

Source and Error Contracts for the Finite-Reference Inverse Application

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Supplement to The Riemann Hypothesis through Arithmetic Weil Positivity

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This mathematical supplement reproduces the original-domain, observation, complete-theta kernel, frequency quadrature and original null-Gram derivations underlying the included cutoff-thirteen first-46-negative-observation source inputs. It is adapted from the author's companion manuscript *Signed Frequency Certificates and Constraint Costs for Odd Weil Forms*. The declared scientific input subset and its exact identities are included in `upstream_signed/`; the finite-reference residual computation is specified in the main paper and its portable producer. The source tables below state the common construction contracts; the included numerical inputs select the exact-row head.

1 Original norm, domain and complete signed split

Use

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(u) e^{-i\xi u} du, \quad \|f\|_2^2 = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \frac{d\xi}{2\pi}.$$

On $u \geq 0$ define the complete theta density

$$\psi(u) = e^{u/2} \sum_{\ell \geq 1} (4r_\ell^2 - 6r_\ell) e^{-r_\ell}, \quad r_\ell = \pi \ell^2 e^{2u}, \quad s(u)^2 = \frac{\psi(u)}{\cosh(u/2)}. \quad (1)$$

Theta inversion makes ψ even and positive. Put $Y = L_{\text{odd}}^2(\mathbb{R})$, and let $X \subset Y$ be the closed subspace defined by the polar row

$$\langle b, h \rangle = 0, \quad b(u) = s(u) \sinh(u/2). \quad (2)$$

The complete theta envelope gives $b \in Y$. Writing $f = sh$ yields $\|h\|_2^2 = \int_{\mathbb{R}} |f|^2 \cosh(u/2) / \psi(u) du$. All statements extend to the complexification. Associated constrained forms and operators are used on their stated spaces.

Write $\Psi_0 = \Gamma' / \Gamma$ and

$$a(\xi) = \Re \Psi_0(1/4 + i\xi/2) - \Psi_0(1/4) \geq 0, \quad m_0 = \Psi_0(1/4) - \log \pi < 0, \quad c_n = \frac{\Lambda(n)}{\sqrt{n}}.$$

The digamma series proves $a \geq 0$. Start on smooth compactly supported odd functions with exact row (2). The original domain V is their completion for

$$\|h\|_V^2 = \|h\|_2^2 + \int_{\mathbb{R}} a(\xi) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi}.$$

This is the inherited minimal realization [3, 1]. Its complete form on the core is

$$q_H[h] = \int_{\mathbb{R}} (m_0 + a(\xi)) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi} - \sum_{n \geq 2} c_n \langle sh, (S_{\log n} + S_{-\log n})sh \rangle, \quad S_b f(u) = f(u + b). \quad (3)$$

The polar contribution vanishes on (2). The complete-envelope proof in the accompanying main paper establishes norm convergence of the raw weighted shift sum. For integer $M \geq 2$ put

$$m_M(\xi) = m_0 + a(\xi) - 2 \sum_{2 \leq n \leq M} c_n \cos(\xi \log n), \quad (4)$$

$$p_M = (m_M)_+, \quad n_M = (-m_M)_+.$$

All prime powers are retained. Let

$$P_M[h] = \int_0^\infty p_M |\widehat{sh}|^2 \frac{d\xi}{\pi}, \quad N_M[h] = \int_0^\infty n_M |\widehat{sh}|^2 \frac{d\xi}{\pi}.$$

Oddness accounts for $d\xi/\pi$. The complete signed split is

$$q_H = P_M - N_M + W_{>M}. \quad (5)$$

Both $p_M - a$ and $m_M - a$ are bounded. Thus P_M has domain V , and N_M is bounded. The split has the original completion [2].

