

The Riemann Hypothesis through Arithmetic Weil Positivity

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Luminary Oddities

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Abstract

We develop a quantitative covariance formulation of the Riemann hypothesis through arithmetic Weil positivity. In the original odd weighted realization, RH is equivalent to $\liminf_{M \rightarrow \infty} \sqrt{M}(\gamma_M - 1)_+ = 0$, where γ_M is the complete covariance norm at ridge $1/M$ after any sequence of finite genuine-null restrictions. We prove that the complete covariance norm differs from one by $O_{b,\varepsilon}(M^{-1/2+\varepsilon})$ for every $\varepsilon > 0$ at each prescribed ridge M^{-b} , $b > 0$ fixed, uniformly over all finite genuine-null restrictions. The estimate includes every negative-source frequency and all mixed terms. Its proof combines a complete arithmetic source majorant, theta-tail bounds, the Vinogradov–Korobov prime-number error and a uniform actual-inverse construction. Genuine derivative nulls then provide complete source coordinates and signed Schur certificates. A complete complementary cap bounded by a fixed $q < 1$, together with source-normalized physical lift cost $o(\sqrt{M})$, suffices for RH. We prove uniform small-set inverse bounds throughout $[0, e^{A\sqrt{\log M}}]$ for every fixed $A > 0$, unconditionally. Factorial physical lift budgets and actual inverse-weighted exterior and null-spectrum bounds quantify the construction. These results retain all mixed terms and give complete signed criteria for vanishing original-norm allowances and the Riemann hypothesis. A second cofinal RH criterion uses explicit $M^{o(1)}$ -dimensional Schur matrices retaining the exact infinite receiving inverse, with a proved complete receiving floor and only $O(\log M)$ positive Green responses. The response reduction preserves the original-norm vanishing-allowance scale. A finite-sine core gives joint Schur error $O(\sqrt{\log M}/M^2)$ in physical core coordinates.

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1 RH through arithmetic Weil positivity

The Weil criterion expresses the Riemann hypothesis as positivity of a complete quadratic form. We study that criterion in a fixed odd reciprocal-theta realization, retaining its original norm and minimal closed form domain. The purpose is to make the approach to positivity quantitative: a complete signed lower bound with an allowance tending to zero establishes RH, and genuine operator-null projections preserve this implication exactly.

The paper connects this endpoint to the complete arithmetic covariance. Positive averaging of the prime cutoff gives

$$H = P_M - N_M + W_M, \quad N_M = E_M E_M^*, \quad \|W_M\| \leq d_{M,1},$$

where P_M is the actual positive receiver and E_M contains the entire negative observation. For a finite genuine-null subspace K and ridge $\kappa > 0$, write

$$\gamma_M(K; \kappa) = \|E_{M,K}^* (P_{M,K} + \kappa I)^{-1} E_{M,K}\|.$$

The subscript denotes same-form restriction to $K^\perp$ and exact projection of the source. Section 2 specifies the original realization; Section 5 fixes the source metric.

The RH condition and the quantitative theorem. For any sequence of finite genuine-null spaces K_M , the original positivity criterion takes the exact form

$$\text{RH} \iff \liminf_{M \rightarrow \infty} \sqrt{M} (\gamma_M(K_M; M^{-1}) - 1)_+ = 0. \quad (1)$$

Corollary 7.2 proves this characterization using the original-domain Weil criterion. Our principal unconditional estimate is

$$\sup_K |\gamma_M(K; M^{-b}) - 1| \leq \frac{C_b \sqrt{\log M} \exp(48\sqrt{b \log M})}{\sqrt{M}}, \quad b > 0 \text{ fixed}, \quad M \geq M_b. \quad (2)$$

The supremum includes every finite genuine-null choice, including moving and growing ones. Theorem 7.1 proves this complete-source bound. Equivalently, for each $\varepsilon > 0$ its right-hand side is $O_{b,\varepsilon}(M^{-1/2+\varepsilon})$. Together, Theorem 7.1 and Corollary 7.2 give a complete covariance estimate and an exact characterization of RH in the original weighted realization.

The proof retains the actual arithmetic positive operator throughout. A complete hard-cutoff source majorizes the possible negative sign of H . The classical Vinogradov–Korobov prime-number error admits a growing comparison band, and an exact polar Green identity supplies a uniform preimage estimate on its entire moving coefficient unit ball. Complete theta tails and the ridge subtraction pay the return to the original form. No finite frequency sampling substitutes for the full covariance.

Complete signed certificates and the RH criterion. Genuine derivative nulls generate a dense system in the complete negative source at each fixed M (Theorem 8.2). Their exact signed identity cancels the artificial ridge charge before estimation. Theorems 8.3 and 9.1 turn this identity into full trace and regularized approximation certificates, including every mixed block. The explicit factorial norm bound in Proposition 9.2 controls the physical lift of growing null families. Uniform small-set inverse estimates then allow collars around poorly conditioned frequencies while retaining the entire exterior. These results provide complete signed certificates and physical lift estimates within the covariance formulation.

Corollary 8.5 gives the following sufficient criterion for RH:

$$\|Q_M G_M Q_M\| \leq q < 1, \quad \|V_M^* V_M\| = o(\sqrt{M}),$$

for a single sequence of source-normalized genuine-null heads V_M ; Q_M projects onto their complete source complement. Corollary 8.5 proves that these resources imply RH, using the exact mixed residual rather than discarding the complement. The growing-band theorem also improves to every measurable $I \subset [0, e^{A\sqrt{\log M}}]$, for each fixed $A > 0$, with inverse cap at most $C_A |I|(1 + \log \log M)$. This unconditional extension combines the same prime-error input with exact Fourier translation of a Gevrey cutoff.

The positive-majorant and raw-exterior obstructions in Section 11 explain why the signed structure matters. Actual inverse-weighted exterior and null-compressed spectral tests further quantify the complementary cap, physical lift and source geometry entering the signed criterion. The accompanying technical supplement gives the independent hard-cutoff and finite-coordinate implementations, their complete source/error contracts, and the two numerical applications. None is an input to the principal averaged-covariance theorem or its RH criterion. The main paper contains all realization-specific proofs needed for those results; the code and literal scientific data document the separate computational applications.

An explicit cofinal RH reduction. Theorem 12.1 supplies a second exact cofinal criterion for RH on a declared growing support. Its explicit projected-polar DN core has dimension $M^{o(1)}$, its complete receiving floor is proved, and only $O(\log M)$ positive Green responses are retained. A whole relative-form estimate preserves original physical $o(M^{-3/2})$ allowances, while a joint Schur comparison pays every core and mixed response term at $M^{-2+o(1)}$; a finite-sine alternative improves this to $O(\sqrt{\log M}/M^2)$ with profile Gram condition below $4L_M$. The resulting finite core matrices retain the exact infinite-dimensional receiving inverse; their signed positivity is the condition isolated by this criterion. The main paper gives both directions of the RH equivalence; Appendix B supplies the full Green, domain, core-energy and mixed-error proofs.

Relation to earlier work. The original weighted realization, null compression and certified lower bounds were developed in [2, 3, 4]; complete hard-cutoff source estimates also appear in [5]. Their necessary realization-specific arguments are given here, with shared elementary estimates collected in Section 3. The present whole-source rate strengthens the origin estimate and complementary criterion by placing an unconditional bound on the full covariance. Mellin smoothing, inverse preimages, Schur elimination and regularization are classical tools [6, 7, 8, 9]. Related exact-form constructions use truncated preimages and theta-tail payments [10, Section 7], and collective exact-Gram covariance with pole compensation [11, Sections 6–7]. Here their quantitative use retains one original domain, true source metrics and simultaneous control of finite genuine-null restrictions. The classical explicit formula and Weil criterion remain the number-theoretic foundations [12, 13, 14].

The covariance estimate is unconditional; its RH implication uses the joint complementary-gap and physical-lift hypotheses.

Organization. Sections 2–4 establish the original form and arithmetic realization. Sections 5–7 prove the actual inverse estimates and complete covariance theorem. Sections 8–10 construct the signed null-source certificates and exterior controls. Section 11 treats their feasibility. Section 12 proves the explicit cofinal finite-response RH criterion. The appendices prove the odd Weil equivalence, complete Green reduction, reference inverse identities and origin payments. The indexed technical supplement develops alternative hard-cutoff and finite-source certificates, conditioning, and reproducible numerical applications, with supporting analytic source proofs included in that document.

2 The positivity endpoint and the finite reduction

The Riemann hypothesis has a positivity endpoint in a fixed odd Weil realization. We retain this realization throughout the complete covariance estimates and the constructive signed certificates. We first specify the original norm and domain; the later reductions return to precisely these objects.

Use $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u) e^{-iu\xi} du$ and Plancherel measure $d\xi/(2\pi)$. For $u \geq 0$ define

$$\psi(u) = e^{u/2} \sum_{n \geq 1} (4r_n^2 - 6r_n) e^{-r_n}, \quad r_n = \pi n^2 e^{2u}, \quad s^2 = \psi / \cosh(u/2), \quad (3)$$

with the even continuation given by theta inversion. Let $\mathcal{K}$ be the odd subspace of $L^2(\mathbb{R})$ satisfying

$$\langle h, s \sinh(u/2) \rangle = 0.$$

Write $\Psi = \Gamma'/\Gamma$ and

$$a(\xi) = \Re \Psi(1/4 + i\xi/2) - \Psi(1/4) \geq 0, \quad m_0 = \Psi(1/4) - \log \pi.$$

The domain V is the completion of the smooth compact odd polar-free core for

$$\|h\|_V^2 = \|h\|_2^2 + \int_{\mathbb{R}} a(\xi) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi}. \quad (4)$$

The complete form on this domain is the closure of

$$\begin{aligned} q_H[h] &= \int_{\mathbb{R}} (m_0 + a) |\widehat{sh}|^2 \frac{d\xi}{2\pi} \\ &\quad - \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \langle sh, (S_{\log n} + S_{-\log n}) sh \rangle, \quad S_b f(u) = f(u + b). \end{aligned} \quad (5)$$

Inner products are conjugate-linear first. For $f = sh$ the original physical norm is

$$\|h\|_2^2 = \int_{\mathbb{R}} |f(u)|^2 \frac{\cosh(u/2)}{\psi(u)} du.$$

These are the conventions of the weighted-Weil and complementary-receiver predecessors [2, 3, 4].

Notation and proof dependencies. The ambient Hilbert norm is always $\|h\|_2$, and $f = sh$ is its physical profile. Q_b removes the polar row; a further projection removes only a specified finite genuine-null space. The symbols $p_{M,k}, n_{M,k}$ refer to arithmetic averaging, while the explicitly declared symbols in Section 3 and the technical supplement refer to hard cutoffs. After averaging order one is fixed, P_M, N_M, E_M and $R_M(\kappa)$ denote its positive form, negative form, complete source, and actual positive inverse. Scalar hard-cutoff budgets have only their local declared meaning. No projection, ridge or cutoff changes the original minimal form domain. The proofs below are internal; predecessor citations identify the development of the constructions.

Lemma 2.1 (Complete elementary envelopes). *For $r = \pi e^{2|u|}$,*

$$2r^2 e^{-r} \leq s(u)^2 \leq 169 e^{4|u| - \pi e^{2|u|}}. \quad (6)$$

The arithmetic sum in (5) converges in original operator norm. Moreover $0 \leq a(\xi) \leq 17\xi^2$.

Proof. Each theta atom is positive. The first atom, $4r^2 - 6r \geq 2r^2$, and $e^{|u|/2} / \cosh(u/2) \geq 1$ give the lower bound. For the upper, $e^{|u|/2} / \cosh(u/2) \leq 2$ and $\sum_{n \geq 1} n^4 e^{-(n^2-1)r} < 2$: the consecutive ratios are bounded by $16e^{-3r} < 1/2$. Hence $s^2 < 16r^2 e^{-r} < 169 e^{4|u| - r}$. For $b = \log n$, put $x = |u| + |u + b| \geq b$. The upper envelope gives

$$s(u)s(u+b) \leq 169 e^{2x - \pi e^x} \leq 169 n^2 e^{-\pi n}.$$

Thus the norm of the full omitted arithmetic sum is bounded by

$$\delta_M = 338 \sum_{n > M} n^{3/2} \log n e^{-\pi n} \rightarrow 0. \quad (7)$$

This sums all integers and therefore every omitted prime power. In particular, for every integer $M \geq 2$,

$$\delta_M < 451(M+1)^2 e^{-\pi(M+1)}. \quad (8)$$

Indeed $\log n \leq \sqrt{n}$, and the consecutive ratios of $n^2 e^{-\pi n}$ are at most $4e^{-\pi} < 1/4$. Summing the resulting geometric majorant gives the coefficient $1352/3 < 451$. Boundedness of s and this convergence make (4) the original semibounded form topology. Finally the digamma series is

$$a(\xi) = \sum_{n \geq 0} \frac{(\xi/2)^2}{(n+1/4)((n+1/4)^2 + (\xi/2)^2)}.$$

The $n = 0$ coefficient is at most $16\xi^2$, and the others total at most $\xi^2 \sum_{n \geq 1} n^{-3}/4 \leq 3\xi^2/8$. $\square$

Theorem 2.2 (Cofinal complete certificates). *For each j let Π_j be a finite orthogonal projection onto genuine null vectors of the complete associated operator H , and put $P_j = I - \Pi_j$. Suppose a complete signed certificate on the original domain gives*

$$q_H[P_j h] \geq -\epsilon_j \|P_j h\|_2^2 \quad (h \in V), \quad 0 \leq \epsilon_j \rightarrow 0 \quad (9)$$

along an unbounded sequence of indices. Then $q_H \geq 0$ on V , and the Riemann hypothesis holds. Each Π_j may use its own finite genuine-null space.

Proof. For a genuine associated-operator null vector $z \in \text{Dom } H$, $q_H[z, h] = 0$ for every $h \in V$. Consequently finite null projections preserve V , $q_H[P_j h] = q_H[h]$ and $\|P_j h\|_2 \leq \|h\|_2$. For each fixed h , (9) gives $q_H[h] \geq -\epsilon_j \|h\|_2^2$; taking the limit proves positivity. Lemma A.1 supplies the endpoint implication. $\square$

The following arithmetic realization and covariance estimates return to precisely this norm and domain. The finite-coordinate constructions in the technical supplement provide alternative implementations of the same endpoint. Related compact-window reductions retain a Fourier envelope and leading-tail coupling [15], or source-exact positive directions in a block-Schur certificate [16]. Our main estimates act on the complete odd weighted realization; the compact-window and whole-line recipients are distinct.

3 Elementary estimates for the original realization

The estimates used below are proved here in the normalization and minimal form domain of Section 2. They consolidate the elementary inputs of the author's weighted-Weil and joint-cutoff predecessors [2, 3, 4, 5]; those citations record provenance rather than substitute for the proofs.

In this section only, $m_M = m_0 + a - 2 \sum_{2 \leq n \leq M} \Lambda(n) n^{-1/2} \cos(\xi \log n)$, $p_M = (m_M)_+$ and $n_M = (-m_M)_+$ denote the hard-cutoff symbols, with $R_M = 6 + 8(\log 2)\sqrt{M}$. The averaged symbols introduced later carry their averaging indices explicitly. The estimates below do not identify a hard-cutoff symbol with an averaged one.

3.1 Elementary arithmetic and logarithmic symbol bounds

Write $\vartheta_C(N) = \log \text{lcm}(1, \dots, N)$. The quotient for $\vartheta_C(2m) - \vartheta_C(m)$ divides $\binom{2m}{m}$: a prime power in $(m, 2m]$ supplies a positive binomial valuation, and there is at most one such new power per prime. Similarly the quotient for $\vartheta_C(2m+1) - \vartheta_C(m+1)$ divides $\binom{2m+1}{m}$. The two equal odd central binomials together are at most 2^{2m+1} , so either is at most 2^{2m} . Induction, starting at $N = 1$, gives

$$\vartheta_C(N) \leq 2N \log 2.$$

Partial summation, with the bound extended to real arguments by flooring, therefore gives

$$S_M := \sum_{2 \leq n \leq M} \frac{\Lambda(n)}{\sqrt{n}} \leq 4(\log 2)\sqrt{M} - 2 \log 2. \quad (10)$$

This estimate needs no prime number theorem.

The identity $m_0 = -\gamma - \pi/2 - 3 \log 2 - \log \pi$ gives $-6 < m_0 < 0$ and $m_0 + 4 + \log 2 < 0$. Here are elementary admissions for these strict constants. The bounds $\gamma < 1$, $\pi < 22/7$, $\log 2 < 7/10$ and $\log \pi < 7/6$ give $m_0 > -6$; the logarithmic bounds follow from positive exponential Taylor sums. Also $\gamma > H_8 - \log 9 > H_8 - 11/5 = 29/56 > 1/2$, since the first six Taylor terms give $e^{11/10} > 3$. Together with $\pi > 3$, $\log 2 > 2/3$ and $\log \pi > 1$, this gives $m_0 + 4 + \log 2 < -1/3$.

