Technical Supplement to RH: Complete Signed Certificates and Canonical Null Synthesis

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Luminary Oddities

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This document accompanies the main paper of the same title. It retains the alternative hard-cutoff and finite-source methods and their numerical applications. Definitions, complete quantitative receiving proofs and carrier synthesis are in the main text; original-domain proofs are retained in the main paper. Its quantitative genuine-null interpolation is proved in Section 10, and its independent qualitative proof is in Appendix C; these unprefixed references refer to the main PDF. All local sections and equations carry an S prefix. The complete source/error derivations for the numerical application are included here. The final sections preserve the earlier M20 certificate, the full fixed-support ridge comparison and the alternative growing-floor criterion; their verification roles are summarized in the main paper's applications section and appendix guide. The results originate in the author's earlier weighted-Weil and signed-frequency work [21, 19, 18, 23, 20]; the references preserve attribution. These supplementary methods complement the main finite-verifiable criterion and averaged-covariance theorem. Literal data and reproducible code are described in the package's supplementary-material index.

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S1 Complete covariance estimates and cost-paid transfers

The following chapters retain the complete-source covariance arguments and their recipient-dependent critical criteria. Their canonical lifts are proved in the main paper. All source and physical norms remain those defined there.

S2 Elementary estimates for the original realization

The estimates used below are proved here in the normalization and minimal form domain of Section 2. They consolidate the elementary inputs of the author's weighted-Weil and joint-cutoff predecessors [21, 19, 18, 20]; those citations record provenance rather than substitute for the proofs.

In this section only, $m_M = m_0 + a - 2\sum_{2\le n\le M}\Lambda(n)n^{-1/2}\cos(\xi\log n)$, $p_M = (m_M)_+$ and $n_M = (-m_M)_+$ denote the hard-cutoff symbols, with $R_M = 6+8(\log 2)\sqrt{M}$. The averaged symbols

introduced later carry their averaging indices explicitly. The estimates below do not identify a hard-cutoff symbol with an averaged one.

S2.1 Elementary arithmetic and logarithmic symbol bounds

Write $\vartheta_C(N) = \log \operatorname{lcm}(1, \dots, N)$. The quotient for $\vartheta_C(2m) - \vartheta_C(m)$ divides $\binom{2m}{m}$: a prime power in $(m, 2m]$ supplies a positive binomial valuation, and there is at most one such new power per prime. Similarly the quotient for $\vartheta_C(2m+1) - \vartheta_C(m+1)$ divides $\binom{2m+1}{m}$. The two equal odd central binomials together are at most 2^{2m+1}, so either is at most 2^{2m}. Induction, starting at $N = 1$, gives

$$\vartheta_C(N) \le 2N \log 2.$$

Partial summation, with the bound extended to real arguments by flooring, therefore gives

$$S_M := \sum_{2 \le n \le M} \frac{\Lambda(n)}{\sqrt{n}} \le 4(\log 2)\sqrt{M} - 2\log 2. \tag{S1}$$

This estimate needs no prime number theorem.

The identity $m_0 = -\gamma - \pi/2 - 3\log 2 - \log \pi$ gives $-6 < m_0 < 0$ and $m_0 + 4 + \log 2 < 0$. Here are elementary admissions for these strict constants. The bounds $\gamma < 1$, $\pi < 22/7$, $\log 2 < 7/10$ and $\log \pi < 7/6$ give $m_0 > -6$; the logarithmic bounds follow from positive exponential Taylor sums. Also $\gamma > H_8 - \log 9 > H_8 - 11/5 = 29/56 > 1/2$, since the first six Taylor terms give $e^{11/10} > 3$. Together with $\pi > 3$, $\log 2 > 2/3$ and $\log \pi > 1$, this gives $m_0 + 4 + \log 2 < -1/3$.

For $t = |\xi|/2$, the summand $f_t(q) = t^2/[q(q^2 + t^2)]$ is nonnegative and decreasing for $q > 0$. Integral comparison with the digamma series gives

$$\tfrac{1}{2}\log(1 + 4\xi^2) \le a(\xi) \le 4 + \tfrac{1}{2}\log(1 + 4\xi^2). \tag{S2}$$

Using the negative m_0, (S1) and $\frac{1}{2}\log(1 + 4\xi^2) \le \log 2 + \log(2 + |\xi|)$ yields

$$0 \le n_M \le R_M, \quad |p_M - a| \le R_M, \quad p_M(\xi) \le R_M + \log(2 + |\xi|). \tag{S3}$$

The middle bound also holds where $m_M < 0$, since there $a < |m_0| + 2S_M$. For $|\xi| \ge \Omega_M = e^{R_M + 1}$, the lower bound $a(\xi) \ge \log(2|\xi|)$ proves $m_M(\xi) > 0$. Thus the full negative support is bounded without importing a sampled sign pattern at another cutoff. The raw tail satisfies $\delta_M < 1$ for $M \ge 2$, since its decreasing majorant at $M = 2$ is below $4059(3/8)^9 < 1$.

S2.2 Complete theta constants and convolution

We first record the complete strip estimates

$$\|s^2(\cdot \pm i/8)\|_2 < 20, \qquad \|s^2(\cdot \pm i/8)\sinh((\cdot \pm i/8)/2)\|_2 < 32. \tag{S4}$$

For $z = x + iy$, $x \ge 0$, $|y| \le 1/8$ and $r = \pi e^{2x}$, $\cos(2y) > 31/32$ and $4r_\ell^2 + 6r_\ell < 6r_\ell^2$. The consecutive ratios of the majorized theta atoms are below $16e^{-3(31/32)\pi} < 1/8$. Summing *all* atoms gives

$$|\psi(z)| \le 7e^{x/2} r^2 e^{-31r/32}, \qquad |s^2(z)| \le 28 r^2 e^{-31r/32}.$$

The second estimate uses $|\cosh(z/2)| \ge \cos(1/16)\cosh(x/2) > \cosh(x/2)/2$. Evenness covers the other real arm. The squared two-arm integral for the first norm in (S4) is at most $4704(16/31)^4 < 400$. For the second use $|\sinh((x + i/8)/2)|^2 \le e^{|x|} \le r/\pi$; the squared norm is at most $(784 \cdot$

$24/\pi)(16/31)^5 < 1024$. On the real line the stronger atom-ratio bound gives $s^2 \le 16r^2e^{-r}$; its maximum for $r \ge \pi > 3$ is less than $144e^{-3} < 9$, so $\|s\|_\infty < 3$.

Put $w = s^2$. Analyticity and complete double-exponential decay justify shifting the two Fourier contours, including the vertical end pieces. Plancherel and (S4) give

$$\int_{\mathbb{R}} e^{|\xi|/4}|\widehat{w}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi} < 800.$$

Weighted Cauchy–Schwarz implies

$$\int_{\mathbb{R}} |\widehat{w}| \frac{\mathrm{d}\xi}{2\pi} < 33, \qquad \int_{\mathbb{R}} \log(2+|\xi|)|\widehat{w}| \frac{\mathrm{d}\xi}{2\pi} < 210. \tag{S5}$$

Indeed the reciprocal-weight integrals are $4/\pi$ and at most $164/\pi$, using $\log(2+x) \le 1+x$; then $3200/3 < 33^2$ and $131200/3 < 210^2$ suffice. For an odd g with Fourier support in $[-F, F]$, the inequality $2+|\xi| \le (2+|\xi-\zeta|)(2+|\zeta|)$ and weighted Young's inequality now prove

$$\|p_M(D)(wg)\|_2 \le [33(R_M + \log(2+F)) + 210]\|g\|_2. \tag{S6}$$

The logarithmic multiplier is used only on these paid sources, rather than asserted bounded on all L^2.

S2.3 Complete hard-cutoff return

Let Q_b be the orthogonal projection from the ambient odd Hilbert space onto the polar space $\mathcal{K}$ of Section 2. For an integer $m \ge 2$ set

$$b_m(\xi) = m_0 - 2\sum_{2 \le n \le m} \frac{\Lambda(n)}{\sqrt{n}} \cos(\xi \log n).$$

Lemma S2.1 (Hard-cutoff source bounds)**.** *The multiplier a is increasing on $[0,\infty)$ and $a(L) \ge \frac{1}{2}\log(2L)$ for $L \ge 1/2$. Moreover*

$$\|s\|_\infty < 3, \qquad |b_m(\xi)| \le 18\sqrt{m}, \qquad H = Q_b M_s(a+b_m)(D)M_s Q_b + V_m, \quad \|V_m\| \le \frac{640(m+1)^2}{20^{m+1}}. \tag{S7}$$

The operator expression denotes equality of forms on the original minimal domain V; the omitted arithmetic perturbation V_m is bounded.

Proof. For fixed $q > 0$, $t^2/[q(q^2+t^2)]$ increases with $t \ge 0$. The positive digamma series in Lemma 2.1 therefore proves monotonicity. Equation (S2) gives

$$a(L) \ge \tfrac{1}{2}\log(1+4L^2) \ge \tfrac{1}{2}\log(2L),$$

since $1+4L^2 \ge 2L$. The real complete-theta estimate above proves $\|s\|_\infty < 3$. From $|m_0| < 6$ and (S1),

$$|b_m| \le |m_0| + 2S_m < 6 + 8(\log 2)\sqrt{m} \le (6 + 8\log 2)\sqrt{m} < 18\sqrt{m}.$$

The exact identity follows first on the compact polar core by retaining all arithmetic shifts with $n \le m$. Its remainder is the complete paired omitted shift operator from Lemma 2.1. Hence, with $k = m+1$,

$$\|V_m\| \le \delta_m < 451k^2e^{-\pi k} < 640k^2 20^{-k},$$

using $e^\pi > 20$ and $451 < 640$. Compression by Q_b cannot increase its norm. The retained symbol differs from a by a bounded multiplier, while V_m is bounded in the original Hilbert norm. The core identity consequently extends through precisely the specified minimal form closure. At no stage are the paired signed edge pieces closed separately. □

S3 Positive arithmetic smoothing and a cofinal schedule

Write M_s for multiplication by the complete theta weight s in (5). All identities in this section begin on the original polar compact core and extend through its minimal closure V. Put

$$m_R(\xi) = m_0 + a(\xi) - 2\sum_{n\le R} \Lambda(n) n^{-1/2} \cos(\xi \log n), \quad p_R = (m_R)_+, \quad n_R = (-m_R)_+.$$

The cutoff is real, with $W_R = W_{\lfloor R \rfloor}$. For integers $M \ge 2$, $k \ge 1$, average $R = Me^v$ with density $v^{k-1}e^{-v}/(k-1)!$. The survival weights are

$$w_{M,k}(n) = \begin{cases} 1, & n \le M, \\ \dfrac{M}{n} \displaystyle\sum_{j=0}^{k-1} \frac{\log(n/M)^j}{j!}, & n > M. \end{cases} \tag{S8}$$

Let $m_{M,k} = m_0 + a - 2\Re S_{M,k}$, where $S_{M,k}(\xi) = \sum_n \Lambda(n) n^{-1/2+i\xi} w_{M,k}(n)$, and take $p_{M,k} = (m_{M,k})_+$, $n_{M,k} = (-m_{M,k})_+$ *after* averaging. The series is absolutely convergent at fixed M, k.

Theorem S3.1 (Exact smoothed realization and complete remainder)**.** *On the same original domain, and after any finite genuine-null projection,*

$$H = P_{M,k} - N_{M,k} + W_{M,k}, \qquad \|W_{M,k}\| \le d_{M,k} := \frac{451(M+1)e^{-\pi(M+1)}}{M^{k-1}(1-e^{-\pi})^k}. \tag{S9}$$

The bound holds for every integer $M \ge 2$, $k \ge 1$. Moreover

$$\|N_{M,k}\| \le B_{M,k} := 9[6 + 8(\log 2)\sqrt{M}\, 2^k]. \tag{S10}$$

The separately averaged positive and negative receivers have an exact common positive term $J_{M,k}$ such that

$$P_{M,k} = \mathbb{E}P_R - J_{M,k}, \qquad N_{M,k} = \mathbb{E}N_R - J_{M,k}.$$

Proof. Integration of the cutoff indicator gives (S8). The elementary estimate $\sum_{n\le R} \Lambda(n)/\sqrt{n} \le 4(\log 2)\sqrt{R}$ is proved in Section S2. Since $\mathbb{E}e^{v/2} = 2^k$, absolute Fubini applies and gives (S10), using $|m_0| < 6$, $a \ge 0$, $\|s\|_\infty < 3$. Every arithmetic perturbation is bounded at fixed M, k; its positive symbol differs from a by a bounded multiplier. The original minimal form domain is therefore unchanged.

Set $q = e^{-\pi}$ and $j = M + l$. The probability of the floor-cutoff cell is

$$\mu_j \le \frac{[(l+1)/M]^{k-1}}{(k-1)!} \frac{M}{j(j+1)}.$$

Indeed $v \le \log((j+1)/M) \le (l+1)/M$ in that cell. Multiply by the complete bound $451(j+1)^2 q^{j+1}$ from (10). As $M(j+1)/j \le M+1$ and $(l+1)^{k-1} \le (k-1)!\binom{l+k-1}{k-1}$, summing the binomial generating series proves (S9). The remainder is an average of bounded original-Hilbert operators. This extends the core identity through the paired original closure.

Pointwise put $A = \mathbb{E}p_R$, $B = \mathbb{E}n_R$. Then $A - B = m_{M,k}$ and $j = \min(A, B) = A - (m_{M,k})_+ = B - (-m_{M,k})_+ \ge 0$. The bounded operator $J_{M,k} = QM_s j(D) M_s Q$ gives both identities. The positive average is interpreted on V; its difference from the bounded negative average is the original signed expression. □

Corollary S3.2 (Fixed-seed certificate criterion)**.** *Fix an integer $M \geq 2$, $\kappa_k > 0$ and $\eta_k, \tau_k \geq 0$. On the original finite genuine-null complement suppose*

$$0 \preceq N_{M,k} - N_{M,k}^{\mathrm{fin}} \preceq \tau_k I, \qquad N_{M,k}^{\mathrm{fin}} \preceq (1+\eta_k)(P_{M,k} + \kappa_k I).$$

Then on all original V,

$$q_H[h] \geq -\left[d_{M,k} + \kappa_k + \frac{\tau_k + \eta_k B_{M,k}}{1+\eta_k}\right] \|h\|^2. \tag{S11}$$

In particular the complete inequalities with $\kappa_k = \tau_k = 2^{-k}$ and $\eta_k = 4^{-k}$ imply RH.

Proof. Rearrange $N \preceq (1+\eta)(P+\kappa I)+\tau I$ using the bounded N: $P - N \succeq -[\kappa+(\tau+\eta\|N\|)/(1+\eta)]I$. Add the complete remainder and use genuine-null transport as in Theorem 2.2. The displayed schedule tends to zero by (S9)–(S10). □

S3.1 An exact full-symbol evaluator

For $F(s) = -\zeta'(s)/\zeta(s)$ and $F_{>M}(s) = F(s) - \sum_{n \leq M} \Lambda(n) n^{-s}$, absolute differentiation of the Euler series gives

$$S_{M,k}(\xi) = \sum_{n \leq M} \Lambda(n) n^{-1/2+i\xi} + M \sum_{j=0}^{k-1} \frac{(-\partial_s - \log M)^j F_{>M}(s)}{j!}\Big|_{s=3/2-i\xi}. \tag{S12}$$

Thus the full arithmetic symbol is evaluable from a finite head and k derivatives in the absolutely convergent half-plane. A Taylor expansion $F_{>M}(s+x) = \sum_l a_l x^l$ makes its jth summand $\sum_{l=0}^{j} (-1)^l a_l (-\log M)^{j-l}/(j-l)!$. The paired operator remainder and the scalar evaluation error are separate quantities; both have explicit roles in (S11). For example $d_{16,1} < 5.122 \cdot 10^{-20}$, $d_{16,2} < 3.346 \cdot 10^{-21}$ and $d_{16,16} < 8.618 \cdot 10^{-38}$.

For fixed k the prime continuum has amplitude

$$\frac{e^{i\xi \log M}}{(1/2+i\xi)(1/2-i\xi)^k},$$

whose negative phase mean is $2^{k+2}\pi^{-1}(1+4\xi^2)^{-(k+1)/2}$. This fixed-$k$ continuum motivates the local reference for $k = 1$ below. The all-k remainder (S9) and the fixed-k mean have their stated, distinct parameter regimes.

S4 Complete sources and the actual inverse construction

The arithmetic realization of Section S3 is now fixed at averaging order one. We retain the complete negative source and construct preimages for its actual positive receiver. Throughout the main covariance arguments the original odd polar space is denoted by $\mathcal{X} = \mathcal{K}$, with minimal form domain V.

$$H = P_M - N_M + W_M, \qquad N_M = E_M E_M^*, \qquad \|W_M\| \leq \delta_M := d_{M,1},$$
$$R_M(\kappa) = (P_M + \kappa I)^{-1}, \qquad G_M(\kappa) = E_M^* R_M(\kappa) E_M.$$

Here $P_M = P_{M,1}$ and $N_M = N_{M,1}$ in the main argument; the appendices explicitly identify their separate hard-cutoff notation. The source space is $\mathcal{Y}_M = L^2(S_M, d\xi/\pi)$, $S_M = \{\xi > 0 : n_{M,1}(\xi) > 0\}$, with

$$E_M^* h(\xi) = \sqrt{n_{M,1}(\xi)} \int_{\mathbb{R}} s(u) h(u) \sin(\xi u)\, du.$$

For a finite genuine-null subspace K, $P_{M,K}$ is the same-form restriction to $K^\perp$ and $E_{M,K}$ its exactly projected source. All such inverses use the original norm. We write $\gamma_M(K;\kappa) = \|E_{M,K}^*(P_{M,K} + \kappa I)^{-1}E_{M,K}\|$.

An equivalent weighted-source convention is convenient for profiles: for $I \subset [0,\infty)$ and $\phi \in L^2(I, n_{M,1}\,d\xi/\pi)$,

$$g_\phi(u) = \int_I n_{M,1}(\xi)\phi(\xi)\sin(\xi u)\frac{d\xi}{\pi}, \quad E_I\phi = Q(sg_\phi), \quad D_\phi = \int_I n_{M,1}(\xi)|\phi(\xi)|^2\frac{d\xi}{\pi}.$$

Multiplication by $\sqrt{n_{M,1}}$ identifies this convention isometrically with the unweighted source on its positive support. Thus neither notation changes a covariance cap or omits mixed source terms.

Set $L_0 = 1 - 4\partial_u^2$, $B(D) = 8L_0^{-1}$ and

$$\mathcal{C}_Q = \frac{8}{\pi}QM_sL_0^{-1}M_sQ, \qquad \|f\|_{1,4}^2 = \int_{\mathbb{R}}(|f|^2 + 4|f'|^2).$$

The exact polar cancellation below supplies the actual inverse construction. The technical supplement gives separate inverse-form reference calibrations. The following exact polar cancellation is the ingredient needed for the actual arithmetic inverse.

Lemma S4.1 (Exact polar Green cancellation)**.** *On the original odd polar space let* $B(D) = 8(1+4D^2)^{-1}$ *and* $K_a = M_sB(D)\cos(aD)M_s$. *For every* $a \geq 0$,

$$\|Q_bK_aQ_b\|_{\mathrm{HS}} \leq 12e^{2a}e^{-\pi(e^a-1)}. \tag{S13}$$

Consequently, for either sign of the cosine and any further original orthogonal constraint projection Q *inside the polar space,*

$$\|QM_s[B(D)(\pm\cos(UD))_+]M_sQ - \mathcal{C}_Q\|_{\mathrm{HS}} \leq 12M^2e^{-\pi(M-1)}. \tag{S14}$$

Proof. The full kernel of K_a is $s(u)s(v)[e^{-|u-v-a|/2} + e^{-|u-v+a|/2}]$. Subtract the bounded rank-two kernel $2e^{-a/2}s(u)s(v)\cosh((u-v)/2)$ on the whole plane. Its even component vanishes on the odd space; its odd component is $-2e^{-a/2}|b\rangle\langle b|$ and is killed by Q_b. The exact remainder is

$$-2s(u)s(v)\sinh((|u-v|-a)/2)1_{|u-v|\geq a}.$$

Put $A = e^a$. On its support $r_u + r_v \geq 2\pi A$ by AM–GM. Its square is at most $A^{-1}s(u)^2s(v)^2e^{|u|+|v|}$. Split $e^{-(r_u+r_v)} \leq e^{-2\pi(A-1)}e^{-(r_u+r_v)/A}$. After enlarging the remaining integration to the whole plane, each factor is bounded by

$$\int_{\mathbb{R}} 16r^2e^{-r/A}e^{|u|}du \leq \frac{16}{\sqrt{\pi}}A^{5/2}\Gamma(5/2) = 12A^{5/2}.$$

Taking square roots proves (S13), including every atom and both infinite arms. The nonconstant real cosine coefficients of $\cos_+$ have absolute sum $1/2 + 1/\pi < 1$. The right side of (S13) decreases with $A \geq 1$; apply it to each harmonic $a = nU$. The constant coefficient is $1/\pi$, giving (S14). Further original projections contract the bound. □

S5 An actual inverse estimate

The exact polar phase cancellation also pays a positive lower of the *actual* averaged arithmetic receiver. This gives a whole-profile receiving theorem in the operator used by the signed criterion. Write $\overline{P}_M = P_{M,1}$, $\overline{n}_M = n_{M,1}$, $U = \log M$. For $0 < \Omega \le 1$, let E_Ω be the complete source map of Section S4 on $[0, \Omega]$, with the same true profile metric.

Theorem S5.1 (Uniform actual-positive origin bound)**.** *There are constants $C > 0$, M_0 such that for every integer $M \ge M_0$, every $0 < \Omega \le 1$ and every finite exact constraint space inside the original odd polar space,*

$$E_\Omega^*(\overline{P}_{M,Q} + M^{-1}I)^{-1}E_\Omega \preceq C\Omega^3(1 + \log\log M)^3 I. \tag{S15}$$

Here $\overline{P}_{M,Q}$ is the closed-form restriction of the same actual positive operator and the source uses its exact orthogonal projection. In particular, for $\Omega = 1/\log M$, the entire origin cap is $O((\log\log M/\log M)^3)$ and tends to zero. Every schedule $\Omega = o(1/\log\log M)$ gives a vanishing actual cap. For any fixed $a > 0$ the same conclusion holds at ridge M^{-a}, with constants depending on a.

Proof. First use only the original polar constraint Q_b. Normalize one arbitrary complex profile by $D_\phi = 1$, and put $A_\Omega = 16\sqrt{M}\Omega^3/(3\pi)$. Common-profile Cauchy–Schwarz gives

$$|g(u)| \le \sqrt{A_\Omega}|u|, \qquad |g^{(j)}(u)| \le \sqrt{A_\Omega}\Omega^{j-1} \quad (j \ge 1). \tag{S16}$$

All constants below are independent of this profile and Ω.

A phase-safe actual positive lower. For $E_\Lambda(t) = \Psi_\Lambda(t) - t$ define the complete Abel error

$$\overline{e}_M = \frac{3}{\sqrt{M}} + \frac{1}{\sqrt{M}}\int_1^M |E_\Lambda(t)|t^{-3/2}dt + \sqrt{M}\int_M^\infty |E_\Lambda(t)|t^{-5/2}dt.$$

Split the weighted prime integral at M and integrate by parts; the common boundary cancels because $\min(1, M/t)$ is continuous. The derivative moduli are at most $(1/2 + F)t^{-3/2}$ and $(3/2 + F)Mt^{-5/2}$. Together with the lower endpoint this gives

$$\sup_{|\xi|\le F}\left|\frac{S_{M,1}(\xi)}{\sqrt{M}} - \frac{e^{iU\xi}}{1/4 + \xi^2}\right| \le (2 + F)\overline{e}_M.$$

The unconditional quantitative PNT [13, Theorem 1.1] implies $|E_\Lambda(t)| \le C_N t/(\log t)^N$ for every fixed N. Splitting the first integral at $\sqrt{M}$ gives $\overline{e}_M = O_N(U^{-N})$; the second normalized integral has mass2. Take $F = U^5$ and $N = 20$. With $B(\xi) = 8/(1 + 4\xi^2)$,

$$\frac{m_{M,1}(\xi)}{\sqrt{M}B(\xi)} \ge -\cos(U\xi) - \varepsilon_M, \quad \varepsilon_M = \frac{3(1 + 4F^2)}{4\sqrt{M}} + \frac{(1 + 4F^2)(2 + F)}{4}\overline{e}_M \longrightarrow 0.$$

We used $a \ge 0$ and $m_0 \ge -6$. Eventually $\varepsilon_M \le 1/2$. The scalar positive-part inequality therefore gives the closed-form order $\overline{P}_M \succeq A_F$, where

$$A_F = \sqrt{M}Q_b M_s B(D)w(UD)1_{|D|\le F}M_s Q_b, \quad w(t) = (-\cos t - 1/2)_+.$$

This is a bounded positive operator. Its phase mean is $\mu = (\sqrt{3} - \pi/3)/(2\pi) > 0$.

A compact smooth preimage. Choose a fixed Gevrey-2 step χ, equal to one to the left of0 and zero to the right of1, with $|\chi^{(j)}| \le C_0 C_1^j (j!)^2$. Such a step is obtained from $\eta(t) = e^{-1/t}$ for $t > 0$ and0 otherwise, by $\chi(t) = \eta(1-t)/(\eta(t) + \eta(1-t))$. Cauchy's estimate on a disk of radius ct bounds derivatives of η by $j!(C/t)^j e^{-c'/t}$; maximization in t gives $C^j (j!)^2$. The denominator is bounded below near each endpoint, and the reciprocal Leibniz recurrence preserves this bound; on the remaining compact middle use a uniform analytic disk.

Put $R = \frac{1}{2}\log(2U/\pi)$ and $f(u) = \chi(|u| - R)g(u)$. For large U, $R \ge 1$, the cutoff is constant near zero and is smooth and even. Its transition has fixed physical width. Leibniz and (S16), with $\Omega \le 1$, give

$$\|f\|_{1,4}^2 \le C A_\Omega (R+1)^3, \qquad \|f^{(j)}\|_2 \le C\sqrt{A_\Omega}(R+1)^{3/2} C_1^j (j!)^2. \tag{S17}$$

Indeed the undifferentiated g is bounded by $\sqrt{A_\Omega}(R+1)$, each differentiated g by $\sqrt{A_\Omega}$, and the factorial ratios in the Leibniz sum are absorbed by one fixed exponential constant. Define

$$Y = \frac{L_0 f}{8\mu\sqrt{M}\, s}.$$

It is compact smooth odd and exactly polar legal: $\int \sinh(u/2) L_0 f = 0$ by two integrations by parts. On its support $r \le 2e^2 U$; the original lower theta bound gives

$$\|Y\|^2 \le C M^{2e^2-1} A_\Omega (R+1)^3. \tag{S18}$$

Complete residual and energy. Let A_∞ omit the frequency truncation in A_F. The cosine coefficients of w are absolutely summable, because its first derivative has bounded variation. Lemma S4.1 then gives

$$A_\infty = \mu\sqrt{M} Q_b M_s B(D) M_s Q_b + H_M, \qquad \|H_M\| \le h_M = C M^{5/2} e^{-\pi(M-1)}.$$

Since $B(D)L_0 = 8$ and $Q_b Y = Y$, its mean acts on Y as $Q_b(sf)$, and the exact truncation action is

$$(A_\infty - A_F)Y = \mu^{-1} Q_b M_s w(UD) 1_{|D|>F} f.$$

Thus the complete residual $e - A_F Y$ is the sum of $Q_b[s(g-f)]$, $-H_M Y$ and this Fourier exterior. All mixed terms are paid by the squared triangle bound with factor3. The full two-arm theta estimate gives

$$\|Q_b[s(g-f)]\|^2 \le C A_\Omega \int_{2U}^\infty r e^{-r} (\log r)^2 dr \le C A_\Omega U e^{-2U} (\log U)^2.$$

For $j = \lceil U \rceil$, (S17) gives

$$\|1_{|D|>F} f\| \le F^{-j} \|f^{(j)}\| \le C\sqrt{A_\Omega}(R+1)^{3/2} e^{-3U\log U + O(U)}.$$

Consequently $M\|e - A_F Y\|^2$ is Ω^3 times a negligible term; the polynomial trial cost (S18) is dominated by h_M. Also $A_F \preceq A_\infty$ gives

$$\langle Y, A_F Y\rangle \le \frac{\langle L_0 f, f\rangle}{8\mu\sqrt{M}} + h_M \|Y\|^2 \le C\Omega^3 (R+1)^3 + \text{smaller terms}.$$

For complex f, integration by parts gives $\langle L_0 f, f\rangle = \|f\|_{1,4}^2$.

For any bounded positive A and $\kappa > 0$, least squares gives

$$\langle e, (A+\kappa)^{-1} e\rangle \le \langle Y, AY\rangle + \kappa^{-1}\|e - AY\|^2. \tag{S19}$$

One obtains this by choosing $A^{1/2}Y$ as the coefficient in the regularized preimage formula. It is valid when A has a kernel. Apply it to A_F, $\kappa = M^{-1}$, and use inverse order from $A_F \preceq \overline{P}_M$. Equations above prove (S15) uniformly on the entire profile unit ball.

For any additional finite constraint space Z, the variational supremum for the restricted inverse is taken over $V \cap Z^{\perp}$, a subset of the original trial space. For the exactly projected source its value is therefore at most the original inverse energy. This gives the same constant for every such constraint space. The closed restriction is densely defined: approximate in the dense form domain and correct the finitely many row evaluations using fixed form-domain dual vectors. Finally, for fixed ridge exponent $a > 0$, replace $r_R = 2U$ by cU, where $c > a + 1/2$. The source residual costs $C\Omega^3 M^{a+1/2-c} U(\log U)^2$; the other errors stay negligible. This proves the last assertion. □

S6 Complete covariance at prescribed polynomial ridges

The complete negative source admits a quantitative estimate at every fixed polynomial ridge. We first isolate the averaged mean that supplies its lower bound and the signed critical characterizations. The upper bound is a specialization of the uniform, ridge-sensitive estimate proved in Section S10; that proof uses neither the critical characterizations nor their canonical consequences.

Lemma S6.1 (Averaged mean and uniform lower comparison)**.** *For each fixed v in the original polar Hilbert space,*

$$\frac{N_M[v]}{\sqrt{M}} \longrightarrow \mathcal{C}_{Q_b}[v] = \frac{8}{\pi}\langle sv, (1 - 4\partial_u^2)^{-1} sv\rangle.$$

For one fixed nonzero $v \in \operatorname{Dom} H \cap (\ker H)^{\perp}$, there is $C_v < \infty$, independent of K and $0 < \kappa \le 1$, such that

$$\gamma_M(K;\kappa) \ge 1 - C_v M^{-1/2}$$

for all sufficiently large M and every finite genuine associated-operator-null space K, with same-form restriction to $K^{\perp}$ and exactly projected source.

Proof. Write $U = \log M$ and $B(\xi) = 8/(1 + 4\xi^2)$. Take one nonzero $v \in \operatorname{Dom} H \cap (\ker H)^{\perp}$. Such a vector exists since the archimedean form is unbounded above on compact high-frequency packets, whereas its arithmetic perturbation is bounded. It belongs to every genuine-null complement. For completeness, the averaged quadratic mean used here follows directly from the complete Abel formula. On each fixed frequency band, $n_{M,1}/\sqrt{M} - B(\xi)(\cos(U\xi))_+ \to 0$ uniformly. The phase mean of $(\cos)_+$ is $1/\pi$, and its absolutely convergent Fourier series gives weak convergence against every integrable test on that band. The uniform complete bound $n_{M,1} \le 16\sqrt{M}$ pays the exterior of $|\widehat{sv}|^2$ by Plancherel. Hence

$$N_M[v]/\sqrt{M} \longrightarrow \langle v, \mathcal{C}_{Q_b} v\rangle =: c_v > 0.$$

The mean argument applies to every fixed polar v by the same Plancherel bound. For the chosen nonzero vector, strict positivity follows from $s > 0$ and $B(\xi) > 0$. This is the averaged order-two mean. The separate hard-cutoff order-one mean is proved in the technical supplement and is not used here. Therefore

$$\|G_{M,K}(\kappa)\| \ge \frac{N_M[v]}{N_M[v] + q_H[v] - \langle v, W_M v\rangle + \kappa\|v\|^2} \ge 1 - C_v M^{-1/2}$$

uniformly in K and $0 < \kappa \le 1$. □

Corollary S6.2 (Complete covariance at fixed polynomial ridges)**.** *For every fixed $b > 0$ and $\epsilon > 0$, there are $C_{b,\epsilon}, M_{b,\epsilon}$ such that, for $M \geq M_{b,\epsilon}$,*

$$\sup_K \left| \|E_{M,K}^*(P_{M,K} + M^{-b}I)^{-1}E_{M,K}\| - 1 \right| \leq \frac{C_{b,\epsilon}\sqrt{\log M}\exp[(8/\sqrt{\pi} + \epsilon)\sqrt{b\log M}]}{\sqrt{M}}. \tag{S20}$$

The supremum is over all finite genuine operator-null spaces, including moving and growing choices, with same-form restriction to $K^\perp$. The upper bound holds for every exact finite restriction with dense restricted form domain and exactly projected source. Every negative source frequency is included.

Proof. Choose $d = 4/\sqrt{\pi} + \epsilon/2$. Choose any fixed $a > 0$ with $d\sqrt{a} < D/3$, where $D > 0$ is the classical prime-error coefficient in Corollary S10.3. Put $U = \log M$ and $\rho = bU$. Since $F(U)^2/U \to \infty$ for $F(U) = U^{3/5}/(\log U)^{1/5}$, eventually $\rho \leq aF(U)^2$. That corollary gives the upper estimate, and Lemma S6.1 gives the lower estimate. The factor $\sqrt{1 + bU}$ is absorbed into $C_{b,\epsilon}\sqrt{U}$. □

At $\kappa = M^{-1}$, Corollary S6.2 establishes the complete covariance estimate

$$\gamma_M = 1 + O(M^{-1/2+o(1)}),$$

uniformly over finite genuine-null restrictions. Corollary S6.3 gives four signed characterizations. On the original polar recipient, without additional null removal, it is enough that $\sqrt{M}(\gamma_M - 1)_+$ remain bounded along an unbounded sequence. With arbitrary finite genuine-null restrictions, the characterization retains the vanishing condition $\liminf_{M\to\infty} \sqrt{M}(\gamma_M - 1)_+ = 0$. For moving constraints contained in one fixed finite canonical derivative-null span, part (iii) permits finite critical liminf as on the base recipient. Part (iv) admits cost-controlled growing canonical families at a fixed polynomial ridge, with each active span's correction costs paid. Appendices S14 and S17 develop complete signed certificates, exact null cancellation and physical lift estimates within this formulation. Corollary S14.4 proves that a fixed complete complementary gap together with the stated lift bound yields vanishing original-norm allowances and hence the Riemann hypothesis.

Corollary S6.3 (Critical covariance characterizations)**.** *The complete-source criteria have the following recipient-dependent forms.*

1. *On the original polar recipient put $\gamma_M^0(\kappa) = \gamma_M(\{0\};\kappa)$. For every positive schedule $\kappa_M \to 0$ with $\delta_M \leq \kappa_M$ eventually,*

$$H \succeq 0 \quad \Longleftrightarrow \quad \liminf_{M\to\infty} \sqrt{M}\,(\gamma_M^0(\kappa_M) - 1)_+ < \infty. \tag{S21}$$

   Under either condition, $\gamma_M^0(\kappa_M) \leq 1$ for all sufficiently large M. In particular, a complete excess $O(M^{-1/2})$ along an unbounded sequence at ridge M^{-1} implies RH.

2. *Let K_M be any sequence of finite genuine operator-null spaces and put $\gamma_M = \|E_{M,K_M}^*(P_{M,K_M} + M^{-1}I)^{-1}E_{M,K_M}\|$. Then*

$$H \succeq 0 \quad \Longleftrightarrow \quad \liminf_{M\to\infty} \sqrt{M}\,(\gamma_M - 1)_+ = 0.$$

3. *Fix $J \geq 0$ and let $K_0 = \operatorname{span}\{z_0, \ldots, z_J\}$, where the genuine derivative nulls z_j are defined in Lemma 8.1. Let $K_M \subset K_0$ for all sufficiently large M. For every positive $\kappa_M \to 0$ with $d_{M,1} \leq \kappa_M$ eventually,*

$$H \succeq 0 \quad \Longleftrightarrow \quad \liminf_{M\to\infty} \sqrt{M}\,(\gamma_M(K_M;\kappa_M) - 1)_+ < \infty.$$

   Under either condition, $\gamma_M(K_M;\kappa_M) \leq 1$ eventually.

4. *Fix $b > 0$ and a family $K_b(M)$ satisfying the cost and onset admission conditions of Theorem S9.1; this includes the prescribed orders in Corollary S9.2. If $K_M \subset K_b(M)$ eventually, then*

$$H \succeq 0 \quad \Longleftrightarrow \quad \liminf_{M\to\infty} \sqrt{M}\,(\gamma_M(K_M; M^{-b}) - 1)_+ < \infty.$$

Under either condition, $\gamma_M(K_M; M^{-b}) \le 1$ eventually.

All four statements characterize the original Weil positivity criterion and hence RH in the realization of Theorem 2.2.

Proof. (i) If $H \succeq 0$, the signed identity gives $N_M \preceq P_M + W_M \preceq P_M + \kappa_M I$ eventually, proving the complete cap. Conversely, choose one unbounded sequence along which $(\gamma_M^0(\kappa_M) - 1)_+ \le L/\sqrt{M}$ for a fixed finite $L \ge 0$. The full source variational inequality gives, on the original form domain,

$$-q_H[h] + \langle h, W_M h\rangle - \kappa_M \|h\|^2 \le \frac{L}{\sqrt{M}}(P_M[h] + \kappa_M\|h\|^2).$$

Fix a smooth compact odd polar-free h and put $f = sh$. The averaged quadratic mean in Lemma S6.1 and the exact signed identity give

$$\frac{N_M[h]}{\sqrt{M}} \longrightarrow \mathcal{C}_{Q_b}[h], \qquad \frac{P_M[h] + \kappa_M\|h\|^2}{\sqrt{M}} \longrightarrow \mathcal{C}_{Q_b}[h].$$

The two ridge and remainder terms on the left vanish. Thus

$$q_H[h] \ge -L\mathcal{C}_{Q_b}[h] = -\frac{8L}{\pi}\langle f, (1 - 4\partial_u^2)^{-1} f\rangle \ge -\frac{8L}{\pi}\|f\|_2^2.$$

Every compact smooth odd polar-free profile f has $h = f/s$ in the original core, because s is positive and smooth on its support. The same constant therefore gives a uniform unweighted floor across all compact supports. Proposition B.2 proves RH. The sequence and constant were fixed before taking each test limit.

(ii) If $H \succeq 0$, the signed identity gives $R_M^{1/2} N_M R_M^{1/2} \preceq I + (\delta_M - M^{-1})R_M \preceq I$ for all large M, also after any same-form restriction. Conversely, if H is not nonnegative, choose a fixed nonzero $v \in \operatorname{Dom} H \cap (\ker H)^\perp$ with $q_H[v] < 0$ by spectral projection onto a bounded negative interval. This vector belongs to every $K_M^\perp$. Its same-state quotient gives

$$\gamma_M - 1 \ge \frac{-q_H[v] + \langle v, W_M v\rangle - M^{-1}\|v\|^2}{P_M[v] + M^{-1}\|v\|^2} \ge c_v M^{-1/2}.$$

The last inequality uses the fixed positive numerator for large M and $P_M[v] = N_M[v] + q_H[v] - \langle v, W_M v\rangle = O_v(\sqrt{M})$. This contradicts the stated liminf condition, which therefore establishes $H \succeq 0$ on the original domain. The original Weil equivalence in Lemma A.1 now applies. Hence, under the stated critical covariance condition, the Riemann hypothesis holds.

(iii) Theorem S7.1 proves the fixed-span statement using exact genuine-null interpolation with arbitrarily small unweighted profile norm. Its variational restriction argument gives the stated moving-subspace consequence. That proof uses part (i)'s arithmetic-mean argument and the profile-floor theorem independently of the present part.

(iv) Theorem S9.1 applies to the chosen admitted family and pays its moving corrections at their actual cutoff. Its uniform profile-floor argument proves the criterion without invoking this part. For subspaces $K_M \subset K_b(M)$, the same-form variational quotient gives $\gamma_M(K_b(M); M^{-b}) \le \gamma_M(K_M; M^{-b})$. This transfers a bounded critical subsequence to the admitted family; the RH-to-cap direction holds on every same-form restriction. □

S7 Bounded critical covariance on canonical complements

The independent qualitative interpolation in Appendix C returns a constrained profile floor to the original recipient. Here the arithmetic limit is taken for each fixed vector before the profile limit. The proof needs neither Theorem 10.1 nor a support-uniform physical-norm budget.

Theorem S7.1 (Bounded critical excess with canonical restrictions)**.** *Fix a finite J and $K_0 = \operatorname{span}\{z_0, \dots, z_J\}$. Let K_M be subspaces of K_0 for all sufficiently large integers M. Let $\kappa_M > 0$, $\kappa_M \to 0$, and $\delta_M = d_{M,1} \le \kappa_M$ eventually. Use the complete order-one source and the same-form restrictions of Section S4, and put*

$$\gamma_{M,K_M} = \gamma_M(K_M; \kappa_M) = \|E^*_{M,K_M}(P_{M,K_M} + \kappa_M I)^{-1} E_{M,K_M}\|.$$

Then the following are equivalent:

1. *RH;*
2. *$\gamma_{M,K_M} \le 1$ for every sufficiently large M;*
3. *$\liminf_{M\to\infty} \sqrt{M}(\gamma_{M,K_M} - 1)_+ < \infty$.*

Under the negation of RH, $\sqrt{M}(\gamma_{M,K_M} - 1)_+ \to +\infty$.

Proof. First take a fixed $K \subset K_0$. Its finite null projection preserves V, so $V \cap K^\perp$ is dense in its Hilbert recipient. On that recipient $A_{M,K} = P_{M,K} + \kappa_M I$ is positive with bounded inverse. Equality of the norms of $A_{M,K}^{-1/2} E_{M,K}$ and its adjoint gives the full-source variational formula

$$\gamma_{M,K} = \sup_{0 \ne v \in V \cap K^\perp} \frac{N_M[v]}{P_M[v] + \kappa_M \|v\|_2^2}. \tag{S22}$$

Indeed $A_{M,K}^{-1/2}$ maps the entire recipient onto its form domain, and $\|E^*_{M,K} v\|^2 = N_M[v]$ there. The denominator is strictly positive. All forms on the right are the original forms on the common test; no source frequencies are removed.

Suppose the liminf is finite for this fixed K. Choose one unbounded sequence M_l and one finite $L \ge 0$ for which $(\gamma_{M_l,K} - 1)_+ \le L/\sqrt{M_l}$. Then for every $v \in V \cap K^\perp$,

$$N_{M_l}[v] \le (1 + L/\sqrt{M_l})(P_{M_l}[v] + \kappa_{M_l}\|v\|_2^2). \tag{S23}$$

The fixed-vector mean in the proof of Corollary S6.2 holds for every fixed physical L^2 vector, and in particular every fixed $v \in V$:

$$\frac{N_M[v]}{\sqrt{M}} \longrightarrow C[v] := \frac{8}{\pi}\langle sv, (1 - 4\partial_u^2)^{-1} sv\rangle \le \frac{8}{\pi}\|sv\|_2^2. \tag{S24}$$

The full-domain mean is Lemma S6.1; its complete-symbol and Plancherel argument applies to every fixed physical vector. The final inequality in (S24) follows from the multiplier bound $(1+4\xi^2)^{-1} \le 1$.

Now fix $v \in V \cap K^\perp$ and write $W_M[v] = \langle v, W_M v\rangle$. The exact identity (S9) gives

$$P_M[v] + \kappa_M\|v\|^2 = N_M[v] + q_H[v] - W_M[v] + \kappa_M\|v\|^2.$$

Consequently this expression divided by $\sqrt{M}$ tends to $C[v]$. Rearranging (S23) gives

$$-q_H[v] + W_{M_l}[v] - \kappa_{M_l}\|v\|^2 \le \frac{L}{\sqrt{M_l}}(P_{M_l}[v] + \kappa_{M_l}\|v\|^2).$$

Since v is fixed, its ridge and remainder terms vanish. We obtain

$$q_H[v] \geq -LC[v] \geq -\frac{8L}{\pi}\|sv\|_2^2 \qquad (v \in V \cap K^\perp).$$

Lemma C.1 transfers this floor with the same constant to the original recipient and proves RH. The arithmetic sequence and L were chosen first; the arithmetic limit was then taken for each fixed v. Only after obtaining the floor on the entire restricted domain are the null corrections and their profile limit used. Their physical norms need no uniform bound in this order of limits.

Conversely, RH gives $q_H \geq 0$ by Lemma A.1. The signed identity yields $N_M \preceq P_M + W_M \preceq P_M + \kappa_M I$ eventually, on V. Restriction and (S22) prove the cap for any of the stated recipients. The cap immediately implies finite liminf. This proves the equivalence for fixed K.

For the moving family, eventual containment gives $V \cap K_0^\perp \subset V \cap K_M^\perp$. The same-form quotient in (S22), at the same ridge, therefore gives

$$\gamma_{M,K_0} \leq \gamma_{M,K_M}.$$

Finite liminf for the moving family gives finite liminf for fixed K_0, hence RH by the fixed-space result. The reverse implication already holds on every restriction. Finally, in the absence of RH the fixed-K_0 nonnegative critical cost has no finite liminf, so it tends to $+\infty$; the displayed comparison proves the same for K_M. □

The fixed ambient finite span is the hypothesis that permits arbitrary motion of the subspaces in Theorem S7.1. It imposes a common bound on the derivative orders throughout the tail of the family. The interpolation theorem supplies arbitrarily small profile errors with exact constraints; its use here is through the successive fixed-vector arithmetic and profile limits above.

S8 From covariance growth to zero-free half-planes

The critical criterion uses a bounded normalized excess along an unbounded sequence. A bound that grows with the cutoff has a quantitative consequence when it holds at sufficiently many cutoffs. The transfer below keeps the arithmetic term in the unweighted profile norm and pays the reciprocal-theta cost only for the ridge and remainder.

We first use the original polar recipient $\mathcal{K}$ and the order-one averaged realization. Put

$$\gamma_M^0(\kappa) = \gamma_M(\{0\};\kappa), \qquad \eta_M = (\gamma_M^0(\kappa_M) - 1)_+, \qquad w(L) = \sup_{|u|\leq L} s(u)^{-2}.$$

For a compact polar-free profile f, write $\mathcal{Q}[f] = q_H[f/s]$, as in Appendix B.

Proposition S8.1 (A support-explicit covariance floor)**.** *For $M \geq 2$ and $\kappa_M > 0$, every real odd smooth polar-free profile f supported in $[-L, L]$ satisfies*

$$\mathcal{Q}[f] \geq -\left[(\kappa_M + d_{M,1})w(L) + \frac{16\sqrt{M}\,\eta_M}{1+\eta_M}\right]\|f\|_2^2. \tag{S25}$$

Moreover

$$w(L) \leq \frac{\exp(\pi e^{2L})}{2\pi^2 e^{4L}}. \tag{S26}$$

Proof. For the order-one survival weight, the elementary prime-power estimate used in Theorem S3.1 gives

$$\sum_{n\ge 2}\frac{\Lambda(n)}{\sqrt{n}}\min(1,M/n)\le 8(\log 2)\sqrt{M}.$$

Since $a\ge 0$ and $|m_0|<6$, the negative part of the complete averaged symbol obeys

$$0\le n_{M,1}(\xi)\le 6+16(\log 2)\sqrt{M}\le 16\sqrt{M},\qquad M\ge 2.$$

Plancherel therefore yields $N_M[h]\le 16\sqrt{M}\|sh\|_2^2$ on the original recipient. The covariance inequality is the form order $N_M\preceq(1+\eta_M)(P_M+\kappa_M I)$. Substitute $P_M=H+N_M-W_M$ and rearrange to obtain

$$q_H[h]\ge\langle h,W_Mh\rangle-\kappa_M\|h\|_2^2-\frac{\eta_M}{1+\eta_M}N_M[h].$$

For $h=f/s$ the compact support and positivity of s place h in the original core, and $\|h\|_2^2\le w(L)\|f\|_2^2$. The remainder bound proves (S25). The theta envelope $s(u)^2\ge 2r^2e^{-r}$, $r=\pi e^{2|u|}$, and the monotonicity of e^r/r^2 for $r\ge\pi>2$ give (S26). □

The quantitative canonical lift of Theorem 10.1 also pays the physical norm of exact null corrections at a prescribed cutoff. It gives the same zero-location exponent with canonical constraints.

Theorem S8.2 (Logarithmic covariance growth and controlled sampling)**.** *Fix $b>0$, $A\ge 0$ and a finite canonical derivative-null span K_J. Let $K_M\subset K_J$ for all sufficiently large M, and set $c_M=\sqrt{M}(\gamma_M(K_M;M^{-b})-1)_+$.*

1. *If, for some finite $C\ge 0$,*

$$c_M\le C(\log M)^A \tag{S27}$$

for every sufficiently large integer M, every nontrivial zero satisfies $\Re\rho\le 1/2+A$.

2. *If the same cap holds on one strictly increasing unbounded integer sequence M_l with*

$$q=\limsup_{l\to\infty}\frac{\log\log M_{l+1}}{\log\log M_l}<\infty,$$

then $\Re\rho\le 1/2+qA$. A cap with every $A>0$ and its own constant on this same sequence implies RH.

3. *At $A=0$, a bounded cap on an arbitrary unbounded subsequence implies RH, without a spacing hypothesis.*

Proof. Correct all of K_J using Theorem 10.1, say with $\theta=1$. Every sufficiently large supported profile has a legal $v=f/s-k$ on every recipient $K_M^\perp$, with

$$\|sv\|_2\le 2\|f\|_2,\qquad \|v\|_2^2\le C\exp[D_1e^{2L}]\|f\|_2^2.$$

Here the physical norm of f/s is paid by (S26), whose logarithm is at most $\pi e^{2L}+O(1)$, and is absorbed by the displayed budget. The complete-source quotient (S22) gives $N_M[v]\le(1+\eta_M)(P_M[v]+M^{-b}\|v\|_2^2)$, where $\eta_M=(\gamma_M(K_M;M^{-b})-1)_+$. The same rearrangement as in Proposition S8.1 holds on the full restricted form domain, giving

$$\mathcal{Q}[f]\ge-(M^{-b}+d_{M,1})\|v\|_2^2-\frac{16\sqrt{M}\,\eta_M}{1+\eta_M}\|sv\|_2^2. \tag{S28}$$

The scalar bound $N_M[v] \le 16\sqrt{M}\|sv\|_2^2$ applies also to these noncompact corrected vectors. All terms use the same original forms.

Choose $M = \lceil \exp(D_2 e^{2L}) \rceil$ with $bD_2 > D_1$. Then $D_2 e^{2L} \le \log M \le D_2 e^{2L} + 1$, and $d_{M,1} \le M^{-b}$ eventually. The first debit tends to zero uniformly over the supported profile unit ball, and the second is at most $64 c_M \|f\|_2^2$. The all-integer cap gives a floor

$$\mathcal{Q}[f] \ge -C' e^{2AL}\|f\|_2^2 - O(1)\|f\|_2^2.$$

Proposition S22.1 with $a = 2A$ therefore yields $\Re\rho \le 1/2 + A$ from this single schedule.

For sampled caps put $U_l = \log M_l$ and fix $q' > q$. The first sample with $U_l \ge D_2 e^{2L}$ satisfies eventually

$$U_l \le U_{l-1}^{q'} < \{D_2 e^{2L}\}^{q'}.$$

The lower bound pays the ridge and the upper bound gives covariance cost $O(e^{2q'AL})$. Apply the support proposition with $a = 2q'A$, then let $q' \downarrow q$ in the resulting closed boundaries. Repeating with every $A > 0$ and using zero symmetry proves RH. For $A = 0$ use Theorem S7.1, which already admits arbitrary cofinal sampling. Conversely RH gives an eventual cap of one on all these same-form recipients by the signed identity. □

Corollary S8.3 (Subpower logarithmic criterion)**.** *Fix $b > 0$ and a family K_M contained eventually in one fixed finite canonical span. RH is equivalent to the following assertion: for every $\varepsilon > 0$ there is a finite $C_\varepsilon \ge 0$ such that*

$$\sqrt{M}\,(\gamma_M(K_M; M^{-b}) - 1)_+ \le C_\varepsilon (\log M)^\varepsilon$$

for every sufficiently large integer M. In particular an eventual upper bound $C(\log\log M)^p$, for fixed $p \ge 0$, implies RH.

Proof. Under RH, the signed identity and $d_{M,1} \le M^{-b}$ give $\gamma_M(K_M; M^{-b}) \le 1$ eventually. Conversely apply Theorem S8.2 with every $A = \varepsilon > 0$ and use zero symmetry. The displayed example satisfies every such cap. □

S8.1 Why the physical payment matters

The qualitative route takes the arithmetic limit for each fixed corrected vector. The quantitative lift instead permits support-dependent corrections and pays their physical norm at the same arithmetic cutoff through (S28). It therefore preserves the zero-location exponent under canonical restrictions. The required arithmetic covariance cap remains a hypothesis.

The endpoint budget has exponent $D_J e^{2L}$, with uniform order witnesses in Theorem 10.1. Its single-row consequence matches the ambient logarithmic cost order. The exact genuine-null leading coefficient remains distinct from the ambient leading coefficient. Section S9 pays each active list's constants and onset, including for prescribed nonmonotone order schedules.

S9 Cost-admitted growing canonical families

The uniform estimate in Theorem 10.1 allows the canonical order to grow at a prescribed rate. Each correction must pay its physical cost at the same cutoff as its covariance bound. We first state that requirement for arbitrary finite-list witnesses. All forms and covariances below are complete and use the original recipient.

Fix $b > 0$. For each $J \geq 0$, choose witnesses $D_J, L_J \geq 1$ such that for every smooth compact odd polar-free profile supported in $[-L, L]$, $L \geq L_J$, the genuine null correction satisfies

$$\|sk\|_2 \leq \|f\|_2, \qquad \|k\|_2^2 \leq \exp(D_J e^{2L})\|f\|_2^2. \tag{S29}$$

Let $J(M)$ be any nonnegative integer schedule, not necessarily monotone, and let

$$K(M) \subset \operatorname{span}\{z_0, \dots, z_{J(M)}\}. \tag{S30}$$

Put

$$U = \log M, \quad E_M = \max\{1, \pi, D_{J(M)}\}, \quad \ell_M = \tfrac{1}{2} \log \frac{bU}{8E_M}. \tag{S31}$$

The admission conditions are

$$\ell_M \geq L_{J(M)} \quad \text{eventually}, \qquad \ell_M \to \infty. \tag{S32}$$

Theorem S9.1 (Critical and logarithmic criteria for cost-admitted families)**.** *Suppose* (S32) *holds. At ridge M^{-b}, set $c_M = \sqrt{M}(\gamma_M(K(M); M^{-b}) - 1)_+$. Eventually every smooth compact odd polar-free profile supported in $[-\ell_M, \ell_M]$ satisfies*

$$\mathcal{Q}[f] \geq -[8e^{-7b \log M/8} + 64c_M]\|f\|_2^2. \tag{S33}$$

Consequently

$$\text{RH} \iff \gamma_M(K(M); M^{-b}) \leq 1 \text{ eventually} \iff \liminf_{M \to \infty} c_M < \infty. \tag{S34}$$

If RH fails, $c_M \to \infty$. Suppose additionally that $r = \liminf 2\ell_M / \log\log M > 0$. An eventual cap $c_M \leq C(\log M)^A$, $A \geq 0$, at all integers gives $\Re\rho \leq 1/2 + A/r$ for every nontrivial zero. If the cap holds only on an increasing unbounded sequence M_l with

$$q = \limsup_l \frac{\log\log M_{l+1}}{\log\log M_l} < \infty,$$

the conclusion is $\Re\rho \leq 1/2 + qA/r$. A bounded cap on an arbitrary unbounded subsequence implies RH without a spacing condition. In either regime, caps with every $A > 0$ and its own constant imply RH.