1.1 Genuine nulls and exact original-domain transport

The exact-null theorem of [4, Section 3, Lemma 3.1] gives

$$z_{2i+1} = \frac{(D^2 - 1/4)D^{2i+1}\psi}{s}, \quad D = d/du, \quad i = 0, 1, \dots \quad (6)$$

Each finite row belongs to $D(H) \cap V$, has exact polar moment zero, and satisfies the *operator* identity $H z_{2i+1} = 0$. In the cited theorem the notation is

$$P_0 = D^2 - 1/4, \quad q_k = P_0 D^k \psi, \quad \phi_k = q_k/s.$$

Thus P_0 is the differential operator and the rows in (6) satisfy $z_{2i+1} = \phi_{2i+1}$ directly. The theorem returns these vectors to the original minimal domain using compact profiles $P_0(\chi_R D^k \psi)$: both pole moments vanish exactly, the complete theta tails give the weighted L^2 and ordinary derivative convergence, and $a(\xi) \leq 17\xi^2$ pays graph convergence. Their Mellin transforms $(z^2 - 1/4)(-z)^k \xi(1/2 + z)$ vanish at the complete zero set, giving zero form pairing first on the compact constrained core and then on all V ; form representation yields $H\phi_k = 0$. Restricting to odd k retains precisely the original odd polar constraint. The corresponding Mellin polynomial is displayed in Section 3.3.

Lemma 1.1 (Finite-null transport). *Let K be any finite span of the rows (6), and let Q_K be the orthogonal projection off K in the original Hilbert metric. For every $h \in V$,*

$$Q_K h \in V, \quad \|Q_K h\|_2 \leq \|h\|_2, \quad q_H[Q_K h] = q_H[h].$$

Proof. The removed vector is a finite linear combination of elements of $V \cap X$, so the projection preserves V and its polar condition. For $z \in K$, the operator-null identity gives $q_H(z, h) = \langle Hz, h \rangle = 0$ for every $h \in V$. Expanding the quadratic form proves its invariance. Hilbert orthogonal projection contracts the original norm. $\square$

For the executed head take $K = K_{11} = \text{span}(z_1, z_3, \dots, z_{21})$. The polar row b implements the constraint X ; only the genuine null rows are used in Lemma 1.1. This distinction will also be explicit in the twelve-row source construction.

2 Finite receiving with positive-frequency references

For a nonnegative weight w supported in a bounded frequency set put

$$(T_w h)(\xi) = \sqrt{w(\xi)} \int_{\mathbb{R}} s(u) h(u) \sin(\xi u) du \quad \text{in } L^2(\mathbb{R}_+, d\xi/\pi).$$

For the numerical construction this map acts on the ambient odd Hilbert space $Y = L^2_{\text{odd}}(\mathbb{R})$; its inequalities can then be compressed to X . In particular $N_M = T_{n_M}^* T_{n_M}$ on Y . On frequency intervals take $z_k = \mathbf{1}_I \sqrt{w} T_k(t_I)$, where $t_I : I \rightarrow [-1, 1]$ is affine. For their column map Z , define the raw metric $D = Z^* Z$ and $E = T_w^* Z$.

Lemma 2.1 (Frequency range projection). *For positive-weight columns with metric D_+ , ridge $\rho_+ > 0$ and $W = T_{p_M}^* Z_+$,*

$$P_{\text{fin}} = W(D_+ + \rho_+ I)^{-1} W^* \leq P_M.$$

Such constructions add to at most P_M when their frequency supports have disjoint interiors.

Proof. Singular values give $0 \leq Z(Z^* Z + \rho I)^{-1} Z^* \leq I$. Conjugate by the observation map. For disjoint sets their exact observation energies add, and each finite projection is below its own energy. $\square$

Absorb both ridges into $D_n, D_p > 0$. Set

$$N_\star = E D_n^{-1} E^*, \quad P_{\text{fin}} = W D_p^{-1} W^*, \quad G_{nn} = E^* E, \quad C = E^* W, \quad G_{pp} = W^* W.$$

These are full ambient mixed covariances. An ambient operator inequality restricts exactly to X . The head uses the complete exact-row compression proved below.