For $t = |\xi|/2$, the summand $f_t(q) = t^2/[q(q^2 + t^2)]$ is nonnegative and decreasing for $q > 0$. Integral comparison with the digamma series gives

$$\frac{1}{2} \log(1 + 4\xi^2) \leq a(\xi) \leq 4 + \frac{1}{2} \log(1 + 4\xi^2). \quad (11)$$

Using the negative m_0 , (10) and $\frac{1}{2} \log(1 + 4\xi^2) \leq \log 2 + \log(2 + |\xi|)$ yields

$$0 \leq n_M \leq R_M, \quad |p_M - a| \leq R_M, \quad p_M(\xi) \leq R_M + \log(2 + |\xi|). \quad (12)$$

The middle bound also holds where $m_M < 0$, since there $a < |m_0| + 2S_M$. For $|\xi| \geq \Omega_M = e^{R_M+1}$, the lower bound $a(\xi) \geq \log(2|\xi|)$ proves $m_M(\xi) > 0$. Thus the full negative support is bounded without importing a sampled sign pattern at another cutoff. The raw tail satisfies $\delta_M < 1$ for $M \geq 2$, since its decreasing majorant at $M = 2$ is below $4059(3/8)^9 < 1$.

3.2 Complete theta constants and convolution

We first record the complete strip estimates

$$\|s^2(\cdot \pm i/8)\|_2 < 20, \quad \|s^2(\cdot \pm i/8) \sinh((\cdot \pm i/8)/2)\|_2 < 32. \quad (13)$$

For $z = x + iy$, $x \geq 0$, $|y| \leq 1/8$ and $r = \pi e^{2x}$, $\cos(2y) > 31/32$ and $4r_\ell^2 + 6r_\ell < 6r_\ell^2$. The consecutive ratios of the majorized theta atoms are below $16e^{-3(31/32)\pi} < 1/8$. Summing *all* atoms gives

$$|\psi(z)| \leq 7e^{x/2}r^2e^{-31r/32}, \quad |s^2(z)| \leq 28r^2e^{-31r/32}.$$

The second estimate uses $|\cosh(z/2)| \geq \cos(1/16) \cosh(x/2) > \cosh(x/2)/2$. Evenness covers the other real arm. The squared two-arm integral for the first norm in (13) is at most $4704(16/31)^4 < 400$. For the second use $|\sinh((x + i/8)/2)|^2 \leq e^{|x|} \leq r/\pi$; the squared norm is at most $(784 \cdot 24/\pi)(16/31)^5 < 1024$. On the real line the stronger atom-ratio bound gives $s^2 \leq 16r^2e^{-r}$; its maximum for $r \geq \pi > 3$ is less than $144e^{-3} < 9$, so $\|s\|_\infty < 3$.

Put $w = s^2$. Analyticity and complete double-exponential decay justify shifting the two Fourier contours, including the vertical end pieces. Plancherel and (13) give

$$\int_{\mathbb{R}} e^{|\xi|/4} |\widehat{w}(\xi)|^2 \frac{d\xi}{2\pi} < 800.$$

Weighted Cauchy–Schwarz implies

$$\int_{\mathbb{R}} |\widehat{w}| \frac{d\xi}{2\pi} < 33, \quad \int_{\mathbb{R}} \log(2 + |\xi|) |\widehat{w}| \frac{d\xi}{2\pi} < 210. \quad (14)$$

Indeed the reciprocal-weight integrals are $4/\pi$ and at most $164/\pi$, using $\log(2 + x) \leq 1 + x$; then $3200/3 < 33^2$ and $131200/3 < 210^2$ suffice. For an odd g with Fourier support in $[-F, F]$, the inequality $2 + |\xi| \leq (2 + |\xi - \zeta|)(2 + |\zeta|)$ and weighted Young's inequality now prove

$$\|p_M(D)(wg)\|_2 \leq [33(R_M + \log(2 + F)) + 210] \|g\|_2. \quad (15)$$

The logarithmic multiplier is used only on these paid sources, rather than asserted bounded on all L^2 .

3.3 Complete hard-cutoff return

Let Q_b be the orthogonal projection from the ambient odd Hilbert space onto the polar space $\mathcal{K}$ of Section 2. For an integer $m \geq 2$ set

$$b_m(\xi) = m_0 - 2 \sum_{2 \leq n \leq m} \frac{\Lambda(n)}{\sqrt{n}} \cos(\xi \log n).$$

Lemma 3.1 (Hard-cutoff source bounds). *The multiplier a is increasing on $[0, \infty)$ and $a(L) \geq \frac{1}{2} \log(2L)$ for $L \geq 1/2$. Moreover*

$$\|s\|_\infty < 3, \quad |b_m(\xi)| \leq 18\sqrt{m}, \quad H = Q_b M_s(a + b_m)(D) M_s Q_b + V_m, \quad \|V_m\| \leq \frac{640(m+1)^2}{20^{m+1}}. \quad (16)$$

The operator expression denotes equality of forms on the original minimal domain V ; the omitted arithmetic perturbation V_m is bounded.

Proof. For fixed $q > 0$, $t^2/[q(q^2 + t^2)]$ increases with $t \geq 0$. The positive digamma series in Lemma 2.1 therefore proves monotonicity. Equation (11) gives

$$a(L) \geq \frac{1}{2} \log(1 + 4L^2) \geq \frac{1}{2} \log(2L),$$

since $1 + 4L^2 \geq 2L$. The real complete-theta estimate above proves $\|s\|_\infty < 3$. From $|m_0| < 6$ and (10),

$$|b_m| \leq |m_0| + 2S_m < 6 + 8(\log 2)\sqrt{m} \leq (6 + 8\log 2)\sqrt{m} < 18\sqrt{m}.$$

The exact identity follows first on the compact polar core by retaining all arithmetic shifts with $n \leq m$. Its remainder is the complete paired omitted shift operator from Lemma 2.1. Hence, with $k = m + 1$,

$$\|V_m\| \leq \delta_m < 451k^2e^{-\pi k} < 640k^220^{-k},$$

using $e^\pi > 20$ and $451 < 640$. Compression by Q_b cannot increase its norm. The retained symbol differs from a by a bounded multiplier, while V_m is bounded in the original Hilbert norm. The core identity consequently extends through precisely the specified minimal form closure. At no stage are the paired signed edge pieces closed separately. $\square$

4 Positive arithmetic smoothing and a cofinal schedule

Write M_s for multiplication by the complete theta weight s in (3). All identities in this section begin on the original polar compact core and extend through its minimal closure V . Put

$$m_R(\xi) = m_0 + a(\xi) - 2 \sum_{n \leq R} \Lambda(n)n^{-1/2} \cos(\xi \log n), \quad p_R = (m_R)_+, \quad n_R = (-m_R)_+.$$

The cutoff is real, with $W_R = W_{\lfloor R \rfloor}$. For integers $M \geq 2$, $k \geq 1$, average $R = Me^v$ with density $v^{k-1}e^{-v}/(k-1)!$. The survival weights are

$$w_{M,k}(n) = \begin{cases} 1, & n \leq M, \\ \frac{M}{n} \sum_{j=0}^{k-1} \frac{\log(n/M)^j}{j!}, & n > M. \end{cases} \quad (17)$$

Let $m_{M,k} = m_0 + a - 2\Re S_{M,k}$, where $S_{M,k}(\xi) = \sum_n \Lambda(n)n^{-1/2+i\xi} w_{M,k}(n)$, and take $p_{M,k} = (m_{M,k})_+$, $n_{M,k} = (-m_{M,k})_+$ after averaging. The series is absolutely convergent at fixed M, k .

Theorem 4.1 (Exact smoothed realization and complete remainder). *On the same original domain, and after any finite genuine-null projection,*

$$H = P_{M,k} - N_{M,k} + W_{M,k}, \quad \|W_{M,k}\| \leq d_{M,k} := \frac{451(M+1)e^{-\pi(M+1)}}{M^{k-1}(1-e^{-\pi})^k}. \quad (18)$$

The bound holds for every integer $M \geq 2$, $k \geq 1$. Moreover

$$\|N_{M,k}\| \leq B_{M,k} := 9[6 + 8(\log 2)\sqrt{M}2^k]. \quad (19)$$

The separately averaged positive and negative receivers have an exact common positive term $J_{M,k}$ such that

$$P_{M,k} = \mathbb{E}P_R - J_{M,k}, \quad N_{M,k} = \mathbb{E}N_R - J_{M,k}.$$

Proof. Integration of the cutoff indicator gives (17). The elementary estimate $\sum_{n \leq R} \Lambda(n)/\sqrt{n} \leq 4(\log 2)\sqrt{R}$ is proved in Section 3. Since $\mathbb{E}e^{v/2} = 2^k$, absolute Fubini applies and gives (19), using $|m_0| < 6$, $a \geq 0$, $\|s\|_\infty < 3$. Every arithmetic perturbation is bounded at fixed M, k ; its positive symbol differs from a by a bounded multiplier. The original minimal form domain is therefore unchanged.

Set $q = e^{-\pi}$ and $j = M + l$. The probability of the floor-cutoff cell is

$$\mu_j \leq \frac{[(l+1)/M]^{k-1}}{(k-1)!} \frac{M}{j(j+1)}.$$

Indeed $v \leq \log((j+1)/M) \leq (l+1)/M$ in that cell. Multiply by the complete bound $451(j+1)^2 q^{j+1}$ from (8). As $M(j+1)/j \leq M+1$ and $(l+1)^{k-1} \leq (k-1)! \binom{l+k-1}{k-1}$, summing the binomial generating series proves (18). The remainder is an average of bounded original-Hilbert operators. This extends the core identity through the paired original closure.

Pointwise put $A = \mathbb{E}p_R$, $B = \mathbb{E}n_R$. Then $A - B = m_{M,k}$ and $j = \min(A, B) = A - (m_{M,k})_+ = B - (-m_{M,k})_+ \geq 0$. The bounded operator $J_{M,k} = QM_s j(D)M_s Q$ gives both identities. The positive average is interpreted on V ; its difference from the bounded negative average is the original signed expression. $\square$

Corollary 4.2 (Fixed-seed certificate criterion). *Fix an integer $M \geq 2$, $\kappa_k > 0$ and $\eta_k, \tau_k \geq 0$. On the original finite genuine-null complement suppose*

$$0 \leq N_{M,k} - N_{M,k}^{\text{fin}} \preceq \tau_k I, \quad N_{M,k}^{\text{fin}} \preceq (1 + \eta_k)(P_{M,k} + \kappa_k I).$$

Then on all original V ,

$$q_H[h] \geq - \left[d_{M,k} + \kappa_k + \frac{\tau_k + \eta_k B_{M,k}}{1 + \eta_k} \right] \|h\|^2. \quad (20)$$

In particular the complete inequalities with $\kappa_k = \tau_k = 2^{-k}$ and $\eta_k = 4^{-k}$ imply RH.

Proof. Rearrange $N \preceq (1+\eta)(P+\kappa I) + \tau I$ using the bounded N : $P - N \succeq -[\kappa + (\tau + \eta\|N\|)/(1+\eta)]I$. Add the complete remainder and use genuine-null transport as in Theorem 2.2. The displayed schedule tends to zero by (18)–(19). $\square$

4.1 An exact full-symbol evaluator

For $F(s) = -\zeta'(s)/\zeta(s)$ and $F_{>M}(s) = F(s) - \sum_{n \leq M} \Lambda(n)n^{-s}$, absolute differentiation of the Euler series gives

$$S_{M,k}(\xi) = \sum_{n \leq M} \Lambda(n)n^{-1/2+i\xi} + M \sum_{j=0}^{k-1} \frac{(-\partial_s - \log M)^j F_{>M}(s)}{j!} \Big|_{s=3/2-i\xi}. \quad (21)$$

Thus the full arithmetic symbol is evaluable from a finite head and k derivatives in the absolutely convergent half-plane. A Taylor expansion $F_{>M}(s+x) = \sum_l a_l x^l$ makes its j th summand $\sum_{l=0}^j (-1)^l a_l (-\log M)^{j-l}/(j-l)!$. The paired operator remainder and the scalar evaluation error are separate quantities; both have explicit roles in (20). For example $d_{16,1} < 5.122 \cdot 10^{-20}$, $d_{16,2} < 3.346 \cdot 10^{-21}$ and $d_{16,16} < 8.618 \cdot 10^{-38}$.

For fixed k the prime continuum has amplitude

$$\frac{e^{i\xi \log M}}{(1/2 + i\xi)(1/2 - i\xi)^k},$$

whose negative phase mean is $2^{k+2}\pi^{-1}(1+4\xi^2)^{-(k+1)/2}$. This fixed- k continuum motivates the local reference for $k = 1$ below. The all- k remainder (18) and the fixed- k mean have their stated, distinct parameter regimes.

5 Complete sources and the actual inverse construction

The arithmetic realization of Section 4 is now fixed at averaging order one. We retain the complete negative source and construct preimages for its actual positive receiver. Throughout the main covariance arguments the original odd polar space is denoted by $\mathcal{X} = \mathcal{K}$, with minimal form domain V .

$$H = P_M - N_M + W_M, \quad N_M = E_M E_M^*, \quad \|W_M\| \leq \delta_M := d_{M,1},$$

$$R_M(\kappa) = (P_M + \kappa I)^{-1}, \quad G_M(\kappa) = E_M^* R_M(\kappa) E_M.$$

Here $P_M = P_{M,1}$ and $N_M = N_{M,1}$ in the main argument; the appendices explicitly identify their separate hard-cutoff notation. The source space is $\mathcal{Y}_M = L^2(S_M, d\xi/\pi)$, $S_M = \{\xi > 0 : n_{M,1}(\xi) > 0\}$, with

$$E_M^* h(\xi) = \sqrt{n_{M,1}(\xi)} \int_{\mathbb{R}} s(u) h(u) \sin(\xi u) du.$$

For a finite genuine-null subspace K , $P_{M,K}$ is the same-form restriction to $K^\perp$ and $E_{M,K}$ its exactly projected source. All such inverses use the original norm. We write $\gamma_M(K; \kappa) = \|E_{M,K}^* (P_{M,K} + \kappa I)^{-1} E_{M,K}\|$.

An equivalent weighted-source convention is convenient for profiles: for $I \subset [0, \infty)$ and $\phi \in L^2(I, n_{M,1} d\xi/\pi)$,

$$g_\phi(u) = \int_I n_{M,1}(\xi) \phi(\xi) \sin(\xi u) \frac{d\xi}{\pi}, \quad E_I \phi = Q(sg_\phi), \quad D_\phi = \int_I n_{M,1}(\xi) |\phi(\xi)|^2 \frac{d\xi}{\pi}.$$

Multiplication by $\sqrt{n_{M,1}}$ identifies this convention isometrically with the unweighted source on its positive support. Thus neither notation changes a covariance cap or omits mixed source terms.

Set $L_0 = 1 - 4\partial_u^2$, $B(D) = 8L_0^{-1}$ and

$$\mathcal{C}_Q = \frac{8}{\pi} Q M_s L_0^{-1} M_s Q, \quad \|f\|_{1,4}^2 = \int_{\mathbb{R}} (|f|^2 + 4|f'|^2).$$

The inverse-form interpretation and explicit numerical reference bounds are proved in Appendices C–D. The following exact polar cancellation is the ingredient needed for the actual arithmetic inverse.

Lemma 5.1 (Exact polar Green cancellation). *On the original odd polar space let $B(D) = 8(1 + 4D^2)^{-1}$ and $K_a = M_s B(D) \cos(aD) M_s$. For every $a \geq 0$,*

$$\|Q_b K_a Q_b\|_{\text{HS}} \leq 12e^{2a} e^{-\pi(e^a - 1)}. \quad (22)$$

Consequently, for either sign of the cosine and any further original orthogonal constraint projection Q inside the polar space,

$$\|Q M_s [B(D)(\pm \cos(UD))_+] M_s Q - \mathcal{C}_Q\|_{\text{HS}} \leq 12M^2 e^{-\pi(M-1)}. \quad (23)$$

Proof. The full kernel of K_a is $s(u)s(v)[e^{-|u-v-a|/2} + e^{-|u-v+a|/2}]$. Subtract the bounded rank-two kernel $2e^{-a/2}s(u)s(v)\cosh((u-v)/2)$ on the whole plane. Its even component vanishes on the odd space; its odd component is $-2e^{-a/2}|b\rangle\langle b|$ and is killed by Q_b . The exact remainder is

$$-2s(u)s(v)\sinh((|u-v|-a)/2)1_{|u-v|\geq a}.$$

Put $A = e^a$. On its support $r_u + r_v \geq 2\pi A$ by AM–GM. Its square is at most $A^{-1}s(u)^2s(v)^2e^{|u|+|v|}$. Split $e^{-(r_u+r_v)} \leq e^{-2\pi(A-1)}e^{-(r_u+r_v)/A}$. After enlarging the remaining integration to the whole plane, each factor is bounded by

$$\int_{\mathbb{R}} 16r^2 e^{-r/A} e^{|u|} du \leq \frac{16}{\sqrt{\pi}} A^{5/2} \Gamma(5/2) = 12A^{5/2}.$$

Taking square roots proves (22), including every atom and both infinite arms. The nonconstant real cosine coefficients of $\cos_+$ have absolute sum $1/2 + 1/\pi < 1$. The right side of (22) decreases with $A \geq 1$; apply it to each harmonic $a = nU$. The constant coefficient is $1/\pi$, giving (23). Further original projections contract the bound. $\square$

6 An actual inverse estimate

The exact polar phase cancellation also pays a positive lower of the *actual* averaged arithmetic receiver. This gives a whole-profile receiving theorem in the operator used by the signed criterion. Write $\bar{P}_M = P_{M,1}$, $\bar{n}_M = n_{M,1}$, $U = \log M$. For $0 < \Omega \leq 1$, let E_Ω be the complete source map of Section 5 on $[0, \Omega]$, with the same true profile metric.