Proof. Use the active list's lift at $L = \ell_M$. Then $v = f/s - k$ lies in $V \cap K(M)^\perp$, $q_H[v] = \mathcal{Q}[f]$, and $\|sv\|_2 \leq 2\|f\|_2$. Eventually $\ell_M \geq 1$, so (S26) gives

$$D_{J(M)} e^{2\ell_M} \leq bU/8, \qquad \log w(\ell_M) \leq \pi e^{2\ell_M} \leq bU/8. \tag{S35}$$

Hence

$$\|v\|_2^2 \leq 2\{w(\ell_M) + e^{bU/8}\}\|f\|_2^2 \leq 4e^{bU/8}\|f\|_2^2. \tag{S36}$$

Apply (S28) to this generally noncompact vector. The bound $N_M[v] \leq 16\sqrt{M}\|sv\|_2^2$ is global, and $d_{M,1} \leq M^{-b}$ eventually, independently of the list. The physical debit is at most $8e^{-7bU/8}\|f\|_2^2$ and the source debit at most $64c_M\|f\|_2^2$, proving the floor.

If $\liminf c_M < \infty$, choose an unbounded subsequence with $c_M \leq C$ for one finite C. Every fixed compact admissible profile eventually lies in its windows, so its unchanged form value satisfies $\mathcal{Q}[f] \geq -64C\|f\|_2^2$. One constant works over all supports. Proposition B.2 proves RH. Only scalar bounds are passed to a limit; the corrected vectors need not converge. Conversely, RH and the

signed identity give $N_M \preceq P_M + d_{M,1} I \preceq P_M + M^{-b} I$ eventually on every recipient. This proves the equivalences and the divergence assertion.

Fix $0 < r' < r$. Eventually $\ell_M \geq (r'/2) \log\log M$. For support radius L choose

$$M(L) = \lceil \exp(\exp(2L/r')) \rceil . \tag{S37}$$

Then $\ell_{M(L)} \geq L$ and $\log M(L) \leq e^{2L/r'} + 1$. The floor and cap give $\mathcal{Q}[f] \geq -C' e^{2AL/r'} \|f\|_2^2$ uniformly over supported profiles. Proposition S22.1 gives $\Re\rho \leq 1/2 + A/r'$. Let $r' \uparrow r$. For samples fix $q' > q$ and choose the first sample with $U_l = \log M_l \geq e^{2L/r'}$. Eventually $U_l \leq U_{l-1}^{q'} < e^{2q'L/r'}$. The lower threshold pays the support, and the upper threshold pays the cap. Apply the same proposition and let $q' \downarrow q$, $r' \uparrow r$. Bounded sampling follows from the critical criterion directly. Separate applications for every $A > 0$, followed by zero symmetry, prove the last assertion. □

If $\delta = \limsup \log E_M / \log\log M < 1$ and the onset condition is paid, the definition gives $r = 1 - \delta$. Thus physical cost, not dimension alone, determines the loss in the zero-location exponent.

Corollary S9.2 (Prescribed order growth at a fixed ridge)**.** *Write $W = \log\log M$ and $n(M) = J(M) + 1$. Every prescribed schedule with*

$$n(M) = o(W) \tag{S38}$$

is cost-admitted with $r = 1$. In particular one may prescribe $J(M) = o(W)$ without monotonicity. The critical equivalence and zero boundaries $1/2 + A$ and $1/2 + qA$ are retained. For each fixed $\mu > 2$, corrections are also admitted on support $L = W/\mu$, with $\|k\|_2^2 \leq M^{o(1)} \|f\|_2^2$.

Proof. By Theorem 10.1, one constant C_a gives $D_J \leq e^{C_a(J+1)}$ and $L_J \leq C_a(J+1)$, using the fixed tolerance $\theta = 1$ throughout. Thus $\log E_M = o(W)$ and $\ell_M = W/2 - o(W)$ eventually exceeds $L_{J(M)} = o(W)$. In particular the onset is paid for the prescribed list itself, not by selecting a slower subsequence. This proves admission and $r = 1$. At $L = W/\mu$ the onset is again paid and the logarithm of the squared-norm budget is

$$D_{J(M)} e^{2L} \leq \exp(C_a n(M) + 2W/\mu) = (\log M)^{2/\mu + o(1)} = o(\log M).$$

The reciprocal-theta cost has the same subpower property, so M^{-b} pays both physical terms. □

A small positive multiple of the boundary order scale is also admitted, with a symbolic exponent loss. Enlarge C_a to cover both witnesses. If $J(M) = \lfloor cW \rfloor$, $c > 0$, and $\delta = C_a c < 1/3$, then $n(M)/W \to c$. The alternative radius

$$\ell_M^* = \tfrac{1}{2}\{W - C_a n(M) - \sqrt{W}\}$$

has $2\ell_M^*/W \to 1 - \delta$ and pays the onset because $(1 - \delta)/2 > \delta$. Also $D_{J(M)} e^{2\ell_M^*} \leq U e^{-\sqrt{W}} = o(U)$ and $\pi e^{2\ell_M^*} \leq \pi U e^{-\sqrt{W}} = o(U)$. The proof above applies eventually and gives boundaries $1/2 + A/(1 - \delta)$ and $1/2 + qA/(1 - \delta)$. The constant C_a is not numerically evaluated here; these sufficient rates do not assert an optimal order law.

S10 Uniform ridge-sensitive covariance windows

Write $U = \log M$ and $\rho = \log(1/\kappa)$. Retaining the paired arithmetic tail and the full logarithmic archimedean lower bound makes the source price depend on ρ itself. We prove a uniform estimate on a power-input window and then establish its classical-input version. Throughout, the recipient is the original odd polar Hilbert space, with its original norm and minimal closed form domain V, and the averaging parameter is one. All restricted sources are exactly projected.

Theorem S10.1 (Uniform ridge-sensitive window)**.** *Suppose, for fixed* $1/2 < \sigma < 1$ *and* $A \geq 0$*, that*

$$|\Psi_\Lambda(x) - x| \leq C_P x^\sigma (1 + \log x)^A \qquad (x \geq 2). \tag{S39}$$

Fix $d > 4/\sqrt{\pi}$ *and* $a > 0$ *with* $d\sqrt{a} < (1-\sigma)/3$*. Then, for every sufficiently large integer* M*, simultaneously for* $0 \leq \rho \leq a(\log M)^2$*,*

$$\sup_K |\gamma_M(K; e^{-\rho}) - 1| \leq \frac{C\sqrt{1+\rho}\exp(2d\sqrt{\rho})}{\sqrt{M}}. \tag{S40}$$

One constant and one onset work throughout this interval. They may depend on d, a, σ, A, C_P *and the fixed realization, but not on* ρ*,* K *or its dimension. The two-sided estimate holds for all finite genuine associated-operator-null spaces* K*, including moving and growing choices. The upper estimate holds for every exact finite same-form restriction with dense restricted form domain. Every negative source frequency is retained. A choice of* d *exists for every*

$$0 < a < \frac{\pi(1-\sigma)^2}{144}.$$

S10.1 The complete Abel error and its decay coefficient

For $E_\Lambda(x) = \Psi_\Lambda(x) - x$, let $\bar{e}_M$ denote the complete Abel error of Section S5, including both integrals and the term $3/\sqrt{M}$. In particular, Ψ_Λ here counts prime powers and is distinct from the digamma function Ψ.

Lemma S10.2 (Two-arm Abel convolution)**.** *Put* $r(v) = e^{-v}|E_\Lambda(e^v)|$ *for* $v \geq 0$*. For* $U = \log M$*,*

$$\bar{e}_{e^U} = 3e^{-U/2} + \int_0^\infty e^{-|v-U|/2} r(v)\, dv. \tag{S41}$$

Suppose r *is locally integrable and, eventually,* $r(v) \leq Ce^{-H(v)}$*, where* H *is differentiable, eventually nondecreasing,* $H(v) \to \infty$*,* $H(v) = o(v)$ *and* $H'(v) \to 0$*. Then*

$$\limsup_{U\to\infty} e^{H(U)} \bar{e}_{e^U} \leq 4C. \tag{S42}$$

The factor four is sharp for this envelope class.

Proof. In each nonnegative integral defining $\bar{e}_M$, substitute $t = e^v$. The integral below M becomes $\int_0^U e^{-(U-v)/2} r(v)\, dv$, and the integral above M becomes $\int_U^\infty e^{-(v-U)/2} r(v)\, dv$. This proves the identity and retains the endpoint contribution.

Fix $0 < \eta < 1/2$ and take v_0 sufficiently large that the envelope holds and $0 \leq H' \leq \eta$ on $[v_0, \infty)$. For $v_0 \leq v \leq U$, $H(U) - H(v) \leq \eta(U - v)$, so the normalized left integral is at most $C/(1/2 - \eta)$. Monotonicity bounds the normalized right integral by $2C$. The compact contribution and the separate endpoint term are at most

$$e^{-U/2 + H(U)} \left(3 + \int_0^{v_0} e^{v/2} r(v)\, dv\right) = o(1).$$

Take the limsup and then let $\eta \downarrow 0$. If $r(v) = Ce^{-H(v)}$ eventually, each fixed offset from U has $H(U+w) - H(U) \to 0$. The same η bound dominates the left offsets by an integrable exponential, and monotonicity dominates the right ones. Dominated convergence therefore gives the limit $4C$ for this model envelope. This equality is not required of the actual prime error. □

Set

$$F(U)=\frac{U^{3/5}}{(\log U)^{1/5}},\qquad D=\frac{5}{2}\left(\frac{5}{3}\right)^{1/5}\left(\frac{2000}{161967}\right)^{3/5},\qquad H_D(U)=D\left(1+\frac{\log\log U}{15\log U}\right)F(U). \tag{S43}$$

The classical input [9, Theorem 1.1] gives $r(v)\le 0.239e^{-H_D(v)}$ for $v\ge 3$. Since

$$\frac{F'(U)}{F(U)}=\frac{3/5-1/(5\log U)}{U},\qquad \left(\frac{\log\log U}{15\log U}\right)'=\frac{1-\log\log U}{15U(\log U)^2},$$

H_D is eventually increasing, $H_D'=o(1)$ and $H_D=o(U)$. Lemma S10.2 yields

$$\bar e_M\le(0.956+o(1))e^{-H_D(\log M)}=O(e^{-DF(\log M)}). \tag{S44}$$

The secondary term remains inside the exponential. The same lemma preserves the coefficient 0.1853 in the older Johnston–Yang estimate [13, Theorem 1.4]: apply it to $H(U)=0.1853F(U)-1.801\log U$.

For the power input (S39), the two integrals instead give

$$\bar e_M\le C_{\sigma,A}M^{\sigma-1}(1+\log M)^A. \tag{S45}$$

Indeed, on the left use $(1+\log t)^A\le(1+U)^A$ and $\int_1^M t^{\sigma-3/2}dt\le C_\sigma M^{\sigma-1/2}$. On the right put $t=Me^w$ and use $1+U+w\le(1+U)(1+w)$; the remaining integral is bounded by $\int_0^\infty e^{-(3/2-\sigma)w}(1+w)^A\,dw<\infty$. The compact initial range and $3/\sqrt{M}$ are absorbed because $\sigma>1/2$.

S10.2 Proof of the uniform covariance estimate

Proof of Theorem S10.1.

A negative-source majorant with its actual bandwidth. For each $\rho\ge 0$ put $\kappa=e^{-\rho}$ and choose the least integer $m\ge 2$ such that

$$451(m+1)^2e^{-\pi(m+1)}\le\kappa/4. \tag{S46}$$

Lemma S2.1 and its proof give, as forms on V,

$$H=Q_bs(a+b_m)(D)sQ_b+V_m,\qquad \|V_m\|\le 451(m+1)^2e^{-\pi(m+1)},$$
$$b_m(\xi)=m_0-2\sum_{2\le n\le m}\frac{\Lambda(n)}{\sqrt{n}}\cos(\xi\log n).$$

This is the complete paired omitted shift operator, before any separation of its edge pieces. Since $k^2e^{-\pi k}$ decreases for $k\ge 3$, minimality in (S46) gives

$$m=\rho/\pi+O(\log(2+\rho))\qquad(\rho\to\infty).$$

For example, compare the logarithms at the last failing and first successful integers: $\pi(m+1)-2\log(m+1)=\rho+O(1)$. Define

$$S_m=\sum_{2\le n\le m}\frac{\Lambda(n)}{\sqrt{n}},\qquad D_m=-m_0+2S_m.$$

The classical prime number theorem and partial summation give

$$S_m = \frac{\Psi_\Lambda(m)}{\sqrt{m}} + \frac{1}{2}\int_1^m \Psi_\Lambda(t) t^{-3/2}\,dt = 2\sqrt{m} + o(\sqrt{m}).$$

To obtain the last error, split at a fixed large t_0, use an arbitrarily small relative PNT error above t_0, and then let that relative error tend to zero. Thus, for the fixed $d > 4/\sqrt{\pi}$, uniformly for $\rho \ge 0$,

$$0 < D_m \le d\sqrt{\rho} + C_d, \qquad D_m \le C_d\sqrt{1+\rho}.$$

Bounded ρ intervals contain only finitely many auxiliary cutoffs. Because $b_m \ge -D_m$ and $a(\xi) \ge \frac{1}{2}\log(1+4\xi^2)$ by (S2),

$$0 \le v_m := (-(a+b_m))_+ \le D_m, \quad \operatorname{supp} v_m \subset [-L_m, L_m], \quad L_m = \tfrac{1}{2}\sqrt{e^{2D_m}-1} \le C_d e^{d\sqrt{\rho}}. \tag{S47}$$

On the unitary odd Fourier space put $T_m = v_m(D)^{1/2}$, $E = Q_b s T_m$ and $S = \|s\|_\infty < 3$. Choose an even Gevrey-2 cutoff χ_R, equal to one on $[-R, R]$, zero off $[-R-1, R+1]$, and with $\|\chi_R^{(j)}\|_\infty \le C_0 C_1^j (j!)^2$ independently of R. Set

$$X_0 = \log(104 S D_m/\kappa), \quad R = \tfrac{1}{2}\log(2X_0), \quad E_0 = Q_b s \chi_R T_m.$$

These expressions are defined also at $\rho = 0$: the elementary bounds $-m_0 > 5$ and $S \ge \sqrt{2\pi^2 e^{-\pi}}$ imply $R > 1$. The complete theta envelope gives

$$t_R := \sup_{|u|\ge R} s(u) \le 26 X_0 e^{-3X_0} \le 13 e^{-X_0} = \kappa/(8SD_m).$$

Consequently $\|E - E_0\| \le t_R\sqrt{D_m}$ and $\|EE^* - E_0E_0^*\| \le 2SD_m t_R \le \kappa/4$. Scalar sign decomposition and the paired tail therefore give

$$H \succeq -E_0E_0^* - (\kappa/2)I \qquad \text{on } V. \tag{S48}$$

The entire coefficient ball in the actual inverse. Choose one τ with

$$d\sqrt{a} < \tau < \min\{1/4, (1-\sigma)/3\}, \qquad T = M^\tau.$$

This is possible since $\sigma > 1/2$. By (S45), the exact phase debit of Section S5 satisfies

$$\varepsilon_M(T) = \frac{3(1+4T^2)}{4\sqrt{M}} + \frac{(1+4T^2)(T+2)}{4}\bar{e}_M = o(1).$$

Thus the actual positive form P_M dominates the bounded positive operator

$$A_T = \sqrt{M} Q_b s B(D) w(UD) 1_{|D|\le T} s Q_b, \quad B(\xi) = \frac{8}{1+4\xi^2}, \quad w(t) = (-\cos t - 1/2)_+.$$

Write $\mu = (\sqrt{3} - \pi/3)/(2\pi) > 0$ for the phase mean. The same complete harmonic estimate as in Section S5 gives

$$A_\infty = \mu\sqrt{M} Q_b s B(D) s Q_b + J_M, \qquad \|J_M\| \le C M^{5/2} e^{-\pi(M-1)}, \qquad A_T \preceq A_\infty.$$

For any complex coefficient $\|\phi\| \le 1$, put $g = T_m\phi$, $f = \chi_R g$, $e = E_0\phi$. Band limitation, Plancherel and Leibniz' rule give

$$\|f\|_{1,4}^2 := \|f\|_2^2 + 4\|f'\|_2^2 \le C D_m(1+L_m^2) \le C_d\sqrt{1+\rho}\, e^{2d\sqrt{\rho}}, \qquad \|f^{(j)}\|_2 \le C\sqrt{D_m}(L_m + Cj^2)^j.$$

The constants are independent of the physical radius R: the transition has fixed width, and all estimates use global L^2 norms of g. For $j = \lceil U^2 \rceil$, $\rho \le aU^2$ implies $L_m \le C_d M^{d\sqrt{a}}$ and hence, uniformly,

$$\|1_{|D|>T} f\|_2 \le T^{-j} \|f^{(j)}\|_2 \le C\sqrt{1+U}\, e^{-cU^3}. \tag{S49}$$

Here $\log[(L_m + Cj^2)/T] \le -cU$ eventually by the fixed gap $\tau - d\sqrt{a} > 0$.

Use the compact smooth odd trial

$$Y = \frac{(1 - 4\partial_u^2) f}{8\mu\sqrt{M}\, s}.$$

Division by s is legal on its compact support; integration by parts against $\sinh(u/2)$ proves its polar moment is exactly zero. On $|u| \le R + 1$, the lower theta bound $s(u)^2 \ge 2r_u^2 e^{-r_u}$, $r_u = \pi e^{2|u|}$, and $r_u \le CX_0$ give

$$\|Y\|^2 \le CM^{-1} e^{CX_0} D_m (1 + L_m^4) \le e^{C_a U^2}, \qquad X_0 = \rho + O(\log(2+\rho)).$$

This pays the original Hilbert norm on the moving support. The exact mean action and the complete residual are

$$\mu\sqrt{M} Q_b s B(D) s Q_b Y = e, \qquad e - A_T Y = -J_M Y + \mu^{-1} Q_b s w(UD) 1_{|D|>T} f.$$

The squared triangle inequality pays the mixed term. In view of $\kappa^{-1} \le e^{aU^2}$, (S49) and the full harmonic bound, for each fixed N both $\kappa^{-1}\|e - A_T Y\|^2$ and $\|J_M\|\|Y\|^2$ are $o(M^{-N})$ uniformly. Moreover

$$\langle Y, A_T Y \rangle \le CM^{-1/2} \|f\|_{1,4}^2 + \|J_M\|\|Y\|^2.$$

The kernel-safe trial inequality (S19) and inverse order from $P_M \succeq A_T$ now prove, on the entire coefficient ball,

$$\|E_0^* (P_M + \kappa I)^{-1} E_0\| \le \frac{C\sqrt{1+\rho} e^{2d\sqrt{\rho}}}{\sqrt{M}}. \tag{S50}$$

The negligible terms are absorbed uniformly because the right-hand scale is at least $M^{-1/2}$.

Return to the complete signed source. Write $R_M = (P_M + \kappa I)^{-1}$. The exact identity $N_M = P_M + W_M - H$, with $N_M = E_M E_M^*$ and $\|W_M\| \le \delta_M = d_{M,1} = O(Me^{-\pi M})$, gives

$$R_M^{1/2} N_M R_M^{1/2} \preceq I + R_M^{1/2} E_0 E_0^* R_M^{1/2} + (\delta_M - \kappa/2) R_M.$$

The last term is nonpositive eventually throughout the window. This is a form comparison on V; it retains every mixed source direction. Equality of the physical and source operator norms gives the complete upper bound. The inverse of any same-form restriction, extended by zero, is below the base inverse by the variational principle, so exactly projected sources satisfy the same upper bound without commuting a projection through an inverse. Lemma S6.1 gives the lower bound uniformly for genuine-null restrictions. Finally, the interval $4/\sqrt{\pi} < d < (1-\sigma)/(3\sqrt{a})$ is nonempty exactly when $a < \pi(1-\sigma)^2/144$. □

Corollary S10.3 (A classical-input window)**.** *Let F and D be as in* (S43)*. Fix $d > 4/\sqrt{\pi}$ and $a > 0$ with $d\sqrt{a} < D/3$. Uniformly for all sufficiently large integers M and $0 \le \rho \le aF(\log M)^2$,*

$$\sup_K |\gamma_M(K; e^{-\rho}) - 1| \le \frac{C_{a,d}\sqrt{1+\rho} e^{2d\sqrt{\rho}}}{\sqrt{M}},$$

with the same source, domain and restriction conventions as Theorem S10.1. A choice of d is possible for every $0 < a < \pi D^2/144$.

Proof. Use (S44) in place of (S45), choose $d\sqrt{a} < \tau < D/3$, and set $T = e^{\tau F(U)}$. Then $T^3\bar{e}_M = O(e^{-(D-3\tau)F(U)}) \to 0$ and $T^2/\sqrt{M} \to 0$, since $F(U) = o(U)$. The auxiliary construction is unchanged and gives $L_m \le C_d e^{d\sqrt{a}F(U)}$. For $j = \lceil U^2 \rceil$, now

$$\log \frac{L_m + Cj^2}{T} \le -cF(U), \qquad \|1_{|D|>T} f\|_2 \le C\sqrt{1+F(U)}e^{-cU^2F(U)}.$$

Here $F(U)/\log U \to \infty$. Since $U^2F(U)$ dominates both $F(U)^2$ and U, this exterior pays $\kappa^{-1} \le e^{aF(U)^2}$ to every inverse power of M. The trial norm is at most $e^{C_a F(U)^2}$ and is paid by the complete harmonic remainder. Retaining $D_m(1+L_m^2)$ in the energy gives the displayed dependence on ρ. The signed return pays $\delta_M \le \kappa/2$ uniformly, and the same fixed-vector lower comparison applies. The strict coefficient condition is precisely the existence of the indicated d and τ. □

S10.3 Zero-free inputs and the power window

If ζ has no zeros in $\Re s > \beta$, $1/2 < \beta < 1$, then for each fixed buffered exponent $\beta < \sigma < 1$, (S39) holds with $A = 2$. Indeed, choose $\beta < \sigma_0 < \sigma$ and use truncated Perron inversion for $-\zeta'/\zeta$ at $c = 1 + 1/\log x$, initially with x half-integral and height $T = x^2$. Zero-free disks a fixed distance inside $\Re s > \beta$, Borel–Carathéodory and Cauchy's estimate applied to the Euler-normalized logarithm give $\zeta'/\zeta(s) = O_{\sigma_0}(\log(3 + |\Im s|))$ in the required strip, away from the pole. Shift to $\Re s = \sigma_0$. The pole contributes x; the vertical integral is $O_{\sigma_0}(x^{\sigma_0}\log^2 T)$ and the horizontal and truncation errors are $O_{\sigma_0}(x \log^2 x/T)$. Bracketing arbitrary x by adjacent half-integers and using monotonicity of Ψ_Λ gives the claimed bound. The margin $\sigma > \beta$ has not been removed. Consequently every

$$0 < a < \frac{\pi(1-\beta)^2}{144}$$

is admissible after choosing a fixed $\sigma > \beta$ sufficiently close to β and then $4/\sqrt{\pi} < d < (1-\sigma)/(3\sqrt{a})$.

For the zeta specialization of the imported $\beta = 7/8$ result [17, Theorem 1.1], for example, take $a = 10^{-6}$, $\sigma = 9/10$, $d = 5/2$ and $\tau = 7/250$. These choices give the simultaneous upper error $O((1+U)M^{-1/2+1/200})$ for $e^{-10^{-6}U^2} \le \kappa \le 1$. Alternatively, an endpoint prime-error input can be used at its closed zero boundary: [9, Theorem 1.4] gives, for $3/4 \le \beta < 1$, if every nontrivial zeta zero satisfies $\Re z \le \beta$,

$$|E_\Lambda(x)| < 0.04934 x^{3/4}(\log x)^2 + C_\beta x^\beta \quad (x > e^3).$$

At $\beta = 7/8$ the tabulated coefficient is $C_\beta = 0.02636425$; for $13/16 < \beta < 7/8$ the applicable coefficient is 7.396193. Each valid endpoint estimate supplies (S39) with $\sigma = \beta$, after enlarging a constant on the initial compact range. It is a separate input from the buffered-contour deduction.

Both windows eventually contain every fixed polynomial ridge. The classical result controls ρ on the scale $U^{6/5}/(\log U)^{2/5}$, whereas a fixed power saving controls the scale U^2. These are regularization estimates for the same complete receiver; their proofs use the stated prime-error inputs.

Corollary S10.4 (Signed criterion throughout the ridge window)**.** *Under the hypotheses of Theorem S10.1, let $e^{-a(\log M)^2} \le \kappa_M \le 1$ with $\kappa_M \to 0$, and let K_M be any sequence of finite genuine operator-null spaces. Then*

$$\text{RH} \iff \liminf_{M\to\infty} \sqrt{M}\,(\gamma_M(K_M;\kappa_M) - 1)_+ = 0.$$

Proof. If $H \succeq 0$, the signed identity and $d_{M,1} \le \kappa_M$ give $\gamma_M(K_M;\kappa_M) \le 1$ for large M. Conversely, a fixed negative vector in $\operatorname{Dom} H \cap (\ker H)^{\perp}$ belongs to every $K_M^{\perp}$. Its same-state quotient, as in Corollary S6.3, has positive numerator $-q_H[v] + \langle v, W_M v\rangle - \kappa_M\|v\|^2$ bounded below and denominator $P_M[v] + \kappa_M\|v\|^2 = O_v(\sqrt{M})$. It therefore forces a positive excess at least $c_v M^{-1/2}$. □

Proposition S10.5 (Majorant cost under vanishing regularization)**.** *Let $0 < \kappa_M \le 1$ tend to zero, and construct E_0 as in the proof of Theorem S10.1, with*

$$m_M = \min\{m \ge 2 : 451(m+1)^2 e^{-\pi(m+1)} \le \kappa_M/4\}.$$

There is $c > 0$ such that, for all sufficiently large M,

$$\|E_0^*(P_M + \kappa_M I)^{-1} E_0\| \ge \frac{c\sqrt{m_M}}{\sqrt{M}(\log m_M)^3}.$$

In particular, its product with $\sqrt{M}$ tends to infinity. At $\kappa_M = e^{-a(\log M)^2}$ the lower bound is $c_a(\log M)/[\sqrt{M}(\log\log M)^3]$.

Proof. Use the fixed compact smooth odd profile f in Supplementary Proposition S30.1, with exact polar moment zero and nonzero first moment, and put $y = f/s$. Now $m_M \to \infty$ and the physical cutoff radius also tends to infinity, so $E_0^* y = T_{m_M} f$ for large M. On $|\xi| \le 1/(2\log m_M)$, the same prime-number estimate gives $v_{m_M}(\xi) \ge c\sqrt{m_M}$, while $|\widehat{f}(\xi)| \ge c|\xi|$. Hence

$$\|E_0^* y\|^2 \ge c\sqrt{m_M}/(\log m_M)^3.$$

The signed identity, $\|N_M\| = O(\sqrt{M})$ and $\kappa_M \le 1$ give $P_M[y] + \kappa_M\|y\|^2 \le C_y\sqrt{M}$ independently of m_M. The inverse variational quotient proves the bound. Its normalized right side tends to infinity; substituting $m_M \sim a(\log M)^2/\pi$ gives the last assertion. □

This cost belongs to the auxiliary positive majorant E_0 used in the upper estimate. The signed characterization concerns the positive excess of the complete covariance of E_M.

S10.4 Why the cutoff must move

The following fixed-cutoff comparison explains the role of that simultaneous limit in the same averaged positive receiver.

Proposition S10.6 (Fixed-cutoff inverse divergence)**.** *Fix $M \ge 2$ and put $O_M = \{\xi > 0 : n_{M,1}(\xi) > 0\}$. Let $g_1, \dots, g_r$ be linearly independent elements of $L^2(O_M)$, extended by zero, and put*

$$F_i(u) = \int_{O_M} g_i(\xi)\sin(\xi u)\,\frac{d\xi}{\pi}.$$

Set $\mathcal{E} = [Q_b(sF_1), \dots, Q_b(sF_r)]$ on the original polar space. Then

$$\lambda_{\min}\Big[\mathcal{E}^*(P_M + \kappa I)^{-1}\mathcal{E}\Big] \longrightarrow \infty \qquad (\kappa \downarrow 0).$$

In particular this applies to independent actual negative-source columns whose coefficient profiles have compact support inside O_M. All data in this statement, including M and the columns, are fixed before κ tends to zero.

Proof. The absolutely convergent averaged arithmetic series is bounded at fixed M. Its symbol is continuous, negative at zero and tends to infinity with the digamma term. Thus O_M is a nonempty bounded open set, and $0 \le p_{M,1}(\xi) \le C_M(1+\xi^2)$.

By density choose smooth compact frequency functions $\phi_j \in C_c^\infty(O_M)$ whose pairings with the independent g_i form an invertible matrix. Their inverse odd transforms f_j are Schwartz and have complete positive energy zero. Put $q_j = (1-4\partial_u^2)^{-1} f_j$, also odd Schwartz with the same frequency support. For a smooth compact even cutoff χ equal to one near zero, define

$$f_{j,R} = (1-4\partial_u^2)[\chi(u/R)q_j], \qquad h_{j,R} = f_{j,R}/s.$$

Each $h_{j,R}$ is compact smooth odd. Integration by parts gives $\int \sinh(u/2) f_{j,R}(u)\, du = 0$ exactly, so these are original polar-core tests. Since $f_{j,R} \to f_j$ in H^1, their entire positive-energy Gram $A_R = [P_M(h_{i,R}, h_{j,R})]$ tends to zero. This uses the complete symbol bound, with every mixed entry retained. Their original Hilbert Gram $D_R = [\langle h_{i,R}, h_{j,R}\rangle]$ is finite at each fixed R. The source-pairing matrix $B_R = \mathcal{E}^*[h_{1,R}, \dots, h_{r,R}]$ tends to an invertible matrix; hence its least singular value is at least some $b > 0$ for large R.

The positive-form variational principle gives

$$\mathcal{E}^*(P_M + \kappa I)^{-1}\mathcal{E} \succeq B_R(A_R + \kappa D_R)^{-1} B_R^*.$$

For any $\varepsilon > 0$, first fix R with $A_R \preceq \varepsilon I$ and the stated source-pairing floor, then take κ small enough that $\kappa D_R \preceq \varepsilon I$. The right side is at least $b^2/(2\varepsilon) I$. This proves the limit for every sufficiently small κ. Actual columns are obtained by taking $g_i = \sqrt{n_{M,1}}\, \varphi_i$ in the sine factorization of E_M; compact support inside O_M makes these legal coefficient profiles. □

Proposition S10.6 and Theorem S10.1 describe two different parameter regimes for the same receiver. At a fixed cutoff every fixed independent negative-source Gram eventually diverges. Along the quantified joint window, the complete covariance norm converges to one, uniformly over the whole source and genuine-null restrictions. Increasing the arithmetic cutoff therefore supplies a regularization corridor with a proved rate and a specified lower ridge scale.

S11 Alternative receivers and verification interfaces

The following constructions supply alternative sources, comparison criteria and worked instances for the main argument. The main paper gives the fixed-exponent receiving proof and the original-domain localization separately.

S12 Supporting verification criteria

The complete-source covariance theorem is accompanied by independent supported certification and high-frequency verification criteria. They are retained with their own domains, norms and proof hypotheses; their detailed constructions are grouped in the appendices.

An arithmetic first-moment interface. Theorem B.6 uses the unweighted odd-half profile norm. At its prescribed spectral cutoff, the directed finite premise $H_N \succeq I/5$ implies the whole supported arithmetic lower bound $\mathcal{A}_{L,2} \succeq 52I/405$. The first-commutator relative-energy argument pays every mixed term and the infinite receiving return. Thus the finite test uses signed arithmetic first moments and the known DN diagonal, instead of assembling a receiving action Gram. Appendix B states its uniform-semiboundedness connection to the original Weil criterion; Appendix S21 proves the complete tail payment. The resource scales are separated in Appendix 4.

Complete-response and high-frequency alternatives. Appendix S20 retains the complete Green response and gives bounded regularization in profile coordinates. Appendix S23 instead fixes averaging seed 16 and increases its order k. Given a global zero-free half-plane $\Re\rho \le b < 1$, its complete high-covariance criterion admits excess $O(\theta^k)$ for fixed $1/2 < \theta < R < 3/2 - b$. The independent classical-input criterion uses excess $O(2^{-k}\exp(ck/\log k))$, $0 < c < 1/48$. Theorems S23.1 and S23.3 retain those complete upper bounds as hypotheses. Their original-domain return uses the exact full-source null lift, constrained inverse and Gevrey estimates proved in Appendix S24.

Evaluated and continuous comparisons. The M24 calculation in Appendix S13, subsection S13.1 verifies the complete supported construction at its stated SD instance. Appendix S25 gives a continuous-ridge band at that same support, with its derivative primitives and full numerical proof in the supplement. The earlier M20 and arithmetic benchmarks document their own complete acceptance tests. These examples expose the receiving and enclosure mechanisms at specified instances; the growing-support assertion uses the unbounded-sequence condition of Theorem 3.2.

S13 Complete supported certificates

The evaluated moving-window instance below uses the original norm and the complete receiving space. It exhibits the constraint inverse, complete output integrals and directed acceptance margin of the SD reference family. We first state its common construction, so its proof is independent of the earlier numerical instance retained in the supplement, Section S48.

Fix an admitted support $L > 0$, integer Green rank $J \ge 0$, ridge $\kappa > 0$ and integer DN prefix length $N \ge 2$. Let $W/2 \le W_{\mathrm{low}} \le W$ be the certified theta floor and put

$$S = D_L + R_{L,J} + m_0 - C_L^{\mathrm{ar}} + \kappa W_{\mathrm{low}}.$$

The arithmetic operator retains every active prime power and both killed translation orientations. Write $p(x) = \sinh(x/2)$ and $p_q = \langle p, e_q\rangle$. The columns

$$C = (e_j - (p_j/p_0)e_0)_{1\le j<N}, \qquad F = [p, C], \qquad Q = \mathrm{proj}_{\ker F^*}$$

give the full low-band polar-free intersection and its infinite receiver; the raw core Gram is $G_C = C^*C$.

For the reflection-symmetric weight $\phi(u) = e^{-u/2} + e^{-(2L-u)/2}$, applying $2|fg| \le |f|^2\phi(y)/\phi(x) + |g|^2\phi(x)/\phi(y)$ to every arithmetic edge gives $m_0 - C_L^{\mathrm{ar}} \succeq m_0 - \mathfrak{r}$, where $\mathfrak{r}$ is the complete finite row sum. A receiving vector's first N DN coefficients are parallel to the low-band pole projection. The polar equation bounds their squared norm fraction by $\delta_N \le 9L^3/[\pi^4(N+1/2)^3]$. Hence

$$D_L|_Q \succeq \ell_N I, \qquad \ell_N = a(k_N) - \delta_N\{a(k_N) - a(k_0)\}.$$

Certified cell bounds $r_{\mathrm{up}} \ge \mathfrak{r}$ and $\alpha \le \ell_N + m_0$, together with $R_{L,J} \succeq 0$, prove the whole receiving inequality

$$T = S_Q \succeq d_Q, \qquad d(x) = \kappa W_{\mathrm{low}}(x) + \alpha - r_{\mathrm{up}}(x) \ge d_* > 0.$$

Here the admitted cell bounds include the uniform constant $d_* > 0$.

Set $G = F^*d^{-1}F$ and

$$Z = d^{-1} - d^{-1}FG^{-1}F^*d^{-1}.$$

The complete constrained inverse satisfies $T^{-1} \preceq Z$ on $\mathrm{ran}\, Q$. If an invertible upper-triangular dyadic conditioner R obeys $\|R^*GR - I\| < \delta$, then $G^{-1} \succeq RR^*/(1+\delta)$. Here $\widehat{S}$ denotes the full

ambient expression defining S, before restriction to $p^{\perp}$. With $E = \widehat{S} - d$, retain all output functions before forming

$$B_E = C^* EC, \qquad K_E = (EC)^* d^{-1}(EC), \qquad H = F^* C + F^* d^{-1} EC.$$

The identical ridge cancels symbolically in E. The complete zero-trial Schur lower is

$$K_{\mathrm{core}} \succeq H^* G^{-1} H - B_E - K_E \succeq A := \frac{H^* R R^* H}{1+\delta} - B_E - K_E. \tag{S51}$$

This is the constrained receiving comparison of Appendix 6; all mixed terms and the polar row are included. Positive directed pivots of an invertible congruence of A therefore prove $S \succeq 0$ by completion of squares against the full positive receiver. Finally $R_L - R_{L,J} \succeq 0$, $W - W_{\mathrm{low}} \geq 0$ and $f = sh$, $sWs = I$, give $q_H[h] \geq -\kappa \|h\|_2^2$ on the stated original supported domain. Instance-specific cell and integral verification is indexed in the technical supplement.

S13.1 The M24 complete moving-window certificate

The common construction gives the following evaluated moving-window instance in the original norm. Its reference and constraint metric are computed at this support; its constants are specific to M24. Put

$$M = 2^{24}, \qquad \kappa = 2^{-24}, \qquad L = \tfrac{1}{2}\log(4\log M/\pi) + \tfrac{1}{10}, \qquad J = 88, \qquad N = 660.$$

The exact full-theta floor and common partition are specified in the companion M24 scientific data and verified in the technical supplement. Let

$$S_{24} = D_L + R_{L,88} + m_0 - C_L^{\mathrm{ar}} + \kappa W_{\mathrm{low}} \quad \text{on } p^{\perp}, \qquad p(x) = \sinh(x/2).$$

Theorem S13.1 (Evaluated M24 complete supported certificate)**.** *The form S_{24} is nonnegative on the original minimal closed odd supported polar-free profile domain. Consequently every original odd polar-free test supported in $(-L, L)$ satisfies*

$$q_H[h] \geq -2^{-24}\|h\|_2^2.$$

Proof. Let C have the 659 columns $e_j - (p_j/p_0)e_0$, $1 \leq j \leq 659$, set $F = [p, C]$, and let Q project onto $\ker F^*$. The common pole-leakage and complete arithmetic row bounds, with the M24 cell data, give

$$T = (S_{24})_Q \succeq d_Q, \qquad d = \kappa W_{\mathrm{low}} + \alpha - r_{\mathrm{up}} > 1/200.$$

This is an inequality on the whole infinite receiving space. The 7,828 common cells retain all 14 active prime powers and both translated channels; the 88 retained Green features are positive. The supplement gives their complete reference and weighted-Gram checks.

Write $G = F^* d^{-1} F$. The exact upper-triangular dyadic conditioner R, with positive diagonal, satisfies

$$\|R^* G R - I\| < 4.747 \times 10^{-74} < \delta, \qquad \delta = 10^{-60}.$$

Hence $G^{-1} \succeq RR^*/(1+\delta)$. With the complete output integrals B_E, K_E, H of (S51), the same constrained-inverse comparison gives the zero-trial Schur lower

$$K_{24} \succeq A_{24} := \frac{H^* R R^* H}{1+\delta} - B_E - K_E.$$

The data and replay in the supplementary section *Verification of the second moving-window certificate* give all 659 positive directed pivots after the invertible raw-core congruence $P_C = G_C^{-1} R_{uu}^{-*}$, where $G_C = C^*C$ and R_{uu} is the lower-right 659×659 core block of the M24 conditioner R. Thus $A_{24} \succ 0$. Completion of squares against the positive whole receiver proves $S_{24} \succeq 0$. Finally $R_L - R_{L,88} \succeq 0$ and $W - W_{\mathrm{low}} \geq 0$ imply $H_f + \kappa W \succeq S_{24}$; the exact physical congruence $f = sh$, $sWs = I$, proves the stated bound. □

The complete evaluated cases are summarized in Table 1.

Table 1: Complete supported zero-trial certificates in the SD family.

Case	Support half-width	Constraints	Core	Allowance
M20	1.535391914 . . .	329	328	2^{-20}
M24	1.626552692 . . .	660	659	2^{-24}

The M24 case extends the evaluated support and reduces this family's supported allowance by a factor of sixteen. The M20 construction and its full certificate are retained in Supplementary Section S48. Both cases use the support/reference family of Section S13; the quantified trial-size bound for the distinct schedule (12) is not transferred to these examples. The two evaluated instances are complete supported certificates; the cofinal assertion of Theorem 3.2 retains its unbounded-sequence quantifier.

What the calculation verifies. The 659 accepted core directions are accompanied by the entire infinite receiver, whose floor is stated before the finite pivots are used. The constraint inverse is enclosed in its full metric, and the complete output integrals supply the mixed return. This is why the finite acceptance proves an inequality for every test on the stated support. The SD support and cell schedule here are distinct from the PS schedule (12); the example validates this declared complete construction.

S14 Genuine nulls and complete source certificates

Genuine null vectors provide coordinates adapted to the signed form. We first establish completeness at each fixed arithmetic cutoff, then quantify the full covariance error of a finite null head.

The genuine derivative nulls and their original-domain proof are given in Lemma 8.1.

Theorem S14.1 (Complete null-generated source)**.** *Let*

$$z_i = \frac{\psi^{(2i+3)} - \psi^{(2i+1)}/4}{s}, \qquad i \geq 0,$$

be the genuine operator nulls of the original realization. For every fixed M with nonzero negative source,

$$\overline{E_M^* \operatorname{span}\{z_i : i \geq 0\}} = \mathcal{Y}_M.$$

Moreover E_M is Hilbert–Schmidt. If Π_r projects onto $\operatorname{span}\{E_M^* z_0, \ldots, E_M^* z_{r-1}\}$, then, for each fixed M and $\kappa > 0$,

$$\|(I - \Pi_r) G_M(\kappa)(I - \Pi_r)\| \longrightarrow 0.$$

Proof. The complete averaged prime multiplier is bounded for fixed M, whereas the archimedean multiplier tends to infinity. Hence S_M lies in a bounded interval $[0, F_M]$. The derivative rule gives, up to one fixed nonzero normalization factor,

$$E_M^* z_i(\xi) = v_M(\xi)(-1)^i \xi^{2i}, \qquad v_M(\xi) = \sqrt{n_{M,1}(\xi)}\, \xi(\xi^2 + 1/4)\widehat{\psi}(\xi).$$

The nonzero entire function $\widehat{\psi}$ has discrete real zeros, so $v_M \neq 0$ almost everywhere on S_M. It belongs to $L^2(S_M)$. If $f \in \mathcal{Y}_M$ is orthogonal to every displayed function, the finite complex measure $\overline{f} v_M \, d\xi/\pi$ annihilates every polynomial in ξ^2. Such polynomials are uniformly dense in $C([0, F_M])$, by polynomial approximation after the homeomorphism $\xi \mapsto \xi^2$. The measure is therefore zero, and $f = 0$ almost everywhere.

The source kernel satisfies

$$\|E_M\|_{\mathrm{HS}}^2 = \int_{S_M} n_{M,1}(\xi) \|Q_b[s \sin(\xi\,\cdot)]\|^2 \frac{d\xi}{\pi} < \infty.$$

Thus $G_M(\kappa)$ is positive trace class. The proved density gives $\Pi_r \to I$ strongly, and compactness gives the asserted norm limit. □

The theorem concerns the source images of known nulls. It does not identify the full physical kernel of H. Its density assertion is on the base polar recipient; imposing further physical constraints can alter that assertion.

Theorem S14.2 (Signed remainder and complete trace certificate)**.** *Fix M, $\kappa > 0$ and finitely many independent derivative-null columns Z. Define*

$$K = Z^* N_M Z, \quad V_Z = Z K^{-1/2}, \quad J = E_M^* V_Z, \quad Q = I - JJ^*, \quad \ell = \|V_Z^* V_Z\|.$$

Then $K \succ 0$, $J^*J = I$, and

$$J^* G_M J \preceq \big[1 + (\delta_M + \delta_M^2/\kappa)\ell\big] I, \tag{S52}$$

$$\tau := \|E_M Q\|_{\mathrm{HS}}^2 = \operatorname{Tr} N_M - \operatorname{Tr}(K^{-1} Z^* N_M^2 Z), \tag{S53}$$

$$\|G_M\| \leq 1 + (\delta_M + \delta_M^2/\kappa)\ell + \tau/\kappa. \tag{S54}$$

For an increasing initial derivative family, $\tau \to 0$ at fixed M. If Z has r columns, then

$$\ell \geq \lambda_r(N_M)^{-1} \geq r/\operatorname{Tr} N_M,$$

where the positive eigenvalues are arranged in decreasing order.

Proof. The nonvanishing common factor in Theorem S14.1 shows that every finite source polynomial family is independent, hence $K \succ 0$. Suppress M and write $A = P + \kappa I$, $R = A^{-1}$, $F = (W - \kappa I)V_Z$. Genuine operator nullity gives $NV_Z = AV_Z + F$, so exact expansion gives

$$GJ = J + E^* RF, \qquad J^* G J = I + V_Z^*(W - \kappa I)V_Z + F^* RF.$$

Using $R \preceq \kappa^{-1} I$ only after combining these terms yields

$$V_Z^*(W - \kappa I)V_Z + F^* RF \preceq V_Z^*(-W + W^2/\kappa)V_Z \preceq (\delta + \delta^2/\kappa) V_Z^* V_Z.$$

This proves (S52). In particular the ridge does not produce a separate $\kappa\ell$ charge.

Expanding Q and using cyclicity of the trace gives (S53); all operators under the trace are trace class or finite rank. Also $\|QGQ\| \le \kappa^{-1}\|EQ\|_{\mathrm{HS}}^2 = \tau/\kappa$. For a positive two-block operator with diagonal norms a, b, positive-form Cauchy–Schwarz bounds its quadratic form by $(\sqrt{a}\|x\|+\sqrt{b}\|y\|)^2$, and consequently its norm by $a+b$. This proves (S54), including the mixed block. Hilbert–Schmidt approximation and source density give $\tau \to 0$. Finally $V_Z^* N V_Z = I$ makes the Rayleigh quotient of N on $\operatorname{ran} V_Z$ at least $1/\ell$. Min–max and $r\lambda_r(N) \le \operatorname{Tr} N$ prove the last two inequalities. □

Proposition S14.3 (Directional Schur alternative)**.** *With the same notation, put $b = \|QG_MQ\|$, $r_F = \|F^*R_MF\|$ and $\varepsilon = (\delta_M + \delta_M^2/\kappa)\ell$. For $\lambda = 1+\eta > b$, the condition*

$$\varepsilon + \frac{br_F}{\lambda - b} \le \eta$$

implies $G_M \preceq \lambda I$.

Proof. The mixed block is $QG_MJ = QE_M^*R_MF$ and its squared norm is at most br_F. Complete the square in the Q coordinate of $\lambda I - G_M$, using $\lambda I - QG_MQ \succeq (\lambda - b)I$, and then apply (S52). □

Corollary S14.4 (A fixed complementary gap sufficient for RH)**.** *At $\kappa = M^{-1}$, suppose that along an unbounded sequence of M, finite genuine-null heads in Proposition S14.3 satisfy*

$$b_M = \|Q_MG_MQ_M\| \le q < 1, \qquad \ell_M = \|V_M^*V_M\| = o(\sqrt{M}),$$

where q is fixed and the complement contains every remaining source frequency. Then the original Weil form is nonnegative and RH holds.

Proof. The actual remainder δ_M decreases exponentially. Since $R_M \preceq \kappa^{-1}I$ and $F_M = (W_M - \kappa I)V_M$,

$$r_{F,M} \le \frac{(\kappa + \delta_M)^2}{\kappa}\ell_M.$$

Put $\varepsilon_M = (\delta_M + \delta_M^2/\kappa)\ell_M$ and $\eta_M = \varepsilon_M + qr_{F,M}/(1-q)$. The two assumed resources imply $\eta_M = o(M^{-1/2})$. With $\lambda_M = 1 + \eta_M$, one has $\lambda_M - b_M \ge 1 - q$; hence Proposition S14.3 gives $G_M \preceq (1+\eta_M)I$, including the mixed block. The original-domain absorption bound is

$$q_H[h] \ge -\left[\delta_M + M^{-1} + \frac{\eta_M B_{M,1}}{1+\eta_M}\right]\|h\|^2.$$

Since $B_{M,1} = O(\sqrt{M})$, its allowance tends to zero. Theorem 2.2 applies: under the stated complete complementary-gap and lift hypotheses, the Riemann hypothesis holds. □

The corollary uses a fixed gap below one, with both resources evaluated on the same head. The matrix-valued signed Schur comparison can retain additional cancellation lost in these scalar sufficient bounds.

S15 Critical-zero augmentation and complete source complements

Finite critical-zero evaluations can be included in the complete source while retaining the original positive receiver. This gives a complementary criterion at the bounded critical scale, with explicit head and mixed costs. Throughout this section the recipient is the original complex odd polar

space $\mathcal{X} = \mathcal{K}$, with form domain V. On arbitrary complex $L^2(\mathbb{R})$ inputs we use the orthogonal projection onto $\mathcal{K}$,

$$P_{\mathrm{odd}} f(u) = \tfrac{1}{2}(f(u) - f(-u)), \qquad Q_b f = P_{\mathrm{odd}} f - b\frac{\langle b, P_{\mathrm{odd}} f\rangle}{\|b\|_2^2}, \qquad b = s\sinh(u/2).$$

It agrees with the earlier polar projection on odd inputs. We use averaging order one and write

$$H = P_M - N_M + W_M, \quad N_M = E_M E_M^*, \quad \|W_M\| \le \delta_M, \quad A_M = P_M + \kappa_M I, \quad R_M = A_M^{-1}.$$

Here $\kappa_M > 0$, $\kappa_M \to 0$, and $\delta_M \le \kappa_M$ eventually.

For a finite set Γ of distinct positive ordinates of actual zeros $1/2 + i\gamma$, let m_γ be their full multiplicities. Set

$$\begin{aligned} k_\gamma &= Q_b[s e^{-i\gamma u}] = -iQ_b[s\sin(\gamma u)], \\ K_\Gamma &= \sum_{\gamma\in\Gamma} 2m_\gamma |k_\gamma\rangle\langle k_\gamma|, & H_\Gamma &= H - K_\Gamma, \\ E_{M,\Gamma}(c,d) &= E_M c + \sum_{\gamma\in\Gamma} \sqrt{2m_\gamma}\, d_\gamma k_\gamma, & \mathcal{Y}_{M,\Gamma} &= \mathcal{Y}_M \oplus \mathbb{C}^{|\Gamma|}. \end{aligned}$$

The finite-dimensional summand has its standard Euclidean metric; the continuous summand retains $d\xi/\pi$. With $f = sh$, conjugate-linear-first inner products and oddness give

$$\langle k_\gamma, h\rangle = \int f(u) e^{i\gamma u}\, du, \qquad K_\Gamma[h] = 2\sum_{\gamma\in\Gamma} m_\gamma |\widehat{f}(\gamma)|^2.$$

The factor two counts the two ordinates $\pm\gamma$. Thus

$$N_{M,\Gamma} := E_{M,\Gamma} E_{M,\Gamma}^* = N_M + K_\Gamma, \qquad H_\Gamma = P_M - N_{M,\Gamma} + W_M. \tag{S55}$$

Since $s \in L^2$, each K_Γ is bounded positive finite rank. For each finite Γ, bounded perturbation preserves the form domain V and gives $\mathrm{Dom}\, H_\Gamma = \mathrm{Dom}\, H = \mathrm{Dom}\, P_M$.

Theorem S15.1 (Augmented critical criterion)**.** *Let $M_j \to \infty$ and let Γ_j be any finite sets as above. On the original polar recipient put*

$$G_j = E_{M_j,\Gamma_j}^* R_{M_j} E_{M_j,\Gamma_j}, \qquad g_j = \|G_j\|.$$

The following assertions are equivalent:

1. *RH;*
2. *$g_j \le 1$ for all sufficiently large j;*
3. *$\liminf_{j\to\infty} \sqrt{M_j}(g_j - 1)_+ < \infty$.*

The finite sets may vary freely in height, cardinality and nesting.

Proof. Under RH, the original compact-core explicit formula (131), with the polar term zero, is the positive sum

$$q_H[h] = 2\sum_{\gamma>0} m_\gamma |\widehat{sh}(\gamma)|^2.$$

Subtracting the terms in Γ_j leaves a nonnegative form. For each fixed j, boundedness of K_{Γ_j} and core approximation in V extend this inequality to $q_{H_{\Gamma_j}} \geq 0$ on V. Equation (S55) then gives $N_{M_j,\Gamma_j} \preceq P_{M_j} + \delta_{M_j} I \preceq A_{M_j}$ eventually. Congruence by $R_{M_j}^{1/2}$ proves $g_j \leq 1$: the source norm equals the norm of $R_{M_j}^{1/2} N_{M_j,\Gamma_j} R_{M_j}^{1/2}$ by the identity $\|B^*B\| = \|BB^*\|$, applied to $B = R_{M_j}^{1/2} E_{M_j,\Gamma_j}$.

Conversely, positivity of K_{Γ_j} implies

$$\gamma_{M_j}^0(\kappa_{M_j}) = \|R_{M_j}^{1/2} N_{M_j} R_{M_j}^{1/2}\| \leq \|R_{M_j}^{1/2}(N_{M_j} + K_{\Gamma_j}) R_{M_j}^{1/2}\| = g_j.$$

Assertion 3 therefore supplies the bounded critical subsequence in Corollary S6.3(i), which implies RH. Assertion 2 implies assertion 3 directly. □

Theorem S15.2 (Complete augmented source complement)**.** *At each index of Theorem S15.1, choose finitely many columns Z_j in* $\operatorname{Dom} H_{\Gamma_j}$ *such that*

$$H_{\Gamma_j} Z_j = 0, \qquad D_j := Z_j^* N_{M_j,\Gamma_j} Z_j \succ 0.$$

Define, on the same augmented source and with the same receiver,

$$V_j = Z_j D_j^{-1/2}, \quad J_j = E_{M_j,\Gamma_j}^* V_j, \quad Q_j^{\mathrm{s}} = I - J_j J_j^*, \quad b_j = \|Q_j^{\mathrm{s}} G_j Q_j^{\mathrm{s}}\|,$$
$$\ell_j = \|V_j^* V_j\|, \quad F_j = (W_{M_j} - \kappa_{M_j} I) V_j, \quad r_{F,j} = \|F_j^* R_{M_j} F_j\|,$$
$$\varepsilon_j = (\delta_{M_j} + \delta_{M_j}^2/\kappa_{M_j})\ell_j.$$

Then $J_j^* J_j = I$, and the complete norm obeys

$$0 \leq (g_j - 1)_+ - (b_j - 1)_+ \leq \varepsilon_j + \sqrt{b_j r_{F,j}}, \qquad r_{F,j} \leq \frac{(\kappa_{M_j} + \delta_{M_j})^2}{\kappa_{M_j}} \ell_j. \tag{S56}$$

If, for these same columns,

$$\sqrt{M_j}\,\varepsilon_j \longrightarrow 0, \qquad M_j r_{F,j} \longrightarrow 0, \tag{S57}$$

then the following assertions are equivalent:

1. *RH;*
2. *$b_j \leq 1$ for all sufficiently large j;*
3. *$\liminf_{j\to\infty} \sqrt{M_j}(b_j - 1)_+ < \infty$.*

Proof. Suppress the index and write $E = E_{M,\Gamma}$ and $N = EE^*$. The source-Gram hypothesis gives $V^* N V = I$, hence $J^* J = I$. The columns lie in $\operatorname{Dom} P_M$, as established above. Their genuine H_Γ-nullity and (S55) consequently give the operator identities

$$NV = AV + F, \qquad GJ = J + E^* R F,$$
$$J^* G J = I + V^*(W - \kappa I)V + F^* R F, \qquad Q^{\mathrm{s}} G J = Q^{\mathrm{s}} E^* R F.$$

These retain both the signed head and the full mixed block. Use $R \preceq \kappa^{-1} I$ after combining the head terms:

$$\begin{aligned} V^*(W - \kappa I)V + F^* R F &\preceq V^*\big[(W - \kappa I) + \kappa^{-1}(W - \kappa I)^2\big]V \\ &= V^*(-W + \kappa^{-1} W^2)V \preceq \varepsilon I. \end{aligned}$$

The same inverse bound separately yields the stated bound for r_F. Factoring the mixed map gives

$$Q^{\mathrm{s}}E^*RF = (Q^{\mathrm{s}}E^*R^{1/2})(R^{1/2}F), \qquad \|Q^{\mathrm{s}}GJ\|^2 \le br_F.$$

In the orthogonal splitting $\operatorname{ran} J \oplus \operatorname{ran} Q^{\mathrm{s}}$, the diagonal blocks have upper bounds $1+\varepsilon$ and b; the off-diagonal operator has norm at most $\sqrt{br_F}$. Therefore

$$b \le \|G\| \le \max(1+\varepsilon, b) + \sqrt{br_F},$$

which proves (S56) after taking positive parts.