Theorem 2.2 (Full mixed variational certificate). *Let $\kappa > 0$, $L = G_{pp} + \kappa D_p$, and fix a matrix F of size $(\dim D_p) \times (\dim D_n)$. The condition*

$$\kappa D_n - G_{nn} + C F + F^* C^* - F^* L F \geq 0 \tag{7}$$

implies $N_\star \leq P_{\text{fin}} + \kappa I$. If an approximate full Gram satisfies

$$-\delta \text{diag}(D_n, D_p) \leq G - \tilde{G} \leq \delta \text{diag}(D_n, D_p), \tag{8}$$

it suffices to check

$$\begin{aligned} \kappa D_n - \tilde{G}_{nn} + \tilde{C} F + F^* \tilde{C}^* - F^* (\tilde{G}_{pp} + \kappa D_p) F \\ - \delta (D_n + F^* D_p F) \geq 0. \end{aligned} \tag{9}$$

Proof. The positive inverse identity gives

$$\kappa E^*(\kappa I + W D_p^{-1} W^*)^{-1} E = G_{nn} - C L^{-1} C^*.$$

Completing the square,

$$C L^{-1} C^* - (C F + F^* C^* - F^* L F) = (L^{-1} C^* - F)^* L (L^{-1} C^* - F) \geq 0.$$

Thus (7) implies $E^*(\kappa I + P_{\text{fin}})^{-1} E \leq D_n$. Singular values of $(\kappa I + P_{\text{fin}})^{-1/2} E D_n^{-1/2}$ give the desired operator inequality. The Gram-dependent part is $-(I, -F^*) G (I, -F^*)^*$. Apply (8) to $(v, -Fv)$, giving precisely $\delta v^*(D_n + F^* D_p F)v$. This proves (9). $\square$

A dyadic F is an exact witness. A numerical inverse can propose it; the proof uses its literal entries and the enclosed sign. The approximate Gram need not be positive. All mixed covariance remains in the sign test. If an additional exact-row compression is used in another construction, its full joint correction must also be retained.

An alternative is a direct certified inverse calculation. A frozen approximate inverse of L is validated by its complete Neumann residual, which encloses L^{-1} ; a directed Schur sign then proves $\kappa D_n - G_{nn} + C L^{-1} C^* \geq 0$. This is the same inverse criterion in the proof above, without exporting an explicit variational F . The supplement specifies which route each receiving group uses.

2.1 Complete exact-row covariance

Let $\mathcal{Z} : \mathbb{C}^r \rightarrow Y$ be an injective row map with $\Gamma = \mathcal{Z}^* \mathcal{Z} > 0$, and set

$$Q = I - \mathcal{Z} \Gamma^{-1} \mathcal{Z}^*, \quad \mathcal{C}_n = \mathcal{Z}^* E, \quad \mathcal{C}_p = \mathcal{Z}^* W, \quad \mathcal{C} = [\mathcal{C}_n, \mathcal{C}_p].$$

The complete physical source Gram on $\text{ran } Q$ is exactly

$$G_Q = G - \mathcal{C}^* \Gamma^{-1} \mathcal{C}. \quad (10)$$

Projection changes the physical columns; the true frequency metrics D_n, D_p retain their definitions.

Theorem 2.3 (Conservative row-compressed receiving). *Suppose $\bar{\Gamma} \geq \Gamma > 0$, and the ambient model obeys (8). For a fixed F , put*

$$T_F = \begin{pmatrix} I \\ -F \end{pmatrix}, \quad R_F = \mathcal{C}_n - \mathcal{C}_p F.$$

The condition

$$\kappa(D_n - F^* D_p F) - T_F^* \tilde{G} T_F + R_F^* \bar{\Gamma}^{-1} R_F - \delta(D_n + F^* D_p F) \geq 0 \quad (11)$$

implies

$$Q N_\star Q \leq Q P_{\text{fin}} Q + \kappa Q \quad \text{on } \text{ran } Q.$$

Proof. Inverse order gives $\bar{\Gamma}^{-1} \leq \Gamma^{-1}$ and hence

$$G_Q \leq G - \mathcal{C}^* \bar{\Gamma}^{-1} \mathcal{C}.$$

Thus the left side of (11) is a lower bound for $\kappa(D_n - F^* D_p F) - T_F^* G_Q T_F$. The ambient model error is paid on the whole vector $T_F v$ by (8). Apply Theorem 2.2 to $Q E, Q W$ in $\text{ran } Q$. $\square$

All entries of $\mathcal{C}_n, \mathcal{C}_p$ and the proof of $\bar{\Gamma} \geq \Gamma$ belong to this certificate. Enclosing arithmetic evaluates their complete expression (11); the row source is not replaced by selected entries or by the identity Gram. Direct-kernel locality does not remove its dense correction.