Theorem 6.1 (Uniform actual-positive origin bound). *There are constants $C > 0$, M_0 such that for every integer $M \geq M_0$, every $0 < \Omega \leq 1$ and every finite exact constraint space inside the original odd polar space,*

$$E_\Omega^*(\bar{P}_{M,Q} + M^{-1}I)^{-1}E_\Omega \preceq C\Omega^3(1 + \log \log M)^3 I. \quad (24)$$

Here $\bar{P}_{M,Q}$ is the closed-form restriction of the same actual positive operator and the source uses its exact orthogonal projection. In particular, for $\Omega = 1/\log M$, the entire origin cap is $O((\log \log M / \log M)^3)$ and tends to zero. Every schedule $\Omega = o(1/\log \log M)$ gives a vanishing actual cap. For any fixed $a > 0$ the same conclusion holds at ridge M^{-a} , with constants depending on a .

Proof. First use only the original polar constraint Q_b . Normalize one arbitrary complex profile by $D_\phi = 1$, and put $A_\Omega = 16\sqrt{M}\Omega^3/(3\pi)$. Common-profile Cauchy–Schwarz gives

$$|g(u)| \leq \sqrt{A_\Omega}|u|, \quad |g^{(j)}(u)| \leq \sqrt{A_\Omega}\Omega^{j-1} \quad (j \geq 1). \quad (25)$$

All constants below are independent of this profile and Ω .

A phase-safe actual positive lower. For $E_\Lambda(t) = \Psi_\Lambda(t) - t$ define the complete Abel error

$$\bar{e}_M = \frac{3}{\sqrt{M}} + \frac{1}{\sqrt{M}} \int_1^M |E_\Lambda(t)| t^{-3/2} dt + \sqrt{M} \int_M^\infty |E_\Lambda(t)| t^{-5/2} dt.$$

Split the weighted prime integral at M and integrate by parts; the common boundary cancels because $\min(1, M/t)$ is continuous. The derivative moduli are at most $(1/2 + F)t^{-3/2}$ and $(3/2 + F)Mt^{-5/2}$. Together with the lower endpoint this gives

$$\sup_{|\xi| \leq F} \left| \frac{S_{M,1}(\xi)}{\sqrt{M}} - \frac{e^{iU\xi}}{1/4 + \xi^2} \right| \leq (2 + F)\bar{e}_M.$$

The unconditional quantitative PNT [17, Theorem 1.1] implies $|E_\Lambda(t)| \leq C_N t/(\log t)^N$ for every fixed N . Splitting the first integral at $\sqrt{M}$ gives $\bar{e}_M = O_N(U^{-N})$; the second normalized integral has mass 2. Take $F = U^5$ and $N = 20$. With $B(\xi) = 8/(1 + 4\xi^2)$,

$$\frac{m_{M,1}(\xi)}{\sqrt{M}B(\xi)} \geq -\cos(U\xi) - \varepsilon_M, \quad \varepsilon_M = \frac{3(1 + 4F^2)}{4\sqrt{M}} + \frac{(1 + 4F^2)(2 + F)}{4}\bar{e}_M \rightarrow 0.$$

We used $a \geq 0$ and $m_0 \geq -6$. Eventually $\varepsilon_M \leq 1/2$. The scalar positive-part inequality therefore gives the closed-form order $\bar{P}_M \succeq A_F$, where

$$A_F = \sqrt{M}Q_b M_s B(D)w(UD)1_{|D| \leq F} M_s Q_b, \quad w(t) = (-\cos t - 1/2)_+.$$

This is a bounded positive operator. Its phase mean is $\mu = (\sqrt{3} - \pi/3)/(2\pi) > 0$.

A compact smooth preimage. Choose a fixed Gevrey-2 step χ , equal to one to the left of 0 and zero to the right of 1, with $|\chi^{(j)}| \leq C_0 C_1^j (j!)^2$. Such a step is obtained from $\eta(t) = e^{-1/t}$ for $t > 0$ and 0 otherwise, by $\chi(t) = \eta(1 - t)/(\eta(t) + \eta(1 - t))$. Cauchy's estimate on a disk of radius ct bounds derivatives of η by $j!(C/t)^j e^{-c'/t}$; maximization in t gives $C^j (j!)^2$. The denominator is bounded below near each endpoint, and the reciprocal Leibniz recurrence preserves this bound; on the remaining compact middle use a uniform analytic disk.

Put $R = \frac{1}{2} \log(2U/\pi)$ and $f(u) = \chi(|u| - R)g(u)$. For large U , $R \geq 1$, the cutoff is constant near zero and is smooth and even. Its transition has fixed physical width. Leibniz and (25), with $\Omega \leq 1$, give

$$\|f\|_{1,4}^2 \leq C A_\Omega (R + 1)^3, \quad \|f^{(j)}\|_2 \leq C \sqrt{A_\Omega} (R + 1)^{3/2} C_1^j (j!)^2. \quad (26)$$

Indeed the undifferentiated g is bounded by $\sqrt{A_\Omega} (R + 1)$, each differentiated g by $\sqrt{A_\Omega}$, and the factorial ratios in the Leibniz sum are absorbed by one fixed exponential constant. Define

$$Y = \frac{L_0 f}{8\mu\sqrt{M}s}.$$

It is compact smooth odd and exactly polar legal: $\int \sinh(u/2) L_0 f = 0$ by two integrations by parts. On its support $r \leq 2e^2 U$; the original lower theta bound gives

$$\|Y\|^2 \leq C M^{2e^2-1} A_\Omega (R + 1)^3. \quad (27)$$

Complete residual and energy. Let A_∞ omit the frequency truncation in A_F . The cosine coefficients of w are absolutely summable, because its first derivative has bounded variation. Lemma 5.1 then gives

$$A_\infty = \mu\sqrt{M}Q_b M_s B(D)M_s Q_b + H_M, \quad \|H_M\| \leq h_M = C M^{5/2} e^{-\pi(M-1)}.$$

Since $B(D)L_0 = 8$ and $Q_b Y = Y$, its mean acts on Y as $Q_b(sf)$, and the exact truncation action is

$$(A_\infty - A_F)Y = \mu^{-1}Q_b M_s w(UD)1_{|D| > F} f.$$

Thus the complete residual $e - A_F Y$ is the sum of $Q_b[s(g - f)]$, $-H_M Y$ and this Fourier exterior. All mixed terms are paid by the squared triangle bound with factor 3. The full two-arm theta estimate gives

$$\|Q_b[s(g - f)]\|^2 \leq C A_\Omega \int_{2U}^\infty r e^{-r} (\log r)^2 dr \leq C A_\Omega U e^{-2U} (\log U)^2.$$

For $j = \lceil U \rceil$, (26) gives

$$\|1_{|D|>F} f\| \leq F^{-j} \|f^{(j)}\| \leq C \sqrt{A_\Omega} (R + 1)^{3/2} e^{-3U \log U + O(U)}.$$

Consequently $M\|e - A_F Y\|^2$ is Ω^3 times a negligible term; the polynomial trial cost (27) is dominated by h_M . Also $A_F \preceq A_\infty$ gives

$$\langle Y, A_F Y \rangle \leq \frac{\langle L_0 f, f \rangle}{8\mu\sqrt{M}} + h_M \|Y\|^2 \leq C \Omega^3 (R + 1)^3 + \text{smaller terms}.$$

For complex f , integration by parts gives $\langle L_0 f, f \rangle = \|f\|_{1,4}^2$.

For any bounded positive A and $\kappa > 0$, least squares gives

$$\langle e, (A + \kappa)^{-1} e \rangle \leq \langle Y, AY \rangle + \kappa^{-1} \|e - AY\|^2. \quad (28)$$

One obtains this by choosing $A^{1/2} Y$ as the coefficient in the regularized preimage formula. It is valid when A has a kernel. Apply it to A_F , $\kappa = M^{-1}$, and use inverse order from $A_F \preceq \bar{P}_M$. Equations above prove (24) uniformly on the entire profile unit ball.

For any additional finite constraint space Z , the variational supremum for the restricted inverse is taken over $V \cap Z^\perp$, a subset of the original trial space. For the exactly projected source its value is therefore at most the original inverse energy. This gives the same constant for every such constraint space. The closed restriction is densely defined: approximate in the dense form domain and correct the finitely many row evaluations using fixed form-domain dual vectors. Finally, for fixed ridge exponent $a > 0$, replace $r_R = 2U$ by cU , where $c > a + 1/2$. The source residual costs $C \Omega^3 M^{a+1/2-cU} (\log U)^2$; the other errors stay negligible. This proves the last assertion. $\square$

7 Complete covariance at prescribed polynomial ridges

The complete source admits an unconditional norm estimate at the ridge M^{-1} used in the signed return. More generally, the following holds at every fixed polynomial ridge. Its proof combines the original hard-cutoff realization of Lemma 3.1 with the actual arithmetic preimage and the Vinogradov–Korobov prime-number error [17].

Theorem 7.1 (Complete covariance at polynomial ridges). *For every fixed $b > 0$, there are C_b, M_b such that, for $M \geq M_b$,*

$$\sup_K \left| \|E_{M,K}^* (P_{M,K} + M^{-b} I)^{-1} E_{M,K}\| - 1 \right| \leq \frac{C_b \sqrt{\log M} \exp(48\sqrt{b \log M})}{\sqrt{M}}. \quad (29)$$

The supremum is over all finite genuine operator-null spaces, including moving and growing choices, with same-form restriction to $K^\perp$. The upper bound holds for every exact finite restriction with dense restricted form domain. Every negative source frequency is included.

Proof. Write $U = \log M$ and $\kappa = M^{-b}$. Use a second arithmetic cutoff m , distinct from the averaged-receiver parameter M . The original minimal realization has

$$H = Q_b s(a + b_m)(D) s Q_b + V_m, \quad \|V_m\| \leq d_m := \frac{640(m+1)^2}{20^{m+1}}, \quad |b_m| \leq 18\sqrt{m}.$$

Here $b_m(\xi) = m_0 - 2 \sum_{2 \leq n \leq m} \Lambda(n) n^{-1/2} \cos(\xi \log n)$, while $a \geq 0$ is increasing on the positive half-line and $a(L) \geq \frac{1}{2} \log(2L)$ for $L \geq \frac{1}{2}$. Lemma 3.1 proves these complete paired form identities in the present manuscript; the raw remainder is never separated into its edge pieces. Put

$$k = \max \left\{ 4, \left\lceil \frac{\log(2560/\kappa)}{\log 10} \right\rceil \right\}, \quad m = k - 1, \quad D_0 = 18\sqrt{m}, \quad L = e^{2D_0}.$$

Since $k^2 \leq 2^k$, $d_m \leq \kappa/4$. The multiplier $v_m = -(a + b_m)_+$ lies in $[0, D_0]$ and is supported in $[-L, L]$. On the unitary odd Fourier profile space put $T_m = v_m(D)^{1/2}$ and $E = Q_b s T_m$. Scalar sign decomposition gives $H \succeq -EE^* - d_m I$ on the original form domain.

Let $S = \|s\|_\infty < 3$, and choose an even Gevrey-2 cutoff χ_R equal to one on $[-R, R]$ and zero off $[-R-1, R+1]$, with a fixed-width transition and bounds $\|\chi_R^{(j)}\|_\infty \leq C_0 C_1^j (j!)^2$. Set

$$X_0 = \log \frac{104SD_0}{\kappa}, \quad R = \frac{1}{2} \log(2X_0), \quad E_0 = Q_b s \chi_R T_m.$$

The complete theta envelope gives

$$\sup_{|u| \geq R} s(u) \leq 26X_0 e^{-3X_0} \leq 13e^{-X_0} = \frac{\kappa}{8SD_0}.$$

Consequently $\|EE^* - E_0 E_0^*\| \leq \kappa/4$ and

$$H \succeq -E_0 E_0^* - (\kappa/2)I. \quad (30)$$

This constructs a source majorant of the negative sign of H without applying a non-monotone negative-part operation to an operator inequality.

We next bound the actual inverse on the entire coefficient unit ball of E_0 . Theorem 1.4 of [17] gives $|\Psi(x) - x| \leq Cx(\log x)^{1.801} \exp[-0.1853(\log x)^{3/5}/(\log \log x)^{1/5}]$ for $x \geq 23$. In the complete Abel error, split $M^{-1/2} \int_1^M |\Psi(t) - t| t^{-3/2} dt$ at $\sqrt{M}$; its lower part is $O(M^{-1/4})$, and its upper part and $M^{1/2} \int_M^\infty |\Psi(t) - t| t^{-5/2} dt$ obey the displayed stretched-exponential error. Thus, for every fixed $A > 0$,

$$\bar{e}_M = O_A(e^{-A\sqrt{U}}).$$

Take $T = \exp[(50\sqrt{b} + 2)\sqrt{U}]$. The exact arithmetic phase debit

$$\varepsilon_M(T) = \frac{3(1 + 4T^2)}{4\sqrt{M}} + \frac{(1 + 4T^2)(T + 2)}{4} \bar{e}_M$$

tends to zero. Hence, for large M , the actual positive form dominates

$$A_T = \sqrt{M} Q_b s B(D) w(UD) 1_{|D| \leq T} s Q_b, \quad B(\xi) = \frac{8}{1 + 4\xi^2}, \quad w(t) = (-\cos t - 1/2)_+.$$

If μ is the positive mean of w , its full-frequency continuation is $A_\infty = \mu\sqrt{M} Q_b s B(D) s Q_b + J_M$, with $\|J_M\| \leq CM^{5/2} e^{-\pi(M-1)}$.

For $\|\phi\| \leq 1$ put $g = T_m\phi$, $f = \chi_R g$ and $e = E_0\phi$. Plancherel and product differentiation give, uniformly over this entire ball,

$$\|f\|_{1,4}^2 \leq CD_0(1 + L^2), \quad \|f^{(j)}\|_2 \leq C\sqrt{D_0}(L + Cj^2)^j.$$

Here $\|f\|_{1,4}^2 = \|f\|_2^2 + 4\|f'\|_2^2$. Since $m/U \rightarrow b/\log 10$, eventually $D_0 \leq 12\sqrt{bU}$ and $L \leq e^{24\sqrt{bU}}$. Choosing $j = \lceil U \rceil$ yields

$$\|1_{|D|>T}f\|_2 \leq C\sqrt{D_0}e^{-c_b U^{3/2}}.$$

The compact smooth odd trial $Y = (1 - 4\partial_u^2)f/(8\mu\sqrt{M}s)$ is exactly polar legal by integration by parts against $\sinh(u/2)$. On its support the lower theta bound $s(u)^2 \geq 2r^2e^{-r}$, $r = \pi e^{2|u|}$, gives

$$\|Y\|^2 \leq CM^{-1}e^{CX_0}D_0(1 + L^4).$$

This pays division by s on the moving support. Since $X_0 = bU + O_b(\log U)$, this growth is harmless against the full harmonic error. The exact mean action and residual are

$$\mu\sqrt{M}Q_b s B(D) s Q_b Y = e, \quad e - A_T Y = -J_M Y + \mu^{-1}Q_b s w(UD)1_{|D|>T}f.$$

Their complete squared residual, including the cross term, is smaller than every fixed inverse power of M after multiplication by κ^{-1} . Also

$$\langle Y, A_T Y \rangle \leq CM^{-1/2}\|f\|_{1,4}^2 + \|J_M\|\|Y\|^2.$$

The kernel-safe trial inequality (28) therefore proves

$$\|E_0^* R_M(\kappa) E_0\| \leq \frac{C_b \sqrt{U} e^{48\sqrt{bU}}}{\sqrt{M}}. \quad (31)$$

The proof is uniform over the full coefficient unit ball, with no finite-column or dimension factor.

Finally $N_M = P_M + W_M - H$ and (30) give

$$R_M^{1/2} N_M R_M^{1/2} \preceq I + R_M^{1/2} E_0 E_0^* R_M^{1/2} + (\delta_M - \kappa/2) R_M.$$

The last term is nonpositive for large M . Equality of the source and physical operator norms gives the required complete upper bound. For any exact same-form restriction its inverse, extended by zero, is at most the base inverse. Restricting the form inequality thus proves the upper uniformly without commuting projections with an inverse.

For the lower take one nonzero $v \in \text{Dom } H \cap (\ker H)^\perp$. Such a vector exists since the archimedean form is unbounded above on compact high-frequency packets, whereas its arithmetic perturbation is bounded. It belongs to every genuine-null complement. For completeness, the averaged quadratic mean used here follows directly from the complete Abel formula. On each fixed frequency band, $n_{M,1}/\sqrt{M} - B(\xi)(\cos(U\xi))_+ \rightarrow 0$ uniformly. The phase mean of $(\cos)_+$ is $1/\pi$, and its absolutely convergent Fourier series gives weak convergence against every integrable test on that band. The uniform complete bound $n_{M,1} \leq 16\sqrt{M}$ pays the exterior of $|\widehat{sv}|^2$ by Plancherel. Hence

$$N_M[v]/\sqrt{M} \longrightarrow \langle v, \mathcal{C}_{Q_b} v \rangle =: c_v > 0.$$

Strict positivity follows from $s > 0$ and $B(\xi) > 0$. This is the averaged order-two mean. The separate hard-cutoff order-one mean is proved in the technical supplement and is not used here. Therefore

$$\|G_{M,K}(\kappa)\| \geq \frac{N_M[v]}{N_M[v] + q_H[v] - \langle v, W_M v \rangle + \kappa\|v\|^2} \geq 1 - C_v M^{-1/2}$$

uniformly in K and $0 < \kappa \leq 1$. This completes the proof. $\square$

At $\kappa = M^{-1}$, Theorem 7.1 establishes the complete covariance estimate

$$\gamma_M = 1 + O(M^{-1/2+o(1)}),$$

uniformly over finite genuine-null restrictions. Corollary 7.2 characterizes RH through

$$\liminf_{M \rightarrow \infty} \sqrt{M} (\gamma_M - 1)_+ = 0.$$

Sections 8–10 develop complete signed certificates, exact null cancellation and physical lift estimates within this formulation. Corollary 8.5 proves that a fixed complete complementary gap together with the stated lift bound yields vanishing original-norm allowances and hence the Riemann hypothesis.