Under RH, Theorem S15.1 gives $G_j \preceq I$ eventually, so $b_j \le 1$. Conversely, assertion 3 provides a subsequence on which $\sqrt{M_j}(b_j-1)_+$ is bounded and $b_j \le 2$ eventually. On this subsequence, (S56) gives

$$\sqrt{M_j}(g_j-1)_+ \le \sqrt{M_j}(b_j-1)_+ + \sqrt{M_j}\varepsilon_j + \sqrt{2M_j r_{F,j}}.$$

The last two terms tend to zero by (S57). Theorem S15.1 now proves RH. The implication from assertion 2 to assertion 3 is immediate. □

Here Q_j^{s} acts on the complete direct-sum source. Thus b_j includes every continuous and discrete source interaction left after the specified head. The quantitative hypotheses concern genuine nulls of H_{Γ_j} and their actual source Gram. Together with those hypotheses, the bounded critical complementary excess is exactly the positivity criterion on the original polar recipient.

S16 Inverse-Gram budgets for complete source complements

An estimate for the actual source-normalized inverse Gram sharpens the mixed payment in Theorem S15.2. The following formulation specifies the two finite-head resources needed for that improvement and transfers logarithmic complementary costs to the original polar recipient.

Use the finite actual critical-zero augmentation, genuine operator-null columns and positive source Gram of Theorem S15.2, now at $\kappa_M = M^{-1}$. At each admitted integer M write

$$\mathsf{E}_M = E_{M,\Gamma_M}, \quad R_M = (P_M + M^{-1}I)^{-1}, \quad G_M = \mathsf{E}_M^* R_M \mathsf{E}_M, \quad g_M = \|G_M\|.$$

If $D_M = Z_M^* \mathsf{E}_M \mathsf{E}_M^* Z_M \succ 0$, put

$$V_M = Z_M D_M^{-1/2}, \quad J_M = \mathsf{E}_M^* V_M, \quad Q_M^{\mathrm{s}} = I - J_M J_M^*, \quad b_M = \|Q_M^{\mathrm{s}} G_M Q_M^{\mathrm{s}}\|,$$
$$\ell_M = \|V_M^* V_M\|, \qquad \iota_M = \|V_M^* R_M V_M\|, \qquad U = \log M.$$

The positive receiver P_M and original physical recipient are unchanged. In particular Q_M^{s} is a projection on the complete augmented source, including its continuous and discrete coordinates.

Proposition S16.1 (Actual inverse-Gram payment)**.** *Suppose a scalar resource $E_M^{\mathrm{bud}} \ge 0$ bounds both actual Grams:*

$$\ell_M \le M^{-1/2} e^{E_M^{\mathrm{bud}}}, \qquad \iota_M \le M^{-1} e^{E_M^{\mathrm{bud}}}. \tag{S58}$$

Define

$$a_M = (\delta_M + M\delta_M^2) e^{E_M^{\mathrm{bud}}}, \qquad c_M = \delta_M M^{3/4} e^{E_M^{\mathrm{bud}}/2}.$$

Then

$$0 \le \sqrt{M}\,[(g_M-1)_+ - (b_M-1)_+] \le a_M + \sqrt{b_M}\,[e^{E_M^{\mathrm{bud}}/2 - U} + c_M]. \tag{S59}$$

Also $\gamma_M^0(M^{-1}) \le g_M$. If $E_M^{\mathrm{bud}} = O(U)$, then $a_M + c_M = o(M^{-p})$ for every fixed $p > 0$.

Proof. All head columns belong to $\operatorname{Dom} P_M$ by the genuine H_{Γ_M}-null hypothesis and the bounded perturbation argument in the preceding section. Set $F_M = (W_M - M^{-1}I)V_M$. The triangle inequality in the actual inverse metric gives

$$\begin{aligned}\|R_M^{1/2}F_M\| &\le M^{-1}\sqrt{\iota_M} + \delta_M\sqrt{M\ell_M} \\ &\le M^{-3/2}e^{E_M^{\mathrm{bud}}/2} + \delta_M M^{1/4}e^{E_M^{\mathrm{bud}}/2}.\end{aligned}$$

The second term uses $R_M \preceq MI$; no commutation with W_M is required. Retaining the signed head cancellation from Theorem S15.2 gives

$$J_M^* G_M J_M \preceq [1 + (\delta_M + M\delta_M^2)\ell_M]I, \qquad \|Q_M^{\mathrm{s}} G_M J_M\| \le \sqrt{b_M}\,\|R_M^{1/2}F_M\|.$$

Indeed the exact head terms combine before the inverse bound as $(W_M - M^{-1}I) + M(W_M - M^{-1}I)^2 = -W_M + MW_M^2$. The complete two-block norm estimate (S56), multiplied by $\sqrt{M}$, now proves (S59). Positive-operator comparison $N_M \preceq N_M + K_{\Gamma_M}$ proves the BASE norm comparison. Finally $\delta_M = d_{M,1} = O(Me^{-\pi M})$ by (S9), whereas $e^{E_M^{\mathrm{bud}}}$ is at most a fixed power of M under the stated growth hypothesis. This proves the remainder assertion. □

For bounded b_M, the critical head/mixed fee is consequently at most $\sqrt{b_M}\,e^{E_M^{\mathrm{bud}}/2-U} + o(1)$. The following sufficient budgets distinguish its endpoints:

critical fee	sufficient resource budget
$o(1)$	$E_M^{\mathrm{bud}} - 2U \longrightarrow -\infty$
$O(1)$	$E_M^{\mathrm{bud}} \le 2U + O(1)$
$O(U^A),\ A \ge 0$	$E_M^{\mathrm{bud}} \le 2U + 2A\log U + O(1).$

In the first row $E_M^{\mathrm{bud}} = O(U)$ follows from its nonnegativity and the displayed condition. A fixed margin $E_M^{\mathrm{bud}} \le (2-\sigma)U$, $0 < \sigma < 2$, gives fee $O(M^{-\sigma/2})$. A pointwise strict inequality below $2U$ alone does not give the first row's limit.

Corollary S16.2 (Bounded and logarithmic complementary costs)**.** *Retain the actual resources* (S58)*.*

1. *On any sequence $M_j \to \infty$ with $E_{M_j}^{\mathrm{bud}} \le 2\log M_j + O(1)$, RH is equivalent to*

$$\liminf_{j\to\infty} \sqrt{M_j}(b_{M_j} - 1)_+ < \infty.$$

Under RH, $b_{M_j} \le 1$ eventually.

2. *Suppose, for every sufficiently large integer M, and fixed $A \ge 0$, $C \ge 0$ and finite C_0,*

$$\sqrt{M}(b_M - 1)_+ \le CU^A, \qquad E_M^{\mathrm{bud}} \le 2U + 2A\log U + C_0. \tag{S60}$$

Then $\sqrt{M}(\gamma_M^0(M^{-1}) - 1)_+ = O(U^A)$ and every nontrivial zeta zero satisfies $\Re\rho \le 1/2 + A$. If instead these hypotheses hold on a strictly increasing integer sequence M_j with

$$q = \limsup_{j\to\infty} \frac{\log\log M_{j+1}}{\log\log M_j} < \infty,$$

the conclusion is $\Re\rho \le 1/2 + qA$.

If the hypotheses in part 2 hold for every $A > 0$, with their own constants and one fixed finite sampling bound q in the sampled case, then RH follows.

Proof. For part 1, choose a subsequence on which the complementary critical cost is bounded. On this subsequence $b_M \le 2$ eventually. Equation (S59) gives a bounded critical cost for g_M, so Theorem S15.1 proves RH. Conversely RH gives $G_M \preceq I$ eventually by that theorem, hence $b_M \le 1$ by compression.

For part 2, the complementary cap again gives $b_M \le 2$ eventually. The resource bound implies $E_M^{\mathrm{bud}} = O(U)$ and $e^{E_M^{\mathrm{bud}}/2-U} \le e^{C_0/2}U^A$. Thus (S59) gives the explicit endpoint estimate

$$\sqrt{M}(\gamma_M^0(M^{-1}) - 1)_+ \le (C + \sqrt{2}e^{C_0/2})U^A + o(1).$$

Apply Theorem S8.2 at ridge exponent one; its support-explicit floor pays the original reciprocal-theta norm and complete remainder. For the sampled conclusion apply Theorem S8.2(ii). Finally let $A \downarrow 0$ and use functional symmetry to obtain RH. □

Under the fixed-margin budget, the same argument improves the last display to $CU^A + o(1)$. At the endpoint budget in (S60), the additional $O(U^A)$ fee can enlarge the constant; the logarithmic exponent is preserved.

Lemma S16.3 (Conditional all-integer selection)**.** *Suppose that for each integer $l \ge 2$ a finite actual set Γ_l is supplied, together with finite numbers $u_l, E_l \ge 0$ and head columns $Z_{M,l}$ for every integer M with $\log M \ge u_l$. Assume those columns satisfy the genuine operator-null, positive source-Gram and domain hypotheses above, and* (S58) *with $E_M^{\mathrm{bud}} = E_l$ on their entire admitted range. For the growing-rank conclusion, assume there are integers $r_l \to \infty$ such that*

$$\operatorname{rank} Z_{M,l} \ge r_l \quad \textit{for every integer } M \textit{ with } \log M \ge u_l.$$

Fix $0 < \sigma < 2$ and put

$$T_l = \max_{2 \le k \le l} \{k, u_k, E_k/(2-\sigma)\}.$$

For every integer M with $\log M \ge T_2$, choose $l(M) = \max\{l \ge 2 : T_l \le \log M\}$. This supplies heads at every such integer, with $l(M) \to \infty$ and $E_{l(M)} \le (2-\sigma)\log M$. Under the stated uniform rank hypothesis, $\operatorname{rank} Z_{M,l(M)} \ge r_{l(M)} \to \infty$.

Proof. Each T_l is finite and the sequence is nondecreasing. Since $T_l \ge l$, the defining set is nonempty and finite at every admitted M. For each fixed l, all sufficiently large M satisfy $\log M \ge T_l$, proving $l(M) \to \infty$. Its definition pays $u_{l(M)}$ and $E_{l(M)}/(2-\sigma)$ at every index, including jumps. No comparison between different heads is needed. The uniform admitted-range rank bound gives $\operatorname{rank} Z_{M,l(M)} \ge r_{l(M)}$ at each selected index; since $l(M) \to \infty$ and $r_l \to \infty$, the ranks tend to infinity. □

All finite onsets and Gram bounds in this selection are supplied hypotheses. A logarithmic cap for its complete b_M, when available, therefore has the all-integer scope in Corollary S16.2. The construction uses no growth estimate for the selected zero heights.

S17 Uniform covariance on small sets in a growing band

The null-source representation can be poorly conditioned near zeros of $\widehat{\psi}$. The actual positive inverse controls small frequency sets there without division by that transform. An exact Fourier shift extends this control beyond every fixed power of $\log M$.

Theorem S17.1 (Growing-band observation bound)**.** *Put $U = \log M$, and let $L_M \to \infty$ satisfy*

$$\frac{\sqrt{L_M}}{U} \longrightarrow \infty, \qquad L_M^3 \bar{e}_M \longrightarrow 0, \tag{S61}$$

where $\bar{e}_M$ is the complete Abel error of Theorem S5.1. For all sufficiently large M and every measurable $I \subset [0, L_M]$,

$$\|E_I^*(P_M + M^{-1}I)^{-1}E_I\| \le C|I|(1 + \log\log M). \tag{S62}$$

The constant and onset may depend on the admitted schedule, but not on the number or locations of the components of I. The upper bound survives additional exact finite constraints, with the same-form restricted receiver, dense restricted form domain and exactly projected source.

Proof. Take $T = 2L_M$. The complete arithmetic phase debit is

$$\varepsilon_M(T) = \frac{3(1+4T^2)}{4\sqrt{M}} + \frac{(1+4T^2)(T+2)}{4}\bar{e}_M.$$

Since $\bar{e}_M \ge 3/\sqrt{M}$, both terms tend to zero by (S61). The scalar inequalities in the actual-origin proof consequently give, for all sufficiently large M,

$$P_M \succeq A_T := \sqrt{M}Q_b s B(D) w(UD) 1_{|D|\le T} s Q_b, \qquad n_{M,1}(\xi) \le \frac{12\sqrt{M}}{1+4\xi^2} \quad (0 \le \xi \le T),$$

where $B(\xi) = 8/(1+4\xi^2)$, $w(t) = (-\cos t - 1/2)_+$ and $\mu > 0$ is its mean. These are inequalities for the actual arithmetic receiver and observation, before theta sandwiching.

Put $e_\xi = Q_b[s\sin(\xi\,\cdot)]$ and $R_0 = \frac{1}{2}\log(2U/\pi)$. Let χ_{R_0} be the even fixed-width Gevrey-2 cutoff equal to one on $[-R_0, R_0]$ and zero outside $[-R_0 - 1, R_0 + 1]$. Define

$$f_\xi(u) = \chi_{R_0}(u)\sin(\xi u), \qquad Y_\xi = \frac{(1 - 4\partial_u^2)f_\xi}{8\mu\sqrt{M}\, s}.$$

This trial is compact smooth, odd and exactly polar legal: integration by parts transfers $1 - 4\partial_u^2$ to $\sinh(u/2)$ and annihilates it. Uniformly for $0 \le \xi \le L_M$, product differentiation and the complete two-arm theta envelopes give

$$\begin{aligned} \|f_\xi\|_{1,4}^2 &\le C(1+\xi^2)(1+\log U), \\ \|s(\sin(\xi\,\cdot) - f_\xi)\|^2 &\le CUe^{-2U}, \\ \|Y_\xi\|^2 &\le CM^{2e^2-1}(1+\xi^2)^2(1+\log U). \end{aligned}$$

The first bound uses a support of length $O(1 + \log U)$ and a fixed transition width. For the last bound the support satisfies $r = \pi e^{2|u|} \le 2e^2 U$, so the original lower envelope $s^2 \ge 2r^2 e^{-r}$ controls division by s.

Every positive-order derivative of χ_{R_0} is supported on two fixed-width translated intervals. Thus $\|\chi_{R_0}^{(j)}\|_1 \le C_0 C_1^j (j!)^2$ for $j \ge 1$, independently of R_0. Integration by parts with j proportional to $\sqrt{|\omega|}$ gives $|\widehat{\chi}_{R_0}(\omega)| \le Ce^{-c\sqrt{|\omega|}}$ for large $|\omega|$, uniformly in R_0. The exact shift identity

$$\widehat{f_\xi}(\omega) = \frac{\widehat{\chi}_{R_0}(\omega - \xi) - \widehat{\chi}_{R_0}(\omega + \xi)}{2i}$$

therefore proves the complete exterior bound

$$\|1_{|D|>T} f_\xi\|_2 \le C e^{-c\sqrt{L_M}}, \qquad 0 \le \xi \le L_M.$$

Indeed both shifted frequencies have modulus at least L_M there; the polynomial factor arising on integration is absorbed by reducing c.

The full-frequency continuation satisfies

$$A_\infty = \mu\sqrt{M} Q_b s B(D) s Q_b + J_M, \qquad \|J_M\| \le C M^{5/2} e^{-\pi(M-1)}.$$

The polar Green identity gives its exact mean action $Q_b(sf_\xi)$. Consequently

$$e_\xi - A_T Y_\xi = Q_b[s(\sin(\xi\,\cdot) - f_\xi)] - J_M Y_\xi + \mu^{-1} Q_b s w(UD) 1_{|D|>T} f_\xi.$$

A squared triangle bound pays all three terms and their mixed products. At ridge M^{-1} the physical error costs $O(U/M)$; the Fourier error costs $O(M e^{-2c\sqrt{L_M}})$, smaller than every fixed inverse power of M by (S61). Also that admission and $\bar e_M \ge 3/\sqrt{M}$ imply $L_M = o(M^{1/6})$, so the displayed trial norm is polynomial in M and is absorbed by the harmonic error. Finally

$$\langle Y_\xi, A_T Y_\xi\rangle \le C M^{-1/2} \|f_\xi\|_{1,4}^2 + \|J_M\| \|Y_\xi\|^2.$$

The kernel-safe least-squares estimate (S19) and $A_T \preceq P_M$ now prove, uniformly on the whole band,

$$\langle e_\xi, (P_M + M^{-1})^{-1} e_\xi\rangle \le C M^{-1/2}(1+\xi^2)(1+\log U). \tag{S63}$$

The negligible errors are absorbed uniformly; at $\xi = 0$ the source is zero. Integrating in the exact source metric gives

$$\|R_M^{1/2} E_I\|_{\mathrm{HS}}^2 = \int_I n_{M,1}(\xi) \langle e_\xi, R_M e_\xi\rangle \frac{d\xi}{\pi} \le C|I|(1+\log U).$$

This trace bounds the full operator norm, retaining every profile combination. Variational restriction gives the last assertion. □

Corollary S17.2 (Unconditional expanded bands)**.** *For every fixed $A > 0$, estimate* (S62) *holds for all measurable $I \subset [0, e^{A\sqrt{\log M}}]$ and all sufficiently large M. Constants and onset may depend on A; no uniformity for a growing A is asserted.*

Proof. The complete Abel estimate proved in Corollary S6.2, using the classical Vinogradov–Korobov prime-number error [13], gives $\bar e_M = O_B(e^{-B\sqrt{U}})$ for every fixed $B > 0$. For $L_M = e^{A\sqrt{U}}$ choose $B > 3A$. Then $L_M^3 \bar e_M = O_B(e^{-(B-3A)\sqrt{U}}) \to 0$ and $\sqrt{L_M}/U = e^{A\sqrt{U}/2}/U \to \infty$. Theorem S17.1 applies in the original norm and domain. □

Corollary S17.3 (Complete collar and null-source certificate)**.** *Use the null head of Theorem S14.2, set $\kappa = M^{-1}$, and let P_I be the source multiplication projection onto $I \subset [0, L_M]$ for a schedule admitted by Theorem S17.1 (in particular $L_M = e^{A\sqrt{\log M}}$ for fixed $A > 0$). Define*

$$N_{\mathrm{out}} = E_M(I - P_I)E_M^*, \qquad \tau_{\mathrm{out}} = \operatorname{Tr} N_{\mathrm{out}} - \operatorname{Tr}(V_Z^* N_{\mathrm{out}}^2 V_Z), \qquad c_I = C|I|(1+\log\log M).$$

If

$$(\delta_M + \delta_M^2/\kappa)\ell + \left(\sqrt{c_I} + \sqrt{\tau_{\mathrm{out}}/\kappa}\right)^2 \le \eta, \tag{S64}$$

then $G_M(\kappa) \preceq (1+\eta)I$, and on the entire original domain

$$q_H[h] \geq -\left[\delta_M + \kappa + \frac{\eta B_{M,1}}{1+\eta}\right] \|h\|^2.$$

In particular, a cofinal family satisfying (S64) *with $\eta = o(M^{-1/2})$ gives vanishing original-norm allowances and RH.*

Proof. Let $Q = I - JJ^*$ as above. Without commuting Q and P_I, cyclicity gives

$$\|E_M(I-P_I)Q\|_{\mathrm{HS}}^2 = \operatorname{Tr} N_{\mathrm{out}} - \operatorname{Tr}(V_Z^* N_{\mathrm{out}}^2 V_Z).$$

Here V_Z uses the full N_M normalization. The triangle inequality in actual inverse energy gives

$$\|QG_MQ\| \leq \left(\|R_M^{1/2}E_MP_IQ\| + \|R_M^{1/2}E_M(I-P_I)Q\|\right)^2 \leq \left(\sqrt{c_I} + \sqrt{\tau_{\mathrm{out}}/\kappa}\right)^2.$$

Combine this with (S52) and the positive-block estimate in Theorem S14.2. The original-domain lower bound follows from Corollary S3.2, with complete negative source and no finite-source omission. Since $B_{M,1} = O(\sqrt{M})$ and $\delta_M \to 0$, the stated rate gives the endpoint. □

All frequencies outside I, including those above L_M, occur in τ_{out}. The mixed collar/exterior cost is the cross term in the square in (S64). Thus the displayed criterion specifies a complete quantitative construction, with growing-rank conditioning and the complete trace remainder evaluated in the same source metric.

S18 Regularized null certificates and physical lift budgets

The null-source construction can avoid an unregularized source-Gram inverse. The price becomes an explicit approximation error and the physical norm of its constructed lift. In this section write $d = \|W_M\|$ and $\mathcal{D} = d + d^2/\kappa$.

Theorem S18.1 (Regularized complete certificate)**.** *Let $Z : \mathbb{C}^r \to \mathcal{X}$ be a physical isometry with range in* $\operatorname{Dom} H \cap \ker H$*. Put $A = E_M^* Z$, $K = A^*A$, and, for $t > 0$,*

$$J = A(K+tI)^{-1}A^*, \qquad T = Z(K+tI)^{-1}A^*.$$

Then $0 \preceq J \preceq I$, $J = E_M^* T$, and

$$\|G_M\| \leq 1 + \frac{\mathcal{D}}{4t} + \frac{\tau_t}{\kappa}, \qquad \tau_t = \operatorname{Tr} N_M - \operatorname{Tr}\big((K+tI)^{-1}K(K+tI)^{-1}Z^*N_M^2Z\big). \tag{S65}$$

If a Hilbert–Schmidt map $X : \mathcal{X} \to \mathbb{C}^r$ satisfies $\|E_M^* - AX\|_{\mathrm{HS}} \leq \epsilon$ and $\|X\|_{\mathrm{HS}} \leq L_0$, then

$$\|G_M\| \leq 1 + \frac{2\epsilon^2}{\kappa} + \sqrt{\frac{2\mathcal{D}}{\kappa}}L_0. \tag{S66}$$

Every trace and approximation here uses the complete source metric.

Proof. The singular-value multipliers of T are $\sigma/(\sigma^2+t)$; hence $\|T\|^2 \le 1/(4t)$. The exact signed-null identity, applied to Tf, gives

$$\langle Jf, G_M Jf\rangle \le \|Jf\|^2 + \mathcal{D}\|Tf\|^2 \le (1+\mathcal{D}/(4t))\|f\|^2.$$

Use the positive partition $N_M = E_M J^2 E_M^* + E_M(I-J^2)E_M^*$. After inverse congruence its first term has norm $\|JG_MJ\|$, and its second is at most $\kappa^{-1}\operatorname{Tr}(E_M(I-J^2)E_M^*)$. Expanding J^2 and using cyclicity proves (S65). Completing the finite-dimensional square gives

$$\operatorname{Tr}(E_M(I-J)E_M^*) = \inf_X\{\|E_M^* - AX\|_{\mathrm{HS}}^2 + t\|X\|_{\mathrm{HS}}^2\}.$$

Since $I - J^2 \preceq 2(I-J)$, the excess in (S65) is at most $\mathcal{D}/(4t) + 2(\epsilon^2 + tL_0^2)/\kappa$. For $\mathcal{D}, L_0 > 0$ optimize at $t = \sqrt{\mathcal{D}\kappa/(8L_0^2)}$; the zero cases follow by limits. □

For independent raw columns Z_0, replace $K + tI$ by $Z_0^* N_M Z_0 + tZ_0^* Z_0$. In (S66) the physical lift norm is then $L_0 = \|Z_0 X_0\|_{\mathrm{HS}}$. This preserves the original physical metric without requiring explicit formation of its inverse square root.

The collar estimate of Theorem S17.1 also survives regularization. With P_I denoting source multiplication by $I \subset [0, e^{A\sqrt{\log M}}]$ for a fixed $A > 0$, suppose

$$\|(I-P_I)E_M^* - AX\|_{\mathrm{HS}} \le \epsilon_{\mathrm{out}}, \qquad \|X\|_{\mathrm{HS}} \le L_{\mathrm{out}}.$$

Completing the same square with $(I-P_I)E_M^*$ and using the triangle inequality in inverse energy gives

$$\|G_M\| \le 1 + \frac{\mathcal{D}}{4t} + \left(\sqrt{c_I} + \sqrt{\frac{2(\epsilon_{\mathrm{out}}^2 + tL_{\mathrm{out}}^2)}{\kappa}}\right)^2, \quad c_I = C_A|I|(1+\log\log M). \tag{S67}$$

No projection is commuted with J. The square includes the mixed term, and the exterior includes every frequency outside I.

Proposition S18.2 (Order-uniform original-norm null budget)**.** *For* $z_j = (\psi^{(2j+3)} - \psi^{(2j+1)}/4)/s$,

$$\|z_j\|_2 \le 100\, 64^{2j+3}(2j+3)!, \qquad j \ge 0.$$

Consequently the raw map $Z_0 = (z_0, \dots, z_{r-1})$ *satisfies*

$$\|Z_0\|_{\mathrm{HS}}^2 \le B_r := 10000r\, 64^{4r+2}[(2r+1)!]^2, \qquad \|Z_0X_0\|_{\mathrm{HS}} \le \sqrt{B_r}\|X_0\|_{\mathrm{HS}}. \tag{S68}$$

Proof. Set $\rho = 1/64$ and $r_u = \pi e^{2|u|}$. The complete theta series is holomorphic in $|\operatorname{Im} z| < \pi/4$ and even there. On $|z-u| = \rho$, reflect to the right half-plane if needed. Then $\operatorname{Re}(\pi e^{2z}) > 3r_u/4$ and $|\pi e^{2z}| \le e^{2\rho} r_u$. Its full series gives

$$|\psi(z)| \le 24 e^{|u|/2} r_u^2 e^{-3r_u/4} \sum_{n\ge 1} n^4 e^{-3(n^2-1)\pi/4} < 48 e^{|u|/2} r_u^2 e^{-3r_u/4}.$$

Indeed the $n=2$ summand is below $1/4$ and subsequent ratios are below $1/256$. Cauchy's estimate yields $|\psi^{(k)}(u)| \le 48k!64^k e^{|u|/2} r_u^2 e^{-3r_u/4}$. Divide by the original lower envelope $s^2 \ge 2r_u^2 e^{-r_u}$ and integrate both arms using $du = dr_u/r_u$ and $e^{|u|} = \sqrt{r_u/\pi}$. Thus

$$\|\psi^{(k)}/s\|_2^2 \le \frac{1152}{\sqrt{\pi}}(k!64^k)^2 \int_\pi^\infty r^{3/2} e^{-r/2}\, dr \le 3456\sqrt{2}(k!64^k)^2 < 80^2(k!64^k)^2.$$

The triangle inequality at $k = 2j+3$ and $k-2$ proves the first bound. Summing its squares and using the largest summand proves the second. □

Since $\log B_r = O(r\log(r+1))$, this gives a useful cofinal lift window. At $\kappa = M^{-1}$ the complete averaged remainder has $d_M \le \mathrm{poly}(M)e^{-\pi M}$. If $r_M = o(M/\log M)$ and $\log(1+\|X_{0,M}\|_{\mathrm{HS}}) = o(M)$, then

$$\sqrt{M\mathcal{D}_M}\,\|Z_{0,M}X_{0,M}\|_{\mathrm{HS}} \le e^{-\pi M/2+o(M)}.$$

Thus the lift part of (S66) is paid at every fixed polynomial scale. A complete-source approximation with $\epsilon_M^2 = o(M^{-a-1})$, $1/2 < a < 1$, would then give the excess $o(M^{-a})$ and the vanishing-allowance endpoint. The approximation norm includes the entire source exterior as part of its complete-source error budget.

S19 Unrestricted derivative nulls and exterior response pairing

The derivative nulls in Lemma 8.1 also satisfy an uncompressed distributional equation. This extension concerns the same original null family. We retain the normalization $\mathcal{M}f(z) = \int_{\mathbb{R}} f(x)e^{zx}\,\mathrm{d}x$ and Fourier measure $\mathrm{d}\xi/(2\pi)$, with inner products conjugate-linear in their first argument. Write $\eta_j = sz_j = (\partial_x^2 - 1/4)\psi^{(2j+1)}$. The transform identity and complete derivative tails are those already proved in Lemma 8.1 and Appendix A.

Define the full signed geometric expression by

$$\begin{aligned}\mathcal{A}_{\mathrm{geo}}f(x) = (a(D)+m_0)f(x) - \sum_{n\ge 2}\frac{\Lambda(n)}{\sqrt{n}}\{f(x-\log n)+f(x+\log n)\}\\ + \int_{\mathbb{R}} 2\cosh((x-y)/2)f(y)\,\mathrm{d}y.\end{aligned} \tag{S69}$$

For compact input the prime sum is locally finite. For the theta-tail inputs η_j it converges locally absolutely with every derivative. The expression is interpreted distributionally; no global self-adjoint realization of its infinite prime convolution is assumed. The pole term in (S69) is part of the definition. On odd tests its quadratic form is $-2|\mathcal{M}f(1/2)|^2$, so the expression agrees with the complete Weil zero form in (131), rather than with the pole-free geometric form on an unrestricted test class.

Proposition S19.1 (Unrestricted distributional annihilation)**.** *For every fixed $j \ge 0$ and every $g \in C_c^\infty(\mathbb{R})$,*

$$\langle \eta_j, \mathcal{A}_{\mathrm{geo}}g\rangle = 0.$$

Equivalently $\mathcal{A}_{\mathrm{geo}}\eta_j = 0$ in distributions on $\mathbb{R}$.

Proof. Use the polar-preserving approximants already employed in Lemma 8.1, now before division by s:

$$\eta_{j,R} = (\partial_x^2 - 1/4)(\chi_R\psi^{(2j+1)}),$$

where χ_R is even with fixed-width transition and uniformly bounded derivatives of each fixed order. These functions are compact, smooth, odd, and have both polar moments exactly zero. The complete theta derivative bounds give convergence to η_j in every fixed exponentially weighted derivative L^1 seminorm and in every fixed H^k. In particular repeated integration by parts yields, uniformly in R and $|\Re z| \le 1/2$,

$$|\mathcal{M}\eta_{j,R}(z)| \le C_{j,N}(1+|\Im z|)^{-N} \quad \text{for each fixed } N.$$

For odd compact g, polarize the complete compact explicit formula with $\eta_{j,R}$ in the first inner-product slot. Its zero side is

$$\sum_{\rho} \mathcal{M}g(\rho - 1/2)\overline{\mathcal{M}\eta_{j,R}(1/2 - \overline{\rho})}.$$

The zero count and the uniform strip estimates give an absolutely summable majorant counting every multiplicity. The limit of each term is zero, since

$$\mathcal{M}\eta_j(z) = (z^2 - 1/4)(-z)^{2j+1}\xi(1/2 + z)$$

vanishes at both polar points and at every centered nontrivial zero; the reflected conjugate argument is another such zero. Multiplicity repeats the same vanishing value and introduces no derivative premise.

The geometric limit is also justified directly. The H^2 convergence and $0 \le a(\xi) \le 17\xi^2$ control the archimedean pairing. For x in a fixed compact interval $[-b, b]$, the complete theta tails and the cutoff derivative bounds give, uniformly in R and large n,

$$|\eta_{j,R}(x \pm \log n)| + |\eta_j(x \pm \log n)| \le Cn^C \exp(-c_b n^2).$$

Thus the entire prime pairing is dominated by a convergent series $\sum_n \Lambda(n) n^{C-1/2} \exp(-c_b n^2)$. Scalar terms converge, and polar cross terms vanish exactly. Dominated convergence proves the assertion for odd g without a polar restriction on g. Even g gives zero by parity of every term in (S69); decomposition proves the general case. Symmetry of the distributional expression identifies the equivalent annihilation statement. No finite zero list or prime truncation is used. □

Proposition S19.2 (Complete exterior identity)**.** *Let $I = (-b, b)$ and let u, extended by zero, be odd and belong to the actual minimal supported local geometric operator domain. Suppose that its constrained equation inside I is*

$$\mathcal{A}_{\mathrm{geo}} u = \lambda u + \beta_u p, \qquad p(x) = \sinh(x/2), \quad \lambda \in \mathbb{R}.$$

Then

$$\lambda \langle \eta_j, u \rangle_I = -\langle \eta_j, \mathcal{A}_{\mathrm{geo}} u - \beta_u p \rangle_{\mathbb{R} \setminus I}. \tag{S70}$$

Every prime and polar term in the exterior pairing is retained and absolutely integrable against η_j. The entire exterior response need not belong to ordinary L^2.

Proof. The domain and supported-closure identification in Appendix 6 gives an L^2 archimedean action inside I. Outside I, the off-diagonal kernel of $a(D)$ is exactly

$$-J(x - y), \qquad J(t) = \frac{e^{-|t|/2}}{1 - e^{-2|t|}}, \quad t \ne 0.$$

Indeed the positive digamma series has summands $2/c_j - 2c_j/(c_j^2 + \xi^2)$, and the inverse Fourier transform of its second term is $-e^{-c_j|t|}$ away from the diagonal. Near either endpoint the cross kernel is bounded by a bounded remainder plus $1/[2(s + t)]$, for positive distances s, t to that endpoint. The Carleman part has L^2 norm at most $\pi/2$, by Schur weights $t^{-1/2}$; the remaining local kernel is bounded and the far kernel decays exponentially. Hence the cross map from $L^2(I)$ to $L^2(\mathbb{R} \setminus I)$ is bounded.

These interior and exterior pieces represent the full archimedean distribution, including the endpoints. To see the last detail without assuming a global operator domain, logarithmic symbol

growth gives $a(D)u \in H^{-\epsilon}(\mathbb{R})$ for every $\epsilon > 0$. The assembled interior/exterior action is in L^2. Their difference is supported at the two endpoints. A point-supported distribution is a finite sum of delta derivatives, none of which belongs to $H^{-\epsilon}(\mathbb{R})$ when $0 < \epsilon < 1/2$. The difference therefore vanishes. Fourier symmetry now justifies the archimedean pairing with the Schwartz function η_j.

For arithmetic, change variables to the compact input interval and use the complete theta bound to obtain

$$\sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}} \int_I |u(y)|\,|\eta_j(y \pm \log n)|\,\mathrm{d}y < \infty.$$

This follows from $\|u\|_{L^1(I)} < \infty$ and the same $n^C \exp(-c_b n^2)$ majorant. It pays every far-prime translate and permits reversal of the full sum. The pole grows at most like $e^{|x|/2}$, whereas η_j has every exponential moment, so its pairing is absolute and can also be reversed. Together with the scalar and archimedean parts, Proposition S19.1 gives

$$\langle \eta_j, \mathcal{A}_{\mathrm{geo}} u\rangle = \langle \mathcal{A}_{\mathrm{geo}} \eta_j, u\rangle = 0.$$

Split inside and outside I, substitute the stated local equation, and use the global identity $\langle \eta_j, p\rangle_{\mathbb{R}} = 0$. The interior polar moment need not vanish separately. Moving it to the exterior gives (S70), with the displayed $\beta_u p$ term and sign. □

These propositions extend the derivative nulls to unrestricted tests and complete exterior responses. For packets null for an operator with a subtracted critical-value correction, that correction remains in the equation. An application to quantitative boundary observation uses the exact exterior identities together with an observation norm and remainder estimates for the actual eigenresponses.

S20 Complete-response verification and bounded regularization

The criterion of Theorem 3.2 concerns the original Weil positivity endpoint. We now give a complementary construction for that same endpoint, retaining the complete Green response and bounding the regularization in profile coordinates. The finite-Green construction and its evaluated examples retain their stated definitions. Each matrix in the complementary construction below is assembled from its stated full-response form.

The support-uniform response bound is Lemma 3.1.

Lemma S20.1 (Full-response commutator and effective actions)**.** *Put $H_L = 1 + 8L/\pi^2$. The complete response R_L preserves* $\operatorname{Dom} D_L$ *and*

$$\|[D_L, R_L]\| \leq 600 H_L.$$

The same bound holds uniformly for every finite Green prefix. If a column map V is supported in the first N DN modes and P_m projects onto the first $m \geq 1$ output modes, then

$$\|(I - P_m) R_L V\| \leq H_L \sqrt{N/m}\,\|V\|.$$

Every finite output coefficient can be enclosed effectively. Truncating its exact reflected scalar series at index $J \geq 1$ incurs entry error at most $2/(LJ)$. An additional directed entry error ϵ gives low-output map error at most $\sqrt{mN}\{2/(LJ) + \epsilon\}\|V\|$.

Proof. Direct integration in the normalization of Section 3 gives

$$\langle r_j, e_q\rangle = \sqrt{2/L}(-1)^q \frac{c_j\sqrt{1+e^{-2c_jL}}}{c_j^2+k_q^2}.$$

For $x, y > 0$, the function $c^2/\{(c^2+x^2)(c^2+y^2)\}$ is unimodal, with maximum $(x+y)^{-2}$ and integral $\pi/\{2(x+y)\}$. A spacing-two unimodal sum is bounded by half its integral plus twice its maximum. Consequently

$$|(R_L)_{qr}| \le \frac{\pi}{L(k_q+k_r)} + \frac{8}{L(k_q+k_r)^2} \le \frac{H_L}{q+r+1}.$$

This also bounds every partial sum, since its summands have a common sign. Writing $t = k/2$ and $b_n = n + 1/4$, the digamma series yields

$$ka'(k) = 2t^2 \sum_{n\ge 0} \frac{b_n}{(b_n^2+t^2)^2} \le 9.$$

Indeed, for $t \le 1/4$, the reciprocal-cube sum is at most 72. For $t \ge 1/4$, the unimodal sum has integral $1/(2t^2)$ and maximum at most $1/(2t^3)$, giving $1+2/t \le 9$. Thus $|a(x)-a(y)| \le 9|\log(x/y)|$. Set $x = q + 1/2$, $y = r + 1/2$ and use Schur weights $x^{-1/2}$. The bound $|\log t| \le 4(t^{1/4} + t^{-1/4})$, together with

$$\sum_{r\ge 0} \frac{x^\nu y^{-\nu}}{x+y} \le 2 + \frac{\pi}{\sin \pi\nu}, \qquad \nu = 1/4, 3/4,$$

proves a commutator norm at most $36H_L(4 + 2\pi\sqrt{2}) < 600H_L$. The displayed sum follows by bounding its decreasing summand by its first value and the positive-line integral. Each column R_Le_r belongs to $\operatorname{Dom} D_L$: the coefficient envelope remains square summable after multiplication by $a(k_q) = O_L(\log(2+q))$. On finite DN sums the bounded matrix is therefore the genuine commutator. Graph-norm approximation and closedness of D_L extend the identity and domain preservation to its full domain.

The squared Hilbert–Schmidt output tail is at most $H_L^2 \sum_{q<N}(m+q)^{-1} \le H_L^2 N/m$. For the scalar series, the omitted absolute sum is at most $(4/L)\{c_J^{-2} + (2c_J)^{-1}\} \le 2/(LJ)$. The finite-block error follows from its Frobenius norm. These low- and high-output errors concern orthogonal output spaces and may be combined in quadrature. They do not assert operator-norm convergence of $R_{L,J}$ to R_L on unrestricted inputs. □

Corollary S20.2 (Support-uniform response resources)**.** *Put* $\widehat{H}_L = \min\{1 + 8L/\pi^2, 7\}$. *The full response and every prefix have coefficient envelope* $|(R_L)_{qr}| \le \widehat{H}_L/(q+r+1)$, *and*

$$\|[D_L, R_L]\| \le 600\widehat{H}_L \le 4200, \qquad \|(I-P_m)R_LV\| \le \widehat{H}_L\sqrt{N/m}\,\|V\|.$$

For a prefix the commutator is also bounded by $\min\{14J, 600\widehat{H}_L\}$.

Proof. Keep the exact coefficient series in Lemma S20.1. Write $s = x + y$ and $f(c) = c^2/\{(c^2 + x^2)(c^2+y^2)\}$. If its maximum occurs at $\sqrt{xy} \ge 1/2$, then $s \ge 1$ and $\sup f = 1/s^2 \le 1/s$. Otherwise its maximum on $[1/2, \infty)$ is at $c = 1/2$; with $a = 2x$, $b = 2y$,

$$sf(1/2) = \frac{2(a+b)}{(1+a^2)(1+b^2)} \le 2.$$

Thus the shifted unimodal sum is at most $(\pi/4+4)/s$. The exact reflected coefficient has multiplier at most $4/L$, giving the envelope $(1+16/\pi)/(q+r+1) < 7/(q+r+1)$, also for every prefix. Combine this with the earlier H_L envelope and use the same weighted-Schur, domain and output-tail arguments. The prefix bound $14J$ follows independently from its finite rank-one decomposition in Appendix 6. □

Proposition S20.3 (Complementary capped full-response criterion)**.** *Retain the schedule, core isometry U_M, complete constraints F_M and projection Q_M of Section 3, with $\beta = 1$. Rebuild every operator and matrix below from*

$$w_M = \min(M^{-1}W_{\mathrm{low}}, \beta), \qquad S_M^{\mathrm{cap}} = D_L + R_L + V_L + w_M, \qquad d_M^{\mathrm{cap}} = \beta + w_M.$$

Then the entire receiver obeys $T_M^{\mathrm{cap}} \succeq (d_M^{\mathrm{cap}})_{Q_M} \succeq \beta I$. For the exact isometric constraint map,

$$(2\beta)^{-1} I \preceq F_M^*(d_M^{\mathrm{cap}})^{-1} F_M \preceq \beta^{-1} I.$$

In particular this complete weighted constraint Gram has condition number at most two. Strict finite lower Schur certificates for these rebuilt forms along any unbounded subsequence of M imply RH. Conversely, under RH each admitted M has a strictly positive, effectively verifiable finite Galerkin certificate. The sufficient trial-size exponent remains $m = \exp(M^{1/3+o(1)})$ on the stated schedule.

Proof. The existing core admission proves the receiving lower for $D_L + V_L$; adding the nonnegative complete R_L and w_M gives the asserted floor. The Gram bound follows by compressing $(2\beta)^{-1}I \preceq (d_M^{\mathrm{cap}})^{-1} \preceq \beta^{-1}I$ by the exact isometry F_M. It is not a condition bound for an unnormalized raw constraint list.

In the rest of this proof all blocks are rebuilt: $H = \mathsf{U}_M^* S_M^{\mathrm{cap}} \mathsf{U}_M$, $C = Q_M S_M^{\mathrm{cap}} \mathsf{U}_M$, and $T = T_M^{\mathrm{cap}}$. Put $\alpha_D = a(k_{N_M})$, $K_F = \sqrt{64 + \alpha_D^2}$, $c_D = \alpha_D + \mathfrak{h}$, $G = 1 + (\mathfrak{h} + K_F)/\beta$ and $R_0 = \|p\|^2/|\langle p, e_0\rangle|^2 < 16L$. Define

$$P_* = \alpha_D R_0 + \mathfrak{h}(1 + R_0), \qquad F_1 = \alpha_D^2 + \mathfrak{h}\alpha_D + K_1 + K_F c_D,$$
$$G_2 = F_1 + (\mathfrak{h} + K_F) G c_D + (K_1 + K_F \mathfrak{h}) c_D/\beta.$$

Here $\mathfrak{h}$ and K_1 are the bounds specified next, not constants copied from a saved finite-Green calculation.

Use the same legal constrained DN trials and second-graph derivation of Appendix 6, recomputing its constants for this form. The bounded perturbation norm is at most $\mathfrak{h} = L + \pi/2 + B_M + \beta$. Its commutator budget is

$$K_1 = 600 H_L + [4(2+\pi) + 4(L+\pi/2)] \sum_{2 \le n < e^{2L}} \frac{\Lambda(n)}{\sqrt{n}} + 4(L+\pi/2) n_{\mathrm{cell}} \beta.$$

The last term uses $\mathrm{TV}(w_M) \le 2 n_{\mathrm{cell}} \beta$; no smooth derivative of the step is assumed. Every projection and polar correction in the second-graph constants G_2, P_* is retained. Thus for the actual complete return $X = C^*(T_M^{\mathrm{cap}})^{-1} C$ and its finite Galerkin return X_m,

$$0 \preceq X - X_m \preceq \epsilon_m I, \qquad \epsilon_m = G_2^2 \{a(k_m)^{-3} + P_* a(k_m)^{-4}\}.$$

All these constants are $M^{o(1)}$ on the stated schedule. Choosing $a(k_m) \ge \max\{1, P_*, (72 M G_2^2)^{1/3}\}$ pays $\epsilon_m \le 1/(36M)$ and gives the asserted near-threshold trial size.

Since $0 < w_M \le M^{-1} W$, a passing complete Schur lower implies the original supported allowance $q_H[h] \ge -M^{-1}\|h\|^2$. Support exhaustion proves the forward assertion. Under RH the

original profile form is nonnegative, and $W_{\mathrm{low}} \geq W/2 \geq 1/18$ gives $S_M^{\mathrm{cap}} \succeq I/(18M)$. Its exact Schur complement has the same core lower bound. Hence $H - X_m - \epsilon_m I \succeq I/(36M)$. A two-sided matrix-center error at most $1/(144M)$, subtracted once again to form a directed lower, leaves $I/(72M)$. Lemma S20.1 encloses the exact full-response entries; all arithmetic, coordinate, constraint and inverse errors must also be paid. This proves effective strict recognition, not a total runtime bound or an evaluated new capped certificate. □

S20.1 The complete Hilbert principal channel and a compact remainder

The coefficient formula also separates the infinite principal response from a remainder with a whole-operator approximation bound. Let E be the diagonal unitary $E_{qq} = (-1)^q$ on DN coefficients and let $\mathsf{H}_{qr} = (q + r + 1)^{-1}$ be the Hilbert matrix. It is bounded: the weighted Schur argument of Lemma 3.1 also bounds H, since its entries are at most twice those of that lemma's half-shift comparison matrix.

Proposition S20.4 (Complete principal-part decomposition)**.** *For every $L > 0$ there is a self-adjoint Hilbert–Schmidt operator K_L such that*

$$R_L = \tfrac{1}{2} E \mathsf{H} E + K_L.$$

Put $t_L = e^{-4L}$ and define

$$S_L = e^{-L} \left\{ \frac{4t_L(1+t_L)}{(1-t_L)^3} + \frac{2t_L}{(1-t_L)^2} + \frac{1}{4(1-t_L)} \right\},$$
$$\mathfrak{a}_L = \frac{4L}{\pi^2}, \qquad \mathfrak{b}_L = \frac{2L^3 S_L}{\pi^4}.$$

Then

$$|(K_L)_{qr}| \leq \frac{\mathfrak{a}_L}{(q+r+1)^2} + \frac{\mathfrak{b}_L}{(q+1/2)^2 (r+1/2)^2}, \qquad \|K_L\|_{\mathrm{HS}} \leq \mathfrak{a}_L \sqrt{\zeta(3)} + L^3 S_L/3.$$

For the first-m DN projection, $m \geq 1$, set

$$\eta_{L,m} = \frac{\mathfrak{a}_L}{\sqrt{2}\, m} + \frac{\mathfrak{b}_L \pi^2}{\sqrt{18}\, m^{3/2}}.$$

The entire output and finite-remainder errors satisfy

$$\|(I - P_m) K_L\| \leq \eta_{L,m}, \qquad \left\| R_L - \left(\tfrac{1}{2} E \mathsf{H} E + P_m K_L P_m \right) \right\| \leq 2\eta_{L,m}.$$

Proof. For $x, y > 0$ extend $f(c) = c^2/\{(c^2 + x^2)(c^2 + y^2)\}$ by zero to $c < 0$. It is continuous, nonnegative and unimodal, with maximum $(x + y)^{-2}$, total variation $2/(x + y)^2$, and integral $\pi/\{2(x + y)\}$. The length-two cells centered at $c_j = 2j + 1/2$ partition $[-1/2, \infty)$. On each cell I_j,

$$\left| f(c_j) - \tfrac{1}{2} \int_{I_j} f(c)\, \mathrm{d}c \right| \leq \mathrm{Var}_{I_j}(f).$$

Summing the variations, with absolute convergence, proves the two-sided complete lattice estimate

$$\left| \sum_{j \geq 0} f(c_j) - \frac{\pi}{4(x+y)} \right| \leq \frac{2}{(x+y)^2}.$$

Apply this to the exact reflected response coefficients in Lemma S20.1. The nonexponential part has main term $\pi/\{2L(k_q+k_r)\} = 1/\{2(q+r+1)\}$ and error at most $4/\{L(k_q+k_r)^2\}$. The exponential part is bounded by

$$\frac{2}{Lk_q^2k_r^2}\sum_{j\ge 0} c_j^2 e^{-2c_jL}.$$

Expansion of $c_j^2 = 4j^2+2j+1/4$ and the exact first two geometric moment sums identify the last sum with S_L. Substituting the DN frequencies yields the displayed entry bound for the difference from $\frac{1}{2}E\mathsf{H}E$.

The squared Hilbert–Schmidt sum of $(q+r+1)^{-2}$ is $\sum_{n\ge 1} n/n^4 = \zeta(3)$. The second comparison is rank one, with Hilbert–Schmidt norm $\mathfrak{b}_L\sum_{q\ge 0}(q+1/2)^{-4} = \mathfrak{b}_L\pi^4/6$. Minkowski gives the asserted norm. Thus the coefficient difference defines a self-adjoint Hilbert–Schmidt operator. Equality of the coefficients proves the decomposition on finite vectors and then on the whole space by boundedness and density.

For $r \ge m \ge 1$, decreasing integral comparison gives

$$\sum_{q\ge 0}(r+q+1)^{-4} \le \frac{1}{3r^3}, \qquad \sum_{r\ge m}\frac{1}{3r^3} \le \frac{1}{2m^2}.$$

Convex midpoint comparison also gives $\sum_{r\ge m}(r+1/2)^{-4} \le 1/(3m^3)$. Combine these with the full sum $\sum_{q\ge 0}(q+1/2)^{-4} = \pi^4/6$ and the two entry bounds to obtain $\|(I-P_m)K_L\| \le \|(I-P_m)K_L\|_{\mathrm{HS}} \le \eta_{L,m}$. Finally self-adjointness and $K_L - P_mK_LP_m = (I-P_m)K_L + P_mK_L(I-P_m)$ give the safe factor two in the whole-operator error. □

The approximation retains the exact infinite Hilbert principal channel. It is not a finite-Green approximation of R_L in operator norm. Requested finite K_L entries still need directed coefficient errors, and any solver or inverse for the Hilbert channel needs its own complete error bound. Neither the compact remainder nor its norm controls the signed arithmetic Schur matrix without those further estimates.

S21 Relative source energy and complete tail payment

This appendix proves the whole-tail payment used by Theorem B.6. The spectral bands are proof devices; the finite test in that theorem requires only the first-moment core.

Lemma S21.1 (Elementary prime budget)**.** *For $X \ge 1$,*

$$\sum_{n\le X}\frac{\Lambda(n)}{\sqrt{n}} < 6\sqrt{X}.$$

Consequently $\Sigma_L < 6e^L$.

Proof. For an integer $h \ge 1$, Legendre's valuation formula gives

$$v_p\binom{2h}{h} = \sum_{k\ge 1}\left(\left\lfloor\frac{2h}{p^k}\right\rfloor - 2\left\lfloor\frac{h}{p^k}\right\rfloor\right).$$

Each summand is nonnegative and equals one when $h < p^k \le 2h$. Writing $\Psi_\Lambda(X) = \sum_{n\le X}\Lambda(n)$, this proves $\Psi_\Lambda(2h) - \Psi_\Lambda(h) \le 2h\log 2$. Dyadic telescoping and monotonicity give $\Psi_\Lambda(X) < 4X\log 2$

for $X > 1$. Partial summation yields

$$\sum_{n\le X}\frac{\Lambda(n)}{\sqrt{n}} = \frac{\Psi_\Lambda(X)}{\sqrt{X}} + \frac{1}{2}\int_1^X \Psi_\Lambda(t)t^{-3/2}\,dt \le 8(\log 2)\sqrt{X} - 4\log 2 < 6\sqrt{X}.$$

The case $X = 1$ is immediate, and restriction to the strict active cutoff proves the last assertion. This is the elementary Chebyshev argument. □

Lemma S21.2 (Relative tail-payment lemma)**.** *Let $D \succeq 0$ be self-adjoint with an increasing discrete eigenbasis. Let V be bounded self-adjoint, preserve* $\operatorname{Dom} D$*, and satisfy $\|V\| \le P$, $\|[D,V]\| \le K$, where $P \ge 1$ is rational. The commutator norm is that of its bounded extension from* $\operatorname{Dom} D$*. Let U select the first N modes, $Q = I - UU^*$, and assume that the first omitted diagonal is at least A. Put*

$$H = U^*(D+V)U \succeq \delta I, \quad \delta > 0, \qquad C = QVU.$$

Suppose

$$A \ge \max\{16P, (1152K^2/\delta)^{1/3}\}.$$

For every $\tau \in [A/2, A/2+1]$, the entire receiver satisfies $Q(D+V)Q \succeq \tau I$ on its original form domain, and

$$\tau^{-1}C^*C \preceq \frac{3}{16}H, \qquad H - \tau^{-1}C^*C \succeq \frac{13}{16}H. \tag{S71}$$

Proof. We reproduce the smooth-cutoff proof from *Relative tail payment and spectral enclosures* [22]. Write $V_N = U^*VU$, $D_Q = D|_{\operatorname{ran} Q}$, and $B = Q[D,V]U$. Since U reduces D, domain invariance gives the actual identities

$$[H, V_N] = U^*[D,V]U, \qquad D_QC - CH = B - CV_N. \tag{S72}$$

In particular $C\mathbb{C}^N \subset \operatorname{Dom} D_Q$, $\|B\| \le K$, $\|C\| \le P$ and $D_Q \succeq AI$. The complete receiving operator is $D_Q + QVQ$ on $\operatorname{Dom} D_Q$; no receiving modes are discarded.

An explicit smooth commutator. For $h > 0$ let $b_h = (2h)^{-1}\mathbf{1}_{[-h,h]}$, and set

$$f = \mathbf{1}_{[-3/8,3/8]} * b_{1/16} * b_{1/16}.$$

The two normalized boxes show that $0 \le f \le 1$, $f = 1$ on $[-1/4, 1/4]$, and $f = 0$ outside $[-1/2, 1/2]$. With $\widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-itx}\,\mathrm{d}x$,

$$\widehat{f}(t) = \frac{2\sin(3t/8)}{t}\left(\frac{\sin(t/16)}{t/16}\right)^2.$$

Both $\widehat{f}$ and $t\widehat{f}$ are integrable, and

$$\frac{1}{2\pi}\int_{\mathbb{R}} |t\widehat{f}(t)|\,\mathrm{d}t \le \frac{1}{\pi}\left(\int_0^{16} 2\,\mathrm{d}t + \int_{16}^{\infty} 512t^{-2}\,\mathrm{d}t\right) = \frac{64}{\pi} < 21.$$

For finite self-adjoint matrices L, W, Duhamel's formula gives $\|[W, e^{itL/A}]\| \le |t|\|[W,L]\|/A$. Integration in the Fourier inversion formula therefore proves

$$\|[W, f(L/A)]\| \le 21\|[W,L]\|/A. \tag{S73}$$

This establishes the functional-calculus estimate needed here directly.