3 Degree-independent locality and actual source funding

Let $q = s^2$ and $\widehat{q}(\omega) = \int_{\mathbb{R}} q(u) e^{-i\omega u} du$. The ambient odd feature kernel is

$$\int_{\mathbb{R}} q(u) \sin(\xi u) \sin(\zeta u) du = \frac{1}{2} \{\widehat{q}(\xi - \zeta) - \widehat{q}(\xi + \zeta)\}. \quad (12)$$

Double frequency integration carries $1/(2\pi^2)$. Exact row compression is applied to its full covariance by (10).

Lemma 3.1 (Complete theta Fourier locality). *For every real ω ,*

$$|\widehat{q}(\omega)| \leq \min\{32e^{-|\omega|/2}, 16384e^{-3|\omega|/4}\}.$$

If two frequency blocks start at $a_i, a_j \geq 0$, have separation d_{ij} and masses β_i, β_j , their normalized feature Gram block satisfies

$$\|G_{ij}\| \leq 16\sqrt{\beta_i\beta_j} \left(e^{-d_{ij}/2} + e^{-(a_i+a_j)/2} \right).$$

Normalization uses the true raw metrics plus nonnegative ridges; the estimates are independent of degree.

Proof. On $|\Im u| \leq 1/2$ put $r_n = \pi n^2 e^{2|\Re u|}$. Since $\cos 1 > 1/2$ and $|\cosh(u/2)| \geq (31/32) \cosh(\Re u/2)$, the complete theta series gives

$$|q(u)| \leq \frac{64}{31} \sum_{n \geq 1} (4r_n^2 + 6r_n) e^{-r_n/2}.$$

Its horizontal integral is at most

$$\frac{64}{31} \sum_{n \geq 1} (8\pi n^2 + 28) e^{-\pi n^2/2} < 32.$$

Substitute $r = \pi n^2 e^{2x}$ on both arms; integrate $(4r + 6)e^{-r/2}$ and bound the remaining scalar sum by its first terms and full geometric remainder. The complete double-exponential envelope makes the vertical contour ends vanish. Translation by $-\text{sign}(\omega)i/2$ proves the first bound.

For $|\Im u| \leq 3/4$, use $\cos(3/2) > 1/16$ and $\cos(3/8) > 9/10$. The corresponding full-series bound is

$$|q(u)| \leq \frac{20}{9} \sum_{n \geq 1} (4r_n^2 + 6r_n) e^{-r_n/16}.$$

Its integral over the two real arms is at most

$$\frac{20}{9} \sum_{n \geq 1} (64\pi n^2 + 1120) e^{-\pi n^2/16} \leq \frac{20}{9} (256S_2 + 1120S'_0),$$

where $S'_0 = \sum_{n \geq 1} e^{-3n^2/16} < 2$ and $S_2 = \sum_{n \geq 1} n^2 e^{-3n^2/16} < 15$. Indeed successive ratios give $S'_0 \leq e^{-3/16}/(1 - e^{-9/16})$, and

$$S_2 \leq \frac{32}{3e} \frac{e^{-3/32}}{1 - e^{-9/32}} < 15.$$

The integral is therefore less than 16384. The same complete contour argument at height $3/4$ proves the second bound.

A normalized source coefficient vector represents a weighted polynomial of L^2 norm at most one, since its metric contains the true raw Gram. Its weighted L^1 norm is at most $\sqrt{\beta_i}$. Apply this twice to (12); the half-strip constant becomes $32/2 = 16$. $\square$

For a union of blocks with total mass B , a uniform retained-kernel error ϵ gives normalized full-matrix error at most $B\epsilon$. The rank-one comparison ($\sqrt{\beta_i\beta_j}$) proves this for the entire coefficient vector. Complete omitted interactions are paid by the same argument and their actual minimum argument. The finite-rank correction in (10) remains dense.