Corollary 7.2 (Critical covariance characterization). *Let K_M be any sequence of finite genuine operator-null spaces and put $\gamma_M = \|E_{M,K_M}^*(P_{M,K_M} + M^{-1}I)^{-1}E_{M,K_M}\|$. Then*

$$H \succeq 0 \iff \liminf_{M \rightarrow \infty} \sqrt{M} (\gamma_M - 1)_+ = 0.$$

Thus the critical covariance condition is equivalent to the original Weil positivity criterion and hence to RH in the realization of Theorem 2.2.

Proof. If $H \succeq 0$, the signed identity gives $R_M^{1/2} N_M R_M^{1/2} \preceq I + (\delta_M - M^{-1})R_M \preceq I$ for all large M , also after any same-form restriction. Conversely, if H is not nonnegative, choose a fixed nonzero $v \in \text{Dom } H \cap (\ker H)^\perp$ with $q_H[v] < 0$ by spectral projection onto a bounded negative interval. This vector belongs to every $K_M^\perp$. Its same-state quotient gives

$$\gamma_M - 1 \geq \frac{-q_H[v] + \langle v, W_M v \rangle - M^{-1}\|v\|^2}{P_M[v] + M^{-1}\|v\|^2} \geq c_v M^{-1/2}.$$

The last inequality uses the fixed positive numerator for large M and $P_M[v] = N_M[v] + q_H[v] - \langle v, W_M v \rangle = O_v(\sqrt{M})$. This contradicts the stated liminf condition, which therefore establishes $H \succeq 0$ on the original domain. The original Weil equivalence in Lemma A.1 now applies. Hence, under the stated critical covariance condition, the Riemann hypothesis holds. $\square$

8 Genuine nulls and complete source certificates

Genuine null vectors provide coordinates adapted to the signed form. We first establish completeness at each fixed arithmetic cutoff, then quantify the full covariance error of a finite null head.

Lemma 8.1 (Genuine derivative nulls). *For every $j \geq 0$, $z_j = (\psi^{(2j+3)} - \psi^{(2j+1)})/4/s$ belongs to $\text{Dom } H$ and satisfies $H z_j = 0$. Every finite initial family is linearly independent.*

Proof. The theta Mellin identity of Appendix A gives

$$\mathcal{M}(s z_j)(z) = (z^2 - 1/4)(-z)^{2j+1} \xi(1/2 + z).$$

It vanishes at both polar points and at every centered nontrivial zero. To verify the original domain, use compact even cutoffs χ_R and the polar-preserving approximants $(\partial_u^2 - 1/4)(\chi_R \psi^{(2j+1)})/s$. The complete derivative tails, also quantified in Proposition 9.2, give convergence in L^2 after division by s and convergence of the weighted physical derivative in L^2 . The bound $a(\xi) \leq 17\xi^2$ proves convergence in V . The same complete tails and integration by parts give arbitrary vertical inverse

powers for the Mellin transforms on $|\Re z| \leq 1/2$. The zero-count bound therefore passes the polarized explicit formula through these approximants as in Appendix A. It yields $q_H[z_j, h] = 0$ first on the compact core and then on all V . Form representation gives the asserted operator nullity. Finally a vanishing finite combination would make a nonzero polynomial times $(z^2 - 1/4)\xi(1/2 + z)$ identically zero. Analyticity excludes this. $\square$

Theorem 8.2 (Complete null-generated source). *Let*

$$z_i = \frac{\psi^{(2i+3)} - \psi^{(2i+1)}/4}{s}, \quad i \geq 0,$$

be the genuine operator nulls of the original realization. For every fixed M with nonzero negative source,

$$\overline{E_M^* \text{span}\{z_i : i \geq 0\}} = \mathcal{Y}_M.$$

Moreover E_M is Hilbert–Schmidt. If Π_r projects onto $\text{span}\{E_M^* z_0, \dots, E_M^* z_{r-1}\}$, then, for each fixed M and $\kappa > 0$,

$$\|(I - \Pi_r)G_M(\kappa)(I - \Pi_r)\| \longrightarrow 0.$$

Proof. The complete averaged prime multiplier is bounded for fixed M , whereas the archimedean multiplier tends to infinity. Hence S_M lies in a bounded interval $[0, F_M]$. The derivative rule gives, up to one fixed nonzero normalization factor,

$$E_M^* z_i(\xi) = v_M(\xi)(-1)^i \xi^{2i}, \quad v_M(\xi) = \sqrt{n_{M,1}(\xi)} \xi(\xi^2 + 1/4) \widehat{\psi}(\xi).$$

The nonzero entire function $\widehat{\psi}$ has discrete real zeros, so $v_M \neq 0$ almost everywhere on S_M . It belongs to $L^2(S_M)$. If $f \in \mathcal{Y}_M$ is orthogonal to every displayed function, the finite complex measure $\overline{f} v_M d\xi/\pi$ annihilates every polynomial in ξ^2 . Such polynomials are uniformly dense in $C([0, F_M])$, by polynomial approximation after the homeomorphism $\xi \mapsto \xi^2$. The measure is therefore zero, and $f = 0$ almost everywhere.

The source kernel satisfies

$$\|E_M\|_{\text{HS}}^2 = \int_{S_M} n_{M,1}(\xi) \|Q_b[s \sin(\xi \cdot)]\|^2 \frac{d\xi}{\pi} < \infty.$$

Thus $G_M(\kappa)$ is positive trace class. The proved density gives $\Pi_r \rightarrow I$ strongly, and compactness gives the asserted norm limit. $\square$

The theorem concerns the source images of known nulls. It does not identify the full physical kernel of H . Its density assertion is on the base polar recipient; imposing further physical constraints can alter that assertion.

Theorem 8.3 (Signed remainder and complete trace certificate). *Fix M , $\kappa > 0$ and finitely many independent derivative-null columns Z . Define*

$$K = Z^* N_M Z, \quad V_Z = Z K^{-1/2}, \quad J = E_M^* V_Z, \quad Q = I - J J^*, \quad \ell = \|V_Z^* V_Z\|.$$

Then $K \succ 0$, $J^* J = I$, and

$$J^* G_M J \preceq [1 + (\delta_M + \delta_M^2/\kappa)\ell] I, \tag{32}$$

$$\tau := \|E_M Q\|_{\text{HS}}^2 = \text{Tr } N_M - \text{Tr}(K^{-1} Z^* N_M^2 Z), \tag{33}$$

$$\|G_M\| \leq 1 + (\delta_M + \delta_M^2/\kappa)\ell + \tau/\kappa. \tag{34}$$

For an increasing initial derivative family, $\tau \rightarrow 0$ at fixed M . If Z has r columns, then

$$\ell \geq \lambda_r(N_M)^{-1} \geq r / \operatorname{Tr} N_M,$$

where the positive eigenvalues are arranged in decreasing order.

Proof. The nonvanishing common factor in Theorem 8.2 shows that every finite source polynomial family is independent, hence $K \succ 0$. Suppress M and write $A = P + \kappa I$, $R = A^{-1}$, $F = (W - \kappa I)V_Z$. Genuine operator nullity gives $NV_Z = AV_Z + F$, so exact expansion gives

$$GJ = J + E^*RF, \quad J^*GJ = I + V_Z^*(W - \kappa I)V_Z + F^*RF.$$

Using $R \preceq \kappa^{-1}I$ only after combining these terms yields

$$V_Z^*(W - \kappa I)V_Z + F^*RF \preceq V_Z^*(-W + W^2/\kappa)V_Z \preceq (\delta + \delta^2/\kappa)V_Z^*V_Z.$$

This proves (32). In particular the ridge does not produce a separate $\kappa\ell$ charge.

Expanding Q and using cyclicity of the trace gives (33); all operators under the trace are trace class or finite rank. Also $\|QGQ\| \leq \kappa^{-1}\|EQ\|_{\text{HS}}^2 = \tau/\kappa$. For a positive two-block operator with diagonal norms a, b , positive-form Cauchy–Schwarz bounds its quadratic form by $(\sqrt{a}\|x\| + \sqrt{b}\|y\|)^2$, and consequently its norm by $a + b$. This proves (34), including the mixed block. Hilbert–Schmidt approximation and source density give $\tau \rightarrow 0$. Finally $V_Z^*NV_Z = I$ makes the Rayleigh quotient of N on $\operatorname{ran} V_Z$ at least $1/\ell$. Min–max and $r\lambda_r(N) \leq \operatorname{Tr} N$ prove the last two inequalities. $\square$

Proposition 8.4 (Directional Schur alternative). *With the same notation, put $b = \|QG_MQ\|$, $r_F = \|F^*R_MF\|$ and $\varepsilon = (\delta_M + \delta_M^2/\kappa)\ell$. For $\lambda = 1 + \eta > b$, the condition*

$$\varepsilon + \frac{br_F}{\lambda - b} \leq \eta$$

implies $G_M \preceq \lambda I$.

Proof. The mixed block is $QG_MJ = QE_M^*R_MF$ and its squared norm is at most br_F . Complete the square in the Q coordinate of $\lambda I - G_M$, using $\lambda I - QG_MQ \succeq (\lambda - b)I$, and then apply (32). $\square$

Corollary 8.5 (A fixed complementary gap sufficient for RH). *At $\kappa = M^{-1}$, suppose that along an unbounded sequence of M , finite genuine-null heads in Proposition 8.4 satisfy*

$$b_M = \|Q_MG_MQ_M\| \leq q < 1, \quad \ell_M = \|V_M^*V_M\| = o(\sqrt{M}),$$

where q is fixed and the complement contains every remaining source frequency. Then the original Weil form is nonnegative and RH holds.

Proof. The actual remainder δ_M decreases exponentially. Since $R_M \preceq \kappa^{-1}I$ and $F_M = (W_M - \kappa I)V_M$,

$$r_{F,M} \leq \frac{(\kappa + \delta_M)^2}{\kappa} \ell_M.$$

Put $\varepsilon_M = (\delta_M + \delta_M^2/\kappa)\ell_M$ and $\eta_M = \varepsilon_M + qr_{F,M}/(1 - q)$. The two assumed resources imply $\eta_M = o(M^{-1/2})$. With $\lambda_M = 1 + \eta_M$, one has $\lambda_M - b_M \geq 1 - q$; hence Proposition 8.4 gives $G_M \preceq (1 + \eta_M)I$, including the mixed block. The original-domain absorption bound is

$$q_H[h] \geq - \left[\delta_M + M^{-1} + \frac{\eta_M B_{M,1}}{1 + \eta_M} \right] \|h\|^2.$$

Since $B_{M,1} = O(\sqrt{M})$, its allowance tends to zero. Theorem 2.2 applies: under the stated complete complementary-gap and lift hypotheses, the Riemann hypothesis holds. $\square$

The corollary uses a fixed gap below one, with both resources evaluated on the same head. The matrix-valued signed Schur comparison can retain additional cancellation lost in these scalar sufficient bounds.

9 Regularized null certificates and physical lift budgets

The null-source construction can avoid an unregularized source-Gram inverse. The price becomes an explicit approximation error and the physical norm of its constructed lift. In this section write $d = \|W_M\|$ and $\mathcal{D} = d + d^2/\kappa$.

Theorem 9.1 (Regularized complete certificate). *Let $Z : \mathbb{C}^r \rightarrow \mathcal{X}$ be a physical isometry with range in $\text{Dom } H \cap \ker H$. Put $A = E_M^* Z$, $K = A^* A$, and, for $t > 0$,*

$$J = A(K + tI)^{-1} A^*, \quad T = Z(K + tI)^{-1} A^*.$$

Then $0 \preceq J \preceq I$, $J = E_M^* T$, and

$$\|G_M\| \leq 1 + \frac{\mathcal{D}}{4t} + \frac{\tau_t}{\kappa}, \quad \tau_t = \text{Tr } N_M - \text{Tr}((K + tI)^{-1} K (K + tI)^{-1} Z^* N_M^2 Z). \quad (35)$$

If a Hilbert–Schmidt map $X : \mathcal{X} \rightarrow \mathbb{C}^r$ satisfies $\|E_M^* - AX\|_{\text{HS}} \leq \epsilon$ and $\|X\|_{\text{HS}} \leq L_0$, then

$$\|G_M\| \leq 1 + \frac{2\epsilon^2}{\kappa} + \sqrt{\frac{2\mathcal{D}}{\kappa}} L_0. \quad (36)$$

Every trace and approximation here uses the complete source metric.

Proof. The singular-value multipliers of T are $\sigma/(\sigma^2 + t)$; hence $\|T\|^2 \leq 1/(4t)$. The exact signed-null identity, applied to Tf , gives

$$\langle Jf, G_M Jf \rangle \leq \|Jf\|^2 + \mathcal{D} \|Tf\|^2 \leq (1 + \mathcal{D}/(4t)) \|f\|^2.$$

Use the positive partition $N_M = E_M J^2 E_M^* + E_M (I - J^2) E_M^*$. After inverse congruence its first term has norm $\|J G_M J\|$, and its second is at most $\kappa^{-1} \text{Tr}(E_M (I - J^2) E_M^*)$. Expanding J^2 and using cyclicity proves (35). Completing the finite-dimensional square gives

$$\text{Tr}(E_M (I - J) E_M^*) = \inf_X \{ \|E_M^* - AX\|_{\text{HS}}^2 + t \|X\|_{\text{HS}}^2 \}.$$

Since $I - J^2 \preceq 2(I - J)$, the excess in (35) is at most $\mathcal{D}/(4t) + 2(\epsilon^2 + tL_0^2)/\kappa$. For $\mathcal{D}, L_0 > 0$ optimize at $t = \sqrt{\mathcal{D}\kappa/(8L_0^2)}$; the zero cases follow by limits. $\square$

For independent raw columns Z_0 , replace $K + tI$ by $Z_0^* N_M Z_0 + tZ_0^* Z_0$. In (36) the physical lift norm is then $L_0 = \|Z_0 X_0\|_{\text{HS}}$. This preserves the original physical metric without requiring explicit formation of its inverse square root.

The collar estimate of Theorem 10.1 also survives regularization. With P_I denoting source multiplication by $I \subset [0, e^{A\sqrt{\log M}}]$ for a fixed $A > 0$, suppose

$$\|(I - P_I) E_M^* - AX\|_{\text{HS}} \leq \epsilon_{\text{out}}, \quad \|X\|_{\text{HS}} \leq L_{\text{out}}.$$

Completing the same square with $(I - P_I) E_M^*$ and using the triangle inequality in inverse energy gives

$$\|G_M\| \leq 1 + \frac{\mathcal{D}}{4t} + \left(\sqrt{c_I} + \sqrt{\frac{2(\epsilon_{\text{out}}^2 + tL_{\text{out}}^2)}{\kappa}} \right)^2, \quad c_I = C_A |I| (1 + \log \log M). \quad (37)$$

No projection is commuted with J . The square includes the mixed term, and the exterior includes every frequency outside I .

Proposition 9.2 (Order-uniform original-norm null budget). *For $z_j = (\psi^{(2j+3)} - \psi^{(2j+1)})/4/s$,*

$$\|z_j\|_2 \leq 100 \cdot 64^{2j+3} (2j+3)!, \quad j \geq 0.$$

Consequently the raw map $Z_0 = (z_0, \dots, z_{r-1})$ satisfies

$$\|Z_0\|_{\text{HS}}^2 \leq B_r := 10000r \cdot 64^{4r+2} [(2r+1)!]^2, \quad \|Z_0 X_0\|_{\text{HS}} \leq \sqrt{B_r} \|X_0\|_{\text{HS}}. \quad (38)$$

Proof. Set $\rho = 1/64$ and $r_u = \pi e^{2|u|}$. The complete theta series is holomorphic in $|\text{Im } z| < \pi/4$ and even there. On $|z - u| = \rho$, reflect to the right half-plane if needed. Then $\text{Re}(\pi e^{2z}) > 3r_u/4$ and $|\pi e^{2z}| \leq e^{2\rho} r_u$. Its full series gives

$$|\psi(z)| \leq 24e^{|u|/2} r_u^2 e^{-3r_u/4} \sum_{n \geq 1} n^4 e^{-3(n^2-1)\pi/4} < 48e^{|u|/2} r_u^2 e^{-3r_u/4}.$$

Indeed the $n = 2$ summand is below $1/4$ and subsequent ratios are below $1/256$. Cauchy's estimate yields $|\psi^{(k)}(u)| \leq 48k!64^k e^{|u|/2} r_u^2 e^{-3r_u/4}$. Divide by the original lower envelope $s^2 \geq 2r_u^2 e^{-r_u}$ and integrate both arms using $du = dr_u/r_u$ and $e^{|u|} = \sqrt{r_u/\pi}$. Thus

$$\|\psi^{(k)}/s\|_2^2 \leq \frac{1152}{\sqrt{\pi}} (k!64^k)^2 \int_{\pi}^{\infty} r^{3/2} e^{-r/2} dr \leq 3456\sqrt{2} (k!64^k)^2 < 80^2 (k!64^k)^2.$$

The triangle inequality at $k = 2j + 3$ and $k - 2$ proves the first bound. Summing its squares and using the largest summand proves the second. $\square$

Since $\log B_r = O(r \log(r+1))$, this gives a useful cofinal lift window. At $\kappa = M^{-1}$ the complete averaged remainder has $d_M \leq \text{poly}(M) e^{-\pi M}$. If $r_M = o(M/\log M)$ and $\log(1 + \|X_{0,M}\|_{\text{HS}}) = o(M)$, then

$$\sqrt{M \mathcal{D}_M} \|Z_{0,M} X_{0,M}\|_{\text{HS}} \leq e^{-\pi M/2 + o(M)}.$$

Thus the lift part of (36) is paid at every fixed polynomial scale. A complete-source approximation with $\epsilon_M^2 = o(M^{-a-1})$, $1/2 < a < 1$, would then give the excess $o(M^{-a})$ and the vanishing-allowance endpoint. The approximation norm includes the entire source exterior as part of its complete-source error budget.