Low-head leakage with self-absorption. Put $F = f(H/A)$, $E = \mathbf{1}_{(-\infty,A/2]}(H)$, $X = CF = XE$, and $H_E = H|_{\operatorname{ran} E}$. The support of f gives $F = FE$. By (S73), $\|[V_N, F]\| \le 21K/A$. Multiplying (S72) by F and restricting the source to $\operatorname{ran} E$ gives the exact Sylvester equation

$$D_Q X - X H_E = BF - X V_N E - C[V_N, F]E =: Y.$$

Here $V_N F = F V_N + [V_N, F]$, which fixes the sign and retains the self-leakage term $X V_N E$. The source spectrum is at most $A/2$, whereas the receiving spectrum is at least A. Differentiating $e^{-tD_Q} X e^{tH_E}$ and integrating gives

$$X = \int_0^\infty e^{-tD_Q} Y e^{tH_E}\,\mathrm{d}t, \qquad \|X\| \le \frac{2}{A}\|Y\|.$$

The domain assertion above justifies the derivative: the finite source image lies in $\operatorname{Dom} D_Q$. The terminal term decays at least as $e^{-At/2}$, so the integral converges in operator norm. If E is zero the same statements hold trivially. Consequently

$$\|X\| \le \frac{2}{A}\left(K + P\|X\| + \frac{21PK}{A}\right).$$

Since $P/A \le 1/16$, absorption yields

$$\|Cf(H/A)\| \le \frac{2 + 42P/A}{1 - 2P/A}\frac{K}{A} \le \frac{37K}{7A} \le \frac{6K}{A}. \tag{S74}$$

For $K > 0$ the last inequality is strict; for $K = 0$ the leakage is zero. No positive lower bound for K has been assumed.

Relative payment of the complete return. Let $E_0 = \mathbf{1}_{(-\infty,A/4]}(H)$. The positive-head premise and the plateau of f give $FE_0 = E_0$, hence $\|CE_0\| \le 6K/A$. On $\operatorname{ran} E_0^\perp$, $H > A/4$. Splitting at this projection and using the triangle inequality gives

$$\|CH^{-1/2}\| \le \frac{6K}{A\sqrt{\delta}} + \frac{2P}{\sqrt{A}}.$$

This common operator norm retains all low/high head cross terms. Using $(x+y)^2 \le 2x^2 + 2y^2$ and $\tau \ge A/2$,

$$\frac{\|CH^{-1/2}\|^2}{\tau} \le \frac{144K^2}{\delta A^3} + \frac{16P^2}{A^2} \le \frac{1}{8} + \frac{1}{16} = \frac{3}{16}.$$

Congruence by $H^{1/2}$ proves (S71). Finally the full receiver is at least $(A - P)I$, and

$$A - P - (A/2 + 1) = A/2 - P - 1 \ge 7P - 1 \ge 0.$$

Thus it dominates every τ in the stated interval. This proves all assertions on the original domain. In particular for a head coordinate u and any tail form vector v,

$$\langle u, Hu\rangle + 2\operatorname{Re}\langle v, Cu\rangle + \tau\|v\|^2 = \langle u, (H - \tau^{-1}C^*C)u\rangle + \tau\|v + \tau^{-1}Cu\|^2 \ge 0.$$

Since the complete tail form dominates $\tau\|v\|^2$, no truncated-tail positivity assumption is needed. □

This is an application of the donor's existing first-commutator theorem, with the same finite positive-head premise and complete receiving tail. It replaces the older growing-band sufficient cutoff, not the head calculation. Indeed if $n \geq 1$ is minimal with $4^n \geq 1 + P$, then $1024(n + 1)^2(K^2 + P^2) \geq 4096K^2 \geq 1152K^2$, so the new sufficient threshold is no larger. At $K = 0$ its cubic term vanishes; when $16P$ already dominates both thresholds there is no cutoff reduction. The sharper sufficient threshold applies under the same directed positive-head premise. For the coarse arithmetic budgets $P, K = O(e^L)$ and fixed δ, it gives $A = O(e^L)$, retaining the dimension order of the earlier threshold. The complete-Galerkin and whole-source Gram tests remain complementary constructions with their own admissions.

S22 Supported unweighted growth and zero-free half-planes

The fixed-shift criterion of Theorem B.3 uses one uniform unweighted lower bound across supports. This appendix quantifies the corresponding implication when the allowed negative lower bound depends on the support radius. The argument combines the complete Weil explicit formula [26, 5] with the classical singularity principle for nonnegative Laplace transforms. Related eventual-sign criteria for zero-free half-planes appear in Suzuki's shifted screw-function formulation [24, Section 11, Theorem 11.1].

For a smooth compact profile f write

$$\mathcal{M}f(z) = \int_{\mathbb{R}} f(u)e^{zu}\,\mathrm{d}u, \qquad \|f\|_2^2 = \int_{\mathbb{R}} |f(u)|^2\,\mathrm{d}u.$$

On profiles satisfying $\mathcal{M}f(1/2) = \mathcal{M}f(-1/2) = 0$, put $q_{\mathrm{geo}}[f] = \mathcal{Q}[f]$, with the complete archimedean-minus-prime form $\mathcal{Q}$ of Appendix B. The compact explicit formula (131) identifies this form with its complete zero sum. Let $\mathcal{Z}$ be the set of distinct centered nontrivial zeros $\sigma = \rho - 1/2$, and write m_σ for their positive integer multiplicities. The Fourier measure remains $\mathrm{d}\xi/(2\pi)$.

Proposition S22.1 (Supported growth controls zero locations)**.** *Fix $a \geq 0$. Suppose that, for every $\varepsilon > 0$, there are constants $C_\varepsilon \geq 0$ and L_ε such that*

$$q_{\mathrm{geo}}[f] \geq -C_\varepsilon e^{(a+\varepsilon)L}\|f\|_2^2 \tag{S75}$$

for every $L \geq L_\varepsilon$ and every real odd $f \in C_c^\infty(\mathbb{R})$ supported in $[-L, L]$ with $\mathcal{M}f(\pm 1/2) = 0$. Then every nontrivial zeta zero satisfies

$$\Re\rho \leq \frac{1 + a}{2}.$$

In particular, the hypothesis with $a = 0$ implies RH.

Proof. Choose a real even $g \in C_c^\infty(\mathbb{R})$, supported in $[-r, r]$, with both polar moments zero, and put $G = \mathcal{M}g$. For $R > r$ define

$$f_R = \tau_R g - \tau_{-R} g, \qquad \tau_R g(u) = g(u - R).$$

This test is real, odd and polar-free, has support in $[-(R + r), R + r]$, and its disjoint translated supports give

$$\|f_R\|_2^2 = 2\|g\|_2^2, \qquad \mathcal{M}f_R(z) = 2\sinh(Rz)G(z). \tag{S76}$$

The complete zero form on a real even test has summands $G(\sigma)^2$; on a real odd test it has summands $-\mathcal{M}f(\sigma)^2$. Using the symmetry $\sigma \mapsto -\sigma$, define

$$C_g(R) = \sum_{\sigma \in \mathcal{Z}} m_\sigma e^{2R\sigma} G(\sigma)^2 = \sum_{\sigma \in \mathcal{Z}} m_\sigma \cosh(2R\sigma) G(\sigma)^2.$$

Conjugation symmetry makes C_g real for real R. Expansion of $\sinh^2$ in the complete explicit formula yields

$$q_{\mathrm{geo}}[f_R] = 2q_{\mathrm{geo}}[g] - 2C_g(R). \tag{S77}$$

For every fixed integer $n \geq 0$, integration by parts gives

$$|G(z)| \leq A_{g,n}(1 + |\Im z|)^{-n}, \qquad |\Re z| \leq 1/2.$$

Together with the multiplicity-counted zero bound $N_\zeta(T) = O(T \log(T+2))$, this proves $\sum_\sigma m_\sigma |G(\sigma)|^2 < \infty$. Thus the sum defining C_g converges absolutely, locally uniformly for real R, and satisfies $|C_g(R)| \leq A_g e^R$ for $R \geq 0$. In particular it is continuous. Applying (S75) at $L = R + r$ and using (S76)–(S77) gives

$$C_g(R) \leq q_{\mathrm{geo}}[g] + C_\varepsilon \|g\|_2^2 e^{(a+\varepsilon)(R+r)}.$$

After enlargement of a constant on a compact initial interval,

$$C_g(R) \leq K_{g,\varepsilon} e^{(a+\varepsilon)R}, \qquad R \geq 0. \tag{S78}$$

For $\Re s > 1$, absolute convergence permits termwise Laplace integration:

$$F_g(s) = \int_0^\infty e^{-sR} C_g(R)\, \mathrm{d}R = \sum_{\sigma \in \mathcal{Z}} \frac{m_\sigma G(\sigma)^2}{s - 2\sigma}. \tag{S79}$$

The right side converges normally on compact subsets avoiding its discrete pole set. Indeed the high-zero denominators are bounded below by a constant times their heights, whereas G has arbitrary inverse-power vertical decay. It therefore defines a meromorphic continuation with residue $m_{\sigma_0} G(\sigma_0)^2$ at $s = 2\sigma_0$. This continuation is holomorphic on the positive real axis: nontrivial real zeta zeros are absent, as recalled in Appendix A.

We recall the positive-Laplace singularity principle with its elementary proof. If $h \geq 0$ is locally integrable and its Laplace integral has a finite real convergence abscissa α, the transform has a singularity at the real point α. Otherwise, let it continue holomorphically to a disk of radius r about α and set $s_0 = \alpha + r/4$. Its Taylor series at s_0 converges at $s_0 - u$, where $u = r/3$. Differentiation in the convergence half-plane and Tonelli's theorem give

$$\sum_{n \geq 0} \frac{u^n}{n!} \int_0^\infty R^n e^{-s_0 R} h(R)\, \mathrm{d}R = \int_0^\infty e^{-(s_0 - u)R} h(R)\, \mathrm{d}R < \infty.$$

All Taylor terms in this real direction are nonnegative, while $s_0 - u = \alpha - r/12 < \alpha$, a contradiction. The derivatives under the integral are justified by a smaller positive convergence margin.

Fix $b > a$ and choose $\varepsilon > 0$ with $a + \varepsilon < b$. Equation (S78) and continuity allow a constant $A > 0$ such that

$$h_b(R) = Ae^{bR} - C_g(R) \geq 0 \quad (R \geq 0), \qquad h_b(R) \geq (A/2)e^{bR} \quad \text{for large } R.$$

Its Laplace abscissa α is finite and lies between b and $\max\{b, 1\}$. The transform has meromorphic continuation

$$H_b(s) = \frac{A}{s-b} - F_g(s).$$

If $\alpha > b$, this expression is holomorphic near the real point α, contradicting the singularity principle. Hence $\alpha = b$, and the genuine Laplace transform is holomorphic throughout $\Re s > b$. By meromorphic continuation the residue in (S79) must therefore vanish at every pole in that half-plane.

Suppose now that a centered nontrivial zero σ_0 has $2\Re\sigma_0 > a$. Choose b strictly between these two numbers. Let ϕ_t be an even real smooth mass-one mollifier with compact support shrinking to zero, and set $g_t = (\partial_u^2 - 1/4)\phi_t$. Integration by parts gives both polar moments zero and

$$\mathcal{M}g_t(\sigma_0) = (\sigma_0^2 - 1/4)\mathcal{M}\phi_t(\sigma_0) \longrightarrow \sigma_0^2 - 1/4 \neq 0.$$

Fix a sufficiently small t. This one fixed admissible probe has a nonzero residue at $2\sigma_0$, contradicting holomorphy for $\Re s > b$. Multiplicity contributes the positive integer m_{σ_0} to this same residue. We conclude $2\Re\sigma \leq a$ for all centered zeros, which is the asserted closed boundary. For $a = 0$, functional symmetry places every nontrivial zero on the critical line. □

Corollary S22.2 (Fixed-probe growth characterization)**.** *For $a \geq 0$, the following are equivalent:*

1. *Every nontrivial zeta zero has $\Re\rho \leq (1+a)/2$.*
2. *For each fixed real even smooth compact polar-free g and each $\varepsilon > 0$, there is a finite $K_{g,\varepsilon} \geq 0$ such that $C_g(R) \leq K_{g,\varepsilon} e^{(a+\varepsilon)R}$ for every $R \geq 0$.*

Proof. Under the first assertion, absolute summation gives

$$|C_g(R)| \leq e^{aR} \sum_{\sigma \in \mathcal{Z}} m_\sigma |G(\sigma)|^2, \qquad R \geq 0,$$

which proves the second assertion. Conversely, apply the positive-Laplace argument in the preceding proof to a fixed detecting probe for each proposed zero with $2\Re\sigma > a$. The second assertion supplies exactly the one-sided bound (S78) used in that argument, and the nonzero residue excludes that proposed zero. □

The constants in the fixed-probe characterization may depend on the chosen probe and its support. Proposition S22.1 uses a single C_ε for all admitted profiles at every sufficiently large support. These two quantifier patterns specify the respective supported-form and translated-probe verification targets.

S23 Fixed-strip high-frequency criteria for RH

The complete arithmetic form admits a high-frequency receiving criterion at a fixed averaging seed. A fixed zero-free half-plane supplies an exponential precision scale for this criterion. The construction keeps the original Hilbert norm and minimal domain; its finite null constraints are normalized in the full arithmetic source.

Fix $M = 16$ in Theorem S3.1, and abbreviate $P_k = P_{16,k}$, $N_k = N_{16,k}$ and $d_k = d_{16,k}$. All operators below act on the original odd polar space $\mathcal{K}$ of Section 2, with form domain V. Let γ be the first positive critical ordinate. The rigorous indexed zero isolation and count in [10] give

$14.1 < \gamma < 14.2$, simplicity of this pair, and no other nontrivial zero with $|\Im\rho| \le 21$. Write Q_b for the original polar projection, and set

$$\begin{aligned} v_\gamma &= Q_b[s\sin(\gamma\,\cdot)], & Z_\gamma &= 2v_\gamma v_\gamma^*, \\ P_k^{>} &= Q_b M_s[1_{|\xi|>17}(m_{16,k})_+](D)M_s Q_b, & N_k^{>} &= Q_b M_s[1_{|\xi|>17}(-m_{16,k})_+](D)M_s Q_b, \\ A_k &= P_k^{>} + Z_\gamma, & \kappa_k &= (3/4)^k. \end{aligned} \tag{S80}$$

Thus $0 \preceq N_k^{>} \preceq N_k$, and $Z_\gamma[h] = 2|\int s(u)h(u)\sin(\gamma u)\,du|^2$.

For a fixed integer $r \ge 1$, use the genuine derivative nulls of Lemma 8.1 and define, for all sufficiently large k,

$$Z_r = (z_0, \dots, z_{r-1}), \qquad K_{r,k} = Z_r^* N_k Z_r, \qquad \mathcal{F}_{r,k} = \ker(Z_r^* N_k), \qquad S_{r,k} = I - Z_r K_{r,k}^{-1} Z_r^* N_k. \tag{S81}$$

The positive definiteness of $K_{r,k}$ and the domain properties used here are proved below. For $r = 0$ put $\mathcal{F}_{0,k} = \mathcal{K}$ and $S_{0,k} = I$. Define the *complete physical receiving norm*

$$\Gamma_{r,k} = \sup_{0 \ne x \in V \cap \mathcal{F}_{r,k}} \frac{N_k^{>}[x]}{A_k[x] + \kappa_k \|x\|_2^2}. \tag{S82}$$

It uses the entire high source and the inverse of the restricted closed positive form, rather than a finite compression of its outputs.

Theorem S23.1 (Fixed-strip high-frequency criterion)**.** *Suppose every nontrivial zeta zero $\rho = \beta + i\gamma_\rho$ satisfies $\beta \le b$, for some fixed $1/2 \le b < 1$. Choose fixed numbers*

$$\frac{1}{2} < \theta < R < \frac{3}{2} - b.$$

For each fixed $r \ge 0$, RH is equivalent to the existence of a constant C and an unbounded sequence of integers k on which

$$\Gamma_{r,k} \le 1 + C\theta^k. \tag{S83}$$

Under RH, $\Gamma_{r,k} \le 1$ for all sufficiently large k.

Corollary S23.2 (The cited zero-free half-plane)**.** *Assume the zeta zero-free conclusion of [14, Main theorem], with $\beta \le b_{\mathrm{new}} = 0.874957019420098946\ldots$. For every fixed $r \ge 0$, a complete bound $\Gamma_{r,k} \le 1 + C(5/8)^k$ along an unbounded sequence implies RH.*

Proof. The cited statement includes zeta and allows its principal pole at 1. It specifies $b_{\mathrm{new}} = (18\xi+7)/(18\xi+9)$ with $\xi < 0.38859 < 7/18$, so $b_{\mathrm{new}} < 7/8$. Choose any fixed $5/8 < R < 3/2 - b_{\mathrm{new}}$ in Theorem S23.1. □

For the coarser supplied strip $\beta \le 7/8$, the robust rational choice $\theta = 3/5$, $R = 49/80$ has horizontal margin $1/80$ and gives the sufficient precision $C(3/5)^k$. If $b = 7/8$ exactly, $\theta = 5/8$ is an excluded endpoint of this argument.

There is also a criterion requiring only the classical zero-free region.

Theorem S23.3 (Classical high-frequency criterion)**.** *For every fixed $r \ge 0$ and $0 < c < 1/48$, RH is equivalent to a complete bound, along an unbounded integer sequence,*

$$\Gamma_{r,k} \le 1 + C2^{-k}\exp(ck/\log k). \tag{S84}$$

Under RH the same eventual bound $\Gamma_{r,k} \le 1$ holds.

The following estimates provide the original-domain return and null-lift resources for both criteria. Their proofs, including every spectral collar and infinite tail, are given in Appendix S24.

Proposition S23.4 (Paid finite band and full-source null resources)**.** *The positive form A_k has domain V, and*

$$H = A_k - N_k^{>} + W_k^{>}, \qquad \|W_k^{>}\| \le e_k := \epsilon_k + d_k,$$
$$\epsilon_k = 2\cdot 10^{-17}4^{-k} + 72(3/50)^{k+1} + \frac{90}{7}(17/50)^k + 8640000000\,3^{-k}. \tag{S85}$$

In particular $2^k e_k/\kappa_k \to 0$. For each fixed $r \ge 1$, with $D_k = \mathrm{diag}(1, k, \dots, k^{r-1})$,

$$(2^k k^{-3/2})^{-1} D_k K_{r,k} D_k \longrightarrow \mathsf{M}_r \succ 0, \tag{S86}$$

where, for $0 \le i, j < r$ and $c_0 = -i\widehat{\psi}(0)/4 \ne 0$,

$$(\mathsf{M}_r)_{ij} = \frac{8|c_0|^2}{\pi^2}(-1)^{i+j}\int_{\mathbb{R}} t^{2i+2j+2}e^{-2t^2}\,dt.$$

Consequently

$$\lambda_{\min}(K_{r,k}) \ge c_r 2^k k^{-2r+1/2}, \qquad \|S_{r,k}\|^2 = O_r(k^{2r-1/2}), \qquad (\kappa_k + e_k)\|S_{r,k}\|^2 \longrightarrow 0. \tag{S87}$$

The map $S_{r,k}$ preserves V and q_H, has range $\mathcal{F}_{r,k}$, and satisfies $N_k[S_{r,k}h] \le N_k[h]$. The head order is fixed throughout.

An exact constrained inverse retains the directional receiving terms. For $r \ge 1$, let $V_{r,k} = Z_r K_{r,k}^{-1/2}$, $B_k = N_k V_{r,k}$ and $R_k = (A_k + \kappa_k I)^{-1}$. If $R_{r,k}$ denotes the inverse of the restricted form on $\mathcal{F}_{r,k}$, extended by zero, then

$$C_k = B_k^* R_k B_k, \qquad X_k = V_{r,k}^*(A_k + \kappa_k I)V_{r,k},$$
$$R_{r,k} = R_k - R_k B_k C_k^{-1} B_k^* R_k, \qquad 0 < C_k^{-1} \preceq X_k \preceq [1 + (\kappa_k + e_k)\|V_{r,k}^* V_{r,k}\|]I. \tag{S88}$$

Thus $\Gamma_{r,k} = \|(E_k^{>})^* R_{r,k} E_k^{>}\|$, where $N_k^{>} = E_k^{>}(E_k^{>})^*$. The inverse in (S88) is an actual physical restriction; its constraint columns use N_k, not $N_k^{>}$. The one-sided bound on C_k^{-1} does not replace the complete inequality (S83) or (S84).

Proof of Theorems S23.1 and S23.3. Appendix S24 proves that compact original polar tests with Gevrey-2 physical profile form a V-dense class. For each fixed such h, under the strip hypothesis,

$$N_k^{>}[S_{r,k}h] \le C_{h,r,b,R}(1 + k^{2r-1/2})\{k^8 R^{-k} + 2^k e^{-a_{h,r}k^2} + 17^{-k}\} \tag{S89}$$

when $r \ge 1$; for $r = 0$ omit the polynomial lift factor. Its product with θ^k tends to zero. Independently of a fixed strip, the classical estimate is

$$N_k^{>}[S_{r,k}h] \le C_{h,r}(1 + k^{2r-1/2})2^k\{k^4\log(4k^2)e^{-k/(24\log(4k^2))} + e^{-a_{h,r}k}\}. \tag{S90}$$

Its product with $2^{-k}e^{ck/\log k}$ tends to zero for $c < 1/48$. All constants concern fixed tests and a fixed head.

Write the stated positive excess bound as η_k. Apply the complete comparison to $x_k = S_{r,k}h$ and use original nullity and (S85):

$$q_H[h] = q_H[x_k] \ge -(\kappa_k + e_k)\|x_k\|^2 - \frac{\eta_k}{1+\eta_k}N_k^{>}[x_k].$$

The first term tends to zero by (S87); the appropriate fixed-test energy estimate pays the second. Positivity follows on the Gevrey class and then on V by density. The original odd Weil criterion, Lemma A.1, gives RH. Conversely RH implies $N_k^{>} \preceq A_k + e_k I$. Since $e_k < \kappa_k$ eventually, its restriction proves $\Gamma_{r,k} \le 1$ for every fixed r. □

S23.1 Critical complementary caps and full-source precision

Proposition S23.5 (Exact-null complementary allowance)**.** *Suppose on the original domain $H = P - N + W$, where $N = EE^*$ is bounded positive, P is a closed positive form and $\|W\| \le d$. Let V_0 have finitely many genuine null columns, normalized by $V_0^* N V_0 = I$. Put*

$$J = E^* V_0, \quad Q = I - JJ^*, \quad G = E^*(P + \kappa I)^{-1} E, \quad S = I - V_0 J^* E^*, \quad B_Q = \|EQE^*\|.$$

If the complete comparison $QGQ \preceq (1+\eta)Q$ holds, $\eta \ge 0$, then on all original V,

$$q_H[h] \ge -\left\{(\kappa + d)L^2 + \frac{\eta B_Q}{1+\eta}\right\}\|h\|^2, \qquad L^2 = \|S\|^2 \le 1 + \|V_0^* V_0\|\|N\|. \tag{S91}$$

Proof. The source isometry $J^*J = I$ gives $E^*S = QE^*$. Nullity gives $q_H[Sh] = q_H[h]$ and $S(V) \subset V$. On $x = Sh$ the complete cap implies $N[x] \le (1+\eta)(P[x] + \kappa\|x\|^2)$, whereas $N[x] = \|QE^*h\|^2 \le B_Q\|h\|^2$. Rearrange and add W. The norm estimate is the orthogonal-range calculation proving (S104). □

Corollary S23.6 (A fixed-head order-one threshold)**.** *For the actual order-one averaging, fix any initial derivative-null head of positive finite length and normalize it in $N_{M,1}$. At ridge $\kappa = M^{-1}$, a complete complementary cap $\|Q_M G_M Q_M\| \le 1 + \eta_M$ with $\eta_M = o(M^{-1/2})$ on an unbounded sequence implies RH. The physical lift has $L_M^2 = O_r(1)$ without a growing-head hypothesis.*

Corollary S23.7 (Polynomial relaxation for the full source)**.** *Fix a real A. At seed* 16 *and ridge $(3/4)^k$, the complete full-source comparison*

$$N_k \preceq (1 + Ck^A 2^{-k})(P_k + \kappa_k I)$$

on an unbounded integer sequence implies RH.

The full-source assertion uses a fixed null-jet repair of each test. The high-frequency criteria above instead use the original or full-source-normalized physical receiving space.

S24 Proofs for the high-frequency criteria

We use the Fourier convention and original paired closure of Section 2. In this appendix $N_k = N_{16,k}$ is always the full gamma negative source and $N_k^{>}$ its part on $|\xi| > 17$.

S24.1 A complete meromorphic bound and the finite-band return

Let $F = -\zeta'/\zeta$, and let ξ denote the completed entire xi function. Its Hadamard product and functional equation give

$$\frac{\xi'}{\xi}(w) = B + \sum_\rho \frac{w}{\rho(w-\rho)}, \qquad B = \log(2\sqrt{\pi}) - 1 - \gamma_E/2 \in (-1/40, 0), \qquad \sum_\rho |\rho|^{-2} < 1/10. \tag{S92}$$

Here and below the zero sum counts multiplicity. For completeness, logarithmic differentiation at $0, 1$ and the functional equation give $\sum_\rho [\rho(1-\rho)]^{-1} = -2B < 1/20$. Absolute convergence follows from the classical zero count and the order-one product. The rigorous low-zero input in [10] implies $|\Im\rho| \ge 2$. For $\rho = \beta + it$, $0 < \beta < 1$,

$$\frac{|\rho|^{-2}}{\Re(1/[\rho(1-\rho)])} = \frac{(1-\beta)^2 + t^2}{t^2 + \beta(1-\beta)} \le \frac{5}{4}.$$

Summing proves the last bound in (S92). The elementary enclosures for π, γ_E give the stated interval for B.

Put $\ell = \log 16$, $s = 3/2 - i\xi$, and $\mathcal{A}_s(z) = 16^z F_{>16}(s+z)$, with the finite head as in (S12). A pole of F of residue c at ρ has $q = s - \rho$ and contributes

$$\frac{c16^{1-q}}{(1-q)q^k} \tag{S93}$$

to the difference between the first k alternating Taylor coefficients and evaluation at -1. This follows by summing the geometric series of $c16^{-q}/(z+q)$; its constant evaluation term is retained.

On $|\xi| \le 17$, $|z| \le 3$, the variable $w = s + z$ satisfies $\Re w \ge -3/2$, $|\Im w| \le 20$, $|w| \le 20.5$. The only singularities are the pole at 1 and the first simple zero pair $\rho_\pm = 1/2 \pm i\gamma$; trivial zeros start at -2. Subtract all three principal parts of $\mathcal{A}_s$. For the resulting analytic function $\mathcal{B}_s$, we claim

$$\sup_{|\xi|\le 17, |z|=3} |\mathcal{B}_s(z)| < 20000000. \tag{S94}$$

Indeed all remaining zeros have $|\rho| > 21$, so $|\rho|/|\rho - w| < 42$. In (S92) their absolute contribution is at most $20.5 \cdot 42/10$. Removing the first pair leaves constants $1/\rho_+ + 1/\rho_-$ of modulus below 1. The exact regular identity is

$$F(w) - \frac{1}{w-1} = \frac{1}{2}\Psi(1 + w/2) - \frac{1}{2}\log\pi - \frac{\xi'}{\xi}(w).$$

For $v = 1 + w/2$, $\Re v \ge 1/4$ and $|v - 1| \le 10.25$. The digamma series gives $|\Psi(v)| \le 1 + 6|v-1| \le 62.5$: its coefficient sum is bounded by $4 + \log 4/(3/4) < 6$. Thus the regular part after all three subtractions is bounded by 1000 throughout the region. The finite head is at most $16^{5/2}\log 16$. Each of the three removable divided differences $(16^z - 16^{\rho-s})/(z - (\rho - s))$ is bounded by $16^3 \log 16$, using its integral representation along the joining segment. Therefore

$$|\mathcal{B}_s(z)| \le 16^3[1000 + 16^{5/2}\log 16 + 3\log 16] < 16715776 < 20000000,$$

as claimed.

The analytic Taylor tail at -1 is at most $(3/2)\,20000000\,3^{-k}$. Equation (S12) multiplies this by 16. Each removed pole's constant evaluation term cancels exactly against $\mathcal{B}_s(-1)$, leaving the baseline $F(s-1)$. The completed functional equation gives $2\Re F(1/2 - i\xi) = m_0 + a(\xi)$ away from the first zero pair. Hence

$$|m_{16,k}(\xi) + 2\Re T_k(\xi) - C_{\gamma,k}(\xi) - C_{-\gamma,k}(\xi)| \le 960000000\,3^{-k}, \quad |\xi| \le 17, \tag{S95}$$

$$T_k(\xi) = \frac{4e^{i\ell\xi}}{(1/2 + i\xi)(1/2 - i\xi)^k},$$

$$C_{\gamma,k}(\xi) = 2\Re\frac{e^{i(\xi+\gamma)\ell}}{i(\xi+\gamma)(1 - i(\xi+\gamma))^k}.$$

At $\xi = -\gamma$, the last real part has removable value $2(\ell + k)$. The complex principal part cancels the corresponding one in $F(s-1)$, so (S95) extends by continuity through both collars.

Define the gamma tails

$$Q_k(t) = e^{-t}\sum_{j<k}\frac{t^j}{j!}, \qquad P_k(t) = 1 - Q_k(t), \quad t \ge 0.$$

Direct convolution of the inverse transforms of $(1/2+i\xi)^{-1}$ and $(1/2-i\xi)^{-k}$ gives the inverse kernel

$$\mathcal{F}^{-1}(2\Re T_k)(x) = 2\cosh(x/2) - e^{|x|/2}P_k(|x|-\ell)1_{|x|>\ell}. \tag{S96}$$

Both theta-sandwiched terms are bounded. The first is rank two and annihilates the original space: for $f = sh$, oddness gives $\int f\cosh(u/2) = 0$ and polarity gives $\int f\sinh(u/2) = 0$. For the zero pair, the characteristic function of a Gamma$(k,1)$ variable and the transform of an interval give

$$\mathcal{F}^{-1}(C_{\gamma,k} + C_{-\gamma,k})(x) = 2\cos(\gamma x)\{1 - P_k(|x|-\ell)1_{|x|>\ell}\}. \tag{S97}$$

Its full rank term is exactly Z_γ on the odd polar space.

The complete theta envelope (8), with $d = |u-v|$, implies

$$|s(u)s(v)| \le 169e^{2d-\pi e^d}. \tag{S98}$$

To see this, first use $x = |u|+|v| \ge d$ and $e^{2|u|} + e^{2|v|} \ge 2e^x$, then the decrease of $2x - \pi e^x$. Also $P_k(t) \le t^k/k! \le 4^{-k}e^{4t}$. Schur's test bounds the two residual kernels in (S96)–(S97) by $C_{16}4^{-k}$ and $D_{16}4^{-k}$, where

$$C_{16} = \frac{338\,16^{5/2}e^{-16\pi}}{16\pi - 13/2} < 13\cdot 10^{-18}, \qquad D_{16} = \frac{676\,16^2 e^{-16\pi}}{16\pi - 6} < 7\cdot 10^{-18}.$$

For example the first row bound is $338\,16^{-4}4^{-k}\int_\ell^\infty e^{(13/2)d-\pi e^d}\,dd$; the derivative of its exponent is at most $13/2 - 16\pi$. The rational bounds follow from $\pi > 3$, $e^3 > 20$: $C_{16} < 692224/(83\,20^{16}) < 13\cdot 10^{-18}$ and $D_{16} < 173056/(42\,20^{16}) < 7\cdot 10^{-18}$.

Finally the entire model exterior $|\xi| > 17$ satisfies

$$|T_k| \le 4(3/50)^{k+1}, \qquad |C_{\gamma,k}| + |C_{-\gamma,k}| \le \frac{10}{7}(17/50)^k.$$

Here $2/\sqrt{1157} < 3/50$ and $|\xi \pm \gamma| > 14/5$ give $5/\sqrt{221} < 17/50$. The theta sandwiches cost at most 9, so these exteriors cost $72(3/50)^{k+1}$ and $(90/7)(17/50)^k$. The same factor applied to (S95) costs $8640000000\,3^{-k}$. We have proved the full signed operator bound

$$\left\| Q_b M_s [1_{|\xi|\le 17} m_{16,k}](D) M_s Q_b - Z_\gamma \right\| \le \epsilon_k. \tag{S99}$$

Combining with (S9) proves (S85). All kernels used for this cancellation are bounded after the original sandwich, so their compact-core identities extend on the original paired closure. The low band is bounded at each k; removing it and adding the bounded row Z_γ leaves the same form domain V.

S24.2 Moving contours and full-source asymptotics

The following single estimate also pays all high-frequency remainders. If $\mathcal{B}_s$ removes just the actual pole at 1, and the disks $|z| \le R < 1$ have distance at least δ from every nontrivial zero, then (S92) gives, on $|\xi| \le T$,

$$|\xi'/\xi(s+z)| \le 1 + \tfrac{1}{10}(T+3)[1 + (T+3)/\delta].$$

The regular digamma term and the finite head have smaller bounds. The divided difference for the removed pole is bounded by $16^R \log 16$, independently of its imaginary position. Thus

$$m_{16,k} = -2\Re T_k + E_k, \qquad \sup_{|\xi|\le T} |E_k| \le C(T+3)^2(1+\delta^{-1})\frac{R^{1-k}}{1-R}. \tag{S100}$$

The bounded pole baseline is $-4e^{i\ell\xi}/(1/2+i\xi)$ in the arithmetic evaluator; it, the finite head and $|m_0+a(\xi)| \le 6+17T^2$ are included in E_k. This is a finite Taylor-sum estimate and uses no evaluation at -1.

For an unconditional choice use [15, Theorem 1.3], in its weaker form $\beta < 1-[6\log|\gamma_\rho|]^{-1}$ for $|\gamma_\rho| \ge 2$. For $T \ge 4$ put

$$\delta_T = [24\log(4T)]^{-1}, \qquad R_T = 1/2+\delta_T < 3/5.$$

Zeros with $|\gamma_\rho| \le 2T$ have $\beta < 1-4\delta_T$; the remaining zeros have imaginary separation greater than $T-3/5$. Consequently all the disks are at distance at least δ_T from every nontrivial zero. Trivial zeros are outside them. Equation (S100) becomes

$$\sup_{|\xi|\le T} |E_k(\xi)| \le C(T+3)^2\log(4T)2^k e^{-k/(24\log(4T))}, \tag{S101}$$

using $\log(1+2\delta_T) \ge \delta_T$.

For a fixed original vector with Schwartz physical profile $f = sh$, take $T = k$. The negative-part Lipschitz inequality, the complete uniform symbol bound $n_{16,k} \le 6+32(\log 2)2^k$, and the Schwartz Fourier tail prove, for every fixed $D > 0$,

$$N_k[h] = \int_{\mathbb{R}} (2\Re T_k)_+ |\widehat{f}(\xi)|^2 \frac{d\xi}{2\pi} + O_{h,D}(2^k k^{-D}). \tag{S102}$$

The same argument applies to mixed entries of any fixed finite list of such profiles: integrate their product in place of $|\widehat{f}|^2$. No complete source exterior has been removed without this payment.

If $\widehat{f}(\xi) = c_f\xi^m + O(\xi^{m+2})$, $c_f \ne 0$, with m odd, rescale $\xi = t/\sqrt{k}$. The positive pole amplitude is $16\,2^k(1+4\xi^2)^{-(k+1)/2}$, and its phase is $\ell\xi+(k-1)\arctan(2\xi) = 2\sqrt{k}\,t+o(1)$ on compact t sets. The periodic mean of $(\cos)_+$ is $1/\pi$. Its absolutely convergent Fourier series and Gaussian domination therefore give

$$N_k[h] \sim \frac{4\,2^{1/2-m}\Gamma(m+1/2)}{\pi^2}|c_f|^2 2^k k^{-m-1/2}. \tag{S103}$$

Here $\int_{\mathbb{R}} t^{2m}e^{-2t^2}dt = 2^{-m-1/2}\Gamma(m+1/2)$. Gaussian domination follows on a fixed small band from $\log(1+4\xi^2) \ge c\xi^2$; the complement is exponentially smaller. This also proves the upper $O_h(2^k k^{-m-1/2})$ when only the indicated vanishing order is known.

For the raw null columns, differentiation gives exactly

$$\widehat{sz_j}(\xi) = (-1)^j\xi^{2j}F_0(\xi), \qquad F_0(\xi) = -i\xi(\xi^2+1/4)\widehat{\psi}(\xi), \qquad F_0'(0) = c_0 \ne 0.$$

These profiles are Schwartz, by their full theta derivative tails. Apply (S102) entrywise with arbitrary D, then the same rescaling and periodic mean to $(-1)^{i+j}\xi^{2i+2j}|F_0(\xi)|^2$. This gives exactly (S86). The powers k^{i+j} in D_k are paid by choosing $D > i+j+3/2$ before taking the limit; no fixed-test asymptotic is applied to a changing coefficient vector. For any nonzero $a \in \mathbb{C}^r$,

$$a^*\mathsf{M}_r a = \frac{8|c_0|^2}{\pi^2}\int_{\mathbb{R}} t^2e^{-2t^2}\left|\sum_{j<r} a_j(-1)^j t^{2j}\right|^2 dt > 0.$$

Finite matrix-norm convergence proves $K_{r,k} \succeq c_r 2^k k^{-3/2}D_k^{-2}$, and hence the first bound in (S87). Constants and onset concern fixed r.

S24.3 Physical null transport and its actual constrained inverse

Suppress r, k temporarily. Put $V_r = Z_r K_{r,k}^{-1/2}$, $B = N_k V_r$. Then $V_r^* N_k V_r = B^* V_r = I$ and $S = I - V_r B^*$ is an oblique projection onto $\ker B^*$ along $\operatorname{ran} Z_r$. It preserves V and q_H by genuine associated-operator nullity. Expansion gives

$$S^* N_k S = N_k - BB^* \succeq 0. \tag{S104}$$

The positivity follows equally from the source isometry $E_k^* V_r$. If Π is the physical orthogonal projection onto $\operatorname{ran} Z_r$, $S = S(I - \Pi)$ and its correction to $(I - \Pi)h$ lies in the orthogonal space $\operatorname{ran} \Pi$. Therefore

$$\|S\|^2 \le 1 + \|V_r^* V_r\| \|N_k\|, \qquad \|V_r^* V_r\| \le C_r 2^{-k} k^{2r-1/2}.$$

Theorem S3.1 now proves (S87) and pays its product with $\kappa_k + e_k$.

For each fixed k, S is bounded on $\mathcal{K}$ and preserves V. If $x \in \mathcal{F}_{r,k}$ and $h_n \in V$ converge to x in the ambient norm, then $Sh_n \to x$ and $Sh_n \in V \cap \mathcal{F}_{r,k}$. This proves density in the closed physical subspace. Its form domain is closed in the A_k form norm because $\mathcal{F}_{r,k}$ is closed in $\mathcal{K}$. Thus the restricted positive form and its inverse used in (S82) are well-defined.

Let $R = (A_k + \kappa_k I)^{-1}$, $X = V_r^*(A_k + \kappa_k I)V_r$ and $C = B^* R B$. Original nullity and (S85) give

$$X = V_r^* N_k^{>} V_r + V_r^*(\kappa_k I - W_k^{>})V_r \preceq I + (\kappa_k + e_k) V_r^* V_r.$$

Both X, C are positive definite. Restricting the variational supremum for $\langle y, Ry\rangle$ to vectors $V_r a$ proves $R \succeq V_r X^{-1} V_r^*$, whence $C \succeq X^{-1}$. This proves the finite inverse bound in (S88) without a high-only source Gram. Its inverse identity follows because $R - RBC^{-1}B^*R$ maps into $\ker B^*$, annihilates $\operatorname{ran} B$, and solves the weak restricted equation against $V \cap \ker B^*$. Positive form congruence then proves (S82) with the complete source. For example $C \preceq \kappa_k^{-1}\|N_k\| I$; the inverse bound is one-sided and does not assert a condition number near one.

S24.4 Gevrey density and high-source energies

Compact polar tests with Gevrey-2 physical profile are V-dense. To verify this exactly, take a compact smooth odd polar profile f and an even compact Gevrey-2 approximate identity ρ_ε. Set $f_\varepsilon = f * \rho_\varepsilon$ and $h_\varepsilon = f_\varepsilon / s$. One may use the normalized bump $e^{-1/(1-x^2)} 1_{|x|<1}$. Cauchy on a circle of radius $(1 - x^2)/8$, followed by optimization of $t^{-n} e^{-c/t}$, gives its $C^n (n!)^2$ derivative bound and flatness at the endpoints. Moving all derivatives to the mollifier proves the required bound for f_ε. Oddness is preserved. Expanding $\sinh((u+v)/2)$ in the convolution shows that its polar moment equals

$$\left(\int \rho_\varepsilon \cosh(v/2)\, dv\right)\left(\int f \sinh(u/2)\, du\right) + \left(\int \rho_\varepsilon \sinh(v/2)\, dv\right)\left(\int f \cosh(u/2)\, du\right) = 0.$$

On a common compact support $1/s$ is bounded, and $f_\varepsilon \to f$ in H^1. The bound $a(\xi) \le 17\xi^2$ proves $h_\varepsilon \to h$ in the original V norm. Combining with the original compact core proves the asserted density.

For a fixed such profile, $\|f^{(n)}\|_1 \le C H^n (n!)^2$ implies $|\widehat{f}(\xi)| \le C_h e^{-a_h \sqrt{|\xi|}}$ by integration by parts, choosing $n = \lfloor \sqrt{|\xi|/(4H)} \rfloor$. Consequently its entire Fourier energy above T is at most $C_h e^{-a_h' \sqrt{T}}$ after polynomial absorption. The fixed raw null profiles have an even stronger bound:

$$|\widehat{sz_j}(\xi)| \le C_j e^{-|\xi|/128}. \tag{S105}$$

Indeed the complete theta series is holomorphic in $|\Im u| < \pi/4$. On $|\Im u| \le 1/128$ each fixed derivative is bounded by a polynomial in $e^{|\Re u|}$ times $e^{-c\pi e^{2|\Re u|}}$, from the Gaussian atoms and even continuation. These bounds are integrable on both horizontal arms and tend to zero on the vertical closing segments. The Fourier contour shift by $\pm 1/128$ proves (S105). No division by theta occurs in this step.

Under the fixed strip in Theorem S23.1, take its declared fixed R and $\delta = 3/2 - b - R > 0$. Every zero is horizontally separated from every disk. Equation (S100), with $T = k^4$, gives

$$\sup_{|\xi| \le k^4} |E_k(\xi)| \le C_{b,R} k^8 R^{-k}.$$

On $17 < |\xi| \le k^4$ use the negative-part Lipschitz inequality; the pole amplitude is at most $8\,17^{-k-1}$. Above k^4 use the complete symbol bound $n_{16,k} \le 6 + 32(\log 2)2^k$ and the Gevrey tail. Thus

$$N_k^{>}[h] \le C_{h,b,R}\{k^8 R^{-k} + 2^k e^{-a_h k^2} + 17^{-k}\}. \tag{S106}$$

Without a fixed strip, put $T = k^2$ in (S101); the same complete exterior argument gives

$$N_k^{>}[h] \le C_h 2^k \{k^4 \log(4k^2) e^{-k/(24\log(4k^2))} + e^{-a_h k}\}. \tag{S107}$$

These are fixed-test energies and every frequency is covered.

It remains to justify these estimates on the moving repaired vectors, not to apply a fixed-test limit to them. Write $S_{r,k}h = h - Z_r a_k$. Equation (S104) gives

$$a_k^* K_{r,k} a_k \le N_k[h] \le C 2^k \|h\|^2, \qquad \|a_k\|^2 \le C_{h,r} k^{2r-1/2}.$$

By (S105), each of the finitely many raw null profiles satisfies (S106) or (S107) with a common fixed exponential constant. Positivity bounds their high Gram norm by the sum of its diagonal entries. Finally

$$N_k^{>}[h - Z_r a_k] \le 2N_k^{>}[h] + 2a_k^*(Z_r^* N_k^{>} Z_r) a_k$$

retains every mixed term. This proves (S89)–(S90) and completes the resource and criterion proofs.

S24.5 The order-one threshold and fixed null-jet repair

We supply the remaining proofs without a uniform assumption on the conditioning of a changing finite head.

Proof of Corollary S23.6. Fix an interval $I \subset (0,1)$ on which $\widehat{\psi}$ has no zero. The complete Abel estimate in Section S5 gives, uniformly on I,

$$m_{M,1}(\xi)/\sqrt{M} = -2\cos((\log M)\xi)/(1/4 + \xi^2) + o(1).$$

Consequently $n_{M,1}/\sqrt{M}$ is bounded below by a fixed positive constant where $\cos((\log M)\xi) \ge 3/4$. The source rows of the fixed raw null head are $(-1)^j \xi^{2j} F_0(\xi)$. The indicator of this periodic set has positive mean. Integration against each fixed C^1 entry $|F_0|^2 \xi^{2i+2j}$ converges to its mean times the integral on I: integrate its zero-mean part against a bounded periodic primitive. The limiting polynomial Gram is positive definite. Since the head length is fixed, entry convergence is matrix-norm convergence. Dropping only positive source frequencies outside I proves $Z_r^* N_{M,1} Z_r \succeq c_r \sqrt{M} I$. Thus $\ell_M = \|V_{0,M}^* V_{0,M}\| = O_r(M^{-1/2})$. The bound $\|N_{M,1}\| = O(\sqrt{M})$ in Theorem S3.1 proves $L_M^2 = O_r(1)$. Apply (S91); the remainder $d_{M,1}$ is exponentially small and $\eta_M B_{Q,M} \le \eta_M \|N_{M,1}\| = o(1)$. The cofinal original-domain endpoint proves the conclusion. □

Proof of Corollary S23.7. For a compact original test h the profile $f = sh$ is odd and smooth. Since $F_0'(0) \neq 0$, the first m odd Taylor coefficients of $\widehat{f} - \sum_{j<m} a_j(-1)^j \xi^{2j} F_0(\xi)$ can be set to zero by one triangular system with nonzero diagonal $F_0'(0)$. The coefficients and repaired vector $T_m h = h - \sum_{j<m} a_j z_j$ are fixed, independent of k; genuine nullity preserves q_H and V. The repaired profile is Schwartz and vanishes at the origin to order at least $2m+1$. By (S102) and the Gaussian scaling used in (S103),

$$N_k[T_m h] = O_h(2^k k^{-2m-3/2}).$$

This upper bound also covers a higher vanishing order, without dividing by its leading Taylor coefficient. Apply the complete full-source comparison to $T_m h$ and rearrange as in the proof of Theorem S23.1. Choose the fixed integer m so that $2m + 3/2 > A$. The excess times this source energy tends to zero; $(\kappa_k + d_k)\|T_m h\|^2$ also tends to zero. Thus $q_H[h] \geq 0$ on the compact original core, and closure and Lemma A.1 give RH. □

S25 Complete fixed-support ridge continuation

The M24 certificate also supplies a continuous family at the same fixed support. With $\kappa_0 = 2^{-24}$, the complete derivative-weight integrals and all 659 directions certify the interval

$$\kappa_0 - h \leq \kappa \leq \kappa_0, \qquad h = \frac{1125889145388605}{75557863725914323419136}.$$

Its lower endpoint is approximately $4.47036260082 \times 10^{-8}$. The full constrained reference-inverse Schur lower obeys $\kappa\Gamma_\kappa' \preceq 3\Gamma_\kappa$ throughout this band. The complete derivative identity, downward-loss estimate, capped-return comparison and numerical proof are in Supplementary Section S49. This extension has separate derivative primitives and exact congruences, identified in the scientific data index. Support, constraints, Green model and legal trials stay fixed throughout its comparison.

S26 A cofinal finite-response criterion for RH

The complete covariance criterion admits a second, explicit supported-form reduction. It keeps the original polar condition and physical norm while replacing the infinite family of positive Green responses by logarithmically many features. The receiving complement remains infinite and has a proved floor; its exact inverse occurs in the finite Schur matrix below.

Use the odd-half unitary identification with $L^2(0, L)$ and put

$$p(x) = \sinh(x/2), \qquad \mathcal{F}_L = p^\perp, \qquad e_q(x) = \sqrt{2/L}\sin(k_q x), \quad k_q = (q + 1/2)\pi/L.$$

Let $D_L e_q = a(k_q)e_q$. The complete supported archimedean form is $D_L + R_L$, where

$$R_L = \sum_{j\geq 0} r_j \otimes r_j, \qquad r_j(x) = \frac{e^{-c_j(L-x)} - e^{-c_j(L+x)}}{\sqrt{1 + e^{-2c_j L}}}, \quad c_j = 2j + \tfrac{1}{2}.$$

The Green identity and original-domain closure are proved in the main paper, Appendix 5. Section S27 below proves the relative-response and joint estimates used here. Write $R_{L,J}$ for the first J responses and $V_L = m_0 - C_L^{\mathrm{ar}}$ for the complete bounded arithmetic operator on the supported interval, including every active prime power. Thus $\|V_L\| \leq B_L := 6 + 8(\log 2)e^L$ by (S1). Restrictions

of these forms to $\mathcal{F}_L$ are understood throughout. Precisely, if $I=(-L,L)$ and $\mathcal{U}_L$ is the odd-half unitary map,

$$C_L^{\mathrm{ar}}=\mathcal{U}_L\left[1_I\sum_{2\le n<e^{2L}}\frac{\Lambda(n)}{\sqrt{n}}(S_{\log n}+S_{-\log n})1_I\right]\mathcal{U}_L^*.$$

This transport retains the reflected channel as well as same-side shifts; the endpoint overlap has measure zero. Consequently the complete supported profile form $D_L+R_L+V_L+M^{-1}s^{-2}$ is exactly the original $H+M^{-1}I$ under $f=sh$ and this unitary map.

Theorem S26.1 (Cofinal finite-response criterion)**.** *For integers $M\ge 3$ define*

$$\begin{aligned}&L_M=\tfrac12\log(10\log M/\pi)+2, && \kappa_M=M^{-1},\\ &B_M^{\mathrm{ar}}=6+8(\log 2)e^{L_M}, && J_M=\lceil\tfrac12\log M\rceil,\\ &N_M=\lceil L_Me^{2B_M^{\mathrm{ar}}}/\pi\rceil.\end{aligned}$$

Let U_M be a profile isometry onto $P_{\mathcal{F}_{L_M}}\operatorname{span}(e_0,\dots,e_{N_M-1})$, and let Q_M be its orthogonal complement in $\mathcal{F}_{L_M}$. Form the complete finite-response operator

$$S_{0,M}=D_{L_M}+R_{L_M,J_M}+V_{L_M}+M^{-1}s^{-2}.$$

Set $T_{0,M}=S_{0,M}|_{Q_M}$ and

$$\mathbf{K}_{0,M}=U_M^*S_{0,M}U_M-(Q_MS_{0,M}U_M)^*T_{0,M}^{-1}(Q_MS_{0,M}U_M). \tag{S108}$$

Then $T_{0,M}\ge\frac13A_{Q_M}>0$, where A_{Q_M} is the form restriction of $D_{L_M}+B_M^{\mathrm{ar}}+M^{-1}s^{-2}$. The Riemann hypothesis is equivalent to $\mathbf{K}_{0,M}\succeq 0$ along some unbounded sequence of integers M. Under RH these matrices are strictly positive for every integer $M\ge 2^{20}$. Moreover $N_M=M^{o(1)}$ and $J_M=O(\log M)$.

Proof. Orthogonality to the projected modes implies orthogonality to the original first N_M DN modes. Since $a(k)\ge\log k$ for $k\ge1$, the diagonal form on this complement is at least $2B_M^{\mathrm{ar}}$. The signed arithmetic is bounded below by $-B_M^{\mathrm{ar}}$ and $R_{L_M,J_M}\succeq 0$. Comparing coefficients gives $T_{0,M}\ge\frac13A_{Q_M}$, with every receiving direction retained. The complete form Schur identity therefore makes $\mathbf{K}_{0,M}\succeq 0$ equivalent to $S_{0,M}\succeq 0$.

The omitted responses are positive, so this matrix inequality implies $q_H[h]+M^{-1}\|h\|_2^2\ge 0$ for every original polar-free test supported in $(-L_M,L_M)$. Since $L_M\to\infty$, every smooth compact test is admitted eventually. Taking the unbounded sequence and then the original form closure proves Weil positivity and hence RH by the odd converse.

Conversely, the complete relative-response estimate in Section S27 gives $0\le R_{L_M}-R_{L_M,J_M}\le M^{-2}(D_{L_M}+B_M^{\mathrm{ar}}+M^{-1}s^{-2})$. Its full physical form sandwich, with $\|s\|_\infty<3$, yields under RH

$$sS_{0,M}s\ \ge\ \left[\frac{1-M^{-2}}{M}-\frac{18B_M^{\mathrm{ar}}}{M^2}\right]I>0$$

For an explicit admission, $B_M^{\mathrm{ar}}<6+120\sqrt{\log M}$ by $\log 2<1$, $e^4<64$ and $\pi>3$. This upper divided by M decreases for $M\ge e$. At $M=2^{20}$ it is below $606/2^{20}$, and $2^{-40}+10908/2^{20}<1/2$. The displayed lower is consequently at least $1/(2M)$ for every $M\ge 2^{20}$, proving strict positivity of the Schur matrix. Finally $B_M^{\mathrm{ar}}=O(\sqrt{\log M})$, so $\log N_M=O(\sqrt{\log M})$ and the stated resource bounds follow. □

The matrix in (S108) retains the *exact inverse of the full receiving operator*; it is not a principal compression. Theorem S26.1 isolates its sign as the remaining condition in this route. Section S27 also bounds the complete core–mixed–receiving change caused by the response reduction by $M^{-2+o(1)}$ in physical core coordinates, and preserves original-norm $o(M^{-3/2})$ allowances directly at the full-form level. The core coordinates do not make the transported receiving complement physically orthogonal; quantitative allowance tests shift the complete physical form before taking its Schur complement. The finite-sine alternative in Proposition S27.5 retains the same cofinal RH equivalence, uses only finite DN combinations, and has raw profile Gram condition below $4L_M = O(\log\log M)$. For that core the joint Schur error is $O(\sqrt{\log M}/M^2)$ in physical core coordinates.

S27 Relative Green approximation and exact-response criteria

S27.1 A whole relative-response bound

Let $R_{L,J} = \sum_{j<J} r_j \otimes r_j$ and $E_J = R_L - R_{L,J}$. First on ambient $L^2(0,L)$, for $B \geq 1$, $\kappa > 0$ define the closed positive reference form

$$A = D_L + B + \kappa s^{-2}.$$

Proposition S27.1 (Complete relative tail)**.** *For $0 < \eta < L$, let $\sigma_\eta^2 = \operatorname{ess\,sup}_{L-\eta<x<L} s(x)^2$. Then $0 \leq E_J \leq \epsilon A$, where*

$$\epsilon = \left[\sqrt{C_L \sigma_\eta^2/\kappa} + \sqrt{\frac{e^{-2c_J\eta}}{2Bc_J(1-e^{-4\eta})}}\right]^2. \tag{S109}$$

The inequality includes every omitted response and survives restriction to the original polar-free subspace.

Proof. Write $P_e = 1_{(L-\eta,L)}$ and $P_i = I - P_e$. Inverse order gives $A^{-1} \preceq (B + \kappa s^{-2})^{-1}$, so

$$\|E_J^{1/2} P_e A^{-1/2}\|^2 \leq C_L \|P_e A^{-1} P_e\| \leq C_L \sigma_\eta^2/\kappa.$$

For $d = L - x \geq \eta$, the exact feature obeys $0 \leq r_j(x) \leq e^{-c_j d}$. Tonelli and positivity give

$$\operatorname{Tr}(P_i E_J P_i) \leq \sum_{j \geq J} \frac{e^{-2c_j\eta}}{2c_j} \leq \frac{e^{-2c_J\eta}}{2c_J(1-e^{-4\eta})}.$$

Using $\|A^{-1/2}\| \leq B^{-1/2}$ bounds the interior factor. Add $E_J^{1/2} P_e A^{-1/2}$ and $E_J^{1/2} P_i A^{-1/2}$ and square their norm bound. This proves (S109) as a form inequality. The discontinuous spatial multipliers are used only as bounded L^2 operators on inverse square roots; they are not assumed to preserve the form domain. Apply the positive-series arguments first to partial sums and pass monotonically. No unweighted whole-tail norm convergence is used. □

Corollary S27.2 (Explicit logarithmic response schedule)**.** *Let $U = \log M$, $M \geq e$, and*

$$L = \tfrac{1}{2}\log(10U/\pi) + 2, \quad \kappa = M^{-1}, \quad J = \max(1, \lceil U/2 \rceil).$$

Then $0 \leq E_J \leq M^{-2} A$ for every $B \geq 1$.