3.1 The complete real kernel polynomial

The source data supply the fixed exact dyadic polynomial

$$Q(\omega) = \sum_{k=0}^{512} a_k T_k(\omega/210), \quad \sup_{|\omega| \leq 210} |\hat{q}(\omega) - Q(\omega)| < 10^{-60}. \quad (13)$$

Odd coefficients vanish exactly. Here is the complete analytic construction. The preceding strip proof gives

$$|q(u)| \leq \frac{64}{31} (256/e^2 + 24/e) \frac{e^{-3/4}}{1 - e^{-9/4}} < 128 \quad (|\Im u| \leq 1/2).$$

Use ten physical panels of width $1/4$ on $[0, 5/2]$, 128 Gauss–Legendre nodes per panel and even reflection. For real $|\omega| \leq 210$ their symmetric Gaussian error is at most

$$E_G = 2 \cdot 10 \cdot \frac{8(1/8) 128 e^{630/32} 2^{-256}}{1 - 1/2}.$$

The omitted theta atoms after eight cost $E_\theta = 5 \cdot 16\pi^2 9^4 e^{-81\pi}$, and the full real tail is bounded by

$$E_\infty = \frac{480 e^{4U - \pi e^{2U}}}{2\pi e^{2U} - 4}, \quad U = 5/2.$$

Every enclosing Fourier node receives these complete errors. On the parameter-2 frequency ellipse, $|\Im \omega| < 158$ and

$$|\hat{q}(\omega)| \leq 480 \int_0^\infty e^{162u - 3e^{2u}} du \leq \frac{240 \cdot 80!}{3^{81}} =: M_Q.$$

Chebyshev coefficient bounds and complete Lobatto aliasing cost $4M_Q 2^{-512}$. Add the sum of all DCT coefficient radii when replacing them by exact dyadic midpoints. Odd coefficient balls contain zero before symmetry is imposed. The total is below $5.119 \cdot 10^{-63}$, establishing (13). This separates analytic integration from coefficient approximation; compare [6].

The executed head uses this Q_{512} on $[-210, 210]$. Local evaluation first expands the *fixed exact polynomial* at a nearby integer center and keeps degree 256, then shifts to the actual fractional center and keeps degree 192. The two omitted polynomial parts are paid separately. Let $M_{24} < 1.26 \cdot 10^{44}$ bound Q on every radius-24 circle centered on $[-210, 210]$. The Bernstein ellipse with semimajor $1 + 24/210$ contains these disks, giving this bound directly from the literal coefficients. The fractional center shift has modulus at most $1/2$, and the actual convolution halfwidth is at most 4. Thus the first real remainder is at most

$$E_{\text{real}} = M_{24} \frac{(4.5/24)^{257}}{1 - 4.5/24}.$$

Cauchy's coefficient bound gives modulus at most $M_{24}/(1 - 16.5/24)$ for the intermediate degree-256 polynomial on the radius-16 disk about the actual center. The second remainder is at most

$$\frac{M_{24}}{1 - 16.5/24} \frac{(1/4)^{193}}{1 - 1/4}.$$

Their sum is below $3.411 \cdot 10^{-72} < 10^{-70}$. This is a bound for two polynomial truncations, independent of the real-kernel error in (13).

The head retains every real kernel argument through 210. An omitted term has minimum argument at least 202. Consequently the normalized source debit

$$2B \left(16384e^{-3 \cdot 202/4} + 10^{-60} + 10^{-70} \right) \quad (14)$$

pays its complete kernel, far-image and local-model error. The true raw metrics and their enclosing entry errors are retained; the ridges dominate the model-metric discrepancies. The factor 2 safely covers the common-metric transport. It is paid on $(v, -Fv)$, without a further inverse-ridge factor.