10 Uniform covariance on small sets in a growing band

The null-source representation can be poorly conditioned near zeros of $\hat{\psi}$. The actual positive inverse controls small frequency sets there without division by that transform. An exact Fourier shift extends this control beyond every fixed power of $\log M$.

Theorem 10.1 (Growing-band observation bound). *Put $U = \log M$, and let $L_M \rightarrow \infty$ satisfy*

$$\frac{\sqrt{L_M}}{U} \rightarrow \infty, \quad L_M^3 \bar{e}_M \rightarrow 0, \quad (39)$$

where $\bar{e}_M$ is the complete Abel error of Theorem 6.1. For all sufficiently large M and every measurable $I \subset [0, L_M]$,

$$\|E_I^*(P_M + M^{-1}I)^{-1}E_I\| \leq C|I|(1 + \log \log M). \quad (40)$$

The constant and onset may depend on the admitted schedule, but not on the number or locations of the components of I . The upper bound survives additional exact finite constraints, with the same-form restricted receiver, dense restricted form domain and exactly projected source.

Proof. Take $T = 2L_M$. The complete arithmetic phase debit is

$$\varepsilon_M(T) = \frac{3(1 + 4T^2)}{4\sqrt{M}} + \frac{(1 + 4T^2)(T + 2)}{4} \bar{e}_M.$$

Since $\bar{e}_M \geq 3/\sqrt{M}$, both terms tend to zero by (39). The scalar inequalities in the actual-origin proof consequently give, for all sufficiently large M ,

$$P_M \succeq A_T := \sqrt{M} Q_b s B(D) w(UD) 1_{|D| \leq T} s Q_b, \quad n_{M,1}(\xi) \leq \frac{12\sqrt{M}}{1 + 4\xi^2} \quad (0 \leq \xi \leq T),$$

where $B(\xi) = 8/(1 + 4\xi^2)$, $w(t) = (-\cos t - 1/2)_+$ and $\mu > 0$ is its mean. These are inequalities for the actual arithmetic receiver and observation, before theta sandwiching.

Put $e_\xi = Q_b[s \sin(\xi \cdot)]$ and $R_0 = \frac{1}{2} \log(2U/\pi)$. Let χ_{R_0} be the even fixed-width Gevrey-2 cutoff equal to one on $[-R_0, R_0]$ and zero outside $[-R_0 - 1, R_0 + 1]$. Define

$$f_\xi(u) = \chi_{R_0}(u) \sin(\xi u), \quad Y_\xi = \frac{(1 - 4\partial_u^2) f_\xi}{8\mu\sqrt{M} s}.$$

This trial is compact smooth, odd and exactly polar legal: integration by parts transfers $1 - 4\partial_u^2$ to $\sinh(u/2)$ and annihilates it. Uniformly for $0 \leq \xi \leq L_M$, product differentiation and the complete two-arm theta envelopes give

$$\begin{aligned} \|f_\xi\|_{1,4}^2 &\leq C(1 + \xi^2)(1 + \log U), \\ \|s(\sin(\xi \cdot) - f_\xi)\|^2 &\leq CUe^{-2U}, \\ \|Y_\xi\|^2 &\leq CM^{2e^2-1}(1 + \xi^2)^2(1 + \log U). \end{aligned}$$

The first bound uses a support of length $O(1 + \log U)$ and a fixed transition width. For the last bound the support satisfies $r = \pi e^{2|u|} \leq 2e^2 U$, so the original lower envelope $s^2 \geq 2r^2 e^{-r}$ controls division by s .

Every positive-order derivative of χ_{R_0} is supported on two fixed-width translated intervals. Thus $\|\chi_{R_0}^{(j)}\|_1 \leq C_0 C_1^j (j!)^2$ for $j \geq 1$, independently of R_0 . Integration by parts with j proportional to $\sqrt{|\omega|}$ gives $|\widehat{\chi}_{R_0}(\omega)| \leq C e^{-c\sqrt{|\omega|}}$ for large $|\omega|$, uniformly in R_0 . The exact shift identity

$$\widehat{f}_\xi(\omega) = \frac{\widehat{\chi}_{R_0}(\omega - \xi) - \widehat{\chi}_{R_0}(\omega + \xi)}{2i}$$

therefore proves the complete exterior bound

$$\|1_{|D| > T} f_\xi\|_2 \leq C e^{-c\sqrt{L_M}}, \quad 0 \leq \xi \leq L_M.$$

Indeed both shifted frequencies have modulus at least L_M there; the polynomial factor arising on integration is absorbed by reducing c .

The full-frequency continuation satisfies

$$A_\infty = \mu\sqrt{M} Q_b s B(D) s Q_b + J_M, \quad \|J_M\| \leq CM^{5/2} e^{-\pi(M-1)}.$$

The polar Green identity gives its exact mean action $Q_b(s f_\xi)$. Consequently

$$e_\xi - A_T Y_\xi = Q_b[s(\sin(\xi \cdot) - f_\xi)] - J_M Y_\xi + \mu^{-1} Q_b s w(UD) 1_{|D| > T} f_\xi.$$

A squared triangle bound pays all three terms and their mixed products. At ridge M^{-1} the physical error costs $O(U/M)$; the Fourier error costs $O(Me^{-2c\sqrt{L_M}})$, smaller than every fixed inverse power of M by (39). Also that admission and $\bar{e}_M \geq 3/\sqrt{M}$ imply $L_M = o(M^{1/6})$, so the displayed trial norm is polynomial in M and is absorbed by the harmonic error. Finally

$$\langle Y_\xi, A_T Y_\xi \rangle \leq CM^{-1/2} \|f_\xi\|_{1,4}^2 + \|J_M\| \|Y_\xi\|^2.$$

The kernel-safe least-squares estimate (28) and $A_T \preceq P_M$ now prove, uniformly on the whole band,

$$\langle e_\xi, (P_M + M^{-1})^{-1} e_\xi \rangle \leq CM^{-1/2} (1 + \xi^2) (1 + \log U). \quad (41)$$

The negligible errors are absorbed uniformly; at $\xi = 0$ the source is zero. Integrating in the exact source metric gives

$$\|R_M^{1/2} E_I\|_{\text{HS}}^2 = \int_I n_{M,1}(\xi) \langle e_\xi, R_M e_\xi \rangle \frac{d\xi}{\pi} \leq C|I|(1 + \log U).$$

This trace bounds the full operator norm, retaining every profile combination. Variational restriction gives the last assertion. $\square$

Corollary 10.2 (Unconditional expanded bands). *For every fixed $A > 0$, estimate (40) holds for all measurable $I \subset [0, e^{A\sqrt{\log M}}]$ and all sufficiently large M . Constants and onset may depend on A ; no uniformity for a growing A is asserted.*

Proof. The complete Abel estimate proved in Theorem 7.1, using the classical Vinogradov–Korobov prime-number error [17], gives $\bar{e}_M = O_B(e^{-B\sqrt{U}})$ for every fixed $B > 0$. For $L_M = e^{A\sqrt{U}}$ choose $B > 3A$. Then $L_M^3 \bar{e}_M = O_B(e^{-(B-3A)\sqrt{U}}) \rightarrow 0$ and $\sqrt{L_M}/U = e^{A\sqrt{U}/2}/U \rightarrow \infty$. Theorem 10.1 applies in the original norm and domain. $\square$

Corollary 10.3 (Complete collar and null-source certificate). *Use the null head of Theorem 8.3, set $\kappa = M^{-1}$, and let P_I be the source multiplication projection onto $I \subset [0, L_M]$ for a schedule admitted by Theorem 10.1 (in particular $L_M = e^{A\sqrt{\log M}}$ for fixed $A > 0$). Define*

$$N_{\text{out}} = E_M(I - P_I)E_M^*, \quad \tau_{\text{out}} = \text{Tr } N_{\text{out}} - \text{Tr}(V_Z^* N_{\text{out}}^2 V_Z), \quad c_I = C|I|(1 + \log \log M).$$

If

$$(\delta_M + \delta_M^2/\kappa)\ell + \left(\sqrt{c_I} + \sqrt{\tau_{\text{out}}/\kappa}\right)^2 \leq \eta, \quad (42)$$

then $G_M(\kappa) \preceq (1 + \eta)I$, and on the entire original domain

$$q_H[h] \geq -\left[\delta_M + \kappa + \frac{\eta B_{M,1}}{1 + \eta}\right] \|h\|^2.$$

In particular, a cofinal family satisfying (42) with $\eta = o(M^{-1/2})$ gives vanishing original-norm allowances and RH.

Proof. Let $Q = I - JJ^*$ as above. Without commuting Q and P_I , cyclicity gives

$$\|E_M(I - P_I)Q\|_{\text{HS}}^2 = \text{Tr } N_{\text{out}} - \text{Tr}(V_Z^* N_{\text{out}}^2 V_Z).$$

Here V_Z uses the full N_M normalization. The triangle inequality in actual inverse energy gives

$$\|QG_M Q\| \leq \left(\|R_M^{1/2} E_M P_I Q\| + \|R_M^{1/2} E_M (I - P_I) Q\|\right)^2 \leq \left(\sqrt{c_I} + \sqrt{\tau_{\text{out}}/\kappa}\right)^2.$$

Combine this with (32) and the positive-block estimate in Theorem 8.3. The original-domain lower bound follows from Corollary 4.2, with complete negative source and no finite-source omission. Since $B_{M,1} = O(\sqrt{M})$ and $\delta_M \rightarrow 0$, the stated rate gives the endpoint. $\square$

All frequencies outside I , including those above L_M , occur in τ_{out} . The mixed collar/exterior cost is the cross term in the square in (42). Thus the displayed criterion specifies a complete quantitative construction, with growing-rank conditioning and the complete trace remainder evaluated in the same source metric.

11 Feasibility and complete exterior control

The preceding certificates retain exact signed information. This section identifies the costs of two simpler bounds and locates the earlier scalar complementary criterion within the complete construction.

Proposition 11.1 (Cost of the positive majorant). *For the explicit E_0 in the proof above, with $m \sim b \log M / \log 10$,*

$$\|E_0^* R_M (M^{-b}) E_0\| \geq \frac{c_b \sqrt{m}}{\sqrt{M} (\log m)^3}$$

for all sufficiently large M . Thus this positive-majorant inverse cost is not $o(M^{-1/2})$.

Proof. Choose compact smooth odd f with $\int f(u) \sinh(u/2) du = 0$ but $\int u f(u) du \neq 0$. Two odd bump pairs construct such an f , since u and $\sinh(u/2)$ are linearly independent on a positive interval. Then $y = f/s$ is legal, $|\hat{f}(\xi)| \geq c|\xi|$ near zero, and for large M , $E_0^* y = T_m f$. On $|\xi| \leq 1/(2 \log m)$ every $\cos(\xi \log n) > 1/2$ for $n \leq m$. The prime number theorem gives $\sum_{n \leq m} \Lambda(n)/\sqrt{n} \geq c\sqrt{m}$; boundedness of a near zero therefore yields $v_m(\xi) \geq c\sqrt{m}$ on this interval. Plancherel gives $\|E_0^* y\|^2 \geq c\sqrt{m}/(\log m)^3$. The full bound $\|N_M\| \leq 144\sqrt{M}$ and the signed identity imply $P_M[y] + M^{-b}\|y\|^2 = O_y(\sqrt{M})$. The inverse variational quotient proves the assertion. $\square$

This concerns the particular positive majorant, not the signed covariance excess. It explains why the next construction retains the exact null remainder rather than trying to sharpen this majorant by constants alone.

The technical supplement's arithmetic-gain section supplies a sharper compact arithmetic price, a complete residual optimizer, and matrix tests for whether the resulting improvement covers a signed certificate deficit. These retain the same receiving core and every mixed term.

11.1 The complete exterior in the actual inverse metric

The next estimate identifies source directions that a constructive null head must capture globally. Put $R_M = (P_M + \kappa I)^{-1}$ and $G_M = E_M^* R_M E_M$ on the base original polar recipient.

Proposition 11.2 (Actual exterior payments). *Let $F = F_M \rightarrow \infty$ satisfy*

$$F^4/\sqrt{M} \rightarrow 0, \quad F^3 \bar{e}_M \rightarrow 0,$$

and let Π_F be source multiplication by $1_{[F, 2F]}$. Uniformly for $0 < \kappa \leq 1$, for all sufficiently large M ,

$$\text{Tr}(E_M \Pi_F E_M^*) \geq c\sqrt{M}/F, \tag{43}$$

$$\text{Tr}(\Pi_F G_M \Pi_F) \geq c/F, \quad \|\Pi_F G_M \Pi_F\| \geq c/F^2. \tag{44}$$

These conclusions hold, in particular, for $F = \exp(A\sqrt{\log M})$ with any fixed $A > 0$.

Proof. Write $U = \log M$, $B(\xi) = 8/(1 + 4\xi^2)$ and $e_\xi = Q_b[s \sin(\xi \cdot)]$. The complete Abel formula and $a(\xi) \leq 17\xi^2$ give on $[F, 2F]$

$$\sup \left| \frac{m_M(\xi)}{\sqrt{M}B(\xi)} + \cos(U\xi) \right| \leq C(F^4/\sqrt{M} + F^3\bar{e}_M) = o(1).$$

Thus $n_M \leq C\sqrt{M}/F^2$ there, and $n_M \geq c\sqrt{M}/F^2$ on its subset where $\cos(U\xi) \geq 1/2$. That subset occupies at least a quarter of each fixed-length interval for large M , and at least $F/4$ of $[F, 2F]$.

The exact projected source Gram is

$$\begin{aligned} \langle e_\xi, e_\zeta \rangle &= \frac{1}{2} \int s^2 \cos((\xi - \zeta)u) du - \frac{1}{2} \int s^2 \cos((\xi + \zeta)u) du \\ &\quad - \frac{\langle s \sin(\xi \cdot), b \rangle \langle b, s \sin(\zeta \cdot) \rangle}{\|b\|^2}. \end{aligned}$$

Riemann–Lebesgue therefore gives $\|e_\xi\|^2 \rightarrow \|s\|_2^2/2 > 0$. Differentiating se_ξ shows $\|(se_\xi)'\|^2 \leq C(1 + \xi^2)$. The fixed hard-cutoff decomposition of the original form is $H = A + B_0$ with B_0 bounded and $A[h] = \int a(\xi) |\widehat{sh}(\xi)|^2 d\xi/(2\pi)$. Consequently these sources belong to the original form domain and $q_H[e_\xi] \leq C(1 + \xi^2)$. Since $P_M = N_M + H - W_M$ and $\|N_M\| \leq 144\sqrt{M}$,

$$P_M[e_\xi] + \kappa\|e_\xi\|^2 \leq C\sqrt{M}, \quad \langle e_\xi, R_M e_\xi \rangle \geq \frac{\|e_\xi\|^4}{P_M[e_\xi] + \kappa\|e_\xi\|^2} \geq c/\sqrt{M}.$$

Here $F^2 \leq \sqrt{M}$ eventually. Integrating these estimates against $n_M(\xi)d\xi/\pi$ proves the raw and inverse-weighted trace bounds.

The norm bound requires a separate coherent source. Choose fixed $\ell > 0$ so small that $\int s^2 \cos(tu) du \geq 3\|s\|^2/4$ for $|t| \leq \ell$. The other two Gram terms tend uniformly to zero on $[F, F + \ell]^2$, so $\langle e_\xi, e_\zeta \rangle \geq c_0 > 0$ there for large M . Let $Z = [F, F + \ell] \cap \{\cos(U\xi) \geq 1/2\}$, with $\mu(Z) = \int_Z d\xi/\pi$, and use the unit source $f = 1_Z/\sqrt{\mu(Z)}$. Its coefficients are nonnegative and $\mu(Z)$ is bounded above and below by positive constants. For $v = E_M f$, the full Gram and differentiated Bochner integral give

$$c\sqrt{M}/F^2 \leq \|v\|^2 \leq C\sqrt{M}/F^2, \quad \|(sv)'\| \leq CM^{1/4}.$$

In particular v belongs to the original form domain and $q_H[v] \leq C\sqrt{M}$. The complete positive energy satisfies

$$P_M[v] + \kappa\|v\|^2 \leq 144\sqrt{M}\|v\|^2 + C\sqrt{M} + (\delta_M + \kappa)\|v\|^2 \leq CM/F^2.$$

The inverse variational test now yields $\langle f, G_M f \rangle \geq \|v\|^4/(P_M[v] + \kappa\|v\|^2) \geq c/F^2$. No trace-to-norm inference is used. Finally Theorem 7.1 proves $\bar{e}_M = O_D(e^{-D\sqrt{U}})$ for every fixed $D > 0$. For fixed A , choose $D > 3A$ to admit $F = e^{A\sqrt{U}}$; $F^4/\sqrt{M} \rightarrow 0$ is immediate. $\square$

If a source head is supported below F , its orthogonal complement acts identically on $[F, 2F]$, so its complete inverse remainder has norm at least c/F^2 and trace at least c/F . The same holds for the remainder of a positive-contraction head supported there. In the displayed stretched-exponential range these bounds exceed $M^{-1/2}$ asymptotically. Thus a low-band head cannot pay its entire positive remainder as $o(M^{-1/2})$, even in the actual inverse metric. Global null-polynomial heads and the directional Schur criterion remain available: these are lower bounds on a positive complement, not on $\|G_M\| - 1$.