Proof. Use $\eta = 1$ and the main paper's theta envelope $s(u)^2 \le 169e^{4u-\pi e^{2u}}$. The envelope decreases on $u \ge 0$; since $L > 1$,

$$\sigma_1^2 \le 169(10e^2/\pi)^2 U^2 M^{-10e^2}.$$

For $U \ge 1$, $C_L \le 5U$, $e^2 < 8$, $e^2 > 7$ and $\pi > 3$. The square of the first term in (S109), divided by M^{-2}, is therefore at most

$$845(80/3)^2 U^3 e^{-67U} \le 845(80/3)^2 e^{-67} < 2^{-47} < \tfrac{1}{4}.$$

Here $U^3 e^{-67U}$ decreases for $U \ge 1$, its constant is below 2^{20}, and $e > 2$. For the second squared term use $4J \ge 2U$, $c_J \ge 5/2$ and

$$[5(1 - e^{-4})]^{-1} < \tfrac{1}{4}.$$

Both squared contributions are below $M^{-2}/4$; their square-root sum is below M^{-1}. All constants are elementary admissions, not sampled floating inequalities. □

S27.2 Preserving the original-norm allowance

On the original polar-free supported space write

$$S_f = D_L + R_L + V_L + \kappa s^{-2}, \quad \|V_L\| \le B, \quad S_{f,J} = S_f - E_J.$$

Since $A \preceq S_f + 2BI$, the complete relative bound implies

$$(1 - \epsilon)S_f - 2\epsilon BI \preceq S_{f,J} \preceq S_f. \tag{S110}$$

Congruence by the same s, with $\|s\|_\infty < 3$, gives

$$(1 - \epsilon)S_h - 18\epsilon BI \preceq S_{h,J} \preceq S_h. \tag{S111}$$

At the displayed growing window the elementary arithmetic budget is

$$B_M = 6 + 8(\log 2)\sqrt{(10e^4/\pi)\log M} = O(\sqrt{\log M}).$$

Thus $18B_M M^{-2} = o(M^{-3/2})$. Equation (S111) proves that existence of original physical allowances $S_h \succeq -\delta_M I$, $\delta_M = o(M^{-3/2})$, is equivalent to existence of such allowances for $S_{h,J}$: take $\delta'_M = (1 - M^{-2})\delta_M + 18B_M M^{-2}$ in one direction and use $S_h \succeq S_{h,J}$ in the other. This uses the full form, not merely a core matrix. Nonvanishing extra finite corrections require a complete two-sided budget; the same power-two deduction then needs that total budget to be $o(\sqrt{M})$.

S27.3 An explicit paid core and its energy

Let $P_{\mathcal{F}}$ be the projection onto $p^\perp$ and take

$$N = \lceil Le^{B+\beta}/\pi \rceil, \qquad \operatorname{ran} U = P_{\mathcal{F}} \operatorname{span}(e_0, \dots, e_{N-1}), \qquad U^*U = I,$$

where $\beta > 0$. A vector in the complementary subspace is orthogonal to each original e_q, $q < N$. Since a is increasing and $a(k) \ge \log k$ for $k \ge 1$, its diagonal energy is at least $B + \beta$. Put $Q = I - UU^*$ on $\mathcal{F}$, and let A_Q denote the form restriction of A to Q. For both the complete and partial-response receiving forms,

$$T \succeq cA_Q, \qquad T_0 \succeq cA_Q, \qquad c = \frac{\beta}{\beta + 2B}. \tag{S112}$$

Indeed $(1 - c)D_L + R_{L,J} + (1 - c)\kappa s^{-2} + V_L - cB$ is bounded below by $(1 - c)\beta - 2cB = 0$. Every response and receiving direction is accounted for.

Lemma S27.3 (Projected-polar core control)**.** *For $L \geq 1$, the preceding core has dimension N and*

$$D_{\mathrm{core}} := \|U^* D_L U\| \leq 4L(17k_N^2 + 68)(1/4 + k_N^2)^2. \tag{S113}$$

Its projected-column Gram has condition number at most $2L(1/4 + k_N^2)^2$.

Proof. Direct integration gives

$$p_q = \sqrt{2/L}(-1)^q \frac{\frac{1}{2}\cosh(L/2)}{1/4 + k_q^2}, \qquad \|p\|^2 = (\sinh L - L)/2.$$

Consequently $p \in \operatorname{Dom} D_L$. With $\bar{p} = p/\|p\|$ and $\delta_N = \|(I - P_N)\bar{p}\|^2$,

$$\delta_N \geq |\langle \bar{p}, e_N \rangle|^2 \geq [2L(1/4 + k_N^2)^2]^{-1}.$$

The raw projected-column Gram is $I - vv^*$, $v = P_N \bar{p}$ in DN coordinates; its eigenvalues are 1 and $\delta_N > 0$. This proves the dimension and Gram assertions and gives an exact rank-one orthonormalization formula if desired.

The positive digamma series yields $a(k) \leq 2k^2 \sum c_j^{-3} < 17k^2$: its first coefficient is 16 and the rest is at most $\frac{1}{4}\sum_{j \geq 1} j^{-3} \leq 3/8$. The DN Laplacian form identity gives

$$D_L[\bar{p}] \leq 17 \frac{\|p'\|^2}{\|p\|^2} = 17 \frac{\sinh L + L}{4(\sinh L - L)} < 68 \quad (L \geq 1).$$

The last quotient is below 4 because $L/\sinh L$ decreases and $\sinh 1 > 7/6$. For $w \in \operatorname{ran} P_N$, $v = P_{\mathcal{F}} w$ satisfies

$$\|v\|^2 \geq \delta_N \|w\|^2, \qquad D_L[v] \leq 2(17k_N^2 + 68)\|w\|^2.$$

Combining proves (S113). In particular the projected columns are legal operator-domain vectors. This is the projected DN core, not the smaller intersection of an unprojected finite prefix with $p^\perp$. □

S27.4 Joint core, mixed, and receiving error

Let $S = S_0 + E$, where S_0 retains the first J responses, and $0 \preceq E \preceq \epsilon A$. All operators here act on the same polar-free profile space. Their form domains equal $\operatorname{Dom} A^{1/2}$. Put $d = 1 + C_L/B$. The bounds $-A \preceq S_0 \preceq dA$ and (S112) hold without a smallness assumption on ϵ. Let

$$H_0 = U^* S_0 U, \quad C_0 = Q S_0 U, \quad Z_0 = T_0^{-1} C_0, \quad K_0 = H_0 - C_0^* T_0^{-1} C_0,$$

and define K analogously for S. Inverse pairings are form-dual pairings when needed; A_Q^{-1} is the inverse of the restricted form, not $QA^{-1}Q$.

Proposition S27.4 (Complete joint Schur budget)**.** *With the preceding objects,*

$$0 \preceq K - K_0 \preceq \epsilon(1 + d/c)^2 U^* A U. \tag{S114}$$

If $W = U^* s^{-2} U$ is the complete physical core Gram, then

$$\|W^{-1/2}(K - K_0)W^{-1/2}\| \leq \epsilon(1 + d/c)^2[9(D_{\mathrm{core}} + B) + \kappa]. \tag{S115}$$

On the growing window, $\beta = \omega B$ with fixed $\omega > 0$ gives $c = \omega/(\omega+2)$, $N = M^{o(1)}$ and $D_{\mathrm{core}} = M^{o(1)}$. Hence with the explicit response schedule the right side is $M^{-2+o(1)} = o(M^{-3/2})$.

Proof. The relative form norm of S_0 is at most d. Thus $C_0^* A_Q^{-1} C_0 \preceq d^2 U^* A U$. Coercivity of T_0 and form Cauchy–Schwarz applied to $T_0 Z_0 = C_0$ give $Z_0^* A_Q Z_0 \preceq (d/c)^2 U^* A U$. The triangle inequality in the A energy norm, in every coefficient direction, therefore gives

$$(U - Z_0)^* A (U - Z_0) \preceq (1 + d/c)^2 U^* A U.$$

Variational minimization over the entire tail yields $K \succeq K_0$. Testing S at the exact S_0 minimizer $-Z_0$ gives $K \preceq K_0 + (U - Z_0)^* E (U - Z_0)$, proving (S114). The square retains all core–response, mixed, and receiving terms together.

Since $W \succeq I/9$,

$$W^{-1/2} U^* A U W^{-1/2} \preceq [9(D_{\mathrm{core}} + B) + \kappa] I.$$

The ridge becomes exactly κI here, with no factor $\max s^{-2}$. At the declared window $B = O(\sqrt{\log M})$, $k_N = \exp(O(\sqrt{\log M}))$ and $L = O(\log\log M)$. Lemma S27.3 proves the asserted growth, completing the quantitative comparison. □

The Gram W normalizes physical images of the core columns only. The profile-orthogonal receiving complement generally transports to a nonorthogonal physical complement. Thus (S115) is a finite-coordinate shorted-matrix bound, not a whole-physical-operator norm bound. For a quantitative global allowance, shift the complete physical form first, or pay the corresponding lift/projection stability. Exact nonnegativity transfers by the form Schur identity; the global allowance comparison (S111) does not require a core-to-tail lift inference.

S27.5 A finite-sine alternative with a sharper joint budget

The following alternative combines the same relative-tail estimate with a finite pole-free intersection core. It avoids a pole-profile input action and improves the joint energy bound.

Proposition S27.5 (Stable finite-sine core)**.** *Suppose $L \geq 3$, $B = 6 + 8(\log 2) e^L$, and take*

$$N' = \lceil L e^{2B+1}/\pi \rceil, \qquad \operatorname{ran} U' = \operatorname{span}(e_0, \ldots, e_{N'-1}) \cap p^\perp, \qquad U'^* U' = I.$$

In this proposition Q, K, K_0, W and all receiving and mixed blocks are rebuilt from U'; no block from the preceding projected core is reused. Both full and finite-response receiving forms on this complement satisfy $T, T_0 \succeq A_Q/3$. A finite DN generating list has Gram condition number below $4L$, and $D_{\mathrm{core}} \leq 2B + 7$. On the growing support, for every integer $M \geq 2^{20}$, the joint bound (S115) *therefore improves to*

$$\|W^{-1/2}(K - K_0)W^{-1/2}\| \leq \frac{49[27B + 63 + M^{-1}]}{M^2} = O\left(\frac{\sqrt{\log M}}{M^2}\right). \tag{S116}$$

The cofinal RH criterion also holds with this $(N' - 1)$-dimensional core.

Proof. Let $\delta_n = \|(I - P_n)\bar{p}\|^2$. The exact pole coefficients and $\cosh(L/2)^2/(\sinh L - L) \leq 2$ for $L \geq 3$ give

$$\delta_n \leq \frac{2L^3}{\pi^4 (n + 1/2)^3}, \qquad n \geq 1.$$

For the ratio bound use $\sinh L - L \geq e^L/4$ and $\cosh(L/2)^2 \leq e^L/2$; the former holds at3 and its difference increases. For the series bound, the decreasing summands are bounded by their first

term plus their integral. A vector $f \perp \operatorname{ran} U'$ in $p^{\perp}$ has $P_{N'}f$ proportional to $P_{N'}p$. The exact pole constraint and Cauchy–Schwarz imply $\|P_{N'}f\|^2 \le \delta_{N'}\|f\|^2$. Hence

$$D_L[f] \ge \{a(k_{N'})(1-\delta_{N'}) + a(k_0)\delta_{N'}\}\|f\|^2.$$

Put $X = 2B+1 \ge 2$. Then $e^X \le k_{N'} < e^X + 2$, while the main logarithmic symbol bounds give $X \le a(k_{N'}) < X+6$. Also $\delta_{N'} \le (2/\pi)e^{-3X}$, so its diagonal leakage is below

$$\tfrac{2}{3}(X+6)e^{-3X} \le 16/(3e^6) < 1/12 < 1.$$

It follows that $D_L|_Q \ge 2B$ and therefore $T, T_0 \ge A_Q/3$. On the core itself $D_{\mathrm{core}} \le a(k_{N'}) < 2B+7$.

Let $v_q = \langle e_q, p\rangle$, with $v_0 > 0$, and use the finite list $w_j = v_0 e_j - v_j e_0$, $1 \le j < N'$. Its Gram is $v_0^2 I + v_{\mathrm{tail}} v_{\mathrm{tail}}^*$; thus its condition number is at most

$$\frac{\|p\|^2}{v_0^2} < 2L(1/4 + \pi^2/(4L^2))^2 < 4L.$$

These columns require only finite primitive DN input actions. The complete output polar projection and physical Gram are still retained. Finally $C_L < B$ for this elementary budget, so $d < 2$ and $(1+d/c)^2 < 49$. Insert $D_{\mathrm{core}} \le 2B+7$ in (S115). The specified onset ensures $L_M \ge 3$. The RH equivalence follows by the same exact Schur and full-form argument, with this paid complement. □

S27.6 A finite-feature residual test with all mixed terms

For completeness, a trial Z into the full receiving form domain gives an alternative sufficient certificate. Let $T = S|Q \succeq cA_Q$, $H = U^*SU$, $C = QSU$, and set

$$\begin{aligned} L_Z &= C^*Z + Z^*C - Z^*TZ, \quad & L_0 &= C_0^*Z + Z^*C_0 - Z^*T_0Z, \\ r_0 &= C_0 - T_0Z, & K_r &= r_0^* A_Q^{-1} r_0, \\ G &= (U-Z)^*A(U-Z). \end{aligned}$$

For every $\tau > 0$,

$$H - C^*T^{-1}C \succeq H_0 - L_0 - c^{-1}\{(1+\tau)K_r + (1+\tau^{-1})\epsilon^2 G\}. \tag{S117}$$

Indeed the full residual is $r = r_0 + QE(U-Z)$. Relative positivity of E gives the joint A_Q-dual Gram of the second term at most $\epsilon^2 G$. Weighted Young and inverse order bound $r^*T^{-1}r$ by the displayed error. The exact residual identity and

$$(H - L_Z) - (H_0 - L_0) = (U-Z)^*E(U-Z) \succeq 0$$

then prove (S117). This is a joint matrix inequality, not a collection of diagonal tests. A numerical use must enclose the complete K_r, G, constraints and reference inverse; its cofinal budget includes the actual growth of G/c. The theorem does not assert that this finite-feature signed test has succeeded, nor that the full receiving inverse has become a finite principal compression.

S28 Local inverse covariance for the averaged reference

For $k = 1$ the continuous prime integral is exactly

$$\int_1^{\infty} t^{-1/2+i\xi} \min(1, M/t)\, dt = \frac{M^{1/2+i\xi}}{1/4+\xi^2} - \frac{1}{1/2+i\xi}.$$

Its phase-mean negative receiver, divided by $\sqrt{M}$, is

$$\mathcal{C}_Q = \alpha Q M_s L_0^{-1} M_s Q, \qquad \alpha = 8/\pi, \quad L_0 = 1 - 4d^2/du^2. \tag{S118}$$

Here Q is an original orthogonal projection on the odd space, with a finite constraint space $Z = \ker Q$. Write $\|g\|_{1,4}^2 = \int (|g|^2 + 4|g'|^2)$.

Theorem S28.1 (Full inverse-form source covariance)**.** *For $h \in \operatorname{ran} Q$,*

$$\|\mathcal{C}_Q^{-1/2} h\|^2 = \frac{\pi}{8} \inf_{z \in Z:\, (h+z)/s \in H^1} \|(h+z)/s\|_{1,4}^2, \tag{S119}$$

with value infinity when the set is empty. For any finite odd H^1 profile family g_i and $e_i = Q(sg_i)$ this yields the simultaneous Gram inequality

$$[\langle \mathcal{C}_Q^{-1/2} e_i, \mathcal{C}_Q^{-1/2} e_j \rangle] \preceq \frac{\pi}{8} [\langle g_i, g_j \rangle + 4 \langle g_i', g_j' \rangle].$$

Equality holds when Z is any finite span of the original polar row and canonical derivative-null rows. All statements allow arbitrary finite source and row counts.

Proof. Factor $\mathcal{C}_Q = \alpha A A^*$ with $A = Q M_s L_0^{-1/2}$. The actual-range identity $\operatorname{ran}(AA^*)^{1/2} = \operatorname{ran} A$ and polar decomposition identify the inverse form with the minimum squared preimage norm [7]. Writing $g = L_0^{-1/2} a$ translates $Aa = h$ to $sg - h \in Z$, and $\|a\|^2 = \|g\|_{1,4}^2$. The chosen profiles give the matrix upper on every common coefficient vector.

More exactly, project the profiles orthogonally in H^1, with this norm, off $\{g \in H^1 : sg \in Z\}$. Their projected Gram gives equality in (S119). This kernel is zero for the canonical rows. Indeed $b/s = \sinh(u/2)$, and for $z_{2j+1} = (\psi^{(2j+3)} - \psi^{(2j+1)}/4)/s$,

$$\frac{z_{2j+1}(u)}{s(u)} \sim \frac{(-2\pi)^{2j+3}}{2} e^{(4j+13/2)u} \quad (u \to +\infty).$$

To obtain this, the first theta atom derivative polynomial has leading coefficient $4(-2)^m$ at order m, by the recurrence $P_{m+1} = (1/2 - 2r)P_m + 2rP_m'$. At fixed order the higher atoms contribute a polynomial times e^{-3r} relative to the first. The exponents of these rows and b/s are distinct, so a nonzero finite combination is not in L^2. This proves the stated equality. □

If the full profile Gram is bounded by KD and the full profile-error Gram by $\eta^2 D$ in this same H^1 metric, the resulting inverse-form Grams differ in either order by at most $(\pi/8)(2\sqrt{K}\eta + \eta^2)D$. Expand each common coefficient combination and apply Cauchy–Schwarz before passing to coordinates. The exact profile projection in the proof is a contraction, so the same error transport applies to every finite constraint space.

Lemma S28.2 (Regularized preimage identity)**.** *For $a, \kappa > 0$ and $e = Q(sg)$, $g \in H^1_{\mathrm{odd}}$,*

$$\langle e, (a\mathcal{C}_Q + \kappa I)^{-1} e \rangle = \inf_{f \in H^1_{\mathrm{odd}}} \left\{ \frac{\|f\|_{1,4}^2}{a\alpha} + \frac{\|Q[s(g-f)]\|^2}{\kappa} \right\}. \tag{S120}$$

Proof. For $B = \sqrt{a\alpha} Q M_s L_0^{-1/2}$, minimize $\|z\|^2 + \kappa^{-1}\|e - Bz\|^2$. Its value is $\langle e, (BB^* + \kappa I)^{-1} e \rangle$. Substitute $f = \sqrt{a\alpha} L_0^{-1/2} z$. This is the classical approximate-preimage regularization identity in the present weighted and projected space [25]. □

S29 Explicit reference origin bounds

Let $U = \log M$, $\Omega = L/U$, and use the actual smoothed negative symbol $n_{M,1}$. On the *entire* profile space $\mathcal{Y}^{\mathrm{org}}_{M,L} = L^2([0,\Omega], n_{M,1}(\xi)\,d\xi/\pi)$ put

$$g_\phi(u) = \int_0^\Omega n_{M,1}(\xi)\phi(\xi)\sin(\xi u)\frac{d\xi}{\pi}, \quad E\phi = Q(sg_\phi), \quad D_\phi = \|\phi\|^2_{\mathcal{Y}^{\mathrm{org}}_{M,L}}.$$

The reference $\mathcal{C}_Q$ is exactly (S118).

Theorem S29.1 (Uniform ridged origin cap)**.** *For every $L > 0$, every integer $M \geq 64$ and every finite original constraint space,*

$$E^*(\sqrt{M}\,\mathcal{C}_Q + M^{-1}I)^{-1}E \preceq K(M,L)I_{\mathcal{Y}^{\mathrm{org}}_{M,L}}, \tag{S121}$$

where, with $R = \frac{1}{2}\log(2\log M)$,

$$K(M,L) = \frac{160}{3}L^3\left(\frac{R}{\log M}\right)^3 + \frac{4096\pi^2}{3}L^3M^{3/2-2\pi}.$$

In particular $K(M,L) = O_L((\log\log M/\log M)^3)$, and $K(M,1) < 1/10$ for every integer $M \geq 2^{20}$.

Proof. The elementary prime bound and $\mathbb{E}e^{v/2} = 2$ give $n_{M,1} \leq 6 + 16(\log 2)\sqrt{M} < 16\sqrt{M}$. Thus, for $A_M = \int_0^\Omega n_{M,1}(\xi)\xi^2 d\xi/\pi$,

$$A_M \leq \frac{16\sqrt{M}L^3}{3\pi U^3}, \qquad |g_\phi(u)|^2 \leq D_\phi A_M u^2, \quad |g'_\phi(u)|^2 \leq D_\phi A_M.$$

These common-profile estimates are Cauchy–Schwarz and $|\sin(\xi u)| \leq |\xi u|$. The finite frequency band puts g_ϕ in H^1. Choose even χ_R equal to one on $[-R,R]$, zero outside $[-2R,2R]$, and linear on each intervening interval. Then $|\chi'_R| \leq 1/R$, and direct integration gives

$$\|\chi_R g_\phi\|^2_{1,4} \leq 80 D_\phi A_M R^3, \quad \|Q[s(g_\phi - \chi_R g_\phi)]\|^2 \leq D_\phi A_M T_R, \quad T_R = \int_{|u|>R} s(u)^2u^2du.$$

Indeed $\int_{-2R}^{2R} u^2du = 16R^3/3$ and the derivative square is at most $(56/3)D_\phi A_M R$; multiplication by four gives the constant80 for $R \geq 1$.

The complete theta estimate $s^2 \leq 16r^2e^{-r}$, $r = \pi e^{2|u|}$, and the two-arm Jacobian $du = dr/r$ give

$$T_R \leq 16\int_{r_R}^\infty r^3e^{-r}dr \leq 32r_R^3e^{-r_R}, \qquad r_R = \pi e^{2R} \geq 6.$$

The last integral has logarithmic slope at most $-1/2$. Use the trial $f = \chi_R g_\phi$ in (S120), with $a = \sqrt{M}$, $\kappa = M^{-1}$ and $r_R = 2\pi U$. The two costs are exactly bounded by the two terms of $K(M,L)D_\phi$. This proves the inequality on every profile, hence as a full observation operator and for every common finite source combination. Projection contraction makes it uniform in all finite original constraints. Both terms decrease on $M \geq 64$. Directed scalar evaluation gives $K(2^{20},1) < 0.091767 < 1/10$, completing the explicit threshold. □

Corollary S29.2 (Complete positive continuum-model cap)**.** *Define the positive continuum receiver*

$$P^{\mathrm{cont}}_M = QM_s\left[\frac{8\sqrt{M}}{1+4D^2}(-\cos(UD))_+\right]M_sQ.$$

For the same actual origin observation E,

$$E^*(P_M^{\mathrm{cont}} + M^{-1}I)^{-1}E \preceq 2K(M,L)I.$$

For $L = 1$ this cap is below $1/5$ for every $M \geq 2^{20}$.

Proof. Lemma S4.1 gives error $e_M = 12M^{5/2}e^{-\pi(M-1)} \leq 1/(2M)$ on $M \geq 64$. The gate holds at64 using $\pi > 3$, $e > 2$, and the majorant decreases thereafter. Hence $P_M^{\mathrm{cont}} + M^{-1}I \succeq \frac{1}{2}(\sqrt{M}\mathcal{C}_Q + M^{-1}I)$. Inverse order and Theorem S29.1 finish the proof. □

S30 Feasibility and complete exterior control

The preceding certificates retain exact signed information. This section identifies the costs of two simpler bounds and locates the earlier scalar complementary criterion within the complete construction.

Proposition S30.1 (Cost of the positive majorant)**.** *For the explicit E_0 in the proof above, with $m \sim b\log M/\log 10$,*

$$\|E_0^* R_M(M^{-b})E_0\| \geq \frac{c_b\sqrt{m}}{\sqrt{M}(\log m)^3}$$

for all sufficiently large M. Thus this positive-majorant inverse cost is not $o(M^{-1/2})$.

Proof. Choose compact smooth odd f with $\int f(u)\sinh(u/2)\,du = 0$ but $\int uf(u)\,du \neq 0$. Two odd bump pairs construct such an f, since u and $\sinh(u/2)$ are linearly independent on a positive interval. Then $y = f/s$ is legal, $|\widehat{f}(\xi)| \geq c|\xi|$ near zero, and for large M, $E_0^* y = T_m f$. On $|\xi| \leq 1/(2\log m)$ every $\cos(\xi\log n) > 1/2$ for $n \leq m$. The prime number theorem gives $\sum_{n\leq m}\Lambda(n)/\sqrt{n} \geq c\sqrt{m}$; boundedness of a near zero therefore yields $v_m(\xi) \geq c\sqrt{m}$ on this interval. Plancherel gives $\|E_0^* y\|^2 \geq c\sqrt{m}/(\log m)^3$. The full bound $\|N_M\| \leq 144\sqrt{M}$ and the signed identity imply $P_M[y] + M^{-b}\|y\|^2 = O_y(\sqrt{M})$. The inverse variational quotient proves the assertion. □

This concerns the particular positive majorant, not the signed covariance excess. It explains why the next construction retains the exact null remainder rather than trying to sharpen this majorant by constants alone.

The technical supplement's arithmetic-gain section supplies a sharper compact arithmetic price, a complete residual optimizer, and matrix tests for whether the resulting improvement covers a signed certificate deficit. These retain the same receiving core and every mixed term.

S30.1 The complete exterior in the actual inverse metric

The next estimate identifies source directions that a constructive null head must capture globally. Put $R_M = (P_M + \kappa I)^{-1}$ and $G_M = E_M^* R_M E_M$ on the base original polar recipient.

Proposition S30.2 (Actual exterior payments)**.** *Let $F = F_M \to \infty$ satisfy*

$$F^4/\sqrt{M} \longrightarrow 0, \qquad F^3\bar{e}_M \longrightarrow 0,$$

and let Π_F be source multiplication by $1_{[F,2F]}$. Uniformly for $0 < \kappa \leq 1$, for all sufficiently large M,

$$\mathrm{Tr}(E_M\Pi_F E_M^*) \geq c\sqrt{M}/F, \tag{S122}$$

$$\mathrm{Tr}(\Pi_F G_M \Pi_F) \geq c/F, \qquad \|\Pi_F G_M \Pi_F\| \geq c/F^2. \tag{S123}$$

These conclusions hold, in particular, for $F = \exp(A\sqrt{\log M})$ with any fixed $A > 0$.

Proof. Write $U = \log M$, $B(\xi) = 8/(1+4\xi^2)$ and $e_\xi = Q_b[s\sin(\xi\,\cdot)]$. The complete Abel formula and $a(\xi) \le 17\xi^2$ give on $[F, 2F]$

$$\sup\left|\frac{m_M(\xi)}{\sqrt{M}B(\xi)} + \cos(U\xi)\right| \le C(F^4/\sqrt{M} + F^3\bar{e}_M) = o(1).$$

Thus $n_M \le C\sqrt{M}/F^2$ there, and $n_M \ge c\sqrt{M}/F^2$ on its subset where $\cos(U\xi) \ge 1/2$. That subset occupies at least a quarter of each fixed-length interval for large M, and at least $F/4$ of $[F, 2F]$.

The exact projected source Gram is

$$\langle e_\xi, e_\zeta\rangle = \tfrac{1}{2}\int s^2\cos((\xi-\zeta)u)\,du - \tfrac{1}{2}\int s^2\cos((\xi+\zeta)u)\,du - \frac{\langle s\sin(\xi\,\cdot), b\rangle\langle b, s\sin(\zeta\,\cdot)\rangle}{\|b\|^2}.$$

Riemann–Lebesgue therefore gives $\|e_\xi\|^2 \to \|s\|_2^2/2 > 0$. Differentiating se_ξ shows $\|(se_\xi)'\|^2 \le C(1+\xi^2)$. The fixed hard-cutoff decomposition of the original form is $H = A + B_0$ with B_0 bounded and $A[h] = \int a(\xi)|\widehat{sh}(\xi)|^2 d\xi/(2\pi)$. Consequently these sources belong to the original form domain and $q_H[e_\xi] \le C(1+\xi^2)$. Since $P_M = N_M + H - W_M$ and $\|N_M\| \le 144\sqrt{M}$,

$$P_M[e_\xi] + \kappa\|e_\xi\|^2 \le C\sqrt{M}, \qquad \langle e_\xi, R_M e_\xi\rangle \ge \frac{\|e_\xi\|^4}{P_M[e_\xi] + \kappa\|e_\xi\|^2} \ge c/\sqrt{M}.$$

Here $F^2 \le \sqrt{M}$ eventually. Integrating these estimates against $n_M(\xi)d\xi/\pi$ proves the raw and inverse-weighted trace bounds.

The norm bound requires a separate coherent source. Choose fixed $\ell > 0$ so small that $\int s^2\cos(tu)\,du \ge 3\|s\|^2/4$ for $|t| \le \ell$. The other two Gram terms tend uniformly to zero on $[F, F+\ell]^2$, so $\langle e_\xi, e_\zeta\rangle \ge c_0 > 0$ there for large M. Let $Z = [F, F+\ell] \cap \{\cos(U\xi) \ge 1/2\}$, with $\mu(Z) = \int_Z d\xi/\pi$, and use the unit source $f = 1_Z/\sqrt{\mu(Z)}$. Its coefficients are nonnegative and $\mu(Z)$ is bounded above and below by positive constants. For $v = E_M f$, the full Gram and differentiated Bochner integral give

$$c\sqrt{M}/F^2 \le \|v\|^2 \le C\sqrt{M}/F^2, \qquad \|(sv)'\| \le CM^{1/4}.$$

In particular v belongs to the original form domain and $q_H[v] \le C\sqrt{M}$. The complete positive energy satisfies

$$P_M[v] + \kappa\|v\|^2 \le 144\sqrt{M}\|v\|^2 + C\sqrt{M} + (\delta_M + \kappa)\|v\|^2 \le CM/F^2.$$

The inverse variational test now yields $\langle f, G_M f\rangle \ge \|v\|^4/(P_M[v] + \kappa\|v\|^2) \ge c/F^2$. No trace-to-norm inference is used. Finally Theorem S6.2 proves $\bar{e}_M = O_D(e^{-D\sqrt{U}})$ for every fixed $D > 0$. For fixed A, choose $D > 3A$ to admit $F = e^{A\sqrt{U}}$; $F^4/\sqrt{M} \to 0$ is immediate. □

If a source head is supported below F, its orthogonal complement acts identically on $[F, 2F]$, so its complete inverse remainder has norm at least c/F^2 and trace at least c/F. The same holds for the remainder of a positive-contraction head supported there. In the displayed stretched-exponential range these bounds exceed $M^{-1/2}$ asymptotically. Thus a low-band head cannot pay its entire positive remainder as $o(M^{-1/2})$, even in the actual inverse metric. Global null-polynomial heads and the directional Schur criterion remain available: these are lower bounds on a positive complement, not on $\|G_M\| - 1$.

S30.2 The scalar mixed-band criterion

Proposition S30.3 (Complete mixed-band cap)**.** *For the same positive inverse $R_P = (P + \kappa I)^{-1}$, split the exact observation source as $E = [E_o, E_c]$. If its two diagonal caps are $E_o^* R_P E_o \preceq aI$ and $E_c^* R_P E_c \preceq bI$, and the full mixed block has norm at most ζ, then*

$$\|E^* R_P E\| \le \tfrac{1}{2}[a + b + \sqrt{(a-b)^2 + 4\zeta^2}] \le a + b \quad \textit{when } \zeta \le \sqrt{ab}.$$

Proof. Positive-form Cauchy–Schwarz pays the mixed block by $\sqrt{ab}$. More generally bound each two-vector quadratic form by the real matrix $\left[\begin{smallmatrix} a & \zeta \\ \zeta & b \end{smallmatrix}\right]$ and compute its largest eigenvalue. All mixed observation terms are retained. □

Corollary S30.4 (A complementary-band cofinal criterion for RH)**.** *Let a_M be the bound in* (S15) *at $\Omega = 1/\log M$, and use any finite genuine-null recipient. If, along an unbounded sequence, the complete complementary observation for the same $\overline{P}_M + M^{-1} I$ has cap*

$$b_M \le 1 - a_M + M^{-1},$$

then on the original domain

$$q_H[h] \ge -\left[d_{M,1} + \frac{1}{M} + \frac{B_{M,1}}{M+1}\right] \|h\|^2,$$

and RH holds. A paid mixed block can instead use the sharper condition in Proposition S30.3 at target $1 + 1/M$.

Proof. The proposition gives the complete cap $1 + 1/M$ including all mixed terms. Apply the bounded-negative absorption of Corollary S3.2 with $k = 1$, $\tau = 0$, $\eta = \kappa = 1/M$. Here $B_{M,1} = O(\sqrt{M})$, so the displayed allowance tends to zero; Theorem 2.2 applies. □

S30.3 Why bounded null rank does not fund the scalar criterion

The sufficient scalar comparison in Corollary S30.4 loses directional covariance information. The following result gives its precise rank limitation at the displayed schedule.

Proposition S30.5 (Fixed-rank scalar obstruction)**.** *Put $\Omega = 1/\log M$, $\kappa = M^{-1}$ and split the complete source into E_o on $[0, \Omega]$ and its entire complement E_c. For the same-form recipient obtained by removing a genuine-null space K, put*

$$a = \|E_o^* R_M E_o\|, \qquad b = \|E_c^* R_M E_c\|, \qquad \chi = \|E_o^* R_M E_c\|.$$

For every fixed integer $D \ge 0$ there are positive constants c_D, C_D and a finite threshold, uniform over all K with $\dim K \le D$, such that

$$\chi^2 \ge c_D \Omega^{4D+3}, \qquad b \ge 1 - C_D \Omega^{4D+7} - C_D M^{-3/2}.$$

Consequently the scalar matrix $\left[\begin{smallmatrix} a & \chi \\ \chi & b \end{smallmatrix}\right]$ is not bounded by $(1+1/M)I$ for all sufficiently large M. Any cofinal realization of that scalar target requires the null dimension to eventually exceed every fixed D.

Proof. Let $L_D = \operatorname{span}\{z_0, \dots, z_D\}$, with the genuine nulls of Lemma 8.1. Dimension gives a unit $z = \sum_{i=0}^{D} \alpha_i z_i \in L_D \cap K^{\perp}$. The derivative Fourier rule and $\widehat{\psi}(0) > 0$ show, near zero,

$$|\widehat{sz}(\xi)| \geq c\xi \left| \sum_{i=0}^{D} (-1)^i \alpha_i \xi^{2i} \right|.$$

The complete Abel formula gives $n_{M,1} \geq c\sqrt{M}$ on $[0, \Omega]$ for large M, since $\cos((\log M)\xi) \geq \cos 1 > 0$. The positive moment matrix $G_D = (1/(2i + 2j + 3))_{0 \leq i,j \leq D}$ gives

$$\int_0^{\Omega} \xi^2 \left| \sum_{i=0}^{D} (-1)^i \alpha_i \xi^{2i} \right|^2 d\xi \geq \lambda_{\min}(G_D) \Omega^{4D+3} \|\alpha\|^2.$$

The fixed physical Gram of L_D bounds $\|\alpha\|$ below. Therefore $N_o[z] \geq c_D \sqrt{M} \Omega^{4D+3}$, uniformly in the moving K. All constants here are for fixed D; no uniform conditioning in D is used.

Write $A_o = E_o^* R_M E_o$ and $C_{oc} = E_o^* R_M E_c$. The exact null identity gives

$$C_{oc} E_c^* z = (I - A_o) E_o^* z + E_o^* R_M (W_M - \kappa I) z.$$

Since $R_M^2 \preceq \kappa^{-1} R_M$,

$$\chi \sqrt{N_c[z]} \geq (1 - a) \sqrt{N_o[z]} - (d_{M,1} + \kappa) \sqrt{a/\kappa}.$$

The actual origin upper is $a \leq C\Omega^3 (1 + \log\log M)^3$. Dividing its error term by the lower bound for $\sqrt{N_o[z]}$ gives at most $C_D M^{-3/4} \Omega^{-2D} (1 + \log\log M)^{3/2} + o(1)$, which tends to zero. The complete bound $N_c[z] \leq 144\sqrt{M}$ then proves the stated lower for χ^2. If $N_c[z] = 0$, the displayed identity already contradicts its positive right side for large M.

For the complementary lower choose a unit $z' \in \operatorname{span}\{z_{D+1}, \dots, z_{2D+1}\} \cap K^{\perp}$. Its transform vanishes to order at least $2D + 3$ at zero, so $N_o[z'] \leq C_D \sqrt{M} \Omega^{4D+7}$. The averaged quadratic mean proved in Theorem S6.2 is positive definite on this fixed finite-dimensional span. Polarization makes its convergence uniform on that span's unit sphere; hence $N_M[z'] \geq c'_D \sqrt{M}$ for every allowed moving choice. The exact null quotient yields

$$b \geq \frac{N_c[z']}{P_M[z'] + \kappa} \geq 1 - C_D \Omega^{4D+7} - C_D M^{-3/2}.$$

Finally put $t = 1 + 1/M$. If $b > t$ the scalar target fails immediately. Otherwise its necessary determinant inequality would imply

$$\chi^2 \leq (t - a)(t - b) \leq 2\{M^{-1} + C_D \Omega^{4D+7} + C_D M^{-3/2}\} = o(\Omega^{4D+3}),$$

contradicting the proved lower. □

The proposition concerns scalarization of the blocks. It neither rejects the operator-valued Schur criterion nor establishes sufficiency of growing null rank. The regularized whole-source construction retains the signed correlations that this scalar comparison discards.

S30.4 Spectral tests for a growing genuine-null head

The directional construction has an actual compressed-spectrum test that retains the physical lift cost. It does not replace that spectrum by the unrestricted mean operator.

Proposition S30.6 (Null-compressed spectral and lift tests)**.** *Let $\mathcal{K}$ be the closed physical span of the genuine nulls and $N_{\mathcal{K}}$ the compression of N_M to $\mathcal{K}$. For a rank-r null-valued map V with $V^*N_MV = I$, put*

$$J = E_M^*V, \quad Q = I - JJ^*, \quad \ell = \|V^*V\|, \quad b_c = \|QG_MQ\|, \quad g = \|G_M\|.$$

If $b_c \le q < 1$, then

$$\lambda_{r+1}(N_{\mathcal{K}}) \le \frac{q(\kappa + \delta_M)}{1-q}. \tag{S124}$$

For every eigenvalue $\lambda > 0$ of $N_{\mathcal{K}}$,

$$b_c \ge \left[\sqrt{(1 - (\kappa + \delta_M)/\lambda)_+} - \sqrt{g\lambda\ell}\right]_+^2. \tag{S125}$$

Proof. Closedness of H implies $\mathcal{K} \subset \operatorname{Dom} H$ and $H|_{\mathcal{K}} = 0$. For $z \in \mathcal{K}$ with $E_M^*z \perp \operatorname{ran} J$, the inverse variational test at z gives

$$\langle E_M^*z, G_ME_M^*z\rangle = \langle N_Mz, R_MN_Mz\rangle \ge \frac{N_M[z]^2}{P_M[z] + \kappa\|z\|^2}.$$

Genuine nullity gives $P_M[z] = N_M[z] - \langle z, W_Mz\rangle$. Putting $t = N_M[z]/\|z\|^2$, we obtain $b_c \ge t/(t+\kappa+\delta_M)$ whenever $t > 0$. There are at most r source-orthogonality conditions on z. Compact min–max on $N_{\mathcal{K}}$ proves (S124), with eigenvalue multiplicities retained.

For the second bound choose a unit compressed eigenvector h at λ and put $f = E_M^*h$, so $\|f\|^2 = \lambda$. Since V is $\mathcal{K}$-valued, the compressed eigenidentity gives $V^*N_Mh = \lambda V^*h$, without assuming invariance of $\mathcal{K}$ under N_M. Hence

$$\|JJ^*f\|^2 \le \lambda^2\ell, \qquad \langle f, G_Mf\rangle \ge \frac{\lambda^2}{\lambda + \kappa + \delta_M} \ge \lambda - (\kappa + \delta_M).$$

The triangle inequality in the positive G_M seminorm yields

$$\sqrt{\langle f, G_Mf\rangle} \le \sqrt{g}\,\lambda\sqrt{\ell} + \sqrt{b_c\lambda}.$$

Division by $\sqrt{\lambda}$ and positive parts prove (S125). □

At $\kappa = M^{-1}$, a fixed complementary gap requires the head to capture every null-compressed eigenvalue above a constant multiple of M^{-1}. If actual eigenvalues satisfy $(\kappa + \delta_M)/\lambda_M \to 0$ and $\lambda_M\ell_M \to 0$, then Theorem S6.2 and (S125) give $\liminf b_c \ge 1$. In particular, an actual eigenvalue comparable to $M^{-1/2}$ would exclude a fixed gap with $\ell_M = o(\sqrt{M})$. The spectral and lift tests are evaluated jointly with the signed complement estimate in the complete source metric.

S31 Source-aware positive receiving

The polynomial construction controls complete positive complements in well-conditioned coordinates. A second, compatible finite reduction retains the actual positive and negative parts of the signed symbol. It prices the error on its finite source rather than on the whole inverse.

Fix $M \ge 2$, put $p_M = (m_M)_+$ and $n_M = (-m_M)_+$, and let X_0 be the odd Hilbert space with the polar row and any specified finite exact rows imposed. On this same recipient define the closed positive form

$$P[v] = \int_0^\infty p_M(\xi)|\widehat{sv}(\xi)|^2\frac{\mathrm{d}\xi}{\pi}, \qquad N[v] = \int_0^\infty n_M(\xi)|\widehat{sv}(\xi)|^2\frac{\mathrm{d}\xi}{\pi}.$$

Thus $H_M = P - N$ as forms. In this section P denotes this positive operator, rather than the range projection of Section S34. Every inverse and observation below uses the original Hilbert norm and the same exact-row recipient.

Theorem S31.1 (Complete inverse exhaustion and finite-source convergence)**.** *Let $\kappa > 0$, $R = (P + \kappa I)^{-1}$, and let P_F restrict the complete positive observation to $[0, F]$. Then $R_F = (P_F + \kappa I)^{-1}$ converges strongly to R, while*

$$\|R_F - R\| = \kappa^{-1}$$

for every finite F. The same norm equality holds for every finite-rank positive lower $P_f \preceq P$. For each fixed finite column map E, however,

$$\|E^*(R_F - R)E\| \longrightarrow 0.$$

Proof. The original graph closure is $\{v : sv \in H_{\log}\}$, with the exact rows imposed, where $H_{\log}$ has Fourier weight $1+\log(2+|\xi|)$. Indeed the digamma series makes $1+a$ comparable to this weight, and $p_M - a$ is bounded. The square root of the weight is submultiplicative up to a constant. Weighted Young's inequality therefore bounds multiplication by a smooth cutoff by its weighted Fourier L^1 norm. Dilated even cutoffs have uniformly bounded such norms. Schwartz density, followed by cutoff convergence in H^1, proves cutoff convergence in $H_{\log}$. On each compact interval division by the smooth positive s, then even mollification, supplies odd smooth compact graph approximants. Multiplication back by s is bounded in $H_{\log}$ because s and its derivatives decrease rapidly. Correct the finitely many row moments by fixed compact odd dual functions; their coefficients tend to zero in the graph norm. This proves the asserted minimal-domain identification. Increasing closed-form convergence now gives strong resolvent convergence [1].

For the norm assertion first use the ambient odd space. Choose a nonzero even smooth compactly supported χ and unit odd packets $h_k = \chi \sin(ku)/\|\chi \sin(k\cdot)\|_2$. These are weakly zero. The finite-band observation is Hilbert–Schmidt, since its squared norm is bounded by $\|s\|_2^2 \int_0^F p_M \, \mathrm{d}\xi/\pi$; hence P_F is compact. Consequently $R_F h_k - \kappa^{-1} h_k \to 0$. The digamma asymptotic [16] gives $p_M(\xi) = \log|\xi| + O_M(1)$. Translating a Schwartz Fourier transform and using the integrable logarithmic majorant shows, for each smooth compact g,

$$\|(\log k)^{-1} p_M(D)(g e^{iku}) - g e^{iku}\|_2 \to 0.$$

Apply this to both sine arms of $v_k = (\log k)^{-1} s^{-2} h_k$. These compact smooth functions lie in the operator domain and satisfy $v_k \to 0$ and $(P + \kappa)v_k - h_k \to 0$. Thus $R h_k \to 0$. Inverse order gives $0 \preceq R_F - R \preceq \kappa^{-1} I$, establishing equality. A finite-rank P_f has the same compactness argument.

For independent exact rows Z_0, the constrained inverse extended by zero is

$$R_X = R_Y - R_Y Z_0 (Z_0^* R_Y Z_0)^{-1} Z_0^* R_Y.$$

Its correction is finite rank. Projecting the weakly zero packets onto X_0 changes them by $o(1)$, so the norm argument survives. Redundant rows are first removed. Finally strong convergence is uniform on the unit sphere of a fixed finite-dimensional source range, proving the last assertion. □

S31.1 A full mixed observation residual

Let $P_f \preceq P$ be a bounded finite positive lower, $R_f = (P_f + \kappa I)^{-1}$, and $V = R_f E$. Its actual trial columns are assumed in the original positive form domain. The variational principle gives the full matrix estimate

$$0 \preceq E^* R_f E - E^* R E \preceq V^* (P - P_f) V. \tag{S126}$$

To prove the upper, evaluate the complete variational formula for $\langle Ez, REz\rangle$ at Vz; its value is $\langle Ez, R_f Ez\rangle - (P - P_f)[Vz]$. This proves the matrix order on every coefficient vector.

Here is a computable price including all cross terms. Write $P = T^*T$, where $Tv(\xi) = \sqrt{p_M(\xi)}\widehat{sv}(\xi)$ in $L^2((0,\infty), \mathrm{d}\xi/\pi)$. Suppose

$$P_f = T^*Z(D + \rho I)^{-1}Z^*T, \qquad D = Z^*Z, \quad \rho > 0,$$

and Z is supported in $[0, F_0]$. For $F \geq \max(F_0, 1)$, a trial coefficient matrix B, and $r \geq 1$, define

$$A_{\mathrm{in}} = \int_0^F (TV - ZB)^*(TV - ZB)\frac{\mathrm{d}\xi}{\pi}, \quad D_r = [\langle D^r(sv_i), D^r(sv_j)\rangle],$$

and $C_M = 4 + 2\sum_{n\leq M}\Lambda(n)/\sqrt{n}$. Then

$$V^*(P - P_f)V \preceq A_{\mathrm{in}} + \rho B^*B + \frac{C_M + \log(3F)}{F^{2r}}D_r. \tag{S127}$$

Ridge-square completion gives the first two terms and the full exterior $\int_F^\infty (TV)^*TV\,\mathrm{d}\xi/\pi$. The digamma series, compared with its integral, gives $a(\xi) \leq 4 + \frac{1}{2}\log(1 + 4\xi^2)$, hence $p_M(\xi) \leq C_M + \log(3\xi)$ for $\xi \geq 1$. The quotient by ξ^{2r} decreases there. Parseval proves the exterior bound with the displayed complete derivative Gram. Actual finite-band columns and row corrections have finite D_r: their weighted profiles are s^2 times bounded-band integrals, complete theta derivatives, and $s^2\sinh(u/2)$.

The finite inverse is explicit. For $P_f = WD_p^{-1}W^*$, with its actual metric $D_p \succ 0$, set $L = W^*W + \kappa D_p$. Then

$$V = \kappa^{-1}\{E - WL^{-1}W^*E\}. \tag{S128}$$

On an exactly constrained recipient use the finite inverse-row correction in the proof of Theorem S31.1. This retains the row/source, row/positive and all mixed covariances.

Theorem S31.2 (Paid source-aware cofinal reduction)**.** *For each j let $\kappa_j > 0$ tend to zero and let $\tau_j, \delta_j \geq 0$. Take the recipient to be the original polar-free space with a finite span of genuine operator nulls removed, and on this recipient let*

$$N_{*,j} = E_j D_{n,j}^{-1}E_j^*, \qquad 0 \preceq N_j - N_{*,j} \preceq \tau_j I, \qquad D_{n,j} \succ 0.$$

Suppose $P_{f,j} \preceq P_j$ is a finite positive lower and, for $0 < \gamma_j < 1/2$,

$$E_j^*(P_j + \kappa_j)^{-1}E_j \preceq (1 - 2\gamma_j)D_{n,j}, \qquad V_j^*(P_j - P_{f,j})V_j \preceq \gamma_j D_{n,j}.$$

The second inequality may be certified by (S127)*. If the complete raw arithmetic error is δ_j and $\tau_j + \delta_j \to 0$, then*

$$q_H[h] \geq -(\kappa_j + \tau_j + \delta_j)\|h\|_2^2$$

on the corresponding genuine-null complement. These certificates, with genuine operator nulls restored, imply RH.

Proof. Equation (S126) gives $E_j^*(P_{f,j} + \kappa_j)^{-1}E_j \preceq (1 - \gamma_j)D_{n,j} \preceq D_{n,j}$. The positive Schur complement is equivalent to $N_{*,j} \preceq P_{f,j} + \kappa_j I$. Add the negative approximation and complete raw errors. The original closed-form domain and exact null transport in Theorem 2.2 prove the conclusion. □

Corollary S31.3 (Polynomial excess above one)**.** *On the same original polar-free recipient with finitely many genuine operator nulls removed, suppose instead that* $\eta_M, \tau_M, \delta_M \geq 0$, $\kappa_M > 0$, $0 \preceq P_{f,M} \preceq P_M$, *and the full finite-source cap satisfies*

$$E_M^*(P_{f,M} + \kappa_M I)^{-1} E_M \preceq (1+\eta_M) D_{n,M}, \qquad 0 \preceq N_M - E_M D_{n,M}^{-1} E_M^* \preceq \tau_M I.$$

Let B_M *be an upper bound for* $\|s\|_\infty^2(|m_0| + 2\sum_{n\leq M} \Lambda(n)/\sqrt{n})$ *and let* δ_M *pay the complete raw tail. Then, on the entire original domain,*

$$q_H[h] \geq -\left\{\delta_M + \kappa_M + \frac{\tau_M + \eta_M B_M}{1+\eta_M}\right\} \|h\|_2^2. \tag{S129}$$

In particular complete caps on a cofinal sequence with $\kappa_M = \eta_M = \tau_M = 1/M$ *give original allowances* $O(M^{-1/2})$ *tending to zero, and hence RH.*

Proof. The cap is equivalent to $N_* \preceq (1+\eta)(P_f + \kappa I)$, hence $N \preceq (1+\eta)P + tI$, where $t = (1+\eta)\kappa + \tau$. The exact symbol gives $0 \preceq N \preceq BI$. Writing $q = P - N + W$, $W \succeq -\delta I$, the following identity of forms proves the absorption without bounding the unbounded P:

$$(1+\eta)(q + \varepsilon I) = [(1+\eta)P + tI - N] + \eta[BI - N] + (1+\eta)[W + \delta I], \quad \varepsilon = \delta + \frac{t + \eta B}{1+\eta}.$$

Each summand is nonnegative. Genuine-null transport returns the bound to the whole original domain. The elementary prime-power estimate (S1) and $\|s\|_\infty < 3$ permit the choice $B_M = 9R_M = O(\sqrt{M})$, with R_M from (S133). Substitution yields (S129) and its stated rate. All finite-source, ridge and continuous errors are included in the cap or in τ_M. □

The cap in this alternative may lie above one; its excess is paid by the complete bounded negative part. It uses a different regularization/cutoff schedule from Theorem S35.3. The source-residual enclosure (S126) can fund the finite positive receiver against either target.

S31.2 Complete strip price and integration collar

For $f_i = sv_i$ put $A_\eta = [\langle f_i(\cdot - i\eta), f_j(\cdot - i\eta)\rangle]$, with $\eta = 1/8$. For actual finite observation and genuine-row columns these analytic weighted profiles give

$$\int_F^\infty (TV)^* TV \frac{\mathrm{d}\xi}{\pi} \preceq 2[C_M + \log(3F)] e^{-F/4} A_\eta, \qquad F \geq 4. \tag{S130}$$

Contour shifting gives $\widehat{f(\cdot - i\eta)} = e^{\eta\xi}\widehat{f}$; full Plancherel and the positive-half measure pay the factor two. The scalar multiplier $[C_M + \log(3\xi)]e^{-\xi/4}$ decreases for $\xi \geq 4$, proving the full matrix order.

For completeness, usable whole-source constants are

$$\|s^2(\cdot \pm i/8)\|_2 < 20, \quad \|\psi^{(m)}(\cdot \pm i/8)\|_2 \leq 32\, 16^m m!, \quad \|s^2(\cdot \pm i/8)\sinh((\cdot \pm i/8)/2)\|_2 \leq 32.$$

Here is a direct complete-tail proof. Put $x = |\Re z|$, $r = \pi e^{2x}$. On $|\Im z| \leq 1/4$, the theta atom series and reflection give $|\psi(z)| \leq 7e^{x/2} r^2 e^{-7r/8}$: each atom is at most $6e^{x/2}(\ell^2 r)^2 e^{-7\ell^2 r/8}$ and successive ratios are below $1/8$. On the $1/8$ line, the sharper cosine bound $31/32$ gives $|s^2(z)| \leq 28 r^2 e^{-31r/32}$. The squared two-arm integral is at most $4704(16/31)^4 < 400$. Cauchy's circle of radius $1/16$ bounds the derivative by $14\, 16^m m! e^{x/2} r^2 e^{-3r/4}$; its squared two-arm integral is at most $196 \int_0^\infty r^4 e^{-3r/2}\, \mathrm{d}r < 1024$. The polar bound follows from $|\sinh(z/2)| \leq e^{x/2}$ and the same integral comparison.

An observation feature $e = s\int_0^B a(\xi)\sin(\xi u)\,\mathrm{d}\xi/\pi$ therefore has weighted strip norm at most $20e^{B/8}\int_0^B |a|\,\mathrm{d}\xi/\pi$. A genuine derivative-null feature has weighted profile $\psi^{(2i+3)} - \psi^{(2i+1)}/4$, so the displayed derivative bounds apply. If $sV = gA$ uses b such features and their strip norms are bounded by k_ℓ, then

$$A_\eta \preceq bA^* \operatorname{diag}(k_\ell^2)A.$$

This finite matrix includes the actual inverse metrics and row coefficients. Under the explicit hypothesis $A_\eta \preceq e^{B_j/4+L_j} D_{n,j}$, the exterior is at most

$$2[C_{M_j} + \log(3F_j)]e^{-(F_j-B_j)/4+L_j} D_{n,j}.$$

For $\gamma_j = c\kappa_j^\rho$, $0 < \rho < 1$, it suffices that

$$\frac{F_j - B_j}{4} \geq L_j + \rho\log(1/\kappa_j) + \log\frac{8[C_{M_j} + \log(3F_j)]}{c}.$$

If $L_j = O(U_j)$, $\log B_j = O(U_j)$, $B_j \geq 1$ and $\log C_{M_j} = O(\log U_j)$, with $U_j = \log(1/\kappa_j)$, a sufficiently large fixed C permits $F_j = B_j + CU_j$. This is an integration collar for the chosen receiver. The interior residual in (S127) and the complete cap in Theorem S31.2 retain their stated hypotheses.