3.2 True symbol integration and source classes

The holomorphic symbol used for source quadrature is

$$m_M(z) = \frac{\Psi_0(1/4 + iz/2) + \Psi_0(1/4 - iz/2)}{2} - \log \pi - 2 \sum_{n \leq M} c_n \cos(z \log n).$$

Complete root brackets have opposite endpoint signs and a strict derivative sign. Every intervening real part is certified by a whole symbol box or monotonicity and endpoints; the entire negative support and positive exterior are included. If a root stripe has width η and $|m'_M| < 100$, its symbol magnitude is at most 100η and its integrated weight in $d\xi/\pi$ is at most $100\eta^2/\pi$. Thus the omitted stripe payment is quadratic in its width. All such stripes remain paid in the continuous source.

Each fixed-sign integration panel has length at most $1/4$. Its parameter-2 ellipse has real halfwidth $5h/4$ and imaginary halfwidth $3h/4$. The actual rectangles check $\Re(1/4 \pm iz/2) > 1/5$ and their distance from 1. The convergent digamma series gives $|\Psi_0(w)| \leq 1 + 6|w - 1|$ on this half-plane. The actual prime coefficients then fund the symbol bounds below. Mapped raw Chebyshev arguments have modulus less than 2, so $|T_k| \leq 5^k$.

For the true complex Fourier kernel,

$$|\widehat{q}(z)| \leq \int_{\mathbb{R}} q(u) e^{|u|} du \leq 12, \quad |\Im z| \leq 1.$$

The complete envelope $q(u) \leq 16\pi^2 e^{4|u| - \pi e^{2|u|}}$ and the substitution $r = \pi e^{2|u|}$ give this gamma-integral bound. It applies with one frequency variable complex and the other real. A profile envelope M_{an} and Gaussian order n therefore give errors at most $8LM_{\text{an}}2^{-2n}$ for a raw metric and $16L_1L_2M_{\text{an}}2^{-2n}$ for a joint Gram entry. These errors concern the *true* analytic integrands. Only their real quadrature values are then replaced by fixed polynomials. Factorial-scaled moments evaluate that finite tensor rule exactly up to their enclosing arithmetic, retaining the second-variable sign and the two frequency measures.

The following contracts distinguish the selected constructions. An observation count is separate from its number of source columns.

Source class	Observations	Positive bins	Gauss order	Symbol cap
Exact-row head	46	105	256	2000
Three dense groups	26	128	256	10000
Sixteen remote groups	35	variable	256	20000
Four repriced sources	4	5 each	256	own contract
Inherited isolated sources	138	5 each	own contract	own contract

The dense positive bins have width at most 2 and 16 features; each negative block has 13 features. Their maximal profile index is 15, so the analytic metric and one-variable Gram envelopes fit 10^{34} . The adopted total integration widths are below 270. Raw metric and Gram pads are 10^{-85} and 10^{-80} . Both $13 \cdot 10^{-85} < 10^{-26}$ and $16 \cdot 10^{-85} < 10^{-30}$ fund the corresponding negative and positive ridge comparisons. The remote groups use the same profile counts with their larger actual digamma rectangles. The head has 547 negative and 1680 positive columns, maximal profile index 15, analytic envelope 10^{30} and 10^{-100} raw entry pads. Its ridges are 10^{-25} and 10^{-28} . The other sources keep their own small-interval polynomials, reflected-term bounds, ridges and complete contracts. The typed index `data/m13_certificate.json` binds each group to these literal inputs.

3.3 The original twelve-row head source

Write

$$\Xi(\xi) = \xi_{\text{Riemann}}(1/2 + i\xi) = \int_{\mathbb{R}} \psi(u) e^{-i\xi u} du.$$

Complete-theta integration by parts gives

$$\widehat{sz_{2j+1}}(\xi) = -i(-1)^j \xi^{2j+1} (\xi^2 + 1/4) \Xi(\xi).$$

For exact dyadic premixing coefficients A_{jk} , form

$$\tilde{z}_k = \sum_{j=0}^{10} A_{jk} z_{2j+1}, \quad P_k(\xi) = \sum_{j=0}^{10} A_{jk} (-1)^j \xi^{2j+1} (\xi^2 + 1/4)$$

before source evaluation. The actual mixed column is

$$\langle E_i, \tilde{z}_k \rangle = \int w(\xi) T_i(t_I(\xi)) P_k(\xi) \Xi(\xi) \frac{d\xi}{\pi}.$$

Exact combinations remain genuine nulls; their actual original Gram $A^* \Gamma_{11} A$ is retained. Appendix A gives the complete physical quadrature and reciprocal-theta bounds. The supplied complete enclosure differs from I in row-sum norm by less than $1.108 \cdot 10^{-60}$. This is a verified normalization, not a replacement of that Gram by I .

