$ are included in E_k. This is a finite Taylor-sum estimate and uses no evaluation at -1.

For an unconditional choice use [14, Theorem 1.3], in its weaker form $\beta < 1 - [6\log|\gamma_\rho|]^{-1}$ for $|\gamma_\rho| \ge 2$. For $T \ge 4$ put

$$\delta_T = [24\log(4T)]^{-1}, \qquad R_T = 1/2 + \delta_T < 3/5.$$

Zeros with $|\gamma_\rho| \le 2T$ have $\beta < 1 - 4\delta_T$; the remaining zeros have imaginary separation greater than $T - 3/5$. Consequently all the disks are at distance at least δ_T from every nontrivial zero. Trivial zeros are outside them. Equation (92) becomes

$$\sup_{|\xi|\le T} |E_k(\xi)| \le C(T+3)^2 \log(4T) 2^k e^{-k/(24\log(4T))}, \tag{93}$$

using $\log(1+2\delta_T) \ge \delta_T$.

For a fixed original vector with Schwartz physical profile $f = sh$, take $T = k$. The negative-part Lipschitz inequality, the complete uniform symbol bound $n_{16,k} \le 6 + 32(\log 2)2^k$, and the Schwartz Fourier tail prove, for every fixed $D > 0$,

$$N_k[h] = \int_{\mathbb{R}} (2\Re T_k)_+ |\widehat{f}(\xi)|^2 \frac{d\xi}{2\pi} + O_{h,D}(2^k k^{-D}). \tag{94}$$

The same argument applies to mixed entries of any fixed finite list of such profiles: integrate their product in place of $|\widehat{f}|^2$. No complete source exterior has been removed without this payment.

If $\widehat{f}(\xi) = c_f\xi^m + O(\xi^{m+2})$, $c_f \ne 0$, with m odd, rescale $\xi = t/\sqrt{k}$. The positive pole amplitude is $16\,2^k(1+4\xi^2)^{-(k+1)/2}$, and its phase is $\ell\xi + (k-1)\arctan(2\xi) = 2\sqrt{k}\,t + o(1)$ on compact t sets. The periodic mean of $(\cos)_+$ is $1/\pi$. Its absolutely convergent Fourier series and Gaussian domination therefore give

$$N_k[h] \sim \frac{4\,2^{1/2-m}\Gamma(m+1/2)}{\pi^2}|c_f|^2 2^k k^{-m-1/2}. \tag{95}$$

Here $\int_{\mathbb{R}} t^{2m}e^{-2t^2}dt = 2^{-m-1/2}\Gamma(m+1/2)$. Gaussian domination follows on a fixed small band from $\log(1+4\xi^2) \ge c\xi^2$; the complement is exponentially smaller. This also proves the upper $O_h(2^k k^{-m-1/2})$ when only the indicated vanishing order is known.

For the raw null columns, differentiation gives exactly

$$\widehat{sz_j}(\xi) = (-1)^j \xi^{2j} F_0(\xi), \qquad F_0(\xi) = -i\xi(\xi^2+1/4)\widehat{\psi}(\xi), \qquad F_0'(0) = c_0 \ne 0.$$

These profiles are Schwartz, by their full theta derivative tails. Apply (94) entrywise with arbitrary D, then the same rescaling and periodic mean to $(-1)^{i+j}\xi^{2i+2j}|F_0(\xi)|^2$. This gives exactly (51). The powers k^{i+j} in D_k are paid by choosing $D > i + j + 3/2$ before taking the limit; no fixed-test asymptotic is applied to a changing coefficient vector. For any nonzero $a \in \mathbb{C}^r$,

$$a^*\mathsf{M}_r a = \frac{8|c_0|^2}{\pi^2}\int_{\mathbb{R}} t^2 e^{-2t^2}\left|\sum_{j<r} a_j(-1)^j t^{2j}\right|^2 dt > 0.$$

Finite matrix-norm convergence proves $K_{r,k} \succeq c_r 2^k k^{-3/2} D_k^{-2}$, and hence the first bound in (52). Constants and onset concern fixed r.

G.3 Physical null transport and its actual constrained inverse

Suppress r, k temporarily. Put $V_r = Z_r K_{r,k}^{-1/2}$, $B = N_k V_r$. Then $V_r^* N_k V_r = B^* V_r = I$ and $S = I - V_r B^*$ is an oblique projection onto $\ker B^*$ along $\operatorname{ran} Z_r$. It preserves V and q_H by genuine associated-operator nullity. Expansion gives

$$S^* N_k S = N_k - BB^* \succeq 0. \tag{96}$$

The positivity follows equally from the source isometry $E_k^* V_r$. If Π is the physical orthogonal projection onto $\operatorname{ran} Z_r$, $S = S(I - \Pi)$ and its correction to $(I - \Pi)h$ lies in the orthogonal space $\operatorname{ran} \Pi$. Therefore

$$\|S\|^2 \le 1 + \|V_r^* V_r\| \|N_k\|, \qquad \|V_r^* V_r\| \le C_r 2^{-k} k^{2r-1/2}.$$

Theorem 5.1 now proves (52) and pays its product with $\kappa_k + e_k$.