S31.3 Null leakage in the same positive metric

Let T now be an isometry onto finitely many genuine operator nulls inside a base recipient X_0, before their removal; write $Q_T = I - TT^*$, $C = P + \kappa I$, $S = T^*CT$. This use of T is distinct from the observation map above. Suppose $N_* = ED_n^{-1}E^*$, $0 \preceq N - N_* \preceq \tau I$, and the raw tail has norm at most $\delta < \kappa$. Put

$$A_0 = D_n^{-1}E^*T, \quad B_* = Q_T E A_0, \quad s_0 = \kappa - \delta + \lambda_{\min}(T^*N_*T) > 0.$$

For a finite positive lower $P_f = WD_p^{-1}W^* \preceq P$ on the *base* recipient, define

$$J_B = B_*^*B_* - B_*^*W(W^*W + \kappa D_p)^{-1}W^*B_* \succeq 0.$$

Then, for every $t > 0$,

$$\kappa\|T^*C^{-1}T - S^{-1}\| \leq s_0^{-2}\|(1+t)J_B + (1+1/t)(\tau+\delta)^2 I\|. \tag{S131}$$

Indeed the exact null identity $HT = 0$ gives $PT = NT - W_{\mathrm{tail}}T$. Thus $B = Q_TPT$ differs from B_* by at most $\tau + \delta$, and $S \succeq s_0 I$. The exact trial-residual identity is

$$T^*C^{-1}T - S^{-1} = S^{-1}B^*C^{-1}BS^{-1}.$$

Inverse order, quadratic Young's inequality and (S128) give $\kappa B^*C^{-1}B \preceq \widetilde{J}$, where $\widetilde{J} = (1+t)J_B + (1+1/t)(\tau+\delta)^2 I$. This proves (S131). More sharply the matrix-relative target $\widetilde{J} \preceq \eta S^2$ yields debit at most η by congruence with S^{-1}; no commutation is assumed. The finite lower must be certified before removing T, so all its mixed row covariances remain present.

These null-source tests are compatible with the full-source cap but have distinct recipients. For example $P = \operatorname{diag}(1,1)$, $N = \operatorname{diag}(1,2)$, $T = e_1$ gives $HT = 0$ and zero null debit for every κ, while the second direction has negative signed form. The complete cap in Theorem S31.2 keeps that direction in its test.

Finally a trial inverse from a different positive reference has a paid exact bridge. For $A \succeq 0$, bounded self-adjoint $K = P - A$, $V_A = (A+\kappa)^{-1}Z$ and $G_A = Z^*V_A$, square completion gives

$$Z^*(P+\kappa)^{-1}Z \preceq G_A - V_A^*KV_A + \kappa^{-1}V_A^*K^2V_A.$$

The identity uses residual $(P+\kappa)V_A - Z = KV_A$, followed by $(P+\kappa)^{-1} \preceq \kappa^{-1}I$. It preserves the full covariance of the changed reference.

S31.4 A complete 547-source inverse-action example

The source-aware viewpoint also gives a quantitative action certificate for a fixed finite positive reference. We use the cutoff-thirteen head tuple comprising its first 46 negative observations of the signed-frequency companion [23]: 547 negative columns E, 1680 positive columns W, and the original polar row with eleven genuine complete-operator null rows. Every column is projected through the full original twelve-row Gram. Let $D_n, D_p \succ 0$ be the actual midpoint frequency metrics plus their rational ridges and put

$$\kappa = 5 \cdot 10^{-19}, \quad B = \kappa I + W D_p^{-1} W^*, \quad C = E^* W, \quad L = W^* W + \kappa D_p.$$

The full frequency quadrature uncertainty is funded within half each ridge, so $B \preceq \kappa I + P_{13}$ in the original metric. Use the exact 1680-by-547 dyadic proposal F in the included head data and set

$$R = C^* - LF, \quad Y = \kappa^{-1}(E - WF), \quad Q_F = \kappa^{-1}(E^*E - CF - F^*C^* + F^*W^*WF + \kappa F^* D_p F).$$

Proposition S31.4 (Full finite-reference covariance and action errors)**.** *For this complete tuple there is a saved full mixed covariance upper $H_R \succeq R^* D_p^{-1} R$ satisfying $H_R \prec 9.36 \cdot 10^{-96} D_n$. In the same coefficient metric the following strict bounds hold:*

$$\begin{aligned}
&0 \preceq Q_F - E^* B^{-1} E && \prec 3.745 \cdot 10^{-59} D_n, \\
&(Y - B^{-1}E)^* B (Y - B^{-1}E) && \prec 3.745 \cdot 10^{-59} D_n, \\
&(BY - E)^* (BY - E) && \prec 1.685 \cdot 10^{-57} D_n, \\
&(Y - B^{-1}E)^* (Y - B^{-1}E) && \prec 7.490 \cdot 10^{-41} D_n.
\end{aligned}$$

The reference is the stated bounded finite B; inverse order gives $E^*(P_{13} + \kappa)^{-1} E \preceq E^* B^{-1} E$.

Proof. Woodbury square completion, with the actual rectangular F, gives

$$Q_F - E^* B^{-1} E = \kappa^{-1} R^* L^{-1} R \preceq \kappa^{-2} R^* D_p^{-1} R, \qquad BY - E = \kappa^{-1} W D_p^{-1} R.$$

Moreover $W^* B^{-1} W \preceq D_p$. Thus, for $e = Y - B^{-1}E$,

$$e^* B e \preceq \kappa^{-2} R^* D_p^{-1} R, \qquad e^* e \preceq \kappa^{-3} R^* D_p^{-1} R.$$

The complete positive-frame trace bounds $\|D_p^{-1/2} W^* W D_p^{-1/2}\| < 44.991$, which pays the action residual separately.

Here is the complete numerical error contract. Reconstruct all 2227 ambient sources and all 2227-by-12 original row crosses, then short by the actual enclosed 12-by-12 row Gram. The source construction and complete error derivation in Section S42, adapted from [23], give a joint error $\delta < 4.129 \cdot 10^{-58}$ in metric $\mathrm{diag}(D_n, D_p)$, including both sine images and the complete kernel, root, reflected and local errors. Let R_0 be the modeled residual and $H_0 = R_0^* D_p^{-1} R_0$. Applying that joint error to the full coefficient map $[I; -F]$ proves

$$H_R = 2H_0 + 2\delta^2 (D_n + F^* D_p F) \succeq R^* D_p^{-1} R.$$

All mixed entries are formed before taking the positive normalized trace. The public saved matrix and actual D_n blocks give, by directed arithmetic,

$$\mathrm{Tr}(D_n^{-1} H_R) < 9.36 \cdot 10^{-96}.$$

Since $H_R \succeq 0$, this proves its full relative upper. The preceding identities, the frame trace and the exact powers of κ prove the table.

The source reconstruction used 768-bit directed arithmetic; its separate sufficient head-cap congruence used 2048 bits. The new source reconstruction and residual assembly took approximately 448 seconds. The reader's saved-array verification reparses all 598418 entries of the two stored 547-square matrices, verifies finite symmetric enclosures and recomputes the actual D_n-normalized residual trace; it is a saved-array replay, rather than another source reconstruction. Its slightly widened durable decimal bounds are those stated above. The public supplement contains the full arrays, true frequency-metric blocks, portable readback and exact algebra controls. Full source regeneration uses the included scientific input subset, analytic source/error supplement and portable producer. □

The distinction between the finite reference and the complete positive observation is retained throughout. A two-sided complete-positive inverse comparison uses the additional source-observation residual of (S126); the finite-reference action estimates above supply their own bounded application of the method.

S31.5 A necessary full-cap margin

The full cap also has a quantitative necessary test independent of null-source conditioning. Let $\mathcal{L}$ be a $(D+1)$-dimensional genuine operator-null space, and let the exact recipient have codimension at most D in the polar space. Suppose $N[v] \ge b_D\|v\|^2$ on $\mathcal{L}$, with $b_D > 0$. If

$$N_* \preceq (1-\gamma)(P+\kappa I), \qquad 0 \preceq N - N_* \preceq \tau I,$$

on that recipient, and the complete raw tail has norm at most δ, then necessarily

$$\gamma \le \frac{\kappa+\tau+\delta}{b_D+\kappa+\delta}. \tag{S132}$$

Indeed dimension gives a unit $v \in \mathcal{L}$ in the recipient. The exact null identity gives $P[v] \le N[v] + \delta$. Together with $N_* \succeq N - \tau I$, the cap gives

$$N[v] - \tau \le (1-\gamma)(N[v] + \delta + \kappa),$$

which proves (S132). The lower observation floors developed for genuine derivative nulls in [21] can be inserted without changing this argument. The estimate concerns the entire signed cap, whereas (S131) prices the inverse error on its specified null source.

S32 Complete observation under moving arithmetic cutoffs

The finite-source criteria above admit a complementary reduction on the *entire* negative observation space. Its cutoff is an operator energy, rather than a Fourier window or a finite-rank projection. The useful new input is a complete-positive source estimate uniform in the arithmetic cutoff, the number of source profiles and the number of genuine null rows.

Let X_M be the original odd polar space with any finite subspace K_M of $\ker H$ removed. All projections and norms in this section are original Hilbert ones. Let $P = P_{M,K_M}$ and $N = N_{M,K_M}$ be the restrictions of the complete positive and negative forms to this recipient. Put

$$\begin{aligned} R_M &= 6 + 8(\log 2)\sqrt{M}, \qquad & B_M &= 9R_M, \\ J_M &= 3(84R_M + 310 + 2^{27}), \qquad & C_M^{\mathrm{src}} &= R_M J_M^2. \end{aligned} \tag{S133}$$

Here $\|s\|_\infty < 3$, $0 \le n_M \le R_M$ and $\|N\| \le B_M$. Section S39 proves these constants from the complete theta series and an elementary prime-power bound.

Theorem S32.1 (Uniform complete-positive source moment)**.** *Let $\phi_1, \dots, \phi_r$ be any finite family in*

$$\mathcal{Y}_M = L^2(\{\xi > 0 : n_M(\xi) > 0\}, n_M(\xi)\,\mathrm{d}\xi/\pi),$$

and let $D \succ 0$ dominate their true joint Gram matrix in this space. Writing Q_M for the original orthogonal projection onto X_M, define

$$g_i(u) = \int_0^\infty n_M(\xi)\phi_i(\xi)\sin(\xi u)\frac{\mathrm{d}\xi}{\pi}, \qquad E_i = Q_M(sg_i).$$

Then every E_i belongs to $\operatorname{Dom} P$*, and*

$$E^*E \preceq B_M D, \qquad (PE)^*(PE) \preceq C_M^{\mathrm{src}} D. \tag{S134}$$

The constants are independent of r and $\dim K_M$*. In particular, for $\kappa, \Lambda > 0$,*

$$0 \preceq E^* 1_{(\Lambda,\infty)}(P)(P+\kappa)^{-1}E \preceq \frac{C_M^{\mathrm{src}}}{\Lambda^2(\Lambda+\kappa)} D. \tag{S135}$$

Proof. Section S39 proves, for an odd F-bandlimited g,

$$\|Q_M(sg)\| \le 3\|g\|, \qquad \|PQ_M(sg)\| \le 3[51R_M + 33\log(2+F) + 210 + 2^{27} + \delta_M]\|g\|.$$

The complete negative support lies in $[-\Omega_M, \Omega_M]$, with $\Omega_M = e^{R_M+1}$. For every common coefficient vector z, Plancherel gives $\|g_z\|^2 \le R_M z^* D z$ and $\log(2+\Omega_M) < R_M + 3$. Since $\delta_M < 1$, the displayed graph constant is below J_M. This proves both full matrix inequalities, including all mixed terms. The spectral theorem gives

$$\frac{1_{\lambda>\Lambda}}{\lambda+\kappa} \le \frac{\lambda^2}{\Lambda^2(\Lambda+\kappa)},$$

which proves (S135). The second moment uses $E_i \in \operatorname{Dom} P$ and does not require $E_i \in \operatorname{Dom} P^2$. □

Define the complete observation $T_M : X_M \to \mathcal{Y}_M$ by

$$(T_M h)(\xi) = \int_{\mathbb{R}} s(u)h(u)\sin(\xi u)\,\mathrm{d}u.$$

Thus $T_M^*\phi = Q_M(sg_\phi)$ and $N = T_M^* T_M$. Let

$$\kappa_M = M^{-1}, \quad \Lambda_M = (4MC_M^{\mathrm{src}})^{1/3}, \quad L_M = T_M 1_{[0,\Lambda_M]}(P)(P+\kappa_M)^{-1}T_M^*. \tag{S136}$$

Theorem S32.2 (Complete-observation energy criterion)**.** *For every $M \ge 2$ and every such finite genuine-null choice,*

$$0 \preceq T_M 1_{(\Lambda_M,\infty)}(P)(P+\kappa_M)^{-1}T_M^* \preceq \frac{1}{4M} I_{\mathcal{Y}_M}. \tag{S137}$$

If on an unbounded sequence of integers M the complete low-energy cap

$$L_M \preceq \left(1 + \frac{3}{4M}\right) I_{\mathcal{Y}_M} \tag{S138}$$

holds, then on the entire original domain

$$q_H[h] \ge -\left[\delta_M + \frac{1}{M} + \frac{B_M}{M+1}\right]\|h\|_2^2. \tag{S139}$$

Conversely, $H \succeq 0$ implies (S138) *for every $M \ge 2$ and every finite genuine-null choice. The cofinal energy criterion is therefore equivalent to positivity of this original H and, by Theorem 2.2, to RH. Its original allowance is $O(M^{-1/2})$, and $\Lambda_M = O(M^{5/6})$.*

Proof. Theorem S32.1 applies to every finite tuple of profiles. On a dense profile core it bounds both $T_M^*\phi$ and $PT_M^*\phi$ by fixed multiples of $\|\phi\|_{\mathcal{Y}_M}$. For a convergent profile sequence, both images are Cauchy. Closedness of P gives graph-norm completion to every $\phi \in \mathcal{Y}_M$, with the same bounds. Thus (S135) extends to the entire observation space. The choice of Λ_M gives (S137); its high and low pieces are the exact functional-calculus decomposition of the same resolvent.

Adding (S138) gives $T_M(P+\kappa_M)^{-1}T_M^* \preceq c_M I$, $c_M = 1 + 1/M$. The equivalent form inequality is $N \preceq c_M(P+\kappa_M I)$. Rearrange using the bounded N, rather than introducing an adverse multiple of unbounded P:

$$P - N \succeq -\kappa_M I - (1 - c_M^{-1})N \succeq -\left[\frac{1}{M} + \frac{B_M}{M+1}\right] I.$$

Add the complete original-norm raw tail. Exact genuine-null transport from Theorem 2.2 proves (S139) on all original V. Its allowance tends to zero.

For the converse, $H = P - N + W_M \succeq 0$ gives $N \preceq P + \delta_M I$. The bound (10) satisfies $\delta_M < 1/M$ for all $M \geq 2$. A rational admission at $M = 2$ is

$$2 \cdot 451 \cdot 9(10/27)^9(10/13) < 1,$$

using $\pi > 31/10$, $e > 27/10$ and $e^{3/10} > 13/10$. The consecutive majorant ratio is at most $(8/3)e^{-\pi} < (8/3)(3/8)^3 < 1$. Hence the complete observation cap is at most I, and so is its low-energy summand. □

The spectral space $1_{[0,\Lambda_M]}(P)X_M$ may be infinite dimensional. Theorem S32.2 specifies a complete operator comparison; the finite-source Schur criterion and source-residual enclosures of Section S31 supply separate constructive interfaces. The energy scale does not replace the Fourier support Ω_M or the rank of a chosen positive receiving frame.

Corollary S32.3 (A finite cutoff interval from the M16 certificate)**.** *If a complete original-domain certificate gives $H \succeq -aI$ with $a = 5.289 \cdot 10^{-19}$, as in the signed-frequency companion [23], then for every integer $2 \leq M \leq 10^{18}$ and every finite genuine-null choice,*

$$T_M(P + M^{-1}I)^{-1}T_M^* \preceq I, \qquad L_M \preceq I.$$

Proof. At every arithmetic cutoff the same original bound gives $N_M \preceq P_M + (a + \delta_M)I$. It suffices that $M(a+\delta_M) < 1$. For $2 \leq M \leq 15$, the decreasing majorant used above gives $M\delta_M < 819/1000$, while $Ma < 1/1000$. For $M \geq 16$,

$$M\delta_M \leq 451 \cdot 16 \cdot 17^2(3/8)^{51} < 1/4, \qquad Ma \leq 5289/10000 \quad (M \leq 10^{18}).$$

These rational bounds prove the assertion. The bridge is the complete original-domain inequality, rather than an identification of the M16 finite matrices with matrices at another cutoff. □

This finite-interval consequence and the unbounded-sequence hypothesis in Theorem S32.2 have different quantifiers. The former follows from one fixed global allowance; the latter is the complete low-energy comparison that would make the allowance vanish.

S32.1 True observation compression and a paid omitted trace

The complete energy criterion has a finite observation-space interface. It retains the true source columns in its finite head and pays the entire remaining covariance, including mixed blocks, by

positivity. Keep $\mathcal{Y}_M, T_M, P, \kappa = M^{-1}$ and $\Omega = e^{R_M+1}$ from Section S32, and write $E = T_M^*$ and $\nu = n_M(\xi)\,\mathrm{d}\xi/\pi$ on $[0,\Omega]$. Put

$$W = R_M\Omega/\pi, \quad a_* = 31/32, \quad t(L) = 28e^{-a_* r_L}(r_L/a_* + a_*^{-2}), \quad r_L = \pi e^{2L}.$$

The complete two-arm bound in Section S39 gives

$$\nu([0,\Omega]) \le W, \qquad \|s\|_2^2 \le 28/a_*^2 < 32, \qquad \int_{|u|>L} s(u)^2\,\mathrm{d}u \le t(L).$$

The last integral follows by substituting $r = \pi e^{2|u|}$ in $s^2 \le 28r^2e^{-a_*r}$; both arms together have Jacobian $\mathrm{d}r/r$.

Theorem S32.4 (True finite head and complete observation return)**.** *Choose $L \ge 1$ with $Wt(L) \le 1/(8M^2)$ and an integer $J \ge 1$ such that*

$$N_J = 2J+1 \ge 2e\Omega L, \qquad N_J \ge \log_2(16M\sqrt{W}). \tag{S140}$$

Let Π be the true orthogonal projection in $\mathcal{Y}_M$ onto $\mathrm{span}\{\xi, \xi^3, \dots, \xi^{2J-1}\}$*, and let $U : \mathbb{C}^J \to \mathcal{Y}_M$ be an isometry onto that span. Then*

$$\|E(I-\Pi)\|_{\mathrm{HS}}^2 \le \frac{1}{4M^2}. \tag{S141}$$

For $R_{\mathrm{low}} = 1_{[0,\Lambda_M]}(P)(P+\kappa)^{-1}$ define the true *finite matrix $G_M = (EU)^* R_{\mathrm{low}}(EU)$. The finite-head condition*

$$G_M \preceq \left(1 + \frac{1}{2M}\right) I_J \tag{S142}$$

implies the complete low-energy cap (S138)*. Its validity along an unbounded sequence of admitted cutoffs and finite genuine-null complements is equivalent to original H positivity, with allowance* (S139)*. Sufficient choices have $J = O(\Omega \log M) = \exp(O(\sqrt{M}))$.*

Proof. The ambient sine-source kernel is Hilbert–Schmidt, with square at most $W\|s\|_2^2$. Applying the original physical orthogonal projections preserves this property. Replace its sine on $|u| \le L$ by $p_J(\xi u) = \sum_{j=0}^{J-1}(-1)^j(\xi u)^{2j+1}/(2j+1)!$ and set the approximant to zero outside. The resulting source map E_J has rank at most J and

$$\|E - E_J\|_{\mathrm{HS}}^2 \le W\left[t(L) + 32\left\{\frac{(\Omega L)^{N_J}}{N_J!}\right\}^2\right]. \tag{S143}$$

Indeed p_J is also the sine Taylor polynomial through degree $2J$, so its real remainder is bounded by $|\xi u|^{N_J}/N_J!$. Orthogonal polar and genuine-null projections contract the whole error. The observation dependence of E_J lies in $\operatorname{ran}\Pi$; thus $E_J(I-\Pi) = 0$. The elementary bound $N! \ge (N/e)^N$ and (S140) make the remainder at most $2^{-N_J} \le 1/(16M\sqrt{W})$. Equation (S143) is then at most $1/(4M^2)$, proving (S141).

For any positive $0 \preceq R \preceq \kappa^{-1}I$, the operator $\mathcal{L} = E^*RE$ is positive trace class. Positivity bounds its mixed quadratic term between $x \in \operatorname{ran}\Pi$ and $y \in \ker\Pi$ by the geometric mean of its diagonal terms. Scalar Cauchy–Schwarz consequently gives

$$\|\mathcal{L}\| \le \|\Pi\mathcal{L}\Pi\| + \operatorname{tr}((I-\Pi)\mathcal{L}(I-\Pi)) \le \|\Pi\mathcal{L}\Pi\| + \kappa^{-1}\|E(I-\Pi)\|_{\mathrm{HS}}^2. \tag{S144}$$

This pays all mixed blocks without declaring them zero. Use $R = R_{\mathrm{low}}$: the omitted trace is at most $1/(4M)$, and (S142) gives $\|\mathcal{L}\| \le 1 + 3/(4M)$. Theorem S32.2 now returns the original allowance. Conversely $H \succeq 0$ gives complete observation cap at most I, so every such true compression satisfies (S142).

The choices terminate because $t(L) \to 0$. Since $\log W = O(\sqrt{M})$, a polynomially growing r_L suffices, giving $L = O(\log M)$; the two requirements on N_J then give the asserted sufficient rank. The monomial Gram is positive definite because $m_M(0) < 0$ and hence ν is positive on an interval near zero. This proves existence of U; numerical evaluation must certify its actual observation metric. □

The finite matrix uses EU, not the Taylor-surrogate $E_J U$. This true compression is what permits a quadratic omitted-error payment. The rank bound concerns the observation compression and makes no rank assertion about $1_{[0,\Lambda_M]}(P)$. For upper certification one may instead bound the larger full-resolvent head $V^*(P+\kappa)^{-1}V$, $V = EU$. If trial columns Y lie in $\operatorname{Dom} P$ and $\mathcal{A} = V - (P+\kappa)Y$ is their *complete* action residual, then

$$\|V^*(P+\kappa)^{-1}V - \operatorname{Herm}(V^*Y)\| \le \kappa^{-1}\|V\|\,\|\mathcal{A}\|.$$

This follows from $(P+\kappa)^{-1}V - Y = (P+\kappa)^{-1}\mathcal{A}$. It supplies a paid upper in the true source coordinates. A residual $V - BY$ for a different finite reference still leaves $(P + \kappa - B)Y$ to be enclosed; all sources, norms and operators in the complete head estimate must match.

S33 The hard-cutoff mean and intrinsic receiver rank

The original hard-cutoff receivers have a useful independent description. On the polar space set

$$\mathcal{C}_0 = \frac{4}{\pi} Q_b M_s \beta(D) M_s Q_b, \quad \beta(\xi) = (1+4\xi^2)^{-1/2}, \quad \mathcal{R} = (\ker H)^{\perp}.$$

The two references $\mathcal{C}_0$ and $\mathcal{C}_Q$ have Fourier orders one and two, respectively. The next estimates concern the hard cutoff. For the right-continuous prime-power sum $\Psi_\Lambda(x) = \sum_{n \le x} \Lambda(n)$ write $E_\Lambda(x) = \Psi_\Lambda(x) - x$ and

$$e_\Lambda(M) = \frac{|E_\Lambda(M)|}{M} + \frac{1}{\sqrt{M}} \int_1^M |E_\Lambda(x)| x^{-3/2} dx.$$

Theorem S33.1 (Complete common mean and one-sided rate)**.** *The operators $N_M/\sqrt{M}$ converge strongly to $\mathcal{C}_0$. On* $\operatorname{Dom} H$*, $P_M h/\sqrt{M} \to \mathcal{C}_0 h$, and the selfadjoint positive operators have the same strong-resolvent limit. For $F \ge 1$, $0 < \eta < 1/2$ and $M^\eta \ge \sqrt{(1+2\eta)/(1-2\eta)}$,*

$$\frac{N_M}{\sqrt{M}} \succeq \mathcal{C}_0 - \rho_M I, \tag{S145}$$

$$\rho_M = 18(1+F)e_\Lambda(M) + \frac{9(12+\log(3F))}{\sqrt{M}} + \frac{36}{\sqrt{1+4F^2}} + \frac{30M^{-\eta}}{1-2\eta}.$$

The same one-sided estimate compresses to every finite row complement. Classical PNT permits $F \to \infty$ with $\rho_M \to 0$.

Proof. Put $z = 1/2 + i\xi$, $U = \log M$. Abel summation, with its lower endpoint retained, gives

$$\frac{1}{\sqrt{M}} \sum_{n \le M} \Lambda(n) n^{-1/2+i\xi} = \frac{e^{i\xi U}}{z} + \frac{z-1}{z\sqrt{M}} + e^{i\xi U} \frac{E_\Lambda(M)}{M}$$

$$-\frac{z-1}{\sqrt{M}}\int_1^M E_\Lambda(x)x^{-3/2+i\xi}dx.$$

As $|z-1|=|z|$, the difference from $e^{i\xi U}/z$ on $|\xi|\le F$ is at most $M^{-1/2}+(1+F)e_\Lambda(M)$. The positive-part map is 1-Lipschitz. Using $|m_0|<6$ and $a(\xi)\le 4+\log(3F)$, the normalized negative symbol differs on this interval by at most $d=2(1+F)e_\Lambda(M)+(12+\log(3F))/\sqrt{M}$ from

$$v_M(\xi)=4\beta(\xi)(\cos(\xi U-\arctan(2\xi)))_+.$$

The absolutely convergent Fourier series of $\cos_+$ has constant coefficient $1/\pi$ and nonconstant complex coefficient sum $1/2+1/\pi$. Riemann–Lebesgue gives the local weak multiplier mean $4\beta/\pi$. The normalized symbols are uniformly bounded by (S133); their local weak means therefore extend against every L^1 frequency test. For fixed input, truncate its Fourier profile $\widehat{sh}$ to $[-F,F]$. Multiplication by s after a finite frequency projection is Hilbert–Schmidt, so it converts the weak convergence on that band to strong convergence. The input exterior tends to zero uniformly in M. This proves the negative strong limit.

For a quantitative continuum bound, the positive lth harmonic continues analytically on $|\Im z|<1/2$ as

$$e^{ilUz}(1-2iz)^{(l-1)/2}(1+2iz)^{-(l+1)/2}.$$

Shift to $\Im z=\eta$ on compact smooth physical tests. The multiplier bound is $(1-4\eta^2)^{-1/2}q^l$, $q=M^{-\eta}\sqrt{(1+2\eta)/(1-2\eta)}\le 1$. The two physical factors acquire opposite exponential weights and $\|e^{\eta u}s\|_\infty\|e^{-\eta u}s\|_\infty<9$. Indeed the complete envelope bounds their squares by $169x^{2+\eta}e^{-\pi x}\le 169e^{-\pi}<9$, $x=e^{2|u|}\ge 1$. Vertical contour ends vanish by smooth compact integration by parts; density extends the resulting bounded operators to all inputs. Summing the harmonics costs at most $30M^{-\eta}/(1-2\eta)$. Discard the actual positive negative-symbol exterior beyond F and pay the omitted continuum by $36/\sqrt{1+4F^2}$. The retained symbol error costs $9d$. This proves (S145).

Finally $P_M=N_M+H-W_M$ on $\operatorname{Dom}H$ gives the positive limit there. The resolvent identity at $(\mathcal{C}_0+i)h$ and the uniform resolvent norm bound extend it by density to strong resolvents. PNT makes $e_\Lambda(M)\to 0$ by splitting its integral at a fixed large threshold; for example take $F=\min(M^{1/4},e_\Lambda(M)^{-1/2})$. □

Theorem S33.2 (Intrinsic hard-cutoff receiver rank)**.** *There is $c_*>0$ such that*

$$\lambda_j(P_{\mathcal{R}}\mathcal{C}_0|_{\mathcal{R}})\ge c_*/j,\qquad j\ge 1.$$

Consequently, if a rank-J positive receiver B_M on the complement of any *finite genuine-null space K_M satisfies $N_{M,K_M}\preceq B_M+\kappa_M I$, then*

$$J+1\ge\frac{c_*}{\rho_M+\kappa_M/\sqrt{M}}. \tag{S146}$$

With a classical PNT remainder one may choose $\rho_M=O(e^{-c\sqrt{\log M}})$. Along $\kappa_M\to 0$, the required receiver rank is therefore at least $c_1e^{c_2\sqrt{\log M}}$ for sufficiently large M.

Proof. The operator $\mathcal{C}_0$ is compact: its finite-frequency sandwiches are Hilbert–Schmidt, and the omitted multiplier has norm at most $(36/\pi)\sup_{|\xi|>F}\beta(\xi)\to 0$. Section S40 proves the eigenvalue floor using the original-domain critical-zero representers and complete Schwartz-weight Gram estimates. On its $(J+1)$-dimensional protected span choose a unit vector in $\ker B_M$. It belongs to every genuine-null complement. Combining the floor with (S145) gives (S146). The classical error $|E_\Lambda(x)|\le Axe^{-c\sqrt{\log x}}$ gives $e_\Lambda(M)=O(e^{-c_0\sqrt{\log M}})$ by splitting at $\sqrt{M}$. Choose $F=e^{c_0\sqrt{\log M}/2}$ and fixed $\eta=1/4$. □

For arbitrary finite, possibly nongenuine row spaces of dimension d, an elementary version is also useful. If $\sigma_R = \min_{|u|\le R} s^2$ and $a_R = 4\sigma_R/[\pi\sqrt{1+4\pi^2/R^2}]$, then $\lambda_j(\mathcal{C}_0) \ge a_R/(j+1)$ and $J + d + 2 \ge a_R/(\rho_M + \kappa_M/\sqrt{M})$. Take the first $j+1$ odd Dirichlet sine modes f on $[-R,R]$, set $h = f/s$ and impose the single polar moment. Fourier Jensen gives

$$\langle f, (1+4D^2)^{-1/2} f\rangle \ge \frac{\|f\|^2}{\sqrt{1+4\|f'\|^2/\|f\|^2}},$$

and $\|f'\| \le \pi(j+1)\|f\|/R$. Minmax and codimension interlacing give the displayed assertions. These are representation costs for the hard-cutoff split; the smoothed reference of Section S28 has its own source geometry.

S34 The full residual covariance and the signed return

The finite receiving gate must retain the source covariance between the phase space, the exact rows and the complementary carriers. The following exact Schur identity gives the interface. It is classical finite-rank elimination specialized to the original constrained recipient [27, 8].

Theorem S34.1 (Residual-covariance signed gate)**.** *Let $Z : \mathbb{C}^d \to \mathcal{H}$ have finite-dimensional range containing the exact row map $\mathcal{R}$. Let P be its range projection and $Q = I - P$. For a finite carrier map U, define*

$$G = Z^*Z, \qquad F = U^*Z, \qquad J = U^*QU.$$

Let $C = C^*$ and $a > \|J\|$. Then

$$aI - ZCZ^* - UU^* \succeq 0 \quad \text{on } \ker \mathcal{R}^* \tag{S147}$$

if and only if

$$aG - GCG - aF^*(aI - J)^{-1}F \succeq 0 \quad \text{on } \{c : \mathcal{R}^*Zc = 0\}. \tag{S148}$$

The columns of Z may be dependent. If $J \preceq \overline{J} \prec aI$, substitution of $\overline{J}$ gives a sufficient gate. For any finite matrix V of trial coefficients,

$$J \preceq (U - ZV)^*(U - ZV). \tag{S149}$$

Proof. The Q block of (S147) is $aQ - QUU^*Q \succ 0$. Its exact elimination adds

$$PUU^*P + PUU^*Q(aQ - QUU^*Q)^{-1}QUU^*P = aPU(aI - U^*QU)^{-1}U^*P$$

to the adverse finite block. Every $x \in P\mathcal{H}$ has the form Zc; evaluation gives (S148). Since $\operatorname{ran}\mathcal{R} \subseteq \operatorname{ran} Z$, the entire Q space satisfies the exact rows, so this elimination holds on the stated kernel. Inverse operator monotonicity proves the upper-covariance substitution. Finally $Q(U - ZV) = QU$, and the orthogonal P, Q split of the trial residual proves (S149). □

For injective Z, $J = U^*U - FG^{-1}F^*$. Formula (S149) supplies a complete positive upper without subtracting nearly equal numerical matrices or assuming a projected row Gram floor. It keeps the full residual covariance rather than replacing it by unrelated diagonal carrier bounds.

S34.1 Paid common-source errors

Theorem S37.1 supplies the joint source enclosure, its exact-row shorting and the full congruence payment. The numerical application in Section S38 retains every phase/row/carrier block.

A positive padded covariance $\Gamma \succeq \delta \operatorname{diag}(D_p, D_r)$ has shorted phase block at least δD_p: minimize its quadratic form over the row coefficients. This supplies a lower bound for the padded elimination without requiring a lower bound on the angle between unpadded phase and row spans. The padding is charged in the signed inequality. An original full-row Gram floor applies to its stated receiving map; a complementary projection requires its own analysis or this paid residual construction.

S34.2 Return to the original form

For a finite arithmetic head and $x = \sqrt{\alpha + s^2/5}\, h$, the exact identity is

$$q_{H_M}[h] + \alpha\|h\|_2^2 = \|x\|_2^2 - \int_{\mathbb{R}} (1/5 - m_M(\xi))|\widehat{t_\alpha x}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi}. \tag{S150}$$

The original polar row becomes $t_\alpha \sinh(u/2)$ in this variable. To separate the favorable unbounded part, fix a finite Λ and write $d_M = 1/5 - m_M$ as

$$d_M = d_{\mathrm{ret}} + b - g, \quad d_{\mathrm{ret}} = 1_{|\xi|\le\Lambda} d_M, \quad b = 1_{|\xi|>\Lambda}(d_M)_+, \quad g = 1_{|\xi|>\Lambda}(-d_M)_+ \ge 0.$$

The original graph domain pays each integral. Dropping the favorable $-g$ gives a form upper bound; only d_{ret} and b receive bounded-operator approximations. Suppose these supply the complete upper domination

$$M_{t_\alpha}(1/5 - m_M)(D)M_{t_\alpha} \preceq ZCZ^* + UU^* + \eta I, \qquad \eta < 1,$$

as forms on the transformed original core. For example, a finite signed source approximation, a positive defect approximation and complete norm errors supply this domination after retaining the favorable exterior by its sign. Set $a = 1 - \eta$ and apply Theorem S34.1 on the exact transformed rows. A successful gate gives

$$q_{H_M}[h] \ge -\alpha\|h\|_2^2, \qquad q_H[h] \ge -(\alpha + \delta_M)\|h\|_2^2,$$

where (9) supplies a full raw arithmetic payment. Finite genuine-null deflation is returned as in Theorem 2.2. All identities begin on the exact polar core and extend through the original closure; bounded retained perturbations preserve its domain.

Thus, for $\alpha_j = 2^{-j}$ and any cofinal $M_j \to \infty$, complete signed gates returning (S150) yield the vanishing allowances required by Theorem 2.2. Theorem S35.1 supplies finite receiving spaces, complete positive complements and certified continuous coordinates for every such index. The exact residual gate specifies the signed comparison to be certified in those coordinates.

The coordinated schedule of Theorem S35.3 supplies the explicit original allowance $(1+2^{-37})2^{-j}$. The direct return adds the raw norm error to α_j; any stronger inverse-normalized error requirement used to establish a chosen finite gate is paid within that gate.

S35 A complete conditioned construction

The receiving space will approximate every frequency source at once. Put

$$\alpha_j = 2^{-j}, \qquad t_j = \frac{s}{\sqrt{\alpha_j + s^2/5}}, \qquad j \ge 16.$$

Let ν_j be a positive finite measure supported in $[0,\Omega_j]$, with mass A_j, and define the positive operator on $\mathcal{H}=L^2_{\mathrm{odd}}(\mathbb{R})$ by

$$O_j=\frac{1}{\pi}\int |t_j\sin(\xi\,\cdot)\rangle\langle t_j\sin(\xi\,\cdot)|\,\mathrm{d}\nu_j(\xi). \tag{S151}$$

For an even nonnegative multiplier b, take $\mathrm{d}\nu=b(\xi)\,\mathrm{d}\xi$ on the positive half-line. The identity $\cos(x-y)-\cos(x+y)=2\sin x\sin y$ gives exactly (S151) in the Fourier convention of Section 2. Positive atoms are also allowed.

Theorem S35.1 (Conditioned complete complements)**.** *Choose a computable integer $B_j\ge\max(1,A_j)$ and put*

$$\begin{aligned}
L_j&=j+\log(1+B_j),\\
T_j&=j\log 2+\log(1+B_j)+4\log L_j+20,\qquad R_j=\tfrac12\log(T_j/\pi),\\
E_j&=12\Omega_jR_j+2\log(1+B_j)+4\log L_j+40.
\end{aligned}$$

Choose an even integer N_j with $E_j\le N_j<E_j+4$. Let V_j be t_j times the odd polynomials of degree less than N_j, restricted to $[-R_j,R_j]$, and let Q_j project onto its entire odd orthogonal complement. Then

$$\|Q_jO_jQ_j\|<\frac{1}{2\log L_j},\qquad \dim V_j\le 6\Omega_jR_j+\log(1+B_j)+2\log L_j+22. \tag{S152}$$

For computable supplied parameters, a terminating certified construction gives a dyadic basis transformation whose actual continuous receiving Gram lies between $3I/5$ and $5I/3$. Its condition number is at most $25/9<3$.

Proof. The elementary inequalities

$$L/2<T<3L,\qquad 0<R<\tfrac12\log L$$

hold for $L\ge16$. For the upper bound use $j\log2+\log(1+B)\le L$ and $4\log L+20<2L$; the latter ratio decreases from its strict endpoint value at 16. For $0\le\xi\le\Omega$, real Taylor approximation supplies an odd polynomial p_ξ of degree less than N with

$$|\sin(\xi u)-p_\xi(u)|\le\frac{(\Omega R)^N}{N!}<4^{-N}\quad(|u|\le R),$$

since $N\ge12\Omega R$ and $e<3$. Thus the squared interior error after multiplication by t, where $t^2<5$, is at most $10R16^{-N}$. The complete theta upper bound (8), with $x=e^{2|u|}$, gives the full exterior estimate

$$\begin{aligned}
\int_{|u|>R}t^2\,\mathrm{d}u&\le 169\,2^j\int_{T/\pi}^{\infty}xe^{-\pi x}\,\mathrm{d}x\\
&=\frac{169e^{-20}}{(1+B)L^4}\frac{T+1}{\pi^2}.
\end{aligned} \tag{S153}$$

Orthogonal projection is at least as accurate as this explicit approximation. Integrating the squared error against the whole positive source gives

$$\|QOQ\|\le\mathrm{Tr}(QOQ)\le\frac{A}{\pi}\left(10R16^{-N}+\frac{169e^{-20}(T+1)}{\pi^2(1+B)L^4}\right). \tag{S154}$$

This trace includes every covariance combination and both infinite spatial tails. Because $N \ge 2\log(1+B)+4\log L+40$ and $\log 16 > 2$, $16^{-N} < e^{-80}(1+B)^{-4}L^{-8}$. Multiplying the first term in (S154) by $\log L$ bounds it by $(5/3)e^{-80}(\log L)^2L^{-8} < 1/4$. The second term times $\log L$ is at most $(676/27)e^{-20}L^{-2} < 1/4$, using $A/(1+B) \le 1$, $T+1 < 4L$ and $\pi > 3$. This proves (S152). Upward rounding an outward enclosure of E of width less than one to the next even integer gives the required N without a transcendental equality decision.

For conditioning, the first theta atom and the decrease of r^2e^{-r} for $r > 2$ give, throughout $[-R,R]$,

$$\frac{s^2}{\alpha_j} \ge c_j = \frac{2T^2e^{-20}}{(1+B)L^4}, \qquad \frac{e^{-20}}{2(1+B)L^2} < c_j < 1.$$

Consequently

$$t_j^2 \ge \frac{c_j}{1+c_j/5} > \frac{e^{-20}}{4(1+B)L^2} > \ell_j := \frac{2^{-32}}{(1+B)L^2}. \tag{S155}$$

The final comparison follows from $30\log 2 > 20$. In an ordinary $L^2([-R,R])$-orthonormal odd Legendre basis the true receiving Gram satisfies $\ell_j I \preceq G_j \preceq 5I$.

Choose a positive rational $\ell_j' \le \ell_j$ by outward evaluation. Compute a symmetric rational moment enclosure $\widetilde{G}_j$ with $\|G_j - \widetilde{G}_j\| \le \ell_j'/4$; entrywise error $\ell_j'/(4\dim V_j)$ suffices. On this compact interval the complete theta series has effective uniform derivative tails and its denominator is positive. Certified partitions and increasing precision therefore terminate. The moments here are physical receiving moments, determined by s, t_j, R_j; computability of ν_j is needed only when evaluating the source operator or its mass rather than using its supplied bounds.

Since $\widetilde{G}_j \succeq 3\ell_j' I/4$, increasingly accurate inverse-Cholesky proposals give a dyadic invertible W_j passing the finite check

$$\tfrac{3}{4}I \preceq W_j^*\widetilde{G}_jW_j \preceq \tfrac{5}{4}I.$$

The error bound and $G_j \succeq \ell_j' I$ imply $(3/4)G_j \preceq \widetilde{G}_j \preceq (5/4)G_j$. Hence

$$\tfrac{3}{5}I \preceq W_j^*G_jW_j \preceq \tfrac{5}{3}I.$$

Only the basis changes; the space, its entire complement and the trace bound are unchanged. □

The condition estimate concerns the polynomial phase Gram on V_j. Adjoined exact rows enter the joint signed gate with their actual mixed Gram, as specified in Section S34.

For fixed A, Ω, a fixed B yields $\dim V_j = O(\log j)$ and a complementary norm tending to zero. The next section proves necessity of this order for a nonzero fixed density. Polynomial approximation, finite-band concentration and whitening are classical mechanisms [2, 12, 11]; the explicit theta-dependent wall, whole-tail payment, funded Gram floor and original signed interface specify their use here.

S35.1 Moving arithmetic heads

The construction also applies when the complete arithmetic head grows. Put

$$m_M(\xi) = \Re\Psi(1/4+i\xi/2) - \log\pi - 2\sum_{2\le n\le M}\frac{\Lambda(n)}{\sqrt{n}}\cos(\xi\log n), \qquad b_M = (1/5 - m_M)_+.$$

Every prime power is included. The following elementary bounds, also used in the signed-frequency companion [23], give concrete inputs.

Lemma S35.2. *For $M \geq 2$, the positive-half support and mass of b_M are bounded by*

$$\Omega_M = e^{7+8(\log 2)\sqrt{M}}, \qquad A_M \leq \Omega_M[7 + 8(\log 2)\sqrt{M}].$$

The coordinated choice below turns these inputs into explicit original-domain tail and receiving-space budgets.

Proof. The Chebyshev and Abel argument in Section S2 gives $S_M \leq 4(\log 2)\sqrt{M} - 2\log 2$ by (S1). The complete digamma comparison (S2) and $m_0 > -6$ give

$$m_M(\xi) > -6 + \log(2|\xi|) - 2S_M.$$

Thus outside Ω_M, $m_M > 1 + \log 2 > 1/5$. Everywhere the defect is bounded by $7 + 2S_M < 7 + 8(\log 2)\sqrt{M}$. Integrating this bound over its positive-half support proves the asserted mass. □

Theorem S35.3 (Coordinated arithmetic and receiving budgets)**.** *For $j \geq 16$, put $\kappa_j = 2^{-j}$ and choose an integer M_j with*

$$E_j \leq M_j < E_j + 2, \qquad E_j = \frac{j\log 2 + 3\log(j+2) + 40}{\pi}. \tag{S156}$$

Then the complete original arithmetic remainder satisfies

$$\delta_{M_j} < 2^{-37}\kappa_j. \tag{S157}$$

For the moving positive source b_{M_j}, Theorem S35.1 gives condition number at most $25/9$ and

$$\log\dim V_j \leq 8\log 2\sqrt{\frac{\log 2}{\pi}}\,\sqrt{j} + O(\log j). \tag{S158}$$

The leading coefficient is $2.60467320417428\ldots$. If complete signed head gates return $q_{H_{M_j}}[h] \geq -\kappa_j\|h\|_2^2$ on the original recipient, then

$$q_H[h] \geq -(1 + 2^{-37})2^{-j}\|h\|_2^2. \tag{S159}$$

Consequently such a cofinal family satisfies Theorem 2.2.

Proof. An outward enclosure of E_j of width less than one, followed by upward integer rounding, supplies the stated integer. For $j \geq 16$, $\log(j+2) \leq j/2$, $\log 2 < 1$ and $\pi > 3$ imply

$$M_j + 1 < E_j + 3 < \frac{5j + 98}{6} \leq 3(j+2).$$

Insert this into (10) and use $M_j \geq E_j$:

$$\delta_{M_j} < \frac{4059e^{-40}}{j+2}2^{-j} \leq \frac{4059}{18}\left(\frac{3}{8}\right)^{40}2^{-j} < 2^{-37}2^{-j}.$$

Here $e > 8/3$; the last comparison is rational. Moreover $M_j = (\log 2/\pi)j + O(\log j)$. Choose B_j by upward enclosure of the mass bound in Lemma S35.2. Then $\log(1+B_j) = 8(\log 2)\sqrt{M_j} + O(\log M_j)$, $L_j = j + O(\sqrt{j})$, and $R_j = O(\log j)$. Substitution into (S152) proves (S158). Finally the bounded raw remainder costs at most $\delta_{M_j}\|h\|_2^2$, proving (S159). Finite genuine-null deflations return by Theorem 2.2. □

The constructed rank in (S158) prices the positive receiving space at the final original-norm scale. With $\epsilon_j = (1 + 2^{-37})2^{-j}$, the same estimate reads

$$\log \dim V_j \le \frac{8 \log 2}{\sqrt{\pi}} \sqrt{\log(1/\epsilon_j)} + O(\log \log(1/\epsilon_j)).$$

The arithmetic and positive-complement estimates hold at every index. The signed hypothesis in Theorem S35.3 is the finite gate of Section S34, including its complete approximation and residual-covariance payments.

S36 The necessary fixed-source dimension

This lower bound counts the whole retained odd span, including any adjoined receiving directions.

Theorem S36.1 (Sharp dimension order)**.** *Let $b \ge 0$ be a nonzero bounded density on positive frequencies supported in a fixed $[a, \Omega]$ with $a > 0$, extended evenly as a Fourier multiplier. Fix one integer $B \ge \max(1, A)$ and use the corresponding R_j of Theorem S35.1. Let $O_j = M_{t_j} b(D) M_{t_j}$ on the odd space. Set*

$$A = \int_0^\infty b, \qquad B_2 = \int_0^\infty b^2, \qquad B_\infty = \|b\|_\infty.$$

For any finite spaces W_j with $\|Q_{W_j} O_j Q_{W_j}\| \le \tau_j \to 0$,

$$\liminf_{j \to \infty} \frac{\dim W_j}{R_j - 1} \ge \frac{B_2}{\pi B_\infty^2} > 0.$$

Together with Theorem S35.1, the optimal dimension order for obtaining a complementary norm tending to zero is $\Theta(\log j)$; the constructed rate is the one in (S152).

Proof. Take $L = R_j - 1$. On $|u| \le L$, the first-atom lower bound gives

$$s^2/\alpha_j \ge 2\pi^2 \exp(j \log 2 - T_j/e^2) \longrightarrow \infty.$$

Thus $t_j^2 \ge 4$ there for all sufficiently large j. Compare the full source covariance with the source covariance obtained by replacing t_j by $1_{[-L,L]}$. The former is at least four times the latter in frequency-source space. By equality of the nonzero spectra of U^*U and UU^*, and min–max,

$$\lambda_n(O_j) \ge 4\lambda_n(T_L), \qquad T_L = 1_{[-L,L]} b(D) 1_{[-L,L]} \tag{S160}$$

on odd functions. This comparison is made in the common source space.

Put $C_L(\omega) = 2 \sin(L\omega)/\omega$, continuously defined at zero. The complete sine-carrier formulas give

$$\begin{aligned} \operatorname{Tr} T_L &= \frac{1}{2\pi} \int_0^\infty b(\xi)[2L - C_L(2\xi)] \, \mathrm{d}\xi, \\ \operatorname{Tr} T_L^2 &= \frac{1}{4\pi^2} \int_0^\infty \int_0^\infty b(\xi) b(\eta) [C_L(\xi - \eta) - C_L(\xi + \eta)]^2 \, \mathrm{d}\xi \, \mathrm{d}\eta. \end{aligned}$$

Consequently $\operatorname{Tr} T_L / L \to A/\pi$. The measures $C_L(\omega)^2 \, \mathrm{d}\omega/(4\pi L)$ are a positive approximate identity of mass one. Applied to the continuous autocorrelation of the zero-extended b, this gives the difference-term limit B_2/π in $\operatorname{Tr} T_L^2 / L$. Since $\xi + \eta \ge 2a$, its square term is $O(1)$ and the mixed term is $O(\sqrt{L})$ by Cauchy–Schwarz. Therefore

$$\operatorname{Tr} T_L^2 / L \longrightarrow B_2/\pi, \qquad \|T_L\| \le B_\infty.$$

If $N_j = \dim W_j$, min–max and (S160) give $\lambda_{N_j+1}(T_L) \le \tau_j/4$. Hence

$$\operatorname{Tr} T_L^2 \le N_j B_\infty^2 + (\tau_j/4)\operatorname{Tr} T_L.$$

Divide by L and take the lower limit. Finally $R_j \sim \frac{1}{2}\log j$ for fixed B. □

The first-two-trace argument belongs to classical time–frequency concentration. Its present use matches the explicit complete-theta upper construction, including arbitrary competing odd receiving spans. Compression by the fixed polar row changes a dimension count by at most one; additional growing constraint dimensions are counted with their actual span.

S37 Complete joint sources and stable row elimination

The phase construction controls a positive complement. To evaluate its signed recipient, collect the actual phase, exact-row and carrier profiles in one source map. Put $q = s^2$, $w_\alpha = q/(\alpha + q/5) = t_\alpha^2$. On a finite interval I, write

$$Yc = \sqrt{w_\alpha}\sum_a c_a f_a, \qquad \Gamma = Y^*Y.$$

Supported phase profiles are polynomials on their support cells, the polar profile is $\sinh(u/2)$, and a derivative-row profile is

$$f_k = (\psi^{(k+2)} - \psi^{(k)}/4)/q \quad (k \text{ odd}).$$

Literal carriers have profile $\sqrt{\nu_a/\pi}\sin(\xi_a u)$. An exact column transformation is applied to the entire derivative polynomial before evaluation. The original preimages determine genuine-null status; the transformed rows are receiving coordinates.

Theorem S37.1 (Common-weight full covariance)**.** *Let the finitely many measurable profiles belong to $L^2(I, w_\alpha)$, and let a real positive function p satisfy $|p - w_\alpha| \le \eta w_\alpha$ on I, where $0 < \eta < 1$. Define $\Gamma_p = \int_I f(u)^* f(u) p(u)\,\mathrm{d}u$. Then*

$$(1-\eta)\Gamma \preceq \Gamma_p \preceq (1+\eta)\Gamma. \tag{S161}$$

For a partition into retained and row coefficients, define the shorted form $S[x] = \inf_z (x,z)^*\Gamma(x,z)$, and similarly S_p. Then

$$(1-\eta)S \preceq S_p \preceq (1+\eta)S.$$

These assertions allow dependent columns and redundant rows.

Let $\widehat{Y}_p$ approximate $Y_p = \sqrt{p}\, f$ and let $D \succ 0$ be a specified coefficient metric. If

$$\|(Y_p - \widehat{Y}_p)D^{-1/2}\| \le e, \qquad \|\widehat{Y}_p D^{-1/2}\| \le M,$$

then, with $\delta = 2Me + e^2$,

$$-\delta D \preceq \Gamma_p - \widehat{Y}_p^*\widehat{Y}_p \preceq \delta D, \qquad \Gamma \preceq \frac{\widehat{Y}_p^*\widehat{Y}_p + \delta D}{1-\eta}. \tag{S162}$$

Shorting this entire upper gives an upper for S.

Proof. Integrate the pointwise weight inequalities against $|\sum c_a f_a|^2$. Taking the infimum over the same row coefficients proves the shorted inequalities. For the profile change put $E = Y_p - \widehat{Y}_p$ and expand $Y_p^* Y_p$. The two cross terms have total D-normalized norm at most $2Me$, and E^*E at most e^2. Combining with the left inequality of (S161) proves (S162). Quadratic-form order is preserved by the defining infimum. □

This is a source-specific use of classical positive integration and variational Schur order [27, 11]. In particular, the relative-weight step preserves the exact kernel. A profile approximation is priced separately by (S162). No lower angle between the phase and row spans enters these bounds. For any exact coefficient map W, an additive enclosure $-\delta D \preceq \Gamma - \widehat{\Gamma} \preceq \delta D$ becomes the enclosure $\delta W^* D W$ after congruence. This is the payment used for enrichment or row changes.

S37.1 Positive polynomial integration

Here is a fully specified realization of the weight step. Suppose w_α is analytic on each closed disk of radius d centered in $[-R, R]$, with $|w_\alpha| \le H$, and $w_\alpha \ge w_0 > 0$ on the real interval. These are checkable source hypotheses. Split the interval into cells of half-width $h \le d/8$, also splitting at every supported-profile endpoint. On a cell centered at c, let $L \ge m \ge 1$ and $\zeta_\ell = e^{2\pi i \ell / L}$. Use

$$a_k = \left(\frac{2h}{d}\right)^k \frac{1}{L} \sum_{\ell=0}^{L-1} w_\alpha(c + (d/2)\zeta_\ell)\zeta_\ell^{-k}, \qquad p(c + hx) = \sum_{k<m} a_k x^k.$$

Exact conjugate-paired sampling makes p real. Cauchy's coefficient bound and root-of-unity aliasing give

$$\|w_\alpha - p\|_\infty \le \frac{H(h/d)^m}{1 - h/d} + \frac{H\, 2^{-L}}{(1 - 2^{-L})(1 - h/d)}. \tag{S163}$$

Indeed the discrete coefficient at index $k < L$ contains only Taylor indices $k + vL$; summing their bounds proves the second term. Literal sample and real dyadic coefficient errors add their full uniform payment. Choosing their sum with (S163) below ηw_0 supplies the positive approximant in Theorem S37.1.

For polynomial profile approximants of degree below N, a positive Gauss–Legendre rule of order at least $N+m$ on each cell integrates $p\widehat{f}^*\widehat{f}$ exactly: every entry has degree below $2N+m$. Certified node, weight and evaluation errors are included when the rule is represented by literal numbers. Thus its matrix is a single positive common-factor sum. The rule uses

$$O((R/d + 1)(N + m))$$

points on a uniform grid, plus the support-boundary subdivisions. This argument establishes a constructive rule under its stated strip and profile hypotheses.

S37.2 Complete growing-row tails

A full-line covariance also needs physical tails. Write a premixed row profile on $u \ge 0$ as

$$f_a(u) = c_a \sinh(u/2) + Q_a(u)/q(u), \quad Q_a(u) = e^{u/2} \sum_{n \ge 1} P_a(\pi n^2 e^{2u}) e^{-\pi n^2 e^{2u}}.$$

Use odd reflection on the other arm. Let $D \ge 2$ bound all polynomial degrees, K_a bound each polynomial's absolute coefficient sum, and $F_0 = 1 + \sum_a (|c_a|^2 + K_a^2)$.