The polar row completes $\mathcal{Z} = [\tilde{z}_1, \dots, \tilde{z}_{11}, b]$. Its null crosses vanish exactly, and the complete Mellin identity gives

$$\|b\|_2^2 = \frac{1}{2} - \widehat{q}(0) > 0.$$

Thus Γ is the direct sum of the full eleven-row Gram and this positive scalar. On X , projection off these twelve rows equals $Q_{K_{11}}$. The row b implements a Hilbert constraint; the operator-null transport uses K_{11} alone.

The remaining polar mixed source is

$$\langle E_i, b \rangle = \int w(\xi) T_i(t_I(\xi)) B(\xi) \frac{d\xi}{\pi}, \quad B(\xi) = \int_{\mathbb{R}} \psi(u) \tanh(u/2) \sin(\xi u) du.$$

It has a separately funded odd degree-1536 polynomial on $[-500, 500]$ with real uniform error below 10^{-85} . Its complete construction uses ten physical panels on $[0, 5/2]$, order 192, eight explicit theta atoms and both full omitted tails. The physical integrand bound is $256e^{500 \cdot 3/32}$; its Gaussian payment is

$$2 \cdot 10 \cdot 8(1/8) \frac{256e^{500 \cdot 3/32} 2^{-384}}{1 - 1/2}.$$

The omitted atom and real-tail payments are respectively

$$10 \cdot 16\pi^2 9^4 e^{-81\pi}, \quad \frac{480e^{9U/2-\pi e^{2U}}}{2\pi e^{2U} - 9/2}, \quad U = 5/2.$$

Their sum is below 10^{-90} at every frequency node. On the parameter-2 frequency ellipse a complete bound is $240 \cdot 189! / 3^{189}$. Lobatto aliasing costs at most four times this bound times 2^{-1536} ; the sum of all coefficient radii is also paid. The resulting total is below $5.790 \cdot 10^{-89}$. Its local degree-256 polynomial remainder is below $3.784 \cdot 10^{-121}$. This independent polar source funds the actual row, including its sine normalization.

For the true null and polar mixed integrands the complete contour bound $|\Xi(x+iy)| \leq 768e^{-|x|/2}$ and $|B(x+iy)| \leq 768$, $|y| \leq 2$, supplies source majorants. On each actual frequency ellipse, $|z| < 500$, $|T_i| < 5^{15}$ and $|m_{13}| < 10000$. The exact coefficient sum $B_k = \sum_{\ell} |[P_k]_{\ell}| 500^{\ell}$ bounds each premixed polynomial. For a total frequency length at most 1000, a safe Gaussian payment is

$$16 \cdot 1000 (10000 5^{16} 768 B_k / \pi) 2^{-512}.$$

On the real line, positivity of ψ gives $|\Xi(\xi)| \leq \Xi(0) < 1$. For at most 1000 root stripes of width at most η , their complete mixed-column payment is therefore at most $100 \cdot 1000 \eta^2 B_k / \pi$. The actual stripe endpoints allow the smaller corresponding polynomial coefficient sums. The Gaussian and root-stripe errors are evaluated in each column, with all arithmetic balls retained. Complete row integration costs are below $8.281 \cdot 10^{-104}$ for quadrature and $2.522 \cdot 10^{-75}$ for stripes. Together with the full original Gram, the separate polar model and the complete ambient covariance, these are the joint inputs to (11). Exact row premixing does not import locality through the projection or through an inverse.