11.2 The scalar mixed-band criterion

Proposition 11.3 (Complete mixed-band cap). *For the same positive inverse $R_P = (P + \kappa I)^{-1}$, split the exact observation source as $E = [E_o, E_c]$. If its two diagonal caps are $E_o^* R_P E_o \preceq aI$ and $E_c^* R_P E_c \preceq bI$, and the full mixed block has norm at most ζ , then*

$$\|E^* R_P E\| \leq \frac{1}{2}[a + b + \sqrt{(a - b)^2 + 4\zeta^2}] \leq a + b \quad \text{when } \zeta \leq \sqrt{ab}.$$

Proof. Positive-form Cauchy–Schwarz pays the mixed block by $\sqrt{ab}$. More generally bound each two-vector quadratic form by the real matrix $\begin{bmatrix} a & \zeta \\ \zeta & b \end{bmatrix}$ and compute its largest eigenvalue. All mixed observation terms are retained. $\square$

Corollary 11.4 (A complementary-band cofinal criterion for RH). *Let a_M be the bound in (24) at $\Omega = 1/\log M$, and use any finite genuine-null recipient. If, along an unbounded sequence, the complete complementary observation for the same $\bar{P}_M + M^{-1}I$ has cap*

$$b_M \leq 1 - a_M + M^{-1},$$

then on the original domain

$$q_H[h] \geq - \left[d_{M,1} + \frac{1}{M} + \frac{B_{M,1}}{M+1} \right] \|h\|^2,$$

and RH holds. A paid mixed block can instead use the sharper condition in Proposition 11.3 at target $1 + 1/M$.

Proof. The proposition gives the complete cap $1 + 1/M$ including all mixed terms. Apply the bounded-negative absorption of Corollary 4.2 with $k = 1$, $\tau = 0$, $\eta = \kappa = 1/M$. Here $B_{M,1} = O(\sqrt{M})$, so the displayed allowance tends to zero; Theorem 2.2 applies. $\square$

11.3 Why bounded null rank does not fund the scalar criterion

The sufficient scalar comparison in Corollary 11.4 loses directional covariance information. The following result gives its precise rank limitation at the displayed schedule.

Proposition 11.5 (Bounded-rank scalar obstruction). *Put $\Omega = 1/\log M$, $\kappa = M^{-1}$ and split the complete source into E_o on $[0, \Omega]$ and its entire complement E_c . For the same-form recipient obtained by removing a genuine-null space K , put*

$$a = \|E_o^* R_M E_o\|, \quad b = \|E_c^* R_M E_c\|, \quad \chi = \|E_o^* R_M E_c\|.$$

For every fixed integer $D \geq 0$ there are positive constants c_D, C_D and a finite threshold, uniform over all K with $\dim K \leq D$, such that

$$\chi^2 \geq c_D \Omega^{4D+3}, \quad b \geq 1 - C_D \Omega^{4D+7} - C_D M^{-3/2}.$$

Consequently the scalar matrix $\begin{bmatrix} a & \chi \\ \chi & b \end{bmatrix}$ is not bounded by $(1 + 1/M)I$ for all sufficiently large M . Any cofinal realization of that scalar target requires the null dimension to eventually exceed every fixed D .

Proof. Let $L_D = \text{span}\{z_0, \dots, z_D\}$, with the genuine nulls of Lemma 8.1. Dimension gives a unit $z = \sum_{i=0}^D \alpha_i z_i \in L_D \cap K^\perp$. The derivative Fourier rule and $\widehat{\psi}(0) > 0$ show, near zero,

$$|\widehat{sz}(\xi)| \geq c\xi \left| \sum_{i=0}^D (-1)^i \alpha_i \xi^{2i} \right|.$$

The complete Abel formula gives $n_{M,1} \geq c\sqrt{M}$ on $[0, \Omega]$ for large M , since $\cos((\log M)\xi) \geq \cos 1 > 0$. The positive moment matrix $G_D = (1/(2i + 2j + 3))_{0 \leq i, j \leq D}$ gives

$$\int_0^\Omega \xi^2 \left| \sum_{i=0}^D (-1)^i \alpha_i \xi^{2i} \right|^2 d\xi \geq \lambda_{\min}(G_D) \Omega^{4D+3} \|\alpha\|^2.$$

The fixed physical Gram of L_D bounds $\|\alpha\|$ below. Therefore $N_o[z] \geq c_D \sqrt{M} \Omega^{4D+3}$, uniformly in the moving K . All constants here are for fixed D ; no uniform conditioning in D is used.

Write $A_o = E_o^* R_M E_o$ and $C_{oc} = E_o^* R_M E_c$. The exact null identity gives

$$C_{oc} E_c^* z = (I - A_o) E_o^* z + E_o^* R_M (W_M - \kappa I) z.$$

Since $R_M^2 \preceq \kappa^{-1} R_M$,

$$\chi \sqrt{N_c[z]} \geq (1 - a) \sqrt{N_o[z]} - (d_{M,1} + \kappa) \sqrt{a/\kappa}.$$

The actual origin upper is $a \leq C\Omega^3(1 + \log \log M)^3$. Dividing its error term by the lower bound for $\sqrt{N_o[z]}$ gives at most $C_D M^{-3/4} \Omega^{-2D} (1 + \log \log M)^{3/2} + o(1)$, which tends to zero. The complete bound $N_c[z] \leq 144\sqrt{M}$ then proves the stated lower for χ^2 . If $N_c[z] = 0$, the displayed identity already contradicts its positive right side for large M .

For the complementary lower choose a unit $z' \in \text{span}\{z_{D+1}, \dots, z_{2D+1}\} \cap K^\perp$. Its transform vanishes to order at least $2D + 3$ at zero, so $N_o[z'] \leq C_D \sqrt{M} \Omega^{4D+7}$. The averaged quadratic mean proved in Theorem 7.1 is positive definite on this fixed finite-dimensional span. Polarization makes its convergence uniform on that span's unit sphere; hence $N_M[z'] \geq c'_D \sqrt{M}$ for every allowed moving choice. The exact null quotient yields

$$b \geq \frac{N_c[z']}{P_M[z'] + \kappa} \geq 1 - C_D \Omega^{4D+7} - C_D M^{-3/2}.$$

Finally put $t = 1 + 1/M$. If $b > t$ the scalar target fails immediately. Otherwise its necessary determinant inequality would imply

$$\chi^2 \leq (t - a)(t - b) \leq 2\{M^{-1} + C_D \Omega^{4D+7} + C_D M^{-3/2}\} = o(\Omega^{4D+3}),$$

contradicting the proved lower. □

The proposition concerns scalarization of the blocks. It neither rejects the operator-valued Schur criterion nor establishes sufficiency of growing null rank. The regularized whole-source construction retains the signed correlations that this scalar comparison discards.

11.4 Spectral tests for a growing genuine-null head

The directional construction has an actual compressed-spectrum test that retains the physical lift cost. It does not replace that spectrum by the unrestricted mean operator.

Proposition 11.6 (Null-compressed spectral and lift tests). *Let $\mathcal{K}$ be the closed physical span of the genuine nulls and $N_{\mathcal{K}}$ the compression of N_M to $\mathcal{K}$. For a rank- r null-valued map V with $V^*N_MV = I$, put*

$$J = E_M^*V, \quad Q = I - JJ^*, \quad \ell = \|V^*V\|, \quad b_c = \|QG_MQ\|, \quad g = \|G_M\|.$$

If $b_c \leq q < 1$, then

$$\lambda_{r+1}(N_{\mathcal{K}}) \leq \frac{q(\kappa + \delta_M)}{1 - q}. \quad (45)$$

For every eigenvalue $\lambda > 0$ of $N_{\mathcal{K}}$,

$$b_c \geq \left[\sqrt{(1 - (\kappa + \delta_M)/\lambda)_+} - \sqrt{g\lambda\ell} \right]_+^2. \quad (46)$$

Proof. Closedness of H implies $\mathcal{K} \subset \text{Dom } H$ and $H|_{\mathcal{K}} = 0$. For $z \in \mathcal{K}$ with $E_M^*z \perp \text{ran } J$, the inverse variational test at z gives

$$\langle E_M^*z, G_ME_M^*z \rangle = \langle N_Mz, R_MN_Mz \rangle \geq \frac{N_M[z]^2}{P_M[z] + \kappa\|z\|^2}.$$

Genuine nullity gives $P_M[z] = N_M[z] - \langle z, W_Mz \rangle$. Putting $t = N_M[z]/\|z\|^2$, we obtain $b_c \geq t/(t + \kappa + \delta_M)$ whenever $t > 0$. There are at most r source-orthogonality conditions on z . Compact min-max on $N_{\mathcal{K}}$ proves (45), with eigenvalue multiplicities retained.

For the second bound choose a unit compressed eigenvector h at λ and put $f = E_M^*h$, so $\|f\|^2 = \lambda$. Since V is $\mathcal{K}$ -valued, the compressed eigenidentity gives $V^*N_Mh = \lambda V^*h$, without assuming invariance of $\mathcal{K}$ under N_M . Hence

$$\|JJ^*f\|^2 \leq \lambda^2\ell, \quad \langle f, G_Mf \rangle \geq \frac{\lambda^2}{\lambda + \kappa + \delta_M} \geq \lambda - (\kappa + \delta_M).$$

The triangle inequality in the positive G_M seminorm yields

$$\sqrt{\langle f, G_Mf \rangle} \leq \sqrt{g} \lambda \sqrt{\ell} + \sqrt{b_c \lambda}.$$

Division by $\sqrt{\lambda}$ and positive parts prove (46). $\square$

At $\kappa = M^{-1}$, a fixed complementary gap requires the head to capture every null-compressed eigenvalue above a constant multiple of M^{-1} . If actual eigenvalues satisfy $(\kappa + \delta_M)/\lambda_M \rightarrow 0$ and $\lambda_M \ell_M \rightarrow 0$, then Theorem 7.1 and (46) give $\liminf b_c \geq 1$. In particular, an actual eigenvalue comparable to $M^{-1/2}$ would exclude a fixed gap with $\ell_M = o(\sqrt{M})$. The spectral and lift tests are evaluated jointly with the signed complement estimate in the complete source metric.

12 A cofinal finite-response criterion for RH

The complete covariance criterion admits a second, explicit supported-form reduction. It keeps the original polar condition and physical norm while replacing the infinite family of positive Green responses by logarithmically many features. The receiving complement remains infinite and has a proved floor; its exact inverse occurs in the finite Schur matrix below.

Use the odd-half unitary identification with $L^2(0, L)$ and put

$$p(x) = \sinh(x/2), \quad \mathcal{F}_L = p^\perp, \quad e_q(x) = \sqrt{2/L} \sin(k_q x), \quad k_q = (q + 1/2)\pi/L.$$

Let $D_L e_q = a(k_q) e_q$. The complete supported archimedean form is $D_L + R_L$, where

$$R_L = \sum_{j \geq 0} r_j \otimes r_j, \quad r_j(x) = \frac{e^{-c_j(L-x)} - e^{-c_j(L+x)}}{\sqrt{1 + e^{-2c_j L}}}, \quad c_j = 2j + \frac{1}{2}.$$

The Green identity, its original-domain closure, and the estimates used here are proved in Appendix B. Write $R_{L,J}$ for the first J responses and $V_L = m_0 - C_L^{\text{ar}}$ for the complete bounded arithmetic operator on the supported interval, including every active prime power. Thus $\|V_L\| \leq B_L := 6 + 8(\log 2)e^L$ by (10). Restrictions of these forms to $\mathcal{F}_L$ are understood throughout. Precisely, if $I = (-L, L)$ and $\mathcal{U}_L$ is the odd-half unitary map,

$$C_L^{\text{ar}} = \mathcal{U}_L \left[1_I \sum_{2 \leq n < e^{2L}} \frac{\Lambda(n)}{\sqrt{n}} (S_{\log n} + S_{-\log n}) 1_I \right] \mathcal{U}_L^*.$$

This transport retains the reflected channel as well as same-side shifts; the endpoint overlap has measure zero. Consequently the complete supported profile form $D_L + R_L + V_L + M^{-1}s^{-2}$ is exactly the original $H + M^{-1}I$ under $f = sh$ and this unitary map.

Theorem 12.1 (Cofinal finite-response criterion). *For integers $M \geq 3$ define*

$$\begin{aligned} L_M &= \frac{1}{2} \log(10 \log M / \pi) + 2, & \kappa_M &= M^{-1}, \\ B_M^{\text{ar}} &= 6 + 8(\log 2)e^{L_M}, & J_M &= \lceil \frac{1}{2} \log M \rceil, \\ N_M &= \lceil L_M e^{2B_M^{\text{ar}}} / \pi \rceil. \end{aligned}$$

Let U_M be a profile isometry onto $P_{\mathcal{F}_{L_M}} \text{span}(e_0, \dots, e_{N_M-1})$, and let Q_M be its orthogonal complement in $\mathcal{F}_{L_M}$. Form the complete finite-response operator

$$S_{0,M} = D_{L_M} + R_{L_M, J_M} + V_{L_M} + M^{-1}s^{-2}.$$

Set $T_{0,M} = S_{0,M}|_{Q_M}$ and

$$\mathbf{K}_{0,M} = U_M^* S_{0,M} U_M - (Q_M S_{0,M} U_M)^* T_{0,M}^{-1} (Q_M S_{0,M} U_M). \quad (47)$$

Then $T_{0,M} \geq \frac{1}{3} A_{Q_M} > 0$, where A_{Q_M} is the form restriction of $D_{L_M} + B_M^{\text{ar}} + M^{-1}s^{-2}$. The Riemann hypothesis is equivalent to $\mathbf{K}_{0,M} \succeq 0$ along some unbounded sequence of integers M . Under RH these matrices are strictly positive for every integer $M \geq 2^{20}$. Moreover $N_M = M^{o(1)}$ and $J_M = O(\log M)$.

Proof. Orthogonality to the projected modes implies orthogonality to the original first N_M DN modes. Since $a(k) \geq \log k$ for $k \geq 1$, the diagonal form on this complement is at least $2B_M^{\text{ar}}$. The signed arithmetic is bounded below by $-B_M^{\text{ar}}$ and $R_{L_M, J_M} \succeq 0$. Comparing coefficients gives $T_{0,M} \geq \frac{1}{3} A_{Q_M}$, with every receiving direction retained. The complete form Schur identity therefore makes $\mathbf{K}_{0,M} \succeq 0$ equivalent to $S_{0,M} \succeq 0$.

The omitted responses are positive, so this matrix inequality implies $q_H[h] + M^{-1}\|h\|_2^2 \geq 0$ for every original polar-free test supported in $(-L_M, L_M)$. Since $L_M \rightarrow \infty$, every smooth compact test is admitted eventually. Taking the unbounded sequence and then the original form closure proves Weil positivity and hence RH by the odd converse.

Conversely, the complete relative-response estimate in Appendix B gives $0 \leq R_{L_M} - R_{L_M, J_M} \leq M^{-2}(D_{L_M} + B_M^{\text{ar}} + M^{-1}s^{-2})$. Its full physical form sandwich, with $\|s\|_\infty < 3$, yields under RH

$$s S_{0,M} s \geq \left[\frac{1 - M^{-2}}{M} - \frac{18 B_M^{\text{ar}}}{M^2} \right] I > 0$$

For an explicit admission, $B_M^{\text{ar}} < 6 + 120\sqrt{\log M}$ by $\log 2 < 1$, $e^4 < 64$ and $\pi > 3$. This upper divided by M decreases for $M \geq e$. At $M = 2^{20}$ it is below $606/2^{20}$, and $2^{-40} + 10908/2^{20} < 1/2$. The displayed lower is consequently at least $1/(2M)$ for every $M \geq 2^{20}$, proving strict positivity of the Schur matrix. Finally $B_M^{\text{ar}} = O(\sqrt{\log M})$, so $\log N_M = O(\sqrt{\log M})$ and the stated resource bounds follow. $\square$

The matrix in (47) retains the *exact inverse of the full receiving operator*; it is not a principal compression. Theorem 12.1 isolates its sign as the remaining condition in this route. Appendix B also bounds the complete core-mixed-receiving change caused by the response reduction by $M^{-2+o(1)}$ in physical core coordinates, and preserves original-norm $o(M^{-3/2})$ allowances directly at the full-form level. The core coordinates do not make the transported receiving complement physically orthogonal; quantitative allowance tests shift the complete physical form before taking its Schur complement. The finite-sine alternative in Proposition B.6 retains the same cofinal RH equivalence, uses only finite DN combinations, and has raw profile Gram condition below $4L_M = O(\log \log M)$. For that core the joint Schur error is $O(\sqrt{\log M}/M^2)$ in physical core coordinates.