Proposition S37.2 (Premixed complete row tails)**.** *For $\alpha = 2^{-j}$ and $T = \pi e^{2R} \ge \max(\pi, 4D)$,*

$$\sum_a \int_{|u|>R} w_\alpha |f_a|^2 \, \mathrm{d}u \le 676\, 2^j F_0 T^{2D} e^{-T}. \tag{S164}$$

For $0 < \varepsilon < 1$, a sufficient wall for this trace to be below ε^2 is

$$T \ge 4D, \qquad T \ge 4\{j \log 2 + \log(676 F_0) + 2D \log(8D) + 2\log(1/\varepsilon) + 10\}.$$

Proof. Put $r = \pi e^{2u}$. For $r \geq 4D$, comparison of successive terms gives $|\sum_n P_a(rn^2)e^{-rn^2}| \leq 2K_a r^D e^{-r}$. For example $n^{2D}e^{-r(n^2-1)}$ is decreasing with a geometrically bounded ratio, using $\log n \leq (n^2-1)/2$. The first positive theta atom gives $q \geq 2r^2e^{-r}$, while (8) gives $q \leq 169e^{4u-r}$ and $w_\alpha \leq 2^j q$. Apply $|x+y|^2 \leq 2|x|^2+2|y|^2$, $\sinh^2(u/2) \leq e^u$, $\mathrm{d}u = \mathrm{d}r/(2r)$ and both arms. Enlarging powers to r^{2D} and constants bounds the trace by $338\,2^j F_0 \int_T^\infty r^{2D}e^{-r}\,\mathrm{d}r$. Since $T \geq 4D$, integration by parts or a geometric series gives an upper $2T^{2D}e^{-T}$. Finally $2D\log T \leq T/4 + 2D\log(8D)$ proves the sufficient wall. □

In particular $D = O(j/\log(j+2))$, $\log F_0 = O(j)$ and $\log(1/\varepsilon) = O(j)$ permit $T = O(j)$ and $R = O(\log j)$. If a common strip has $d^{-1} = O(j)$, logarithmic profile bounds $O(j)$, and $\log(w_0^{-1}) = O(j)$, the construction above reaches exponential accuracy with $O(j)$ scaled coefficient precision. For the raw identity metric this also requires logarithmic column count $O(j)$; a different metric additionally pays its inverse norm. High-frequency sine profiles use degree $O(|\xi|h+j)$ on each cell. These statements specify source precision and node counts under the displayed hypotheses.

Complete mixed tails follow from the full source factor: if the row and carrier tail traces are E_r, E_c, their joint tail trace is $E_r + E_c$ and the cross block has norm at most $\sqrt{E_r E_c}$. Supported phase tails vanish outside their support. The full covariance therefore remains a single error-controlled input to Theorem S34.1.

S38 A complete 4473-column source certificate

We give a whole-line application of the common-source construction at $\alpha = 3/10^{12}$. The supplied data specify exact dyadic row coordinates and literal frequency carriers. The polynomial-weight construction of Section S37 is one general realization; this application uses a directly enclosed positive Gaussian rule for the actual theta sources.

Theorem S38.1 (Full mixed physical Gram)**.** *Let Y have the 4080 supported phase columns, 41 complete receiving rows and 352 literal carriers specified below and in the scientific data. Let $\widehat{\Gamma}$ be the exact 2^{-100}-dyadic block-arrow matrix in* `data/joint_centers.npz`*. Then*

$$\|Y^*Y - \widehat{\Gamma}\| < 1.384607318 \cdot 10^{-20} < 2^{-65}.$$

The norm is the raw Euclidean coefficient operator norm. All phase/row, phase/carrier and row/carrier terms, the entire global block, quadrature errors and both infinite physical tails are included. In particular, putting $\delta = 2^{-65}$,

$$Y^*Y \preceq \widehat{\Gamma} + \delta I, \qquad \widehat{\Gamma} + 2\delta I \succeq \delta I.$$

S38.1 Exact columns and a portable construction

The negative half-line is split into 100 cells of half-width $r = 1/80$, centered at $c_i = (-199+2i)/80$, $0 \leq i < 100$. Odd reflection supplies the positive half-line. On the last 68 negative cells ($32 \leq i < 100$), take the 60 local columns

$$t_\alpha(u)\sqrt{\frac{2k+1}{2r}}\,\mathrm{P}_k((u-c_i)/r), \qquad 0 \leq k < 60,$$

where P_k is the Legendre polynomial; outside the paired cell supports they vanish. Their unweighted odd Gram is $2I_{4080}$. Disjoint cell interiors give exact zero off-cell blocks.

The 41 global rows are an exact dyadic triangular mixture of the polar profile and the profiles with odd derivative indices $1, 3, \dots, 79$. The file `row_coordinates.json` gives the literal coefficients and polynomial normalizations. To reconstruct them, start with

$$P_0(x) = 4x^2 - 6x, \qquad P_{\ell+1}(x) = (1/2 - 2x)P_\ell(x) + 2xP'_\ell(x).$$

Then $\psi^{(\ell)}(u) = e^{u/2}\sum_n P_\ell(\pi n^2 e^{2u})e^{-\pi n^2 e^{2u}}$ on $u \geq 0$. Normalize $P_{k+2} - P_k/4$ by its stated largest absolute coefficient and premix the exact polynomials before evaluation. The file `literal_rule.json` gives 352 rational frequencies $2400 < \xi_a < 6000$ and positive dyadic weights ν_a with $\sum_a \nu_a < 2$. Their columns are exactly $t_\alpha\sqrt{\nu_a/\pi}\sin(\xi_a u)$.

The script `generate_rows.py` evaluates all 80,000 negative-arm Gaussian nodes at 512-bit ball precision using 16 theta atoms and a complete polynomial-dependent remainder. It produces double-double source arrays and certified point errors. The global assembly uses those source arrays, computes carrier phases by directed ball arithmetic, and pays all point and fixed-point errors jointly. The exact row coordinates are mathematical inputs; no numerical QR is part of the certificate.

S38.2 Whole quadrature and tail payments

The following explicit bounds specify the analytic contract implemented by `row_budget.py` and `joint_gram.py`. They also permit reproduction without a sampled Gram hypothesis. On each Gaussian cell use the Bernstein ellipse $\rho = 7/5$. Its imaginary height is

$$h = \frac{r}{2}(\rho - \rho^{-1}) < 1/200.$$

The actual theta denominator is bounded in this strip by first-atom comparison on $|\Re u| \leq 7/5$ and by regularization outside. For $v = \pi e^{2z}$ the first scaled atom satisfies $|P_0(v)| \geq 2|v|^2$. The other atoms have relative sum at most $3\sum_{n\geq 2} n^4 e^{-(n^2-1)\Re v} < 1/10$ for $\Re v > 2.99$, giving the lower denominator modulus $9X_-^2/5$ below. Inner-strip phase comparison gives $\Re\psi > 0$; the outer envelope and $\Re\cosh(z/2) > .99$ give receiving denominator real part above $\alpha/2$. The complete strip envelope for ψ is below 100, which also funds the displayed bound $4 \cdot 10^{19}$ for t_α^2. The finite checks are

$$13/400 + \pi e^{14/5}/100 < 1, \qquad 120e^{63/10 - \pi e^{14/5}\cos(1/100)} < \alpha/4.$$

Use even reflection $z \mapsto -z$ for a negative-real portion of the first positive-arm ellipse; the weight is even and the receiving rows are odd. For a positive-arm cell let $x_- = \max(0, c - r(\rho + \rho^{-1})/2)$ and $x_+ = c + r(\rho + \rho^{-1})/2$ be its extreme real coordinates on that ellipse, $X_\pm = \pi e^{2x_\pm}$, and $c_h = \cos(2h)$. A valid majorant is

$$t_2 = \min\{4 \cdot 10^{19},\ (240/\alpha)e^{9x_+/2 - X_- c_h}\}, \qquad d_0^{-1} = 5/(9X_-^2).$$

For a premixed derivative polynomial P_a, let A_a be its absolute-coefficient polynomial. Its complete atom numerator is bounded by

$$B_a = \sum_{n=1}^{16} A_a(n^2 X_+)e^{-(n^2-1)X_- c_h} + 2A_a(289X_+)e^{-288X_- c_h}.$$

The script verifies that the remaining atom ratio is below 1/2. The corresponding row bound is $\sqrt{t_2}\, e^{x_+/2}(|S_{0a}| + d_0^{-1}B_a)$. The local Legendre factor is bounded by $70\rho^{59}$ and the carrier sine factor by e^{6000h}. Positive Gaussian integration of an analytic scalar product has error at most

$$\frac{16r\rho^{-1600}}{1 - \rho^{-1}} \times (\text{its ellipse supremum}).$$

Trace bounds for row blocks and Frobenius bounds for mixed blocks give a complete summed quadrature error below $5.6 \cdot 10^{-185}$.

For the whole physical tail put $Z = \pi e^5$, and define

$$I_0 = e^{-Z}, \qquad I_k = kI_{k-1} + Z^k e^{-Z} = \int_Z^\infty x^k e^{-x}\,\mathrm{d}x.$$

If $A_a(x)^2 = \sum b_{a,k} x^k$, the complete row-tail trace is bounded by

$$E_r = \sum_a \left\{ \frac{4}{\alpha} \sum_{k\geq 2} b_{a,k} I_{k-2} + \frac{240}{\alpha} |S_{0a}|^2 I_2 \right\} < 1.384607235 \cdot 10^{-20}.$$

The full exterior atom sum is bounded by twice its first absolute-polynomial term; the explicit ratio tests at Z in the script verify this for every included degree. The carrier tail has bound

$$E_c = \frac{\sum \nu_a}{\pi} \frac{240}{\alpha} (e^5/3 + 1/9) e^{-3e^5}.$$

The row/carrier mixed tail is at most $\sqrt{E_r E_c}$. Supported phase columns vanish outside their own wall. Adding $E_r + E_c + 2\sqrt{E_r E_c}$ is therefore a conservative full-block tail payment.

Proof of Theorem S38.1. Use the common 800-point positive Gauss–Legendre rule on each of the 100 cells. Complete theta source intervals pay the point uncertainty. Directed 256-bit ball products compute the sampled matrix; an alternative supplied route quantizes the common factors to 2^{-120} and uses exact integer products. Directed products retain the double-double uncertainties in their full matrix intervals. In the fixed-point alternative, the common-factor inequality (S162) pays source uncertainty and factor quantization together. For this factor payment let the phase/global errors be e_p, e_g, and let the represented global trace be T_g. The phase factor norm is at most $\sqrt{10}$ before perturbation, since $t_\alpha^2 < 5$. Thus

$$e = (e_p^2 + e_g^2)^{1/2}, \quad M \leq \{(\sqrt{10} + e_p)^2 + T_g\}^{1/2}, \quad E_{\mathrm{pt}} \leq 2Me + e^2.$$

Center the exact products at 2^{-100}. The block-arrow center-rounding error is at most

$$E_{\mathrm{ctr}} \leq \max(E_p, E_g) + E_{pg},$$

where E_p, E_g are maximal directed row sums of the phase/global center radii and E_{pg} is the Frobenius norm of all mixed radii. This uses the block-diagonal phase norm, the global block norm, and the complete rectangular cross-block Frobenius norm; disjoint-support zeros remain exact. For exact fixed-point products the simpler bound $\{\max(60, 393) + \sqrt{4080 \cdot 393}\} 2^{-100}$ suffices. The portable assembly verifies $E_{\mathrm{pt}} + E_{\mathrm{ctr}} < 8.3 \cdot 10^{-28}$. Add the whole quadrature and tail bounds above to obtain the stated radius. The final two inequalities follow from positive semidefiniteness of Y^*Y and its two-sided enclosure. □

All uncertainty in this theorem concerns the stated literal carriers. A continuous frequency measure is supplied to the original signed return with its own operator-approximation debit. The raw covariance error transports through an enrichment W as $\delta W^* W$. The scientific supplement contains the exact columns, complete source arrays, centers and portable generators, together with a short verification command and controls that reject underfunded error caps.

S39 Complete source bounds with uniform null cost

This section proves the actual-source estimates used in Section S32. The Fourier convention, theta weight, original polar row and minimal realization are those of Section 2. No auxiliary positive reference replaces $P_{M,K}$.

The shared arithmetic, logarithmic and complete-theta estimates used here are proved once in Section S2. In this section P_M again denotes the hard-cutoff positive form.

S39.1 Original polar return and count-independent null return

On the ambient odd space let $\overline{P}_M = M_s p_M(D) M_s$ and $h_0 = sg$. Bandlimited L^2 functions have every derivative in L^2; complete theta multiplication and smooth compact cutoffs put h_0 in the minimal form domain. Equation (S6) and form representation then put it in the ambient operator domain and give

$$\|\overline{P}_M h_0\| \le 3[33(R_M + \log(2+F)) + 210]\|g\|. \tag{S165}$$

For example the compact approximants converge in L^2 and their weighted physical derivatives converge in L^2; $a \le 17\xi^2$ pays convergence in the specified minimal form norm.

The original odd polar row is $b = s\sinh(u/2)$. Put $f_b = sb = s^2\sinh(u/2)$. From (S4), its exponentially weighted Fourier norm is at most $32\sqrt{2}$. Since $\sup_{x\ge 0} x^2 e^{-x/8} = 256/e^2 < 36$,

$$\|f_b''\| < 1728, \qquad \|\overline{A}b\| < 88128.$$

On each interval $1/4 \le |u| \le 1/2$, the first complete theta atom gives $s^2 > 18\cdot 3^{-10}$ and $\sinh^2(u/2) \ge 1/64$. Consequently

$$\|b\|^2 > \frac{9}{64\cdot 3^{10}} > 2^{-20}, \qquad \frac{\|\overline{A}b\|}{\|b\|} < 2^{27}. \tag{S166}$$

Here $r = \pi e^{2|u|} < \pi e < 10$, $r > 3$ and $e < 3$ suffice for the lower bound. The same compact form approximation and representation justify b's operator membership. From (S3), $\overline{P}_M - \overline{A}$ is bounded by B_M. Writing Q_b for the polar projection and $x = Q_b h_0$, therefore,

$$\|P_M x\| \le \|\overline{P}_M h_0\| + (2^{27} + B_M)\|h_0\|.$$

The constrained action is $P_M x = Q_b \overline{P}_M x$; preservation of the polar condition by the ambient action is not assumed.

Now let T be an original Hilbert isometry onto any finite-dimensional $K \subseteq \ker H$. Bounded perturbation in the same minimal realization gives $\operatorname{Dom} H = \operatorname{Dom} P_M$ and

$$P_M T = N_M T - W_M T, \qquad \|P_M T\| \le B_M + \delta_M.$$

For $e = Q_K x$, the operator on the further constrained form domain acts as $P_{M,K} e = Q_K P_M e$, whence

$$\|P_{M,K} e\| \le \|P_M x\| + (B_M + \delta_M)\|x\|.$$

Combining the last displays, $\|x\| \le \|h_0\| \le 3\|g\|$ and (S165) proves the promised count-independent bound

$$\|P_{M,K} Q_K Q_b(sg)\| \le J(M,F)\|g\|, \quad J(M,F) = 3[51R_M + 33\log(2+F) + 210 + 2^{27} + \delta_M]. \tag{S167}$$

This argument uses actual operator-nullity. An arbitrary additional row requires its own graph cost. Independence of $\dim K$ is an analytic norm estimate and does not assert that numerical projection is free.

For arbitrary negative profiles and a common complex coefficient vector,

$$\|g_z\|^2 = \int_0^\infty n_M^2 |\phi_z|^2 \frac{\mathrm{d}\xi}{\pi} \le R_M z^* D z.$$

Their support is bounded by Ω_M, and $J(M, \Omega_M) < J_M$. Equation (S167) proves the complete joint moment in Theorem S32.1. Applying the same inequalities to profile differences and using closedness of $P_{M,K}$ proves the full observation-space graph completion used in Theorem S32.2.

The same graph estimate also pays finite inverse-action trials. If E_0, W_0 are actual negative and positive source families, the latter supported below F_0, and D_n, D_p dominate their true frequency metrics, put $V = \kappa^{-1} Q_M (E_0 - W_0 F_c)$ and $F_b = \max(\Omega_M, F_0)$. Before multiplication by s, the two profiles have disjoint Fourier sign supports. Thus

$$(PV)^*(PV) \preceq \kappa^{-2} J(M, F_b)^2 [R_M D_n + (R_M + \log(2 + F_0)) F_c^* D_p F_c].$$

This supports complete source-action residual accounting. Physical mixed covariance after multiplication by s remains present; disjoint raw frequency sign supports do not make the receiving cross block vanish.

S40 Protected critical representers and their spectral floor

We supply the original-domain and uniform Gram arguments used in Theorem S33.2. The complete polarized formula is

$$q_H(h_1, h_2) = \sum_z m_z \, \mathcal{M}(s h_2)(z) \overline{\mathcal{M}(s h_1)(-\bar{z})}, \qquad \mathcal{M} f(z) = \int f(u) e^{zu} du.$$

All centered nontrivial zeros occur with multiplicity.

Lemma S40.1 (Critical evaluation vectors in the operator range)**.** *For every positive critical ordinate γ, $v_\gamma = Q_b(s \sin(\gamma u))$ belongs to* $\operatorname{ran} H$.

Proof. Put $\sigma = i\gamma$ of exact multiplicity m and $P_0 = d^2/du^2 - 1/4$. For a function whose first r transform jets vanish at σ, define

$$R_\sigma^r f(u) = \int_u^\infty \frac{(t-u)^{r-1}}{(r-1)!} e^{\sigma(t-u)} f(t) dt.$$

Its canceled lower-tail expression is $(-1)^r \int_{-\infty}^u (u-t)^{r-1} e^{\sigma(t-u)} f(t) dt/(r-1)!$. The vanishing jets prove equality, and direct integration gives $\mathcal{M}(R_\sigma^r f)(z) = \mathcal{M} f(z)/(z-\sigma)^r$. With $\eta = R_\sigma^m \psi$, the detector $h_\sigma = P_0 \eta / s$ therefore has transform $(z^2 - 1/4)\xi(1/2 + z)/(z-\sigma)^m$. It evaluates only at σ, where its value A is nonzero.

Here is its original minimal-domain return. Each fixed-order theta derivative is bounded on either arm by a polynomial in $r = \pi e^{2|u|}$ times $e^{|u|/2 - r}$; the first atom gives the matching positive lower for s^2. In the upper primitive use $t = u + v$ and $e^{2v} \ge 1 + 2v$; its complete tail is bounded by a polynomial times $e^{u/2 - r} e^{-rv} v^{m-1}$ beyond a fixed onset. Use the canceled lower primitive on the other arm. Differentiation gives the corresponding fixed-order estimates. For compact even cutoffs, $P_0(\chi_R \eta)$ has both pole moments zero exactly, converges after division by s in L^2, and its derivative converges in ordinary L^2. The bound $a \le 17\xi^2$ gives convergence in V. Weighted integration by parts gives arbitrary vertical inverse powers, so the $O(T \log T)$ zero count passes the full zero formula to the limit, as in Appendix A.

Oddify by $h_o(u) = h_\sigma(u) - h_\sigma(-u)$. Its only evaluations are A and $-A$ at $\pm i\gamma$. The polarized formula gives, in the conjugate-linear-first convention, $q_H(h_o, h) = 2mi\bar{A}\langle v_\gamma, h\rangle$. This is a bounded Hilbert functional; form representation puts $h_o \in \operatorname{Dom} H$ and $Hh_o = -2miAv_\gamma$. In particular $v_\gamma \perp \ker H$. □

Classical positive critical-line proportion [6] and the local total zero bound $N(T+1) - N(T) = O(\log(T+3))$ give a sequence of distinct critical ordinates with any fixed large separation G and $\gamma_j \le Dj$. To see this, greedily retain a point and discard the following interval of length G. Each discarded cluster has total multiplicity $O_G(\log T)$ below height T, whereas the critical-line count is at least $cT\log T$. Thus the number retained below T is at least $c_G T$, even when multiplicities are charged to clusters. Discard any fixed initial interval and enlarge D if necessary.

For an even positive Schwartz weight F, its sine Gram is

$$\tfrac{1}{2}[\widehat{F}(\gamma_i - \gamma_j) - \widehat{F}(\gamma_i + \gamma_j)].$$

Fourier decay bounds the off-diagonal difference row sums by $C_N \sum_{l\neq 0}(1 + G|l|)^{-N}$; the sum-frequency row sums tend to zero with the first retained height. Schur's test therefore gives uniform upper and lower Riesz bounds, simultaneously for $F = s^2$ and s^4. The polar coefficients $c_j = \langle b, s\sin(\gamma_j u)\rangle/\|b\|^2$ have an arbitrarily small *whole* ℓ^2 norm after increasing that height. Hence for $h = \sum_{i\le j} a_i v_{\gamma_i}$ and $f = sh$,

$$l_h\|a\|^2 \le \|h\|^2 \le u_h\|a\|^2, \qquad \|f\|^2 \ge l_f\|a\|^2.$$

Differentiating $f = \sum a_i s^2 \sin(\gamma_i u) - sb\sum a_i c_i$ and applying the upper cosine/sine Gram bounds with weights s^4 and $|(s^2)'|^2$ gives $\|f'\|^2 \le C\gamma_j^2\|a\|^2$ uniformly in j. Fourier Jensen, as displayed after Theorem S33.2, now gives $\langle \mathcal{C}_0 h, h\rangle \ge c_*/j\|h\|^2$ on this j-dimensional span. Lemma S40.1 puts it in $\mathcal{R}$. Compact minmax proves the required floor. Complete theta decay supplies all Schwartz seminorms used here; no constant depends on the number of subsequently removed genuine nulls.

S41 Origin compression and fixed-window separation

This appendix records an additional constraint on the original hard-cutoff origin layer. Put $U = \log M$, $0 < L < \pi/2$ and fix $B < \infty$, $C \ge 0$. On the original polar-only space let N_M^{org} use the negative window $[0, L/U]$, and let $P_M^{[B]}$ use the actual positive window $[0, B/U]$.

Proposition S41.1 (A bounded scaled positive window)**.** *There are compact smooth original polar tests h_M and constants $c, c' > 0$, with an integer k fixed before M, such that eventually*

$$(N_M^{\mathrm{org}} - CP_M^{[B]})[h_M] \ge c\sqrt{M}/U, \quad \|h_M\|^2 \le c' U^{4k+2}.$$

Thus this fixed positive-window budget has an adverse Rayleigh depth at least a constant times $\sqrt{M}/U^{4k+3}$.

Proof. PNT and Abel summation give uniformly on every fixed scaled interval $m_M(t/U)/\sqrt{M} \to -4\cos t$, with positive/negative parts converging to $4(-\cos t)_+$ and $4(\cos t)_+$. Choose $b > \max(B, L, \pi/2)$ and $p_k(t) = t(1 - t^2/b^2)^k$. The positive integral on $[0, B]$ is exponentially small in k, since its support starts at $\pi/2$. The negative integral on $[0, L]$ is at least a positive multiple of k^{-3} by restricting to $[0, 1/(2k)]$. Fix k so the negative integral exceeds C times the positive one by a strict amount.

On a fixed compact interval the odd functions $u, u^3, \dots, u^{2k+1}, \sinh(u/2)$ are independent. A positive even bump and their Gram inverse supply compact smooth odd moment duals. Writing $p_k(t) = \sum_{j\le k} a_j t^{2j+1}$, choose f_M with exact polar moment zero and $\int f_M u^{2j+1} = (-1)^j(2j+1)!U^{2j+1}a_j$. Then $h_M = f_M/s$ is in the original compact polar core, $\|h_M\| = O(U^{2k+1})$, and its sine transform at t/U is $p_k(t) + O(U^{-2})$ uniformly on the two retained intervals. Substitute in the actual observations and use their scaled limits. The fixed strict polynomial gap gives the asserted inequalities. □

There is also a positive finite-frequency compression for any original projection Q. With $Th(\xi) = \langle Q(s\sin(\xi u)), h\rangle$ in the true origin measure, its monomial projection Π_d gives $0 \preceq T^*\Pi_d T \preceq T^*T$. This is a true source projection and keeps its full monomial Gram. A sine Taylor approximation and the complete theta moments give the bound used in the portable controls:

$$\|T^*T - T^*\Pi_d T\| \le 64\beta_M \frac{[\Omega\log(4m)/2]^{2m}}{(m!)^2}, \quad m = 2d+1, \quad \beta_M = \int_0^\Omega n_M\, d\xi/\pi.$$

For completeness put $r_m = \log(4m)/2$. For $x = |u| \le r_m$, $x^{2m} \le r_m^{2m}$. For $x \ge r_m \ge \log 2$, monotonicity of xe^{-2x} gives

$$2m\log(x/r_m) \le 2me^{-2r_m}e^{2x} = \tfrac{1}{2}e^{2x} < \tfrac{\pi}{2}e^{2x}.$$

Thus $x^{2m} \le r_m^{2m}e^{r/2}$ everywhere, and both real arms give

$$\|su^m\|^2 \le 16r_m^{2m}\int_\pi^\infty re^{-r/2}dr = 16(2\pi+4)e^{-\pi/2}r_m^{2m} < 64r_m^{2m}.$$

The real Taylor remainder is $|\xi u|^m/m!$; the squared observation-map error integrates to the displayed bound. The exact projection has no larger error. At $\Omega = L/\log M$, for any fixed $c > (p+1/2)/8$, taking $d = \lceil c\log M/\log\log M\rceil$ pays M^{-p} eventually, by Stirling. This is a compression estimate in the true observation metric, complementary to the complete ridged comparison of Theorem S29.1.

S42 Source and error contracts for the finite-reference application

This section supplies the source construction and analytic error payments used in Section S31.4, in the original norm and minimal form domain of Section 2. It adapts the author's signed-frequency construction [23]; that citation records the origin of the construction, while the necessary arguments are included here. The hard-cutoff symbols in this section are

$$m_M = m_0 + a - 2\sum_{2\le n\le M}\Lambda(n)n^{-1/2}\cos(\xi\log n), \qquad p_M = (m_M)_+, \quad n_M = (-m_M)_+.$$

In the numerical head $M = 13$. Let $Y = L^2_{\mathrm{odd}}(\mathbb{R})$, $b = s\sinh(u/2)$ and $X = \mathcal{K} = b^\perp \cap Y$. The weight, Fourier measure and physical norm are exactly those of (5)–(6). The original realization and its bounded paired hard-cutoff return were established in Lemma S2.1; the genuine operator nulls

$$z_{2j+1} = (\psi^{(2j+3)} - \psi^{(2j+1)}/4)/s$$

are the vectors proved legal and null in Lemma 8.1. Their Mellin normalization and the polar cancellation are proved in Appendix A. Thus these domain and null facts require no additional predecessor hypothesis. Only the finite scientific inputs, including literal dyadic coefficients and row/source moments, are stored as data; their analytic construction and error bounds are given below.

S42.1 Finite receiving with positive-frequency references

For a nonnegative weight w supported in a bounded frequency set put

$$(T_w h)(\xi) = \sqrt{w(\xi)} \int_{\mathbb{R}} s(u)h(u)\sin(\xi u)\,\mathrm{d}u \quad \text{in } L^2(\mathbb{R}_+,\, \mathrm{d}\xi/\pi).$$

For the numerical construction this map acts on the ambient odd Hilbert space $Y = L^2_{\mathrm{odd}}(\mathbb{R})$; its inequalities can then be compressed to X. In particular $N_M = T_{n_M}^* T_{n_M}$ on Y. On frequency intervals take $z_k = \mathbf{1}_I \sqrt{w}\, T_k(t_I)$, where $t_I : I \to [-1,1]$ is affine. For their column map Z, define the raw metric $D = Z^*Z$ and $E = T_w^* Z$.

Lemma S42.1 (Frequency range projection)**.** *For positive-weight columns with metric D_+, ridge $\rho_+ > 0$ and $W = T_{p_M}^* Z_+$,*

$$P_{\mathrm{fin}} = W(D_+ + \rho_+ I)^{-1} W^* \le P_M.$$

Such constructions add to at most P_M when their frequency supports have disjoint interiors.

Proof. Singular values give $0 \le Z(Z^*Z + \rho I)^{-1} Z^* \le I$. Conjugate by the observation map. For disjoint sets their exact observation energies add, and each finite projection is below its own energy. □

Absorb both ridges into $D_n, D_p > 0$. Set

$$N_\star = E D_n^{-1} E^*, \quad P_{\mathrm{fin}} = W D_p^{-1} W^*, \quad G_{nn} = E^*E, \quad C = E^*W, \quad G_{pp} = W^*W.$$

These are full ambient mixed covariances. An ambient operator inequality restricts exactly to X. The head uses the complete exact-row compression proved below.

Theorem S42.2 (Full mixed variational certificate)**.** *Let $\kappa > 0$, $L = G_{pp} + \kappa D_p$, and fix a matrix F of size $(\dim D_p) \times (\dim D_n)$. The condition*

$$\kappa D_n - G_{nn} + CF + F^*C^* - F^*LF \ge 0 \tag{S168}$$

implies $N_\star \le P_{\mathrm{fin}} + \kappa I$. If an approximate full Gram satisfies

$$-\delta\,\mathrm{diag}(D_n, D_p) \le G - \widetilde{G} \le \delta\,\mathrm{diag}(D_n, D_p), \tag{S169}$$

it suffices to check

$$\begin{aligned}\kappa D_n - \widetilde{G}_{nn} + \widetilde{C}F + F^*\widetilde{C}^* - F^*(\widetilde{G}_{pp} + \kappa D_p)F \\ - \delta(D_n + F^* D_p F) \ge 0.\end{aligned} \tag{S170}$$

Proof. The positive inverse identity gives

$$\kappa E^*(\kappa I + W D_p^{-1} W^*)^{-1} E = G_{nn} - C L^{-1} C^*.$$

Completing the square,

$$CL^{-1}C^* - (CF + F^*C^* - F^*LF) = (L^{-1}C^* - F)^* L (L^{-1}C^* - F) \ge 0.$$

Thus (S168) implies $E^*(\kappa I + P_{\mathrm{fin}})^{-1} E \le D_n$. Singular values of $(\kappa I + P_{\mathrm{fin}})^{-1/2} E D_n^{-1/2}$ give the desired operator inequality. The Gram-dependent part is $-(I, -F^*)G(I, -F^*)^*$. Apply (S169) to $(v, -Fv)$, giving precisely $\delta\, v^*(D_n + F^* D_p F)v$. This proves (S170). □

A dyadic F is an exact witness. A numerical inverse can propose it; the proof uses its literal entries and the enclosed sign. The approximate Gram need not be positive. All mixed covariance remains in the sign test. If an additional exact-row compression is used in another construction, its full joint correction must also be retained.

An alternative is a direct certified inverse calculation. A frozen approximate inverse of L is validated by its complete Neumann residual, which encloses L^{-1}; a directed Schur sign then proves $\kappa D_n - G_{nn} + CL^{-1}C^* \geq 0$. This is the same inverse criterion in the proof above, without exporting an explicit variational F. The supplement specifies which route each receiving group uses.

S42.1.1 Complete exact-row covariance

Let $\mathcal{Z} : \mathbb{C}^r \to Y$ be an injective row map with $\Gamma = \mathcal{Z}^*\mathcal{Z} > 0$, and set

$$Q = I - \mathcal{Z}\Gamma^{-1}\mathcal{Z}^*, \qquad \mathcal{C}_n = \mathcal{Z}^*E, \quad \mathcal{C}_p = \mathcal{Z}^*W, \quad \mathcal{C} = [\mathcal{C}_n, \mathcal{C}_p].$$

The complete physical source Gram on $\operatorname{ran} Q$ is exactly

$$G_Q = G - \mathcal{C}^*\Gamma^{-1}\mathcal{C}. \tag{S171}$$

Projection changes the physical columns; the true frequency metrics D_n, D_p retain their definitions.

Theorem S42.3 (Conservative row-compressed receiving)**.** *Suppose $\overline{\Gamma} \geq \Gamma > 0$, and the ambient model obeys* (S169)*. For a fixed F, put*

$$T_F = \begin{pmatrix} I \\ -F \end{pmatrix}, \qquad R_F = \mathcal{C}_n - \mathcal{C}_p F.$$

The condition

$$\kappa(D_n - F^*D_pF) - T_F^*\widetilde{G}T_F + R_F^*\overline{\Gamma}^{-1}R_F - \delta(D_n + F^*D_pF) \geq 0 \tag{S172}$$

implies

$$QN_\star Q \leq QP_{\mathrm{fin}}Q + \kappa Q \quad \textit{on } \operatorname{ran} Q.$$

Proof. Inverse order gives $\overline{\Gamma}^{-1} \leq \Gamma^{-1}$ and hence

$$G_Q \leq G - \mathcal{C}^*\overline{\Gamma}^{-1}\mathcal{C}.$$

Thus the left side of (S172) is a lower bound for $\kappa(D_n - F^*D_pF) - T_F^*G_QT_F$. The ambient model error is paid on the whole vector T_Fv by (S169). Apply Theorem S42.2 to QE, QW in $\operatorname{ran} Q$. □

All entries of $\mathcal{C}_n, \mathcal{C}_p$ and the proof of $\overline{\Gamma} \geq \Gamma$ belong to this certificate. Enclosing arithmetic evaluates their complete expression (S172); the row source is not replaced by selected entries or by the identity Gram. Direct-kernel locality does not remove its dense correction.

S42.2 Degree-independent locality and actual source funding

Let $q = s^2$ and $\widehat{q}(\omega) = \int_{\mathbb{R}} q(u)e^{-i\omega u}\,\mathrm{d}u$. The ambient odd feature kernel is

$$\int_{\mathbb{R}} q(u)\sin(\xi u)\sin(\zeta u)\,\mathrm{d}u = \tfrac{1}{2}\{\widehat{q}(\xi - \zeta) - \widehat{q}(\xi + \zeta)\}. \tag{S173}$$

Double frequency integration carries $1/(2\pi^2)$. Exact row compression is applied to its full covariance by (S171).

Lemma S42.4 (Complete theta Fourier locality)**.** *For every real ω,*

$$|\widehat{q}(\omega)| \le \min\{32e^{-|\omega|/2},\, 16384e^{-3|\omega|/4}\}.$$

If two frequency blocks start at $a_i, a_j \ge 0$, have separation d_{ij} and masses β_i, β_j, their normalized feature Gram block satisfies

$$\|G_{ij}\| \le 16\sqrt{\beta_i\beta_j}\left(e^{-d_{ij}/2} + e^{-(a_i+a_j)/2}\right).$$

Normalization uses the true raw metrics plus nonnegative ridges; the estimates are independent of degree.

Proof. On $|\Im u| \le 1/2$ put $r_n = \pi n^2 e^{2|\Re u|}$. Since $\cos 1 > 1/2$ and $|\cosh(u/2)| \ge (31/32)\cosh(\Re u/2)$, the complete theta series gives

$$|q(u)| \le \frac{64}{31}\sum_{n\ge1}(4r_n^2 + 6r_n)e^{-r_n/2}.$$

Its horizontal integral is at most

$$\frac{64}{31}\sum_{n\ge1}(8\pi n^2 + 28)e^{-\pi n^2/2} < 32.$$

Substitute $r = \pi n^2 e^{2x}$ on both arms; integrate $(4r+6)e^{-r/2}$ and bound the remaining scalar sum by its first terms and full geometric remainder. The complete double-exponential envelope makes the vertical contour ends vanish. Translation by $-\operatorname{sign}(\omega)i/2$ proves the first bound.

For $|\Im u| \le 3/4$, use $\cos(3/2) > 1/16$ and $\cos(3/8) > 9/10$. The corresponding full-series bound is

$$|q(u)| \le \frac{20}{9}\sum_{n\ge1}(4r_n^2 + 6r_n)e^{-r_n/16}.$$

Its integral over the two real arms is at most

$$\frac{20}{9}\sum_{n\ge1}(64\pi n^2 + 1120)e^{-\pi n^2/16} \le \frac{20}{9}(256S_2 + 1120S_0'),$$

where $S_0' = \sum_{n\ge1} e^{-3n^2/16} < 2$ and $S_2 = \sum_{n\ge1} n^2 e^{-3n^2/16} < 15$. Indeed successive ratios give $S_0' \le e^{-3/16}/(1-e^{-9/16})$, and

$$S_2 \le \frac{32}{3e}\,\frac{e^{-3/32}}{1-e^{-9/32}} < 15.$$

The integral is therefore less than 16384. The same complete contour argument at height $3/4$ proves the second bound.

A normalized source coefficient vector represents a weighted polynomial of L^2 norm at most one, since its metric contains the true raw Gram. Its weighted L^1 norm is at most $\sqrt{\beta_i}$. Apply this twice to (S173); the half-strip constant becomes $32/2 = 16$. □

For a union of blocks with total mass B, a uniform retained-kernel error ϵ gives normalized full-matrix error at most $B\epsilon$. The rank-one comparison ($\sqrt{\beta_i\beta_j}$) proves this for the entire coefficient vector. Complete omitted interactions are paid by the same argument and their actual minimum argument. The finite-rank correction in (S171) remains dense.

S42.2.1 The complete real kernel polynomial

The source data supply the fixed exact dyadic polynomial

$$Q(\omega)=\sum_{k=0}^{512}a_kT_k(\omega/210),\qquad \sup_{|\omega|\le210}|\widehat{q}(\omega)-Q(\omega)|<10^{-60}. \tag{S174}$$

Odd coefficients vanish exactly. Here is the complete analytic construction. The preceding strip proof gives

$$|q(u)|\le\frac{64}{31}(256/e^2+24/e)\frac{e^{-3/4}}{1-e^{-9/4}}<128\quad(|\Im u|\le1/2).$$

Use ten physical panels of width $1/4$ on $[0,5/2]$, 128 Gauss–Legendre nodes per panel and even reflection. For real $|\omega|\le210$ their symmetric Gaussian error is at most

$$E_G=2\cdot10\cdot\frac{8(1/8)\,128e^{630/32}\,2^{-256}}{1-1/2}.$$

The omitted theta atoms after eight cost $E_\theta=5\cdot16\pi^2\,9^4e^{-81\pi}$, and the full real tail is bounded by

$$E_\infty=\frac{480e^{4U-\pi e^{2U}}}{2\pi e^{2U}-4},\qquad U=5/2.$$

Every enclosing Fourier node receives these complete errors. On the parameter-2 frequency ellipse, $|\Im\omega|<158$ and

$$|\widehat{q}(\omega)|\le480\int_0^\infty e^{162u-3e^{2u}}\,\mathrm{d}u\le\frac{240\cdot80!}{3^{81}}=:M_Q.$$

Chebyshev coefficient bounds and complete Lobatto aliasing cost $4M_Q2^{-512}$. Add the sum of all DCT coefficient radii when replacing them by exact dyadic midpoints. Odd coefficient balls contain zero before symmetry is imposed. The total is below $5.119\cdot10^{-63}$, establishing (S174). This separates analytic integration from coefficient approximation; compare [3].

The executed head uses this Q_{512} on $[-210,210]$. Local evaluation first expands the *fixed exact polynomial* at a nearby integer center and keeps degree 256, then shifts to the actual fractional center and keeps degree 192. The two omitted polynomial parts are paid separately. Let $M_{24}<1.26\cdot10^{44}$ bound Q on every radius-24 circle centered on $[-210,210]$. The Bernstein ellipse with semimajor $1+24/210$ contains these disks, giving this bound directly from the literal coefficients. The fractional center shift has modulus at most $1/2$, and the actual convolution halfwidth is at most 4. Thus the first real remainder is at most

$$E_{\mathrm{real}}=M_{24}\frac{(4.5/24)^{257}}{1-4.5/24}.$$

Cauchy's coefficient bound gives modulus at most $M_{24}/(1-16.5/24)$ for the intermediate degree-256 polynomial on the radius-16 disk about the actual center. The second remainder is at most

$$\frac{M_{24}}{1-16.5/24}\,\frac{(1/4)^{193}}{1-1/4}.$$

Their sum is below $3.411\cdot10^{-72}<10^{-70}$. This is a bound for two polynomial truncations, independent of the real-kernel error in (S174).

The head retains every real kernel argument through 210. An omitted term has minimum argument at least 202. Consequently the normalized source debit

$$2B\left(16384e^{-3\cdot202/4}+10^{-60}+10^{-70}\right) \tag{S175}$$

pays its complete kernel, far-image and local-model error. The true raw metrics and their enclosing entry errors are retained; the ridges dominate the model-metric discrepancies. The factor 2 safely covers the common-metric transport. It is paid on $(v, -Fv)$, without a further inverse-ridge factor.

S42.2.2 True symbol integration and source classes

The holomorphic symbol used for source quadrature is

$$m_M(z) = \frac{\Psi_0(1/4 + iz/2) + \Psi_0(1/4 - iz/2)}{2} - \log \pi - 2 \sum_{n \le M} c_n \cos(z \log n).$$

Complete root brackets have opposite endpoint signs and a strict derivative sign. Every intervening real part is certified by a whole symbol box or monotonicity and endpoints; the entire negative support and positive exterior are included. If a root stripe has width η and $|m'_M| < 100$, its symbol magnitude is at most 100η and its integrated weight in $\mathrm{d}\xi/\pi$ is at most $100\eta^2/\pi$. Thus the omitted stripe payment is quadratic in its width. All such stripes remain paid in the continuous source.

Each fixed-sign integration panel has length at most $1/4$. Its parameter-2 ellipse has real halfwidth $5h/4$ and imaginary halfwidth $3h/4$. The actual rectangles check $\Re(1/4 \pm iz/2) > 1/5$ and their distance from 1. The convergent digamma series gives $|\Psi_0(w)| \le 1 + 6|w - 1|$ on this half-plane. The actual prime coefficients then fund the symbol bounds below. Mapped raw Chebyshev arguments have modulus less than 2, so $|T_k| \le 5^k$.

For the true complex Fourier kernel,

$$|\widehat{q}(z)| \le \int_{\mathbb{R}} q(u) e^{|u|} \, \mathrm{d}u \le 12, \qquad |\Im z| \le 1.$$

The complete envelope $q(u) \le 16\pi^2 e^{4|u| - \pi e^{2|u|}}$ and the substitution $r = \pi e^{2|u|}$ give this gamma-integral bound. It applies with one frequency variable complex and the other real. A profile envelope M_{an} and Gaussian order n therefore give errors at most $8LM_{\mathrm{an}}2^{-2n}$ for a raw metric and $16L_1L_2M_{\mathrm{an}}2^{-2n}$ for a joint Gram entry. These errors concern the *true* analytic integrands. Only their real quadrature values are then replaced by fixed polynomials. Factorial-scaled moments evaluate that finite tensor rule exactly up to their enclosing arithmetic, retaining the second-variable sign and the two frequency measures.

Table 2 distinguishes the selected constructions. An observation count is separate from its number of source columns. The dense positive bins have width at most 2 and 16 features; each

Table 2: Source integration contracts for the selected constructions.

Source class	Observations	Positive bins	Gauss order	Symbol cap
Exact-row head	46	105	256	2000
Three dense groups	26	128	256	10000
Sixteen remote groups	35	variable	256	20000
Four repriced sources	4	5 each	256	own contract
Inherited isolated sources	138	5 each	own contract	own contract

negative block has 13 features. Their maximal profile index is 15, so the analytic metric and one-variable Gram envelopes fit 10^{34}. The adopted total integration widths are below 270. Raw metric and Gram pads are 10^{-85} and 10^{-80}. Both $13 \cdot 10^{-85} < 10^{-26}$ and $16 \cdot 10^{-85} < 10^{-30}$ fund the corresponding negative and positive ridge comparisons. The remote groups use the same profile counts with their larger actual digamma rectangles. The head has 547 negative and 1680 positive

columns, maximal profile index 15, analytic envelope 10^{30} and 10^{-100} raw entry pads. Its ridges are 10^{-25} and 10^{-28}. The other sources keep their own small-interval polynomials, reflected-term bounds, ridges and complete contracts. The typed index `data/m13_certificate.json` binds each group to these literal inputs.

S42.2.3 The original twelve-row head source

Write

$$\Xi(\xi) = \xi_{\mathrm{Riemann}}(1/2 + i\xi) = \int_{\mathbb{R}} \psi(u) e^{-i\xi u}\,\mathrm{d}u.$$

Complete-theta integration by parts gives

$$\widehat{sz_{2j+1}}(\xi) = -i(-1)^j \xi^{2j+1}(\xi^2 + 1/4)\Xi(\xi).$$

For exact dyadic premixing coefficients A_{jk}, form

$$\widetilde{z}_k = \sum_{j=0}^{10} A_{jk} z_{2j+1}, \qquad P_k(\xi) = \sum_{j=0}^{10} A_{jk}(-1)^j \xi^{2j+1}(\xi^2 + 1/4)$$

before source evaluation. The actual mixed column is

$$\langle E_i, \widetilde{z}_k \rangle = \int w(\xi) T_i(t_I(\xi)) P_k(\xi) \Xi(\xi) \frac{\mathrm{d}\xi}{\pi}.$$

Exact combinations remain genuine nulls; their actual original Gram $A^*\Gamma_{11}A$ is retained. Section S42.3 gives the complete physical quadrature and reciprocal-theta bounds. The supplied complete enclosure differs from I in row-sum norm by less than $1.108 \cdot 10^{-60}$. This is a verified normalization, not a replacement of that Gram by I.

The polar row completes $\mathcal{Z} = [\widetilde{z}_1, \ldots, \widetilde{z}_{11}, b]$. Its null crosses vanish exactly, and the complete Mellin identity gives

$$\|b\|_2^2 = \tfrac{1}{2} - \widehat{q}(0) > 0.$$

Thus Γ is the direct sum of the full eleven-row Gram and this positive scalar. On X, projection off these twelve rows equals $Q_{K_{11}}$. The row b implements a Hilbert constraint; the operator-null transport uses K_{11} alone.

The remaining polar mixed source is

$$\langle E_i, b \rangle = \int w(\xi) T_i(t_I(\xi)) B(\xi) \frac{\mathrm{d}\xi}{\pi}, \qquad B(\xi) = \int_{\mathbb{R}} \psi(u) \tanh(u/2) \sin(\xi u)\,\mathrm{d}u.$$

It has a separately funded odd degree-1536 polynomial on $[-500, 500]$ with real uniform error below 10^{-85}. Its complete construction uses ten physical panels on $[0, 5/2]$, order 192, eight explicit theta atoms and both full omitted tails. The physical integrand bound is $256e^{500\cdot 3/32}$; its Gaussian payment is

$$2 \cdot 10 \cdot 8(1/8)\, \frac{256 e^{500\cdot 3/32}\, 2^{-384}}{1 - 1/2}.$$

The omitted atom and real-tail payments are respectively

$$10 \cdot 16\pi^2\, 9^4 e^{-81\pi}, \qquad \frac{480 e^{9U/2 - \pi e^{2U}}}{2\pi e^{2U} - 9/2}, \quad U = 5/2.$$

Their sum is below 10^{-90} at every frequency node. On the parameter-2 frequency ellipse a complete bound is $240\cdot 189!/3^{189}$. Lobatto aliasing costs at most four times this bound times 2^{-1536}; the sum of all coefficient radii is also paid. The resulting total is below $5.790\cdot 10^{-89}$. Its local degree-256 polynomial remainder is below $3.784\cdot 10^{-121}$. This independent polar source funds the actual row, including its sine normalization.

For the true null and polar mixed integrands the complete contour bound $|\Xi(x+iy)|\le 768e^{-|x|/2}$ and $|B(x+iy)|\le 768$, $|y|\le 2$, supplies source majorants. Here is a direct scalar admission for these constants. On either $\operatorname{Im} u=\pm 1/2$, reflection and $\cos 1>1/2$ give

$$|\psi(u)|\le e^{|\operatorname{Re} u|/2}\sum_{n\ge 1}(4r_n^2+6r_n)e^{-r_n/2},\qquad r_n=\pi n^2e^{2|\operatorname{Re} u|}.$$

Since $e^{5x/2}=(r_n/(\pi n^2))^{5/4}\le r_n^2/(\pi n^2)^{5/4}$ for $x\ge 0$, the two-arm weighted integral is bounded by

$$\pi^{-5/4}\sum_{n\ge 1}n^{-5/2}\int_0^\infty(4r^3+6r^2)e^{-r/2}\,dr<\frac{480}{3}\frac{5}{3}<768.$$

The double-exponential tails allow translation by $-i\operatorname{sign}(x)/2$ in the Fourier integral, giving the stated Ξ bound. On the real physical line, $|\tanh(u/2)|\le 1$ and $|\sin((x+iy)u)|\le e^{2|u|}$; the same larger positive envelope gives the stated polar bound.

On each actual frequency ellipse, $|z|<500$, $|T_i|<5^{15}$ and $|m_{13}|<10000$. The exact coefficient sum $B_k=\sum_\ell|[P_k]_\ell|500^\ell$ bounds each premixed polynomial. For a total frequency length at most 1000, a safe Gaussian payment is

$$16\cdot 1000\,(10000\,5^{16}\,768B_k/\pi)2^{-512}.$$

On the real line, positivity of ψ gives $|\Xi(\xi)|\le\Xi(0)<1$. To justify the strict scalar bound in this normalization, the Mellin identity of Appendix A gives

$$\Xi(0)=\xi(1/2)=\frac{\Gamma(1/4)\eta(1/2)}{8\pi^{1/4}(\sqrt{2}-1)}.$$

The alternating-series upper bound is $\eta(1/2)\le 1-1/\sqrt{2}+1/\sqrt{3}<9/10$. Splitting the gamma integral at one gives $\Gamma(1/4)<4+e^{-1}<35/8$. Finally $\pi^{1/4}>13/10$ and $\sqrt{2}-1>2/5$ yield $(35/8)(9/10)<8(13/10)(2/5)$, proving $\Xi(0)<1$. For at most 1000 root stripes of width at most η, their complete mixed-column payment is therefore at most $100\cdot 1000\,\eta^2B_k/\pi$. The actual stripe endpoints allow the smaller corresponding polynomial coefficient sums. The Gaussian and root-stripe errors are evaluated in each column, with all arithmetic balls retained. Complete row integration costs are below $8.281\cdot 10^{-104}$ for quadrature and $2.522\cdot 10^{-75}$ for stripes. Together with the full original Gram, the separate polar model and the complete ambient covariance, these are the joint inputs to (S172). Exact row premixing does not import locality through the projection or through an inverse.

S42.3 Complete physical source for the original null Gram

The covariance in Section S42.2.3 uses the original norm, including its reciprocal-theta factor. Here we give the bounds underlying its physical quadrature. Set

$$F_j=(D^2-1/4)D^{2j+1}\psi,\qquad G^0_{jk}=2\int_0^\infty\frac{F_j(u)F_k(u)\cosh(u/2)}{\psi(u)}\,\mathrm{d}u,\quad 0\le j,k\le 10.$$

Define the exact polynomial operator $\mathcal{D}Q = \frac{1}{2}Q + 2rQ' - 2rQ$ and

$$Q_j = (\mathcal{D}^2 - 1/4)\mathcal{D}^{2j+1}(4r^2 - 6r), \qquad d_j = \deg Q_j = 2j + 5, \qquad S_j = \sum_\ell |[Q_j]_\ell|.$$

Thus $F_j(z) = e^{z/2}\sum_{n\geq 1} Q_j(\pi n^2 e^{2z})e^{-\pi n^2 e^{2z}}$. All these polynomials have exact rational coefficients.

Consider $z = x + iy$, $x \geq 0$, $|y| \leq .01$, and put $r = \pi e^{2x}$ and $r_c = \pi e^{2z}$. Then $\Re r_c > .99r$ and $|r_c| = r \geq \pi$. The first theta atom satisfies $|4r_c^2 - 6r_c| \geq 2r^2$. Relative to this atom, the sum of the remaining absolute values is at most

$$3\sum_{n\geq 2} n^4 e^{-.99r(n^2-1)} \leq 64e^{-3\cdot.99r} < .1.$$

Indeed the successive-term ratio is at most $(3/2)^4 e^{-5\cdot.99\pi} < 1/4$. Consequently the complete denominator is nonzero and

$$|\psi(z)| \geq \frac{9}{5} r^2 e^{x/2-r}. \tag{S176}$$

Splitting $.99 = .24 + .75$ and maximizing $t^{d_j}e^{-.24t}$ gives

$$C_j = \frac{S_j(d_j/.24)^{d_j}e^{-d_j}}{1 - e^{-3\pi/4}}, \qquad |F_j(z)| \leq C_j e^{x/2-3r/4}.$$

Since $|\cosh(z/2)| \leq e^{x/2}$, it follows that

$$\left|\frac{F_j(z)F_k(z)\cosh(z/2)}{\psi(z)}\right| \leq \frac{5C_jC_k}{9\sqrt{\pi}} r^{-3/2}e^{-r/2} \leq M_{jk} := \frac{5C_jC_k}{9\pi^2}e^{-\pi/2}. \tag{S177}$$

Each F_j is odd and ψ is even, so the Gram integrand is even. This also funds the part of the first quadrature ellipse with negative real part by reflection to $|\Re z|$.

The finite interval $[0, U]$, $U = 3$, is split into 120 equal cells with halfwidth $h = 3/240$. For the Bernstein parameter $\rho = 7/5$, $h(\rho - \rho^{-1})/2 < .01$. Order 512 Gaussian quadrature, including both physical arms, therefore has complete error at most

$$8UM_{jk}\frac{\rho^{-1024}}{1 - \rho^{-1}}.$$

At every real node eight theta atoms are evaluated. The remaining denominator and jet tails are respectively bounded by

$$8e^{u/2}r^2\, 9^4 e^{-81r}, \qquad 2S_j e^{u/2} r^{d_j} 9^{2d_j} e^{-81r}.$$

The successive-term bounds $(10/9)^{2d_j}e^{-19\pi} < 1/2$ fund all omitted atoms. These positive-radius balls are included before the reciprocal of ψ is taken; positivity of the complete real denominator is checked. Finally $r(U) = \pi e^6 > 1200$. Integrating the first bound in (S177), with $\mathrm{d}u = \mathrm{d}r/(2r)$, funds the two-arm physical tail by

$$\frac{10C_jC_k}{9\sqrt{\pi}}\, 1200^{-5/2}e^{-600}.$$

Both errors are added to every raw Gram entry. The exact dyadic map A then gives the complete enclosure A^*G^0A, with all interval radii retained. The row-sum allowance stated in Section S42.2.3 uses this enclosure. The portable physical generator evaluates these formulas anew; it does not substitute an identity matrix for the conditioned Gram.

S42.4 Joint error used by the complete inverse-action application

For the declared head input, let B_{mass} be the sum of the actual zeroth raw-metric entries over all negative and positive source blocks, and let the enclosing integration balls be retained throughout. The complete ambient kernel debit is

$$\delta_{\mathrm{src}} = 2B_{\mathrm{mass}}\left(16384e^{-3\cdot 202/4} + 10^{-60} + 10^{-70}\right) < 4.129\cdot 10^{-58}.$$

The full raw metrics plus their deterministic ridges dominate the true frequency metrics: each block of size d has total entry uncertainty less than half its ridge divided by d. This comparison and the displayed scalar bound are checked by directed arithmetic on the literal input. The original twelve-row Gram and every row/source cross are enclosing balls, not midpoint replacements. They are inserted into (S171) before the residual covariance is formed, so their quadrature, stripe and reciprocal-Gram uncertainties remain in that full expression. Applying the ambient two-sided source error to the entire map $[I; -F]$ gives its debit $\delta_{\mathrm{src}}(D_n + F^* D_p F)$; the squared residual construction in Section S31.4 pays the resulting full mixed error. No locality or identity-Gram hypothesis is applied after row compression. The code `regenerate_inverse_residual.py` reconstructs this complete tuple; the saved-array reader separately checks the retained residual matrices. The proofs and data have distinct roles: scientific literal coefficients specify the finite object, while the preceding arguments supply its complete continuous, kernel and tail error contract.

S43 Arithmetic bounds and complete signed-deficit coverage

This section combines a complete compact arithmetic bound with a direction-sensitive test of its benefit to a signed certificate. Weighted Schur bounds, constrained inverses and quadratic completion are classical tools [27, 8]. We give the relevant derivations in full. All comparisons of receiving blocks below use one fixed original physical metric, constraint space and receiving core. They do not identify a scalar arithmetic saving with a signed matrix margin.