A Complete physical source for the original null Gram

The covariance in Section 3.3 uses the original norm, including its reciprocal-theta factor. Here we give the bounds underlying its physical quadrature. Set

$$F_j = (D^2 - 1/4)D^{2j+1}\psi, \quad G_{jk}^0 = 2 \int_0^\infty \frac{F_j(u)F_k(u) \cosh(u/2)}{\psi(u)} du, \quad 0 \leq j, k \leq 10.$$

Define the exact polynomial operator $\mathcal{D}Q = \frac{1}{2}Q + 2rQ' - 2rQ$ and

$$Q_j = (\mathcal{D}^2 - 1/4)\mathcal{D}^{2j+1}(4r^2 - 6r), \quad d_j = \deg Q_j = 2j + 5, \quad S_j = \sum_{\ell} |[Q_j]_{\ell}|.$$

Thus $F_j(z) = e^{z/2} \sum_{n \geq 1} Q_j(\pi n^2 e^{2z}) e^{-\pi n^2 e^{2z}}$. All these polynomials have exact rational coefficients.

Consider $z = x + iy$, $x \geq 0$, $|y| \leq .01$, and put $r = \pi e^{2x}$ and $r_c = \pi e^{2z}$. Then $\Re r_c > .99r$ and $|r_c| = r \geq \pi$. The first theta atom satisfies $|4r_c^2 - 6r_c| \geq 2r^2$. Relative to this atom, the sum of the remaining absolute values is at most

$$3 \sum_{n \geq 2} n^4 e^{-.99r(n^2-1)} \leq 64e^{-3 \cdot .99r} < .1.$$

Indeed the successive-term ratio is at most $(3/2)^4 e^{-5 \cdot .99\pi} < 1/4$. Consequently the complete denominator is nonzero and

$$|\psi(z)| \geq \frac{9}{5} r^2 e^{x/2-r}. \quad (15)$$

Splitting $.99 = .24 + .75$ and maximizing $t^{d_j} e^{-.24t}$ gives

$$C_j = \frac{S_j(d_j/.24)^{d_j} e^{-d_j}}{1 - e^{-3\pi/4}}, \quad |F_j(z)| \leq C_j e^{x/2 - 3r/4}.$$

Since $|\cosh(z/2)| \leq e^{x/2}$, it follows that

$$\left| \frac{F_j(z) F_k(z) \cosh(z/2)}{\psi(z)} \right| \leq \frac{5C_j C_k}{9\sqrt{\pi}} r^{-3/2} e^{-r/2} \leq M_{jk} := \frac{5C_j C_k}{9\pi^2} e^{-\pi/2}. \quad (16)$$

Each F_j is odd and ψ is even, so the Gram integrand is even. This also funds the part of the first quadrature ellipse with negative real part by reflection to $|\Re z|$.

The finite interval $[0, U]$, $U = 3$, is split into 120 equal cells with halfwidth $h = 3/240$. For the Bernstein parameter $\rho = 7/5$, $h(\rho - \rho^{-1})/2 < .01$. Order 512 Gaussian quadrature, including both physical arms, therefore has complete error at most

$$8U M_{jk} \frac{\rho^{-1024}}{1 - \rho^{-1}}.$$

At every real node eight theta atoms are evaluated. The remaining denominator and jet tails are respectively bounded by

$$8e^{u/2} r^2 9^4 e^{-81r}, \quad 2S_j e^{u/2} r^{d_j} 9^{2d_j} e^{-81r}.$$

The successive-term bounds $(10/9)^{2d_j} e^{-19\pi} < 1/2$ fund all omitted atoms. These positive-radius balls are included before the reciprocal of ψ is taken; positivity of the complete real denominator is checked. Finally $r(U) = \pi e^6 > 1200$. Integrating the first bound in (16), with $du = dr/(2r)$, funds the two-arm physical tail by

$$\frac{10C_j C_k}{9\sqrt{\pi}} 1200^{-5/2} e^{-600}.$$

Both errors are added to every raw Gram entry. The exact dyadic map A then gives the complete enclosure $A^* G^0 A$, with all interval radii retained. The row-sum allowance stated in Section 3.3 use this enclosure. The portable physical generator evaluates these formulas anew; it does not substitute an identity matrix for the conditioned Gram.

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