A The odd original-domain criterion

We give the needed realization-specific reduction to the classical Weil criterion [12, 14]. For $F(z) = \int f(u)e^{zu} du$, the complete explicit formula has zero-side form

$$W[f, g] = \sum_{\rho} F(\rho - 1/2) \overline{G(1/2 - \bar{\rho})}. \quad (48)$$

The sum counts multiplicities. On odd real tests $W[f] = -\sum_{\rho} F(\rho - 1/2)^2$. The physical form in (5) equals $W[f] + 2|F(1/2)|^2$ on odd tests; the polar term vanishes on $\mathcal{K}$.

Lemma A.1 (Odd original-domain criterion). *Nonnegativity of q_H on V is equivalent to RH.*

Proof. The theta Mellin identity, in the normalization (3), is

$$M\psi(z) = \xi(1/2 + z), \quad M\psi'(z) = -z\xi(1/2 + z), \quad M\psi'(1/2) = -1/4.$$

Indeed, with $t = e^{2u}$ and $A(t) = \sum_{n \geq 1} e^{-\pi n^2 t}$, one has $\psi = e^{u/2}(4t^2 A'' + 6tA')$. Two integrations by parts on $0 < t < \infty$, initially for $\Re s > 1$, give

$$M\psi(s - 1/2) = \frac{s(s-1)}{2} \int_0^\infty A(t)t^{s/2-1} dt = \xi(s).$$

Theta inversion gives the entire continuation. For any compact smooth real odd f , put $g = f + 4F(1/2)\psi'$. Its polar moment is zero and its transform agrees with F at every nontrivial zero, so $W[g] = W[f]$.

Here g/s belongs to the original minimal domain. To see this directly, take even smooth cutoffs χ_R and a fixed compact real odd b with $Mb(1/2) = 1$. Set $g_R = \chi_R g - M(\chi_R g)(1/2)b$. Then g_R is in the exact polar compact core. Complete theta derivative envelopes give

$$\|(g_R - g)/s\|_2 \rightarrow 0, \quad \|(g_R - g)'\|_2 \rightarrow 0.$$

For example $|\psi'| \leq 40e^{|u|/2}r^3e^{-r}$ and the lower bound (6) give an integrable upper $800e^{|u|}r^4e^{-r}$ for $|\psi'/s|^2$. The derivative atom polynomials are $-8r^3 + 30r^2 - 15r$ and $16r^4 - 112r^3 + 165r^2 -$

$75r/2$. Their absolute coefficient bounds for $r \geq \pi > 3$, and consecutive atom ratios, give $|\psi''| \leq 160e^{|u|/2}r^4e^{-r}$. These complete tails pay both limits. Bounded cutoff derivatives and the vanishing compact moment correction preserve them. The bound $a \leq 17\xi^2$ now gives convergence in (4).

The same tails give exponential-weighted L^1 convergence of $g_R - g$ and its first derivative. Integration by parts therefore gives

$$\sup_{|\Re z| \leq 1/2} (1 + |\Im z|)|M(g_R - g)(z)| \longrightarrow 0.$$

The zero count implies $\sum_{\rho} (1 + |\Im \rho|)^{-2} < \infty$, so the complete quadratic zero sum converges as well. Passing the compact explicit formula to the limit yields $q_H[g/s] = W[g] = W[f]$. Thus positivity on V implies $W[f] \geq 0$ for every real odd compact smooth f .

Suppose an off-line centered zero $\sigma_0 = \delta + i\gamma$ exists, with $\delta, \gamma > 0$ by symmetry. Nontrivial real zeros are absent because the alternating eta series gives $\zeta(s) < 0$ for real $0 < s < 1$. Fix $\Gamma > \gamma + 1$. An even real polynomial P annihilates every zero orbit in $|\Im \sigma| \leq \Gamma$ except the quartet of σ_0 , and $P(\sigma_0) \neq 0$. Choose real a, b with $iP(\sigma_0)(a + b\sigma_0^2) = 1$, possible because $\Im \sigma_0^2 \neq 0$. Put

$$F_T(z) = iP(z)(a + bz^2)[e^{T(z-i\gamma)^2} - e^{T(z+i\gamma)^2}], \quad T \geq 1.$$

This entire function is odd and real type. It is the bilateral Laplace transform of a real odd Gaussian multiplied by polynomial and trigonometric factors. At each fixed T , even compact cutoffs of that physical function converge in exponential Schwartz seminorms. Repeated integration by parts gives transform convergence with every polynomial vertical decay uniformly on $|\Re z| \leq 1/2$. The zero count $N_{\zeta}(Y) = O(Y \log(Y + 2))$ therefore permits dominated convergence in (48), so $W[f_T] \geq 0$.

At σ_0 ,

$$e^{-T\delta^2}F_T(\sigma_0) = 1 - e^{-4T\gamma^2 + 4iT\delta\gamma} \longrightarrow 1.$$

The target quartet, of multiplicity m , contributes $-4m\Re F_T(\sigma_0)^2 \sim -4me^{2T\delta^2}$. All other zeros in the finite window are annihilated. For exterior zeros put $d_{\rho} = ||\Im \sigma_{\rho}| - \gamma| > 1$. A fixed polynomial envelope bounds their squared transform by

$$C(1 + |\Im \sigma_{\rho}|)^{2D}e^{2T(1/4 - d_{\rho}^2)} \leq Ce^{-3T/2}(1 + |\Im \sigma_{\rho}|)^{2D}e^{-(d_{\rho}^2 - 1)}.$$

The full remaining zero sum is $O(e^{-3T/2})$ by the same zero-count bound. The negative quartet dominates, a contradiction. This proves RH. Conversely, RH makes (48) a sum of squared absolute transforms on the polar core; original form closure extends positivity to V . $\square$

B Complete Green reduction and cofinal Schur control

This section proves the supported-form reduction used by the cofinal RH criterion in the main paper. The positive exponential Green construction is inherited from Li [1]; the contribution here is its complete physical-ridge relative error, joint Schur budget, and cofinal application. All arithmetic sums below include every active prime power. Optional critical carriers are empty, and the original polar restriction is retained.

B.1 Exact realization and domain

On the odd-half unitary space $L^2(0, L)$ put

$$c_j = 2j + \frac{1}{2}, \quad k_q = (q + \frac{1}{2})\pi/L, \quad e_q(x) = \sqrt{2/L} \sin(k_q x), \quad D_L e_q = a(k_q)e_q.$$

Lemma B.1 (Complete Green realization). *The minimal zero-extended supported archimedean form equals $D_L + R_L$, where*

$$R_L = \sum_{j \geq 0} r_j \otimes r_j, \quad r_j(x) = \frac{2 \sinh(c_j x)}{\sqrt{e^{2c_j L} + 1}}, \quad (49)$$

$$0 \preceq R_L, \quad \|R_L\| \leq C_L := L + \pi/2. \quad (50)$$

The associated operator domain is $\text{Dom } D_L$; finite DN spans are an operator graph core.

Proof. The positive digamma difference expansion is

$$a(\xi) = \sum_{j \geq 0} \left(\frac{2}{c_j} - \frac{2c_j}{c_j^2 + \xi^2} \right).$$

For the Dirichlet half-line and Dirichlet–Neumann interval Green functions,

$$G_{\infty, c}(x, y) = \frac{e^{-c|x-y|} - e^{-c(x+y)}}{2c},$$

$$G_{\text{DN}, c}(x, y) = \frac{\sinh(c \min(x, y)) \cosh(c(L - \max(x, y)))}{c \cosh(cL)},$$

direct subtraction gives $2c(G_{\text{DN}, c} - G_{\infty, c}) = r_c \otimes r_c$. For $z = 2L - x - y > 0$, the sum kernel is bounded by

$$\sum_{j \geq 0} e^{-c_j z} = \frac{e^{-z/2}}{1 - e^{-2z}} \leq 1 + \frac{1}{2z}.$$

The constant kernel has norm L ; the Carleman kernel $[2(u + v)]^{-1}$, $u = L - x$, has Schur bound $\pi/2$ with weight $u^{-1/2}$. All positive partial sums therefore converge strongly to the bounded R_L in (50).

The identity first holds on smooth compact odd tests. To identify its closure, extend a finite DN combination oddly across zero and evenly across L , with period $4L$. Cut it off in endpoint layers of width ε and mollify preserving parity. The reflected error has H^t norm $O(\varepsilon^{1/2-t})$ for every fixed $0 < t < 1/2$: rescale each bounded smooth boundary bump in the fractional Sobolev norm. The bound $a(k) \leq C_t(1 + k^2)^t$, from the logarithmic symbol estimate, shows convergence in the diagonal form norm. Conversely smooth compact tests have rapidly decreasing DN coefficients. Finite sequence truncation is a form core of D_L ; hence the original minimal closure is precisely $D_L + R_L$. Boundedness of R_L then gives its operator domain and graph core. This argument retains the supported realization, rather than silently imposing a different endpoint domain. $\square$

At fixed L , the original positive smooth s has positive minimum. Thus $s^{\pm 1}$ are bounded on this interval. The original physical congruence is exactly $S_h = sS_f s$, with physical norm $\|h\|_2$. The polar condition becomes $f \perp p$, $p(x) = \sinh(x/2)$, in profile coordinates. Restrict the closed forms to this subspace, retaining their minimal supported closure; one can enforce the constraint on the smooth approximants by subtracting a fixed compact test with nonzero polar pairing. Finite bounded arithmetic and ridge terms preserve these domain statements. No condition number is assumed uniform as L grows.

B.2 A whole relative-response bound

Let $R_{L,J} = \sum_{j < J} r_j \otimes r_j$ and $E_J = R_L - R_{L,J}$. First on ambient $L^2(0, L)$, for $B \geq 1$, $\kappa > 0$ define the closed positive reference form

$$A = D_L + B + \kappa s^{-2}.$$

Proposition B.2 (Complete relative tail). *For $0 < \eta < L$, let $\sigma_\eta^2 = \text{ess sup}_{L-\eta < x < L} s(x)^2$. Then $0 \leq E_J \leq \epsilon A$, where*

$$\epsilon = \left[\sqrt{C_L \sigma_\eta^2 / \kappa} + \sqrt{\frac{e^{-2c_J \eta}}{2B c_J (1 - e^{-4\eta})}} \right]^2. \quad (51)$$

The inequality includes every omitted response and survives restriction to the original polar-free subspace.

Proof. Write $P_e = 1_{(L-\eta, L)}$ and $P_i = I - P_e$. Inverse order gives $A^{-1} \preceq (B + \kappa s^{-2})^{-1}$, so

$$\|E_J^{1/2} P_e A^{-1/2}\|^2 \leq C_L \|P_e A^{-1} P_e\| \leq C_L \sigma_\eta^2 / \kappa.$$

For $d = L - x \geq \eta$, the exact feature obeys $0 \leq r_j(x) \leq e^{-c_j d}$. Tonelli and positivity give

$$\text{Tr}(P_i E_J P_i) \leq \sum_{j \geq J} \frac{e^{-2c_j \eta}}{2c_j} \leq \frac{e^{-2c_J \eta}}{2c_J (1 - e^{-4\eta})}.$$

Using $\|A^{-1/2}\| \leq B^{-1/2}$ bounds the interior factor. Add $E_J^{1/2} P_e A^{-1/2}$ and $E_J^{1/2} P_i A^{-1/2}$ and square their norm bound. This proves (51) as a form inequality. The discontinuous spatial multipliers are used only as bounded L^2 operators on inverse square roots; they are not assumed to preserve the form domain. Apply the positive-series arguments first to partial sums and pass monotonically. No unweighted whole-tail norm convergence is used. $\square$

Corollary B.3 (Explicit logarithmic response schedule). *Let $U = \log M$, $M \geq e$, and*

$$L = \frac{1}{2} \log(10U/\pi) + 2, \quad \kappa = M^{-1}, \quad J = \max(1, \lceil U/2 \rceil).$$

Then $0 \leq E_J \leq M^{-2} A$ for every $B \geq 1$.

Proof. Use $\eta = 1$ and the main paper's theta envelope $s(u)^2 \leq 169e^{4u - \pi e^{2u}}$. The envelope decreases on $u \geq 0$; since $L > 1$,

$$\sigma_1^2 \leq 169(10e^2/\pi)^2 U^2 M^{-10e^2}.$$

For $U \geq 1$, $C_L \leq 5U$, $e^2 < 8$, $e^2 > 7$ and $\pi > 3$. The square of the first term in (51), divided by M^{-2} , is therefore at most

$$845(80/3)^2 U^3 e^{-67U} \leq 845(80/3)^2 e^{-67} < 2^{-47} < \frac{1}{4}.$$

Here $U^3 e^{-67U}$ decreases for $U \geq 1$, its constant is below 2^{20} , and $e > 2$. For the second squared term use $4J \geq 2U$, $c_J \geq 5/2$ and

$$[5(1 - e^{-4})]^{-1} < \frac{1}{4}.$$

Both squared contributions are below $M^{-2}/4$; their square-root sum is below M^{-1} . All constants are elementary admissions, not sampled floating inequalities. $\square$

B.3 Preserving the original-norm allowance

On the original polar-free supported space write

$$S_f = D_L + R_L + V_L + \kappa s^{-2}, \quad \|V_L\| \leq B, \quad S_{f,J} = S_f - E_J.$$

Since $A \preceq S_f + 2BI$, the complete relative bound implies

$$(1 - \epsilon)S_f - 2\epsilon BI \preceq S_{f,J} \preceq S_f. \quad (52)$$

Congruence by the same s , with $\|s\|_\infty < 3$, gives

$$(1 - \epsilon)S_h - 18\epsilon BI \preceq S_{h,J} \preceq S_h. \quad (53)$$

At the displayed growing window the elementary arithmetic budget is

$$B_M = 6 + 8(\log 2)\sqrt{(10e^4/\pi)\log M} = O(\sqrt{\log M}).$$

Thus $18B_MM^{-2} = o(M^{-3/2})$. Equation (53) proves that existence of original physical allowances $S_h \succeq -\delta_M I$, $\delta_M = o(M^{-3/2})$, is equivalent to existence of such allowances for $S_{h,J}$: take $\delta'_M = (1 - M^{-2})\delta_M + 18B_MM^{-2}$ in one direction and use $S_h \succeq S_{h,J}$ in the other. This uses the full form, not merely a core matrix. Nonvanishing extra finite corrections require a complete two-sided budget; the same power-two deduction then needs that total budget to be $o(\sqrt{M})$.

B.4 An explicit paid core and its energy

Let $P_{\mathcal{F}}$ be the projection onto $p^\perp$ and take

$$N = \lceil Le^{B+\beta}/\pi \rceil, \quad \text{ran } U = P_{\mathcal{F}} \text{span}(e_0, \dots, e_{N-1}), \quad U^*U = I,$$

where $\beta > 0$. A vector in the complementary subspace is orthogonal to each original e_q , $q < N$. Since a is increasing and $a(k) \geq \log k$ for $k \geq 1$, its diagonal energy is at least $B + \beta$. Put $Q = I - UU^*$ on $\mathcal{F}$, and let A_Q denote the form restriction of A to Q . For both the complete and partial-response receiving forms,

$$T \succeq cA_Q, \quad T_0 \succeq cA_Q, \quad c = \frac{\beta}{\beta + 2B}. \quad (54)$$

Indeed $(1 - c)D_L + R_{L,J} + (1 - c)\kappa s^{-2} + V_L - cB$ is bounded below by $(1 - c)\beta - 2cB = 0$. Every response and receiving direction is accounted for.

Lemma B.4 (Projected-polar core control). *For $L \geq 1$, the preceding core has dimension N and*

$$D_{\text{core}} := \|U^*D_L U\| \leq 4L(17k_N^2 + 68)(1/4 + k_N^2)^2. \quad (55)$$

Its projected-column Gram has condition number at most $2L(1/4 + k_N^2)^2$.

Proof. Direct integration gives

$$p_q = \sqrt{2/L}(-1)^q \frac{\frac{1}{2} \cosh(L/2)}{1/4 + k_q^2}, \quad \|p\|^2 = (\sinh L - L)/2.$$

Consequently $p \in \text{Dom } D_L$. With $\bar{p} = p/\|p\|$ and $\delta_N = \|(I - P_N)\bar{p}\|^2$,

$$\delta_N \geq |\langle \bar{p}, e_N \rangle|^2 \geq [2L(1/4 + k_N^2)]^{-1}.$$

The raw projected-column Gram is $I - vv^*$, $v = P_N \bar{p}$ in DN coordinates; its eigenvalues are 1 and $\delta_N > 0$. This proves the dimension and Gram assertions and gives an exact rank-one orthonormalization formula if desired.

The positive digamma series yields $a(k) \leq 2k^2 \sum c_j^{-3} < 17k^2$: its first coefficient is 16 and the rest is at most $\frac{1}{4} \sum_{j \geq 1} j^{-3} \leq 3/8$. The DN Laplacian form identity gives

$$D_L[\bar{p}] \leq 17 \frac{\|p'\|^2}{\|p\|^2} = 17 \frac{\sinh L + L}{4(\sinh L - L)} < 68 \quad (L \geq 1).$$

The last quotient is below 4 because $L/\sinh L$ decreases and $\sinh 1 > 7/6$. For $w \in \text{ran } P_N$, $v = P_{\mathcal{F}} w$ satisfies

$$\|v\|^2 \geq \delta_N \|w\|^2, \quad D_L[v] \leq 2(17k_N^2 + 68)\|w\|^2.$$

Combining proves (55). In particular the projected columns are legal operator-domain vectors. This is the projected DN core, not the smaller intersection of an unprojected finite prefix with $p^\perp$. $\square$

B.5 Joint core, mixed, and receiving error

Let $S = S_0 + E$, where S_0 retains the first J responses, and $0 \preceq E \preceq \epsilon A$. All operators here act on the same polar-free profile space. Their form domains equal $\text{Dom } A^{1/2}$. Put $d = 1 + C_L/B$. The bounds $-A \preceq S_0 \preceq dA$ and (54) hold without a smallness assumption on ϵ . Let

$$H_0 = U^* S_0 U, \quad C_0 = Q S_0 U, \quad Z_0 = T_0^{-1} C_0, \quad K_0 = H_0 - C_0^* T_0^{-1} C_0,$$

and define K analogously for S . Inverse pairings are form-dual pairings when needed; A_Q^{-1} is the inverse of the restricted form, not $QA^{-1}Q$.