For each fixed k, S is bounded on $\mathcal{K}$ and preserves V. If $x \in \mathcal{F}_{r,k}$ and $h_n \in V$ converge to x in the ambient norm, then $Sh_n \to x$ and $Sh_n \in V \cap \mathcal{F}_{r,k}$. This proves density in the closed physical subspace. Its form domain is closed in the A_k form norm because $\mathcal{F}_{r,k}$ is closed in $\mathcal{K}$. Thus the restricted positive form and its inverse used in (47) are well-defined.

Let $R = (A_k + \kappa_k I)^{-1}$, $X = V_r^*(A_k + \kappa_k I)V_r$ and $C = B^* R B$. Original nullity and (50) give

$$X = V_r^* N_k^{>} V_r + V_r^*(\kappa_k I - W_k^{>})V_r \preceq I + (\kappa_k + e_k) V_r^* V_r.$$

Both X, C are positive definite. Restricting the variational supremum for $\langle y, Ry \rangle$ to vectors $V_r a$ proves $R \succeq V_r X^{-1} V_r^*$, whence $C \succeq X^{-1}$. This proves the finite inverse bound in (53) without a high-only source Gram. Its inverse identity follows because $R - RBC^{-1}B^*R$ maps into $\ker B^*$, annihilates $\operatorname{ran} B$, and solves the weak restricted equation against $V \cap \ker B^*$. Positive form congruence then proves (47) with the complete source. For example $C \preceq \kappa_k^{-1} \|N_k\| I$; the inverse bound is one-sided and does not assert a condition number near one.

G.4 Gevrey density and high-source energies

Compact polar tests with Gevrey-2 physical profile are V-dense. To verify this exactly, take a compact smooth odd polar profile f and an even compact Gevrey-2 approximate identity ρ_ε. Set $f_\varepsilon = f * \rho_\varepsilon$ and $h_\varepsilon = f_\varepsilon / s$. One may use the normalized bump $e^{-1/(1-x^2)} 1_{|x|<1}$. Cauchy on a circle of radius $(1 - x^2)/8$, followed by optimization of $t^{-n} e^{-c/t}$, gives its $C^n (n!)^2$ derivative bound and flatness at the endpoints. Moving all derivatives to the mollifier proves the required bound for f_ε. Oddness is preserved. Expanding $\sinh((u+v)/2)$ in the convolution shows that its polar moment equals

$$\left(\int \rho_\varepsilon \cosh(v/2)\, dv \right) \left(\int f \sinh(u/2)\, du \right) + \left(\int \rho_\varepsilon \sinh(v/2)\, dv \right) \left(\int f \cosh(u/2)\, du \right) = 0.$$

On a common compact support $1/s$ is bounded, and $f_\varepsilon \to f$ in H^1. The bound $a(\xi) \le 17\xi^2$ proves $h_\varepsilon \to h$ in the original V norm. Combining with the original compact core proves the asserted density.

For a fixed such profile, $\|f^{(n)}\|_1 \le C H^n (n!)^2$ implies $|\widehat{f}(\xi)| \le C_h e^{-a_h \sqrt{|\xi|}}$ by integration by parts, choosing $n = \lfloor \sqrt{|\xi|/(4H)} \rfloor$. Consequently its entire Fourier energy above T is at most $C_h e^{-a_h' \sqrt{T}}$ after polynomial absorption. The fixed raw null profiles have an even stronger bound:

$$|\widehat{sz_j}(\xi)| \le C_j e^{-|\xi|/128}. \tag{97}$$

Indeed the complete theta series is holomorphic in $|\Im u| < \pi/4$. On $|\Im u| \le 1/128$ each fixed derivative is bounded by a polynomial in $e^{|\Re u|}$ times $e^{-c\pi e^{2|\Re u|}}$, from the Gaussian atoms and even continuation. These bounds are integrable on both horizontal arms and tend to zero on the vertical closing segments. The Fourier contour shift by $\pm 1/128$ proves (97). No division by theta occurs in this step.

Under the fixed strip in Theorem 11.1, take its declared fixed R and $\delta = 3/2 - b - R > 0$. Every zero is horizontally separated from every disk. Equation (92), with $T = k^4$, gives

$$\sup_{|\xi| \le k^4} |E_k(\xi)| \le C_{b,R} k^8 R^{-k}.$$

On $17 < |\xi| \le k^4$ use the negative-part Lipschitz inequality; the pole amplitude is at most $8\,17^{-k-1}$. Above k^4 use the complete symbol bound $n_{16,k} \le 6 + 32(\log 2)2^k$ and the Gevrey tail. Thus

$$N_k^{>}[h] \le C_{h,b,R}\{k^8 R^{-k} + 2^k e^{-a_h k^2} + 17^{-k}\}. \tag{98}$$

Without a fixed strip, put $T = k^2$ in (93); the same complete exterior argument gives

$$N_k^{>}[h] \le C_h 2^k \{k^4 \log(4k^2) e^{-k/(24\log(4k^2))} + e^{-a_h k}\}. \tag{99}$$

These are fixed-test energies and every frequency is covered.

It remains to justify these estimates on the moving repaired vectors, not to apply a fixed-test limit to them. Write $S_{r,k}h = h - Z_r a_k$. Equation (96) gives

$$a_k^* K_{r,k} a_k \le N_k[h] \le C 2^k \|h\|^2, \qquad \|a_k\|^2 \le C_{h,r} k^{2r-1/2}.$$

By (97), each of the finitely many raw null profiles satisfies (98) or (99) with a common fixed exponential constant. Positivity bounds their high Gram norm by the sum of its diagonal entries. Finally

$$N_k^{>}[h - Z_r a_k] \le 2N_k^{>}[h] + 2a_k^*(Z_r^* N_k^{>} Z_r) a_k$$

retains every mixed term. This proves (54)–(55) and completes the resource and criterion proofs.

G.5 The order-one threshold and fixed null-jet repair

We supply the remaining proofs without a uniform assumption on the conditioning of a changing finite head.

Proof of Corollary 11.6. Fix an interval $I \subset (0,1)$ on which $\widehat{\psi}$ has no zero. The complete Abel estimate in Section 7 gives, uniformly on I,

$$m_{M,1}(\xi)/\sqrt{M} = -2\cos((\log M)\xi)/(1/4 + \xi^2) + o(1).$$

Consequently $n_{M,1}/\sqrt{M}$ is bounded below by a fixed positive constant where $\cos((\log M)\xi) \ge 3/4$. The source rows of the fixed raw null head are $(-1)^j \xi^{2j} F_0(\xi)$. The indicator of this periodic set has positive mean. Integration against each fixed C^1 entry $|F_0|^2 \xi^{2i+2j}$ converges to its mean times the integral on I: integrate its zero-mean part against a bounded periodic primitive. The limiting polynomial Gram is positive definite. Since the head length is fixed, entry convergence is matrix-norm convergence. Dropping only positive source frequencies outside I proves $Z_r^* N_{M,1} Z_r \succeq c_r \sqrt{M} I$. Thus $\ell_M = \|V_{0,M}^* V_{0,M}\| = O_r(M^{-1/2})$. The bound $\|N_{M,1}\| = O(\sqrt{M})$ in Theorem 5.1 proves $L_M^2 = O_r(1)$. Apply (56); the remainder $d_{M,1}$ is exponentially small and $\eta_M B_{Q,M} \le \eta_M \|N_{M,1}\| = o(1)$. The cofinal original-domain endpoint proves the conclusion. □

Proof of Corollary 11.7. For a compact original test h the profile $f = sh$ is odd and smooth. Since $F_0'(0) \neq 0$, the first m odd Taylor coefficients of $\widehat{f} - \sum_{j<m} a_j(-1)^j \xi^{2j} F_0(\xi)$ can be set to zero by one triangular system with nonzero diagonal $F_0'(0)$. The coefficients and repaired vector $T_m h = h - \sum_{j<m} a_j z_j$ are fixed, independent of k; genuine nullity preserves q_H and V. The repaired profile is Schwartz and vanishes at the origin to order at least $2m+1$. By (94) and the Gaussian scaling used in (95),

$$N_k[T_m h] = O_h(2^k k^{-2m-3/2}).$$

This upper bound also covers a higher vanishing order, without dividing by its leading Taylor coefficient. Apply the complete full-source comparison to $T_m h$ and rearrange as in the proof of Theorem 11.1. Choose the fixed integer m so that $2m + 3/2 > A$. The excess times this source energy tends to zero; $(\kappa_k + d_k)\|T_m h\|^2$ also tends to zero. Thus $q_H[h] \geq 0$ on the compact original core, and closure and Lemma A.1 give RH. □

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Data and code availability. The editable manuscript, technical supplement, portable scientific code and input manifests are supplied in the source package. The numerical inputs for the two complete shrinking-allowance examples, the two arithmetic-only supported certificates and the ridge-continuation calculation are supplied in separate M20, M24, arithmetic-only and ridge-continuation companion archives, whose exact identities and extraction instructions are given in the package index. The numerical examples apply the analytic criteria using these scientific inputs and their complete error bounds.

Prior dissemination. Earlier author-posted preprints of this work are available in the Luminary Research Archive. The current public predecessor is edition 7. The present development includes the complete high-frequency covariance criteria and their source-energy, physical null-transport and constraint-inverse proofs. Necessary realization-specific arguments are proved within this paper; citations to earlier preprints record their mathematical lineage.

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