Proposition B.5 (Complete joint Schur budget). *With the preceding objects,*

$$0 \preceq K - K_0 \preceq \epsilon(1 + d/c)^2 U^* A U. \quad (56)$$

If $W = U^* s^{-2} U$ is the complete physical core Gram, then

$$\|W^{-1/2}(K - K_0)W^{-1/2}\| \leq \epsilon(1 + d/c)^2 [9(D_{\text{core}} + B) + \kappa]. \quad (57)$$

On the growing window, $\beta = \omega B$ with fixed $\omega > 0$ gives $c = \omega/(\omega+2)$, $N = M^{o(1)}$ and $D_{\text{core}} = M^{o(1)}$. Hence with the explicit response schedule the right side is $M^{-2+o(1)} = o(M^{-3/2})$.

Proof. The relative form norm of S_0 is at most d . Thus $C_0^* A_Q^{-1} C_0 \preceq d^2 U^* A U$. Coercivity of T_0 and form Cauchy–Schwarz applied to $T_0 Z_0 = C_0$ give $Z_0^* A_Q Z_0 \preceq (d/c)^2 U^* A U$. The triangle inequality in the A energy norm, in every coefficient direction, therefore gives

$$(U - Z_0)^* A (U - Z_0) \preceq (1 + d/c)^2 U^* A U.$$

Variational minimization over the entire tail yields $K \succeq K_0$. Testing S at the exact S_0 minimizer $-Z_0$ gives $K \preceq K_0 + (U - Z_0)^* E (U - Z_0)$, proving (56). The square retains all core–response, mixed, and receiving terms together.

Since $W \succeq I/9$,

$$W^{-1/2}U^*AUW^{-1/2} \preceq [9(D_{\text{core}} + B) + \kappa]I.$$

The ridge becomes exactly κI here, with no factor $\max s^{-2}$. At the declared window $B = O(\sqrt{\log M})$, $k_N = \exp(O(\sqrt{\log M}))$ and $L = O(\log \log M)$. Lemma B.4 proves the asserted growth, completing the quantitative comparison. $\square$

The Gram W normalizes physical images of the core columns only. The profile-orthogonal receiving complement generally transports to a nonorthogonal physical complement. Thus (57) is a finite-coordinate shorted-matrix bound, not a whole-physical-operator norm bound. For a quantitative global allowance, shift the complete physical form first, or pay the corresponding lift/projection stability. Exact nonnegativity transfers by the form Schur identity; the global allowance comparison (53) does not require a core-to-tail lift inference.

B.6 A finite-sine alternative with a sharper joint budget

The following alternative combines the same relative-tail estimate with a finite pole-free intersection core. It avoids a pole-profile input action and improves the joint energy bound.

Proposition B.6 (Stable finite-sine core). *Suppose $L \geq 3$, $B = 6 + 8(\log 2)e^L$, and take*

$$N' = \lceil Le^{2B+1}/\pi \rceil, \quad \text{ran } U' = \text{span}(e_0, \dots, e_{N'-1}) \cap p^\perp, \quad U'^*U' = I.$$

In this proposition Q, K, K_0, W and all receiving and mixed blocks are rebuilt from U' ; no block from the preceding projected core is reused. Both full and finite-response receiving forms on this complement satisfy $T, T_0 \succeq A_Q/3$. A finite DN generating list has Gram condition number below $4L$, and $D_{\text{core}} \leq 2B + 7$. On the growing support, for every integer $M \geq 2^{20}$, the joint bound (57) therefore improves to

$$\|W^{-1/2}(K - K_0)W^{-1/2}\| \leq \frac{49[27B + 63 + M^{-1}]}{M^2} = O\left(\frac{\sqrt{\log M}}{M^2}\right). \quad (58)$$

The cofinal RH criterion also holds with this $(N' - 1)$ -dimensional core.

Proof. Let $\delta_n = \|(I - P_n)\bar{p}\|^2$. The exact pole coefficients and $\cosh(L/2)^2/(\sinh L - L) \leq 2$ for $L \geq 3$ give

$$\delta_n \leq \frac{2L^3}{\pi^4(n + 1/2)^3}, \quad n \geq 1.$$

For the ratio bound use $\sinh L - L \geq e^L/4$ and $\cosh(L/2)^2 \leq e^L/2$; the former holds at 3 and its difference increases. For the series bound, the decreasing summands are bounded by their first term plus their integral. A vector $f \perp \text{ran } U'$ in $p^\perp$ has $P_{N'}f$ proportional to $P_{N'}p$. The exact pole constraint and Cauchy–Schwarz imply $\|P_{N'}f\|^2 \leq \delta_{N'}\|f\|^2$. Hence

$$D_L[f] \geq \{a(k_{N'})(1 - \delta_{N'}) + a(k_0)\delta_{N'}\}\|f\|^2.$$

Put $X = 2B + 1 \geq 2$. Then $e^X \leq k_{N'} < e^X + 2$, while the main logarithmic symbol bounds give $X \leq a(k_{N'}) < X + 6$. Also $\delta_{N'} \leq (2/\pi)e^{-3X}$, so its diagonal leakage is below

$$\frac{2}{3}(X + 6)e^{-3X} \leq 16/(3e^6) < 1/12 < 1.$$

It follows that $D_L|_Q \geq 2B$ and therefore $T, T_0 \geq A_Q/3$. On the core itself $D_{\text{core}} \leq a(k_{N'}) < 2B + 7$.

Let $v_q = \langle e_q, p \rangle$, with $v_0 > 0$, and use the finite list $w_j = v_0 e_j - v_j e_0$, $1 \leq j < N'$. Its Gram is $v_0^2 I + v_{\text{tail}} v_{\text{tail}}^*$; thus its condition number is at most

$$\frac{\|p\|^2}{v_0^2} < 2L(1/4 + \pi^2/(4L^2))^2 < 4L.$$

These columns require only finite primitive DN input actions. The complete output polar projection and physical Gram are still retained. Finally $C_L < B$ for this elementary budget, so $d < 2$ and $(1 + d/c)^2 < 49$. Insert $D_{\text{core}} \leq 2B + 7$ in (57). The specified onset ensures $L_M \geq 3$. The RH equivalence follows by the same exact Schur and full-form argument, with this paid complement. $\square$

B.7 A finite-feature residual test with all mixed terms

For completeness, a trial Z into the full receiving form domain gives an alternative sufficient certificate. Let $T = S|Q \succeq cA_Q$, $H = U^*SU$, $C = QSU$, and set

$$\begin{aligned} L_Z &= C^*Z + Z^*C - Z^*TZ, & L_0 &= C_0^*Z + Z^*C_0 - Z^*T_0Z, \\ r_0 &= C_0 - T_0Z, & K_r &= r_0^*A_Q^{-1}r_0, \\ G &= (U - Z)^*A(U - Z). \end{aligned}$$

For every $\tau > 0$,

$$H - C^*T^{-1}C \succeq H_0 - L_0 - c^{-1}\{(1 + \tau)K_r + (1 + \tau^{-1})\epsilon^2G\}. \quad (59)$$

Indeed the full residual is $r = r_0 + QE(U - Z)$. Relative positivity of E gives the joint A_Q -dual Gram of the second term at most ϵ^2G . Weighted Young and inverse order bound $r^*T^{-1}r$ by the displayed error. The exact residual identity and

$$(H - L_Z) - (H_0 - L_0) = (U - Z)^*E(U - Z) \succeq 0$$

then prove (59). This is a joint matrix inequality, not a collection of diagonal tests. A numerical use must enclose the complete K_r , G , constraints and reference inverse; its cofinal budget includes the actual growth of G/c . The theorem does not assert that this finite-feature signed test has succeeded, nor that the full receiving inverse has become a finite principal compression.

C Local inverse covariance for the averaged reference

For $k = 1$ the continuous prime integral is exactly

$$\int_1^\infty t^{-1/2+i\xi} \min(1, M/t) dt = \frac{M^{1/2+i\xi}}{1/4 + \xi^2} - \frac{1}{1/2 + i\xi}.$$

Its phase-mean negative receiver, divided by $\sqrt{M}$, is

$$\mathcal{C}_Q = \alpha Q M_s L_0^{-1} M_s Q, \quad \alpha = 8/\pi, \quad L_0 = 1 - 4d^2/du^2. \quad (60)$$

Here Q is an original orthogonal projection on the odd space, with a finite constraint space $Z = \ker Q$. Write $\|g\|_{1,4}^2 = \int(|g|^2 + 4|g'|^2)$.

Theorem C.1 (Full inverse-form source covariance). *For $h \in \text{ran } Q$,*

$$\|\mathcal{C}_Q^{-1/2}h\|^2 = \frac{\pi}{8} \inf_{z \in Z: (h+z)/s \in H^1} \|(h+z)/s\|_{1,4}^2, \quad (61)$$

with value infinity when the set is empty. For any finite odd H^1 profile family g_i and $e_i = Q(sg_i)$ this yields the simultaneous Gram inequality

$$[\langle \mathcal{C}_Q^{-1/2}e_i, \mathcal{C}_Q^{-1/2}e_j \rangle] \preceq \frac{\pi}{8} [\langle g_i, g_j \rangle + 4\langle g'_i, g'_j \rangle].$$

Equality holds when Z is any finite span of the original polar row and canonical derivative-null rows. All statements allow arbitrary finite source and row counts.

Proof. Factor $\mathcal{C}_Q = \alpha AA^*$ with $A = QM_s L_0^{-1/2}$. The actual-range identity $\text{ran}(AA^*)^{1/2} = \text{ran } A$ and polar decomposition identify the inverse form with the minimum squared preimage norm [7]. Writing $g = L_0^{-1/2}a$ translates $Aa = h$ to $sg - h \in Z$, and $\|a\|^2 = \|g\|_{1,4}^2$. The chosen profiles give the matrix upper on every common coefficient vector.

More exactly, project the profiles orthogonally in H^1 , with this norm, off $\{g \in H^1 : sg \in Z\}$. Their projected Gram gives equality in (61). This kernel is zero for the canonical rows. Indeed $b/s = \sinh(u/2)$, and for $z_{2j+1} = (\psi^{(2j+3)} - \psi^{(2j+1)})/4/s$,

$$\frac{z_{2j+1}(u)}{s(u)} \sim \frac{(-2\pi)^{2j+3}}{2} e^{(4j+13/2)u} \quad (u \rightarrow +\infty).$$

To obtain this, the first theta atom derivative polynomial has leading coefficient $4(-2)^m$ at order m , by the recurrence $P_{m+1} = (1/2 - 2r)P_m + 2rP'_m$. At fixed order the higher atoms contribute a polynomial times e^{-3r} relative to the first. The exponents of these rows and b/s are distinct, so a nonzero finite combination is not in L^2 . This proves the stated equality. $\square$

If the full profile Gram is bounded by KD and the full profile-error Gram by $\eta^2 D$ in this same H^1 metric, the resulting inverse-form Grams differ in either order by at most $(\pi/8)(2\sqrt{K}\eta + \eta^2)D$. Expand each common coefficient combination and apply Cauchy-Schwarz before passing to coordinates. The exact profile projection in the proof is a contraction, so the same error transport applies to every finite constraint space.

Lemma C.2 (Regularized preimage identity). *For $a, \kappa > 0$ and $e = Q(sg)$, $g \in H_{\text{odd}}^1$,*

$$\langle e, (a\mathcal{C}_Q + \kappa I)^{-1}e \rangle = \inf_{f \in H_{\text{odd}}^1} \left\{ \frac{\|f\|_{1,4}^2}{a\alpha} + \frac{\|Q[s(g-f)]\|^2}{\kappa} \right\}. \quad (62)$$

Proof. For $B = \sqrt{a\alpha}QM_s L_0^{-1/2}$, minimize $\|z\|^2 + \kappa^{-1}\|e - Bz\|^2$. Its value is $\langle e, (BB^* + \kappa I)^{-1}e \rangle$. Substitute $f = \sqrt{a\alpha}L_0^{-1/2}z$. This is the classical approximate-preimage regularization identity in the present weighted and projected space [9]. $\square$

D Explicit reference origin bounds

Let $U = \log M$, $\Omega = L/U$, and use the actual smoothed negative symbol $n_{M,1}$. On the *entire* profile space $\mathcal{Y}_{M,L}^{\text{org}} = L^2([0, \Omega], n_{M,1}(\xi) d\xi/\pi)$ put

$$g_\phi(u) = \int_0^\Omega n_{M,1}(\xi) \phi(\xi) \sin(\xi u) \frac{d\xi}{\pi}, \quad E\phi = Q(sg_\phi), \quad D_\phi = \|\phi\|_{\mathcal{Y}_{M,L}^{\text{org}}}^2.$$

The reference $\mathcal{C}_Q$ is exactly (60).

Theorem D.1 (Uniform ridged origin cap). *For every $L > 0$, every integer $M \geq 64$ and every finite original constraint space,*

$$E^*(\sqrt{M}\mathcal{C}_Q + M^{-1}I)^{-1}E \preceq K(M, L)I_{\mathcal{Y}_{M,L}^{\text{org}}}, \quad (63)$$

where, with $R = \frac{1}{2} \log(2 \log M)$,

$$K(M, L) = \frac{160}{3}L^3 \left(\frac{R}{\log M} \right)^3 + \frac{4096\pi^2}{3}L^3 M^{3/2-2\pi}.$$

In particular $K(M, L) = O_L((\log \log M / \log M)^3)$, and $K(M, 1) < 1/10$ for every integer $M \geq 2^{20}$.

Proof. The elementary prime bound and $\mathbb{E}e^{v/2} = 2$ give $n_{M,1} \leq 6 + 16(\log 2)\sqrt{M} < 16\sqrt{M}$. Thus, for $A_M = \int_0^\Omega n_{M,1}(\xi)\xi^2 d\xi/\pi$,

$$A_M \leq \frac{16\sqrt{M}L^3}{3\pi U^3}, \quad |g_\phi(u)|^2 \leq D_\phi A_M u^2, \quad |g'_\phi(u)|^2 \leq D_\phi A_M.$$

These common-profile estimates are Cauchy-Schwarz and $|\sin(\xi u)| \leq |\xi u|$. The finite frequency band puts g_ϕ in H^1 . Choose even χ_R equal to one on $[-R, R]$, zero outside $[-2R, 2R]$, and linear on each intervening interval. Then $|\chi'_R| \leq 1/R$, and direct integration gives

$$\|\chi_R g_\phi\|_{1,4}^2 \leq 80D_\phi A_M R^3, \quad \|Q[s(g_\phi - \chi_R g_\phi)]\|^2 \leq D_\phi A_M T_R, \quad T_R = \int_{|u|>R} s(u)^2 u^2 du.$$

Indeed $\int_{-2R}^{2R} u^2 du = 16R^3/3$ and the derivative square is at most $(56/3)D_\phi A_M R$; multiplication by four gives the constant 80 for $R \geq 1$.

The complete theta estimate $s^2 \leq 16r^2 e^{-r}$, $r = \pi e^{2|u|}$, and the two-arm Jacobian $du = dr/r$ give

$$T_R \leq 16 \int_{r_R}^\infty r^3 e^{-r} dr \leq 32r_R^3 e^{-r_R}, \quad r_R = \pi e^{2R} \geq 6.$$

The last integral has logarithmic slope at most $-1/2$. Use the trial $f = \chi_R g_\phi$ in (62), with $a = \sqrt{M}$, $\kappa = M^{-1}$ and $r_R = 2\pi U$. The two costs are exactly bounded by the two terms of $K(M, L)D_\phi$. This proves the inequality on every profile, hence as a full observation operator and for every common finite source combination. Projection contraction makes it uniform in all finite original constraints. Both terms decrease on $M \geq 64$. Directed scalar evaluation gives $K(2^{20}, 1) < 0.091767 < 1/10$, completing the explicit threshold. $\square$

Corollary D.2 (Complete positive continuum-model cap). *Define the positive continuum receiver*

$$P_M^{\text{cont}} = QM_s \left[\frac{8\sqrt{M}}{1+4D^2} (-\cos(UD))_+ \right] M_s Q.$$

For the same actual origin observation E ,

$$E^*(P_M^{\text{cont}} + M^{-1}I)^{-1}E \preceq 2K(M, L)I.$$

For $L = 1$ this cap is below $1/5$ for every $M \geq 2^{20}$.

Proof. Lemma 5.1 gives error $e_M = 12M^{5/2}e^{-\pi(M-1)} \leq 1/(2M)$ on $M \geq 64$. The gate holds at 64 using $\pi > 3$, $e > 2$, and the majorant decreases thereafter. Hence $P_M^{\text{cont}} + M^{-1}I \succeq \frac{1}{2}(\sqrt{M}\mathcal{C}_Q + M^{-1}I)$. Inverse order and Theorem D.1 finish the proof. $\square$

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