S43.1 A sharp leading arithmetic bound

Put $\ell = \log K > 0$ and, on $L^2([0,\ell])$ with zero extension, let

$$C_K = \sum_{2\le n<K} \frac{\Lambda(n)}{\sqrt{n}}(T_{\log n} + T^*_{\log n}), \qquad (T_t f)(x) = f(x-t).$$

Every active prime power and both shift orientations are retained. This is the complete compact arithmetic block when the arithmetic cutoff is at least K. Define the truncated sums

$$\Theta_K(y) = \sum_{2\le n<K,\ n\le y} \Lambda(n), \qquad A_K(y) = \sum_{2\le n<K,\ n\le y} \frac{\Lambda(n)}{n}, \quad \phi(x) = e^{-x/2} + e^{-(\ell-x)/2}.$$

For $1 < z < K$ put

$$r_K(z) = \frac{\sqrt{K}[\Theta_K(z) + A_K(K/z)] + z[\Theta_K(K/z) + A_K(z)]}{\sqrt{K} + z}, \qquad B_{\mathrm{ES}}(K) = \operatorname*{ess\,sup}_{1<z<K} r_K(z).$$

Proposition S43.1 (Complete arithmetic price)**.** *The two-sided bound $\|C_K\| \le B_{\mathrm{ES}}(K)$ holds. For integer K, the upper is determined by finitely many one-sided values at the rational breakpoints $1, K, n, K/n$. Moreover, using the classical unconditional PNT,*

$$\|C_K\| = (1+o(1))\sqrt{K} \quad (K\to\infty).$$

The same leading asymptotic holds on the centered odd space after any finite-dimensional odd row constraints, with no restriction on their dimension as a function of K.

Proof. Exact summation of the allowed shift overlaps gives

$$(C_K\phi)(x) = e^{-x/2}[\Theta_K(e^x) + A_K(e^{\ell-x})] + e^{-(\ell-x)/2}[\Theta_K(e^{\ell-x}) + A_K(e^x)].$$

Thus $(C_K\phi)/\phi = r_K(e^x)$. For each edge use $2|f(x)f(y)| \le |f(x)|^2\phi(y)/\phi(x) + |f(y)|^2\phi(x)/\phi(y)$. Integration over the full overlaps and summation over both orientations give $|\langle f, C_K f\rangle| \le \int |f(x)|^2 r_K(e^x)\,dx$. The operator is self-adjoint; this proves its norm bound without assuming it positive semidefinite. On every open breakpoint cell the four partial sums are constant, so $r_K(z) = (\sqrt{K}a + zb)/(\sqrt{K} + z)$ has derivative of constant sign. Its supremum is at a one-sided endpoint. Duplicate breakpoints must be merged; a value using the wrong side's sums is not an upper. Interval-enclosed noninteger cutoffs require certified ordering and membership decisions.

For the asymptotic, $\Theta_K(y) = y + o(y)$ on growing arguments, $\Theta_K(y) = O(y)$, and $A_K(K) = O(\log K)$. These follow from the unconditional PNT [13] and partial summation. Excluding $n = K$ changes the endpoint by at most $\log K$. Replacing the two Θ_K terms by their arguments gives exactly $\sqrt{K}$. On $K^{1/4} \le z \le K^{3/4}$ the relative PNT error tends uniformly to zero. For $z \le K^{1/4}$ the small-argument error contributes $O(K^{1/4})$, and the other argument is at least $K^{3/4}$, giving $o(\sqrt{K})$. Reflection $z \mapsto K/z$ handles the other end. The two A_K terms form a convex average bounded by $O(\log K)$. Consequently $r_K/\sqrt{K} \to 1$ uniformly. The Rayleigh quotient at ϕ is its ϕ^2-weighted average, proving the matching lower. The assertion about odd constraints follows from Lemma S43.2 below, applied after centering. □

For completeness, an exact finite lower uses

$$\|\phi\|^2 = 2(1 - K^{-1}) + 2\ell/\sqrt{K},$$

$$\langle \phi, T_{\log n}\phi\rangle = 2(n^{-1/2} - n^{1/2}/K) + \log(K/n)\big(\sqrt{n/K} + 1/\sqrt{Kn}\big).$$

These formulas follow by expanding the four exponential products on $[\log n, \ell]$. The Rayleigh numerator is twice their sum with weights $\Lambda(n)/\sqrt{n}$. No frequency quadrature is involved. Scalar minima of independently valid complete norm uppers are permitted; entrywise minima of Hermitian matrix uppers are not.

Lemma S43.2 (Finite-head invariance of the pure arithmetic norm)**.** *Let C be bounded self-adjoint on the line or a centered interval, commute with reflection, and be an operator-norm limit of finite sums $M_a T_t$ with bounded multipliers, including interval restrictions. For any finite-dimensional row space in the odd Hilbert space, with orthogonal complement projection Q,*

$$\|QCQ|_{\operatorname{ran} Q}\| = \|C\|.$$

Proof. For $U_\omega f = e^{i\omega x} f$, $U_\omega^* M_a T_t U_\omega = e^{-i\omega t} M_a T_t$. Simultaneous recurrence on each finite torus of shift phases and a diagonal choice through the norm approximants give $\omega_j \to \infty$ with $\|U_{\omega_j}^* C U_{\omega_j} - C\| \to 0$ for both signs. To see unbounded recurrence directly, apply compactness to the integer-time orbit of the finite phase vector: arbitrarily close orbit points have differences arbitrarily close to the identity. Either these return times are unbounded, or a bounded subsequence yields an exact return whose multiples are unbounded.

Choose a unit, parity-pure f with $|\langle f, Cf\rangle|$ arbitrarily close to $\|C\|$, possible because the two parity spaces reduce C. Use $v_j = f\cos(\omega_j x)$ if f is odd, and $v_j = f\sin(\omega_j x)$ if f is even. Then v_j is odd, $\|v_j\|^2 \to 1/2$, and $v_j \rightharpoonup 0$. The modulation identity and Riemann–Lebesgue applied to

$\overline{f}\,Cf \in L^1$ give $\langle v_j, Cv_j\rangle \to \langle f, Cf\rangle/2$. Finite rank implies $\|(I-Q)v_j\| \to 0$; boundedness of C preserves the Rayleigh limit after projection. This proves the lower; compression gives the reverse inequality. □

The lemma applies to C_K and, with the complete norm-convergent prime tail proved in the main paper, to the bounded theta-sandwiched prime block. It does not apply to the unbounded full Weil operator or automatically to an inverse-weighted return. The recurrent frequencies may have large archimedean energy. For legal form tests one may first choose smooth compact Rayleigh approximants and use operator-domain rows; this adds no uniform energy or frequency bound.

S43.2 The same-core receiving improvement

Work in the original physical Hilbert norm on a fixed support window. Let Q project off all fixed constraints and the finite receiving core. All restrictions use the original minimal closed form. Core columns lie in the associated operator domain, and their finite projection preserves the form domain; mixed columns are actual operator actions. Suppose the complete finite polar correction has its range in that excluded space, and the actual tail is

$$T = Q[gI + M_s a(D) M_s + M_s V_{\mathrm{ar}} M_s]Q\big|_{\operatorname{ran} Q}, \qquad V_{\mathrm{ar}} = m_0 I - C_K, \quad g > 0.$$

Assume a paid archimedean comparison $QM_s a(D) M_s Q \succeq \gamma Q M_{s^2} Q$ and two valid scalar prices $B_0 \geq B_1 \geq \|V_{\mathrm{ar}}\|$, with $G = B_0 - B_1 \geq 0$. For $\beta_i = \gamma - B_i > 0$ define

$$D_{\beta_i} = Q(gI + \beta_i M_{s^2})Q, \qquad Z_{\beta_i} = D_{\beta_i}^{-1}.$$

Thus $T \succeq D_{\beta_1} \succeq D_{\beta_0}$ and $T^{-1} \preceq Z_{\beta_1} \preceq Z_{\beta_0}$. These inverses act on $\operatorname{ran} Q$, extended by zero elsewhere. If W is an orthonormal map spanning the entire excluded finite space, and $d_\beta = g + \beta s^2$, their explicit formula is

$$Z_\beta = d_\beta^{-1} - d_\beta^{-1} W (W^* d_\beta^{-1} W)^{-1} W^* d_\beta^{-1}.$$

Indeed the right side annihilates W, has range in its orthogonal complement, and multiplication by $Q d_\beta Q$ gives Q there. It is not the generally different compression $Q d_\beta^{-1} Q$.

Put $A_s = QM_{s^2}Q$. The bounded resolvent identity gives

$$Z_{\beta_0} - Z_{\beta_1} = G Z_{\beta_0} A_s^{1/2} (I + G A_s^{1/2} Z_{\beta_0} A_s^{1/2})^{-1} A_s^{1/2} Z_{\beta_0}. \tag{S178}$$

Since $D_{\beta_0} \succeq \beta_0 A_s$, $A_s^{1/2} Z_{\beta_0} A_s^{1/2} \preceq \beta_0^{-1} I$. Hence

$$\frac{G\beta_0}{\beta_0 + G} Z_{\beta_0} A_s Z_{\beta_0} \preceq Z_{\beta_0} - Z_{\beta_1} \preceq G Z_{\beta_0} A_s Z_{\beta_0}, \qquad Z_{\beta_0} - Z_{\beta_1} \preceq \frac{G}{\beta_0 + G} Z_{\beta_0}. \tag{S179}$$

The last assertion is scalar functional calculus for $Gt/(1+Gt)$ on $0 \leq t \leq \beta_0^{-1}$ after conjugating by $Z_{\beta_0}^{1/2}$. These bounds include $G = 0$ and use no cutoff-uniform minimum of s.

For the positive reference $T_0 = Q[M_s a(D) M_s + B_1 M_{s^2} + gI]Q$, the bounded difference satisfies $0 \preceq T_0 - T \preceq 2B_1 A_s$. Therefore

$$c T_0 \preceq T \preceq T_0, \qquad c = \frac{\beta_1}{\beta_1 + 2B_1}.$$

For an already paid core with $\gamma = B_0 + 1$, prices $B_0 = (2 + o(1))\sqrt{K}$ and $B_1 = (1 + o(1))\sqrt{K}$ give $c \to 1/3$. An exact choice for the former price is $|m_0|$ plus twice the sum of $\Lambda(n)/\sqrt{n}$ for $n \leq \sqrt{K}$,

plus its sum for $\sqrt{K} < n < K$. Each killed translation has norm at most one. For $t > \ell/2$, the two end intervals joined by $T_t + T_t^*$ are disjoint; this operator exchanges them isometrically and is zero on the remaining interval, so its norm is one. The unconditional PNT and partial summation give $\sum_{n<x} \Lambda(n)/\sqrt{n} = (2+o(1))\sqrt{x}$, while the $n \le \sqrt{K}$ contribution is $O(K^{1/4})$ by the elementary Chebyshev bound. Hence B_0 has the stated leading term. The latter price follows from Proposition S43.1, with $|m_0|$ added to both and the scalar minimum with B_0 taken if needed at finite K. This restores a uniform relative inverse factor on that same paid core. It does not assert that such a core is small.

For clarity, writing $R_0 = T_0^{-1}$ and $C = R_0^{1/2}(T_0 - T)R_0^{1/2}$ gives $0 \preceq C \preceq (1-c)I$ and $T^{-1} = R_0^{1/2}(I-C)^{-1}R_0^{1/2}$ by coercive form factorization. If $c = 1$, $T = T_0$ and the inverse is exact. For $0 < c < 1$, the Neumann inverse remainder after degree n is bounded by $(1-c)^{n+1}R_0/c$. For $n \ge 1$, Chebyshev residual polynomials on $[c,1]$ give the factor $\delta_n = 1/\mathrm{T}_n((1+c)/(1-c)) \le 2[(1-\sqrt{c})/(1+\sqrt{c})]^n$: for $p_n(\lambda) = (1 - r_n(\lambda))/\lambda$ and $r_n(\lambda) = \mathrm{T}_n((1+c-2\lambda)/(1-c))/\mathrm{T}_n((1+c)/(1-c))$, one has $(1-\delta_n)\lambda^{-1} \le p_n(\lambda) \le (1+\delta_n)\lambda^{-1}$. Full mixed-column congruence retains these inequalities. Approximate reference actions still require their own error enclosures.

S43.3 An optimized complete return and its signed test

Let H_c be the finite signed core and J its complete mixed map into the tail. Every J, E, Y column below takes values in $\operatorname{ran} Q$. Put $X = J^*T^{-1}J$ and $K_{\mathrm{true}} = H_c - X$. For a finite map Y with columns in $\operatorname{Dom} T$, define

$$r = J - TY, \quad L(Y) = J^*Y + Y^*J - Y^*TY, \quad U_Z(Y) = L(Y) + r^*Zr \quad (Z \succeq T^{-1}).$$

Expansion proves the complete two-sided enclosure

$$H_c - L(Y) - r^*Zr \preceq K_{\mathrm{true}} \preceq H_c - L(Y), \qquad X = L(Y) + r^*T^{-1}r. \tag{S180}$$

Thus a positive lower certifies this signed test; a negative direction of the upper certifies its failure. A negative lower alone is inconclusive. All action, interval and omitted-tail errors must be included on their proper sides. No $\operatorname{Dom} T^2$ premise is used.

For independent trial columns $E \subset \operatorname{Dom} T$, set

$$V = J^*ZJ, \quad A_E = (TE)^*Z(TE) - E^*TE, \quad F_E = (TE)^*ZJ - E^*J.$$

Then $U_Z(EC) = V - F_E^*C - C^*F_E + C^*A_EC$. Taking $Z = (1+\varepsilon)Z_\beta$, $\varepsilon > 0$, gives $A_E \succeq \varepsilon E^*TE \succeq \varepsilon g E^*E \succ 0$. Square completion therefore gives the unique optimizer

$$C_{\mathrm{opt}} = A_E^{-1}F_E, \qquad U_{\mathrm{opt}} = V - F_E^*A_E^{-1}F_E \succeq X.$$

Enlarging the trial span cannot increase this matrix upper. For the two prices on the same core, write $Z_i = (1+\varepsilon)Z_{\beta_i}$ and $r_0 = J - TY_0$ at the old optimizer. Evaluation of the new quadratic at Y_0 proves

$$M_{\mathrm{gain}} := U_{\mathrm{opt},0} - U_{\mathrm{opt},1} \succeq r_0^*(Z_0 - Z_1)r_0 \succeq \frac{(1+\varepsilon)G\beta_0}{\beta_0 + G}(sZ_{\beta_0}r_0)^*(sZ_{\beta_0}r_0) =: G_*.$$

This computable gain lower preserves the full residual and every cross term. Its factor saturates; growing scalar savings need not give growing gains in every coefficient direction.

Proposition S43.3 (Exact deficit coverage)**.** *Let $D = U_{\mathrm{opt},0} - H_c$. The improved certificate passes exactly when $M_{\mathrm{gain}} - D \succeq 0$. For any positive lower $G_* \preceq M_{\mathrm{gain}}$, $G_* - D \succeq 0$ is sufficient. Decompose the coefficient space into $\operatorname{ran} G_* \oplus \ker G_*$ and put $A = -D_{NN}$. Then this sufficient test is equivalent to*

$$A \succeq 0, \qquad D_{RN} \ker A = 0, \qquad (G_*)_R \succeq D_{RR} + D_{RN} A^\dagger D_{NR}.$$

When $G_* = 0$ the test is simply $D \preceq 0$; when its kernel is empty only the full positive-definite comparison remains.

Proof. The exact identity is $H_c - U_{\mathrm{opt},1} = M_{\mathrm{gain}} - D$. The displayed block conditions follow by completing the square for $G_* - D$ with bottom block A. Positivity forces the off-diagonal to annihilate $\ker A$; after that compatibility the remaining block is $(G_*)_R - D_{RR} - D_{RN} A^\dagger D_{NR}$. □

A numerical near-kernel cannot be treated as an exact kernel. For a strictly positive G_* the test is $\lambda_{\max}(G_*^{-1/2} D G_*^{-1/2}) \le 1$, with the full matrix-error budget paid. It is not necessary for coverage when the lower gain is strictly smaller than M_{gain}. For example $G_* = \operatorname{diag}(1, 0)$ misses $D = \operatorname{diag}(0, 1/10)$ despite a larger gain trace. For $D = \left[\begin{smallmatrix} 0 & 1/10 \\ 1/10 & 0 \end{smallmatrix}\right]$ the kernel diagonal is zero but the mixed compatibility fails. Both are exact finite interface examples, not Weil counterexamples.

Finally, repricing must conserve the actual positive surplus. If an arithmetic pair has old prices a, d and new price $\varphi = a + d - G$, then

$$\varphi I - (C_n + C_m) = (aI - C_n) + (dI - C_m) - GI.$$

Increasing the baseline by G while retaining both old surplus features counts that gain twice. For instance the true tail $T = 1$ may be written as $1/4 + 3/4$ or $3/4 + 1/4$; combining the new floor with the old surplus falsely gives $3/2$. At $H_c = 4/5, J = 1$ this changes the true Schur value $-1/5$ to the false positive $2/15$. Rebuild the actual blocks after any core change. For a shifted relative-covariance test, shift the core, mixed map and tail together before applying the identities above. These are complete acceptance and failure tests; their success at growing cutoffs still requires the actual joint matrices and all original-norm return payments.

S44 Actual-prime discrepancy and physical packet energy

This section gives an analytic arithmetic input for the finite-verification framework. Mellin smoothing controls the actual prime-power discrepancy; uniform packet conditioning expresses that control in the original unweighted logarithmic energy. The mother function and separation are fixed throughout, while the number and highest frequency of the packets may grow. All Fourier integrals use the main paper's measure $d\xi/(2\pi)$.

S44.1 A fixed-window Mellin estimate

Assume every nontrivial zero of ζ has real part at most b, where $1/2 \le b < 1$. Fix $0 < \eta < 1 - b$ and put $d = b - 1/2 + \eta$. For $W \in C_c^\infty(\mathbb{R})$ write $\widetilde{W}(z) = \int W(u) e^{zu}\, du$.

Lemma S44.1 (Smoothed actual-prime discrepancy)**.** *For each fixed W there is a constant $C_{W,b,\eta}$ such that, for every $T \ge 0$ and $\omega \in \mathbb{R}$,*

$$\left| \sum_{n \ge 2} \Lambda(n) n^{-1/2 - i\omega} W(\log n - T) - e^{(1/2 - i\omega)T} \widetilde{W}(1/2 - i\omega) \right| \le C_{W,b,\eta} e^{dT} \log(2 + |\omega|).$$

The constant is bounded by a b,η-dependent multiple of

$$\mathcal{A}_d(W) = \int_{\mathbb{R}} |\widetilde{W}(d+iv)|[1+\log(2+|v|)]\,dv.$$

The main term is the full integral over $x > 0$. It also equals the integral over $x \geq 1$ when $T + \operatorname{supp} W \subset [0,\infty)$.

Proof. We first pay the logarithmic derivative on the line $\sigma_0 = b + \eta$. Euler summation gives, for $\Re s > 0$, $s \neq 1$,

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{x\}x^{-s-1}\,dx.$$

The remainder converges absolutely and gives polynomial growth $|\zeta(s)| \leq C(1+|\Im s|)$ on a fixed strip away from its pole. For $|t| \geq 4$, the disk centered at $2+it$ with radius $R = 2-b-\eta/4$ is zero-free and avoids the pole. Its analytic logarithm $h = \log\zeta$ has uniformly bounded center value, since the Euler logarithm there is absolutely bounded by $\sum_{n\geq 2}(n^2-1)^{-1} = 3/4$. The growth estimate bounds $\Re h$ above by $C + \log(2+|t|)$ on the disk.

Here is the needed real-part estimate. If f vanishes at the disk center and $\Re f \leq B$, $B > 0$, then $f/(2B-f)$ maps the disk to the closed unit disk. Schwarz's lemma therefore gives $|f| \leq 2Br/(R-r)$ on its radius-r subdisk. Apply this to $h - h(2+it)$ with $r = 2-b-\eta/2$, and then apply Cauchy's derivative estimate on a radius-$\eta/4$ circle about $\sigma_0 + it$. It follows that

$$|\zeta'/\zeta(\sigma_0+it)| \leq C_{b,\eta}\log(2+|t|).$$

The same argument is uniform between real parts σ_0 and $3/2$ at large heights. At bounded height the fixed line is zero-free and avoids 1, so continuity supplies its bound there.

Bilateral Laplace inversion on $\Re z = 1$ and absolute convergence of the Dirichlet series give the exact expression

$$\frac{1}{2\pi i}\int_{\Re z=1} -\frac{\zeta'}{\zeta}(1/2+i\omega+z)e^{zT}\widetilde{W}(z)\,dz.$$

Shift to $\Re z = d$. The sole crossed pole is at $z = 1/2 - i\omega$, with residue $e^{(1/2-i\omega)T}\widetilde{W}(1/2-i\omega)$. At each fixed T,ω, horizontal segments at heights larger than $|\omega|+4$ vanish: repeated integration by parts gives rapid vertical decay of $\widetilde{W}$, while the logarithmic derivative has the bound just proved. The remaining integral is

$$\frac{1}{2\pi}\int_{\mathbb{R}} -\frac{\zeta'}{\zeta}(b+\eta+i(\omega+v))e^{(d+iv)T}\widetilde{W}(d+iv)\,dv.$$

Using $\log(2+|\omega+v|) \leq \log(2+|\omega|)+\log(2+|v|)$ pays its entire vertical tail and proves the estimate. Finally, substitution $x = e^{T+u}$ identifies the exact main integral and its stated support case. □

S44.2 Complete mixed packet matrices

Fix a nonzero real even $\phi \in C_c^\infty(-r,r)$. For $\omega \geq 1$ put

$$g_\omega = -\omega^{-2}(\partial_u^2 - 1/4)[\phi(u)\cos(\omega u)], \qquad \ell_\omega = \log(2+\omega).$$

These functions are real, even and have both exponential moments zero. For a finite frequency list $\omega_i \geq 1$ with separation at least $h > 0$, set

$$H_{ij}(s) = \int_{\mathbb{R}} g_{\omega_i}(x+s)\overline{g_{\omega_j}(x)}\,dx, \qquad L_\ell = \operatorname{diag}(\ell_{\omega_i}).$$

The complete actual-prime discrepancy matrix is

$$\mathfrak{E}_{ij}(T) = \sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}} H_{ij}(\log n - T) - \int_0^\infty x^{-1/2} H_{ij}(\log x - T)\, dx.$$

Proposition S44.2 (Uniform mixed-matrix control)**.** *Under the strip hypothesis above,*

$$\|L_\ell^{-1/2}\mathfrak{E}(T)L_\ell^{-1/2}\| \leq C_{\phi,h,b,\eta} e^{dT}, \qquad T \geq 0.$$

The constant is independent of list length and highest frequency.

Proof. Expand exactly

$$g_\omega = A_\omega \cos(\omega u) + B_\omega \sin(\omega u), \quad A_\omega = \phi - (\phi'' - \phi/4)/\omega^2, \quad B_\omega = 2\phi'/\omega.$$

Thus $g_\omega = \sum_{\sigma=\pm 1} a_{\omega,\sigma} e^{i\sigma\omega u}$, where $a_{\omega,\sigma} = (A_\omega - i\sigma B_\omega)/2$ has fixed support and uniform smooth seminorms. Each H_{ij} is the sum of four terms $e^{i\sigma\omega_i s} W_{ij,\sigma,\tau}(s)$, with

$$W_{ij,\sigma,\tau}(s) = \int a_{\omega_i,\sigma}(x+s)\overline{a_{\omega_j,\tau}(x)} e^{i(\sigma\omega_i - \tau\omega_j)x}\, dx.$$

Every envelope has support in $[-2r, 2r]$. Differentiate in s and integrate by parts in x. For all fixed $m, k \geq 0$,

$$\|\partial_s^m W_{ij,\sigma,\tau}\|_\infty \leq C_{\phi,m,k}(1 + |\sigma\omega_i - \tau\omega_j|)^{-k}.$$

The internal phase is independent of s, and compact smooth support removes all boundary terms. The direct bound handles phase magnitude at most one.

The seminorm $\mathcal{A}_d(W)$ in Lemma S44.1 is bounded by a constant times the L^1 norms of $e^{ds}W$ and its second derivative, using the direct and twice-integrated transform bounds. Hence the same off-resonance decay pays the changing-window constants. Apply the lemma to all four carriers, retaining their unit-modulus centering phases. This gives

$$|\mathfrak{E}_{ij}(T)| \leq C_k e^{dT} \ell_{\omega_i} \big[(1 + |\omega_i - \omega_j|)^{-k} + (1 + \omega_i + \omega_j)^{-k}\big].$$

Write $\Delta = |\omega_i - \omega_j|$. Since $\ell_{\omega_i} \leq \ell_{\omega_j} + \log(1+\Delta)$ and $\ell_{\omega_j} \geq \log 3$, taking $k = 3$ bounds each normalized entry by $Ce^{dT}(1+\Delta)^{-2}$. For an h-separated list both its row sums and column sums are bounded by a multiple of $\sum_{m\geq 0}(1+mh)^{-2}$. The two-sided Schur test proves the induced bound for the entire matrix. The real-even generators also imply $H_{ij} = H_{ji}$ by reflection, though the norm proof does not require it. □

S44.3 The physical logarithmic-energy metric

Put

$$\mathcal{E}(g) = \|g\|_2^2 + \int_{\mathbb{R}} a(\xi)|\widehat{g}(\xi)|^2 \frac{d\xi}{2\pi}.$$

This is the unweighted profile energy. In particular its norm is distinguished from the original theta-weighted norm of h when $f = sh$.

Proposition S44.3 (Uniform physical-energy conditioning)**.** *There are fixed $\Omega_0 \geq 1$, $h_0 > 0$ and $0 < c_\phi \leq C_\phi < \infty$ such that, for every finite list $\omega_i \geq \Omega_0$ separated by at least h_0, and every complex coefficient vector c,*

$$c_\phi \sum_i \ell_{\omega_i}|c_i|^2 \leq \mathcal{E}\left(\sum_i c_i g_{\omega_i}\right) \leq C_\phi \sum_i \ell_{\omega_i}|c_i|^2.$$

These constants are independent of list length and highest frequency.

Proof. The main paper's equation (S2) gives $\frac{1}{2}\log(1+4\xi^2) \le a(\xi) \le \frac{1}{2}\log(1+4\xi^2)+4$. Thus $a(\xi) = \log|\xi| + O(1)$ at infinity, with coefficient one. The carrier amplitudes above converge in every Schwartz Fourier seminorm to $\phi/2$ as $\omega \to \infty$.

Let G be the complete energy Gram. A carrier pairing is

$$\int_{\mathbb{R}} [1+a(\xi)]\widehat{a}_i(\xi-\sigma\omega_i)\overline{\widehat{a}_j(\xi-\tau\omega_j)}\frac{d\xi}{2\pi}.$$

The symbol satisfies $1+a(\xi) \le C\ell_{\omega_i}[1+\log(2+|\xi-\sigma\omega_i|)]$. Split this integral where one of the two distances from its carrier is at least half their separation. Extra Schwartz powers on that factor and integration of the remaining weighted decay give, for every k,

$$\frac{|G_{ij}|}{\sqrt{\ell_{\omega_i}\ell_{\omega_j}}} \le C_k\big[(1+|\omega_i-\omega_j|)^{-k} + (1+\omega_i+\omega_j)^{-k}\big].$$

The logarithmic ratio used in the preceding proposition absorbs the normalization with one extra Schwartz power.

On the diagonal, substitute $\xi = \sigma\omega + v$. Dominated convergence with weight $1+\log(2+|v|)$ gives $\|\phi\|_2^2/4$ from each of the two equal-carrier terms after division by ℓ_ω. Opposite-carrier terms tend to zero by the separation estimate. Therefore $G_{ii}/\ell_{\omega_i} \to m_\phi = \|\phi\|_2^2/2 > 0$. Choose Ω_0 so every normalized diagonal G_{ii}/ℓ_{ω_i} lies in $[m_\phi/2, 2m_\phi]$. Then choose the fixed h_0 so the off-diagonal row and column sums, bounded by a constant times $\sum_{k\ge1}(1+kh_0)^{-2}$, are at most $m_\phi/4$. Schur's bound gives the asserted two-sided inequality, with $c_\phi = m_\phi/4$ and $C_\phi = 9m_\phi/4$. □

Corollary S44.4 (Actual-prime control in physical energy)**.** *Under the strip hypothesis, for the fixed mother, minimum frequency and separation in Proposition S44.3, let* $g_c = \sum_i c_i g_{\omega_i}$ *and* $g_z = \sum_i z_i g_{\omega_i}$*. Then*

$$|c^*\mathfrak{E}(T)z| \le Ce^{dT}\mathcal{E}(g_c)^{1/2}\mathcal{E}(g_z)^{1/2}.$$

There is a fixed T_0*, depending only on the mother support, such that for every* $T \ge T_0$ *this bound can also be paid by the energies of the odd polar-free profiles* $f_{c,T} = \tau_{T/2}g_c - \tau_{-T/2}g_c$ *and* $f_{z,T}$*.*

Proof. Combine the two preceding propositions. For the odd-arm assertion take $T > 2r$. Both moments of g_c vanish by integration by parts, so its translated antisymmetrization is odd and exactly polar-free. Its disjoint arms give $\|f_{c,T}\|^2 = 2\|g_c\|^2$. The off-diagonal archimedean kernel is $-J$, where $J(t) = e^{-|t|/2}/(1-e^{-2|t|})$. If $T-2r \ge 1$, its cross pairing is at most $\epsilon_T\|g_c\|^2$, with $\epsilon_T = (8r/3)e^{-(T-2r)/2}$, by Schur's test and $J(t) \le (4/3)e^{-|t|/2}$. Choose one T_0 so $\epsilon_T \le 1/2$ for $T \ge T_0$. Then

$$\mathcal{E}(g_c) \le \mathcal{E}(f_{c,T}) \le 3\mathcal{E}(g_c).$$

On the actual odd-half supported realization, $D_{\text{full}} = D_L + R_L$ and $0 \preceq R_L \preceq \pi I/2$; the corresponding norm-plus-DN energy is therefore equivalent with factor at most $1+\pi/2$. Finally the self interaction uses only $|t| \le 2r$, a fixed finite prime-power shift sum and a bounded continuous main kernel, so its discrepancy costs at most $C_r\|g_c\|^2$. This pays the self fee separately from the complete long cross matrix. □

The supporting estimate retains the explicit coefficient e^{dT}. It controls all mixed entries on the specified, uniformly conditioned packet families. Its constants fix the mother and frequency separation, providing an arithmetic input in the same physical energy coordinates used for these odd test profiles.

S45 Verification of the complete supported example

This section specifies the computer-assisted inputs and the complete verification for the supported example in main-paper Section S13. The original realization and closed domain are those proved in the main paper. All functions below are in the odd-half unitary profile space. The primary receiving trial is zero.

S45.1 Exact model and positive step reference

Fix

$$M = 2^{20}, \quad \kappa = M^{-1}, \quad L = \tfrac{1}{2}\log(4\log M/\pi) + \tfrac{1}{10}, \quad J = 74, \quad N = 329.$$

The complete active prime powers are

$$2, 4, 8, 16, 3, 9, 5, 7, 11, 13, 17, 19,$$

each with weight $\Lambda(n)/\sqrt{n}$. Here $21 < e^{2L} < 22 < M$; larger translations have zero supported overlap. Both orientations of every remaining translation are retained. Let $r = \pi e^{2x}$, $r_L = \pi e^{2L}$, and

$$P = 6459 = \lceil 100(r_L - \pi) \rceil, \qquad x_i = \tfrac{1}{2}\log\!\left(1 + \frac{(r_L - \pi)i}{\pi P}\right), \quad 0 \le i \le P.$$

The scientific data specify exact positive dyadic heights on these radial cells. The full-theta brackets proved in the main paper give, with

$$W_1(x) = \frac{(1 + e^{-x})e^{\pi e^{2x}}}{4\pi e^{2x}(2\pi e^{2x} - 3)},$$

the bounds $(128/129)W_1 \le W \le W_1$ and $|(\log W)'| < 21\pi e^{2x}$. Thus, on every subcell $[a, b]$, with $v = 21\pi e^{2b}(b - a)$, the directed checks

$$\tfrac{1}{2}W_1(a)e^{v} < w_j < \tfrac{128}{129}W_1(a)e^{-v}$$

prove $W/2 \le W_{\mathrm{low}} \le W$ on the whole cell. They are checked for every cell, including the endpoints.

For completeness, the arithmetic row price is a whole-space form bound. On the full interval of length $\log K = 2L$, use the positive reflection-symmetric Schur weight $\phi(u) = e^{-u/2} + e^{-(2L-u)/2}$. For $z = e^{L+x}$, define the finite sums, with one-sided endpoint values,

$$\Psi(y) = \sum_{\substack{n<K \\ n\le y}} \Lambda(n), \qquad H(y) = \sum_{\substack{n<K \\ n\le y}} \frac{\Lambda(n)}{n}, \qquad K = e^{2L}.$$

Direct summation of $C^{\mathrm{ar}}\phi/\phi$ gives

$$\mathfrak{r}(x) = \frac{\sqrt{K}\{\Psi(z) + H(K/z)\} + z\{\Psi(K/z) + H(z)\}}{\sqrt{K} + z}.$$

Applying $2|fg| \le |f|^2\phi(y)/\phi(x) + |g|^2\phi(x)/\phi(y)$ to each translated edge proves $|\langle f, C^{\mathrm{ar}} f\rangle| \le \int \mathfrak{r}|f|^2$. Restriction to the odd subspace preserves this bound. The common partition adds every positive breakpoint $\log n - L$ or $L - \log n$ to the radial nodes and has 6471 cells. On each open cell the four partial sums are constant, so $\mathfrak{r} = (\sqrt{K}A + zB)/(\sqrt{K} + z)$ has its extrema at the two one-sided endpoints. The dyadic upper $r_{\mathrm{up},j}$ is checked against both.

Let $p = \sinh(x/2)$, $p_q = \langle p, e_q\rangle$, and $\mathcal{V} = \operatorname{span}(e_0, \dots, e_{328}) \cap p^{\perp}$. For Q the complete orthogonal complement of $[p, \mathcal{V}]$, the pole-tail argument gives

$$D_L|_Q \succeq \ell_N I, \qquad \ell_N = a(k_N) - \delta_N\{a(k_N) - a(k_0)\}, \qquad \delta_N \le \frac{9L^3}{\pi^4(N+1/2)^3}.$$

Indeed the low-band part of a receiving vector is parallel to the low-band pole projection; its squared fraction is at most the omitted pole fraction. Splitting the DN diagonal below and above N gives the displayed inequality. The exact pole coefficients are

$$p_q = \frac{\sqrt{2/L}(-1)^q \cosh(L/2)}{2(k_q^2 + 1/4)}.$$

The data fix a dyadic $\alpha \le \ell_N + m_0$, approximately 4.675643352302501, and on each common cell set

$$d_j = \kappa w_j + \alpha - r_{\mathrm{up},j} > 1/200.$$

The complete positive response, the diagonal suffix and the arithmetic row bound therefore prove

$$T = (D_L + R_{L,74} + m_0 - C^{\mathrm{ar}} + \kappa W_{\mathrm{low}})|_Q \succeq d_Q \succeq I/200.$$

This bound covers all receiving directions. Finite inputs and the bounded fixed-window perturbations use the same operator and minimal closed form domains as in the main realization proof.

S45.2 All constraints and the exact inverse direction

Write C for the raw core map with columns $u_j = e_j - (p_j/p_0)e_0$, and let $F = [p, C]$. Its Gram is $G_C = C^*C = I + vv^*$, $v_j = p_j/p_0$. For $G = F^*d^{-1}F$, the full constrained multiplication inverse is

$$Z = d^{-1} - d^{-1}FG^{-1}F^*d^{-1}.$$

Solving $dy = f$ modulo $\operatorname{ran} F$, with $F^*y = 0$, proves this formula and $T^{-1} \preceq Z$ on $\operatorname{ran} Q$. No orthonormality of the raw list is assumed. The archived upper-triangular matrix R_G has exact entries in $2^{-256}\mathbb{Z}$ and positive diagonal. Complete directed integration and the absolute-row-sum bound give

$$\|R_G^* G R_G - I\| < 4 \times 10^{-75} = \delta, \qquad G^{-1} \succeq R_G R_G^*/(1+\delta).$$

The complete 329-direction congruence is checked; all directions are retained. For example, the needed elementary moments on a constant-weight cell follow from

$$e_q e_r = \frac{\cos((q-r)\pi x/L) - \cos((q+r+1)\pi x/L)}{L}, \qquad \int p^2 = \frac{\sinh x - x}{2},$$

and

$$\int p e_q = \frac{\sqrt{2/L}\{\frac{1}{2}\sin(k_q x)\cosh(x/2) - k_q \cos(k_q x)\sinh(x/2)\}}{k_q^2 + 1/4}.$$

The zero-frequency cosine integral uses its exact length limit.

S45.3 Complete output moments and symbolic ridge cancellation

Let $E = \widehat{S}_\star - d$. The identical ridge cancels symbolically; the logarithmic DN diagonal remains and acts exactly on finite DN inputs. On each fully split cell the primitive action on e_q is

$$Ee_q = \sqrt{2/L}\{(A_q + \rho)\sin(k_q x) + B_q \cos(k_q x)\} + \sum_{j<74} r_j(x) V_{jq}, \qquad \rho = r_{\mathrm{up},j},$$

where, for each active $t_n = \log n$,

$$\begin{aligned} A_q &= a(k_q) + m_0 - \alpha - \sum_{n<K} \frac{\Lambda(n)}{\sqrt{n}} (\chi_+ + \chi_-)\cos(k_q t_n), \\ B_q &= -\sum_{n<K} \frac{\Lambda(n)}{\sqrt{n}} (\chi_+ - \chi_-)\sin(k_q t_n), \\ V_{jq} &= \frac{\sqrt{2/L}(-1)^q c_j \sqrt{1 + e^{-2c_j L}}}{c_j^2 + k_q^2}. \end{aligned}$$

Here $\chi_+ = 1_{x+t_n<L}$, $\chi_- = 1_{x-t_n>-L}$. The odd extension supplies the reflected negative argument exactly. The full spatial output of every column is retained.

Products of these actions are integrated by elementary endpoint moments. In addition to trigonometric product-to-sum formulas, put $n_j = \sqrt{1 + e^{-2c_j L}}$. The required primitives are

$$\begin{aligned} \int e_q r_j &= \frac{2\sqrt{2/L} e^{-c_j L}}{n_j (c_j^2 + k_q^2)} \{c_j \cosh(c_j x)\sin(k_q x) - k_q \sinh(c_j x)\cos(k_q x)\}, \\ \int \sqrt{2/L}\cos(k_q x) r_j &= \frac{2\sqrt{2/L} e^{-c_j L}}{n_j (c_j^2 + k_q^2)} \{c_j \cosh(c_j x)\cos(k_q x) + k_q \sinh(c_j x)\sin(k_q x)\}, \\ r_i r_j &= \frac{2e^{-(c_i + c_j)L}}{n_i n_j} \{\cosh((c_i + c_j)x) - \cosh((c_i - c_j)x)\}. \end{aligned}$$

The difference-frequency zero is integrated as x. Stable normalized exponentials evaluate the same formulas. Each region's action coefficients are applied before summing its moments; endpoint jumps include both outer endpoints.

For the 393 primitives $\mathbf{e} = (e_0, \ldots, e_{392})$, the data contain

$$B = \mathbf{e}^* E\mathbf{e}, \quad K = (E\mathbf{e})^* d^{-1} E\mathbf{e}, \quad H_s = \mathbf{e}^* d^{-1} E\mathbf{e}, \quad P_E = p^* d^{-1} E\mathbf{e},$$

and the sine/response, pre-response-action/response and full response/response blocks VS, XR, RR. For example, with E_0 the displayed trigonometric part,

$$K = E_0^* d^{-1} E_0 + XRV + V^* XR^* + V^* RRV.$$

Thus the response Gram is the full spatial Gram, including every cross term. The relation $p = (n_0 e^{L/2}/2) r_0$ supplies the full pole/action row from the same weighted moments.

S45.4 Finite sign and optional receiving enrichment

The zero-trial main-paper matrix uses

$$H = \begin{bmatrix} 0 \\ G_C \end{bmatrix} + \begin{bmatrix} P_E \\ C^* H_s \end{bmatrix} C, \qquad A_\star = \frac{H^* R_G R_G^* H}{1 + \delta} - C^*(B + K)C,$$

where C in these products is its 393-row primitive coefficient map. All 329 weighted constraint rows are applied before finite core conditioning. Since R_G is upper triangular, its core block R_{uu} is invertible, and

$$G_C^{-1} = I - \frac{vv^*}{1+v^*v}, \qquad P_C = G_C^{-1} R_{uu}^{-*}$$

is an invertible congruence. Every received directed pivot of $P_C^* A_\star P_C$ is positive.

The optional trial columns are

$$t_j = e_j - \frac{p_j}{\|p_N\|^2} p_N, \quad 329 \le j \le 392, \qquad p_N = \sum_{q<N} p_q e_q.$$

They are exactly pole-free and core-orthogonal, with Gram $I + v_{\mathrm{tail}} v_{\mathrm{tail}}^* / \|p_N\|^2$. Together with the core they span the full constrained 393-mode prefix. For any exact dyadic coefficient matrix Z_t, the trial is legal. The original complete coefficient lower A_{all} is tested by

$$[I; -Z_t]^* A_{\mathrm{all}} [I; -Z_t] = A_{uu} - A_{ut} Z_t - Z_t^* A_{tu} + Z_t^* A_{tt} Z_t.$$

A proposed inverse only selects the dyadic trial; the original lower is evaluated afterward. The archived trial has denominator 2^{384}.

Table 3 summarizes the finite sign checks for this case. The complete primitive assembly uses

Table 3: Directed sign checks for the M20 supported example.

Candidate	Core dimension	Positive directed pivots
Zero receiving trial	328	328
64 legal receiving trials	328	328

2048-bit directed arithmetic. The 4096-bit replay reconstructs both candidates from all 153664 entries of A_{all}, reconstructs the analytical core conditioner, and checks all 107584 conditioner entries. Its four-block trial assembly and right-associated congruence use a different multiplication order from the producer. Completed-column LDL supplies all 656 positive pivots. These are sign certificates in their actual coordinates.

S45.5 Reproducibility files and scope of checks

The companion scientific archive `worked_example_data.zip` contains four numerical inputs and `INPUTS.json`, whose file hashes identify every exported numerical field. Decimal interval strings and exact dyadic integers are retained losslessly. The principal package contains the portable code under `code/worked_example/`. After extracting the companion archive into `data/worked_example/`, run the following from the principal package root:

```
python code/worked_example/check_reference.py -data data/worked_example
python code/worked_example/check_weighted_gram.py -data data/worked_example
python code/worked_example/compute_signed.py -data data/worked_example -output
signed_rebuilt.json
python code/worked_example/check_signed.py -data data/worked_example
```

The first checks all reference cells, theta floors, arithmetic endpoints and the complete receiving admission. The second regenerates the whole weighted Gram and tests its exact dyadic conditioner. The third reconstructs the complete signed lower from the archived mixed moments; the fourth

independently receives the saved lower and replays both finite signs. `generate_mixed.py` provides full elementary moment regeneration; its input and output options are given in the reader instructions. The code uses Python with python-flint/Arb. The mathematical derivations, whole-domain implications and actual precision checks have the scopes stated above. The code and replays use the same arithmetic library. The zero-trial case is the primary evaluated supported certificate.

S46 Verification of the second moving-window certificate

This section supplies the M24 case of main-paper Section S13.1. All definitions use the same original norm, exact polar constraint and minimal closed supported domain. The receiving trial is zero. The formulas in Section S45 remain valid with the new case-specific inputs listed here.

S46.1 Exact new case and complete reference

Set $M = 2^{24}$, $\kappa = 2^{-24}$, $L = \frac{1}{2}\log(4\log M/\pi) + \frac{1}{10}$, $J = 88$, and $N = 660$. Here $25 < e^{2L} < 26$; the complete active prime powers are

$$2, 4, 8, 16, 3, 9, 5, 25, 7, 11, 13, 17, 19, 23.$$

The same exact dyadic floor construction uses 7,814 radial cells and the common partition has 7,828 cells. The additional endpoints are every interior arithmetic split $\log n - L$ and $L - \log n$. The public reference checker recomputes these exact endpoints, verifies the full prime-power list, checks both one-sided arithmetic row prices and checks $W/2 \le W_{\mathrm{low}} \le W$ on every cell using the complete theta brackets. The new diagonal admission satisfies

$$\alpha \le a(k_N) - \frac{9L^3}{\pi^4(N+1/2)^3}\{a(k_N) - a(k_0)\} + m_0.$$

Every cell has the exactly represented positive reference $d_j = \kappa w_j + \alpha - r_{\mathrm{up},j} > 1/200$. This retains the infinite receiver before any finite sign test.

S46.2 All constraints, mixed actions and numerical sign

With all 660 columns $F = [p, C]$, the full weighted Gram $G = F^* d^{-1} F$ has 435,600 entries. The exactly represented upper dyadic conditioner has denominator 2^{256} and positive diagonal. Its acceptance uses the complete directed congruence, not the midpoint factorization used to propose it:

$$\|R^* G R - I\| < 4.747 \times 10^{-74} < 10^{-65} < 10^{-60}.$$

The final signed inverse bound conservatively uses $\delta = 10^{-60}$. An inconclusive direct interval factorization at an earlier proposal pivot is not used as a negative-Gram inference or as a reason to discard a constraint direction.

The complete continuous mixed integrals use the elementary primitives of Section S45, retaining all five scalar weights, the pole/action row, and all sine, cosine and Green cross terms. For this larger list, integer and half-integer phases at each endpoint are evaluated directly in interval sine and cosine. This replaces a phase recurrence that can inflate enclosure radii; it does not replace the integrals by a finite output projection or numerical quadrature.

The scientific data contain B_E, K_E, H_s, P_E and the full response blocks for 660 primitive inputs. The original ridge cancels symbolically in $E = \widehat{S}_{24} - d$. Assemble the lower $A_{24} = H^* R R^* H/(1+\delta) - B_E - K_E$ as in the main proof. All 659 pivots of $P_C^* A_{24} P_C$ are strictly positive in directed

arithmetic. No additional 64 receiving trials are present in this case. The raw core Gram, all constraint rows and the analytic invertible conditioner are retained.

S46.3 Portable scientific inputs and checks

The M24 companion archive contains four scientific JSON inputs and `INPUTS.json`. Decimal interval strings and exact dyadic integers are retained losslessly. Extract this archive into `data/worked_example_m24/` in the principal source package. The separate `code/worked_example_m24/` programs use only this chosen data folder and a reader-chosen output path:

```
python code/worked_example_m24/check_reference.py -data data/worked_example_m24
python code/worked_example_m24/check_weighted_gram.py -data
data/worked_example_m24
python code/worked_example_m24/compute_signed.py -data data/worked_example_m24
-output signed_m24_rebuilt.json
python code/worked_example_m24/check_signed.py -data data/worked_example_m24
```

The first program checks the whole reference and theta floor. The second regenerates the complete weighted Gram from the reference cells and receives the exact dyadic conditioner. The third reconstructs the signed lower from the supplied full mixed moments. The fourth independently receives the saved lower, reconstructs the analytic core conditioner, and replays completed-column LDL with a different multiplication grouping. `generate_mixed.py` regenerates elementary moments for a requested finite input list; its `-inputs`, `-bits` and output options are documented in the package README. A small-list cross-check, full signed replay and complete source regeneration have different scopes. The general theorem remains analytic; these interval checks certify the stated finite instance within their declared source and error contracts.

S47 Arithmetic certificate transfers in original profile coordinates

These results strengthen the paper's arithmetic verification framework through complete-core representation, certified finite-matrix error bounds, and quantitative transfer of supported lower bounds. They establish a complete-core pullback, a polar-completion theorem under the shared zero-free-strip hypothesis, and a sharper local support-extension theorem. Together, they expand the supporting tools available for the paper's arithmetic matrix criteria while preserving the original operators, norms, constraints and complete mixed terms.

Each subsection states its own support and arithmetic assumptions. The complete-core construction concerns the prescribed DN matrix of Theorem B.6; the polar completion assumes a supplied polar-free raw floor; the local extension starts from a positive whole arithmetic-only bound. These give three distinct interfaces in the same unweighted profile realization.

S47.1 Complete-core representation and certified finite pullbacks

The packet estimates above concern selected separated families. We now construct a complete representation of the ordinary DN core used in Theorem B.6. The representation includes low frequencies, both local parity channels and one polar direction. It supplies a second way of evaluating the same finite matrix, with a paid induced error. Throughout this subsection $L \geq 4$; this restriction does not apply to the other transfer results below.

For a full-line profile set

$$\mathcal{E}(v) = \|v\|_2^2 + \int_{\mathbb{R}} a(\xi)|\widehat{v}(\xi)|^2 \frac{d\xi}{2\pi}, \qquad \|v\|_{\mathcal{E}} = \mathcal{E}(v)^{1/2}. \tag{S181}$$

The symbol is the one in the main paper. Its exact paired representation is

$$\mathcal{E}(v) = \|v\|_2^2 + \frac{1}{2}\iint_{\mathbb{R}^2} J(x-y)|v(x)-v(y)|^2\,dx\,dy, \qquad J(t) = \frac{e^{-|t|/2}}{1-e^{-2|t|}}.$$

Indeed $\int_{\mathbb{R}}(1-\cos(\xi t))J(t)\,dt = a(\xi)$ by the positive digamma series and Tonelli's theorem. Also $1 + a(\xi) \asymp \log(2+|\xi|)$ by the symbol bounds in the main paper. These are unweighted profile coordinates, not the original h norm.

Lemma S47.1 (Compact polar localization and complete frame)**.** *There are fixed smooth compact mothers and a fixed lattice $\omega_k = \pi k/5$, $k \in \mathbb{Z}$, with packets*

$$g_k(u) = -\frac{(\partial_u^2 - 1/4)[\theta(u)e^{i\omega_k u}]}{1+\omega_k^2}, \qquad \ell_k = \log(2+|\omega_k|),$$

having the following properties. Every smooth profile v supported in $(-L, L)$ with both exponential moments zero has an exact expansion

$$v = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} d_{jk}\, g_k(\cdot - j), \qquad \sum_{j,k} \ell_k |d_{jk}|^2 \leq C\mathcal{E}(v).$$

Only $|j| < L+2$ occur. Synthesis is bounded from the entire weighted coefficient space to the energy space:

$$\mathcal{E}\left(\sum_{j,k} d_{jk} g_k(\cdot - j)\right) \leq C\sum_{j,k}\ell_k|d_{jk}|^2.$$

Analysis, synthesis and their identity extend to the minimal energy closure of these profiles. Individual atoms have both moments zero but need not be odd. All k, including zero, are retained.

Proof. Let $P = \partial_u^2 - 1/4$. Its inverse has kernel $-e^{-|u|/2}$, because $Pe^{-|u|/2} = -\delta_0$. Thus $h = P^{-1}v$ has the same support as v: to the right of its support it equals $-e^{-u/2}\int e^{x/2}v(x)\,dx = 0$, and the other moment treats the left. Fourier multiplication gives

$$\sum_{r=0}^{2} \mathcal{E}(h^{(r)}) \le C\mathcal{E}(v).$$

Fix an even smooth partition $\sum_j \eta(u-j) = 1$, with η supported in $(-2,2)$. One explicit choice is to divide $b(u/2)$ by its positive periodic translate sum, where $b(t) = e^{-1/(1-t^2)}$ for $|t| < 1$ and zero otherwise. Fix an even $\theta \in C_c^\infty(-4,4)$ equal to one on $[-2,2]$. Such a cutoff can be specified by the normalized integral of the same bump. All support, derivative and overlap constants below are fixed.

Write $h_j(u) = \eta(u-j)h(u)$ and $v_j = Ph_j$. Each v_j has both moments zero by integration by parts, and $\sum_j v_j = v$ exactly. For any bounded-overlap family of smooth translates b_j, the paired jump formula gives

$$\sum_j \mathcal{E}(b_j z) \le C\mathcal{E}(z).$$

To see this, split each difference into $b_j(x)(z(x)-z(y)) + (b_j(x)-b_j(y))z(y)$ and use $\sum_j |b_j(x) - b_j(y)|^2 \le \min(C, C|x-y|^2)$. Its integral against $J(x-y)$ is finite, both near zero and at infinity. Applying this to $\eta_j, \eta_j', \eta_j''$ and the preceding derivative bound proves both $\sum_j \sum_{r=0}^2 \mathcal{E}(h_j^{(r)}) \le C\mathcal{E}(v)$ and $\sum_j \mathcal{E}(v_j) \le C\mathcal{E}(v)$.

Expand $h_j(j+u) = \sum_k c_{jk}e^{i\omega_k u}$ on the fixed period $(-5,5)$. It vanishes near the period boundary, and $\theta h_j = h_j$. Consequently $d_{jk} = -(1+\omega_k^2)c_{jk}$ gives the stated expansion. For completeness, periodizing J gives Fourier eigenvalues $a(\omega_k)$. Its zero-image term is controlled by full-line energy; all other images are uniformly bounded on the support $[-2,2]$ and contribute at most a fixed multiple of the squared L^2 norm. Apply this comparison to h_j, h_j', h_j'' and use periodic Parseval. Since $(1+\omega^2)^2 \asymp 1 + \omega^2 + \omega^4$, it proves the weighted coefficient bound, including the zero mode.

We also need synthesis, not only an analysis inequality. At one cell put $z = -\sum_k d_k e^{i\omega_k u}/(1+\omega_k^2)$. Periodic Parseval gives

$$\sum_{r=0}^{2} \mathcal{E}_{\mathrm{per}}(z^{(r)}) \le C\sum_k \ell_k |d_k|^2.$$

For any fixed smooth b supported away from the period endpoints, $\mathcal{E}(bz) \le C_b\mathcal{E}_{\mathrm{per}}(z)$. Inside the period this follows from the same difference split, because $J(x-y) \le \sum_m J(x-y+10m)$. For pairs with one point outside it, the fixed gap between $\operatorname{supp} b$ and its endpoints makes the exterior integral uniformly bounded by $C_b\|z\|_2^2$. Apply this to $\theta, \theta', \theta''$ in $P(\theta z)$. The translated outputs have a fixed support overlap. At a pair (x,y), at most twice that overlap many differences are nonzero; Cauchy–Schwarz before the paired jump integral pays their entire sum. This proves the full synthesis bound for finite lists and then for their weighted closure. The analysis and synthesis maps are bounded, so their identity on smooth tests extends to the stated minimal closure. Neither a lower Riesz bound for arbitrary redundant coefficients nor a high-frequency restriction has been used. □

Lemma S47.2 (A uniformly conditioned smooth polar direction)**.** *Fix a nonzero even nonnegative $\phi \in C_c^\infty(-\rho,\rho)$, $r > \rho > 0$, and $L \ge r + \rho + 1$. In full odd coordinates put*

$$p_L(x) = \sinh(x/2), \qquad u_L(x) = \phi(x-(L-r)) - \phi(x+(L-r)).$$

Then u_L is supported in $(-L, L)$, $\|u_L\|_2^2 = 2\|\phi\|_2^2$, and

$$\frac{|\langle p_L, u_L\rangle|}{\|p_L\|_2\|u_L\|_2} \geq c_{\phi,r} > 0, \qquad \mathcal{E}(u_L) + \|a(D)u_L\|_2^2 \leq C_{\phi,r}.$$

Every supported odd profile in the energy domain splits exactly as $v = v_0 + \alpha u_L$, with $v_0 \perp p_L$ and $\mathcal{E}(v_0) + |\alpha|^2 \leq C\mathcal{E}(v)$.

Proof. Set $A_\phi = \int \cosh(t/2)\phi(t)\,dt > 0$. Evenness gives $\langle p_L, u_L\rangle = 2A_\phi \sinh((L-r)/2)$, while $\|p_L\|_2^2 = \sinh L - L \leq e^L/2$. Since $L - r \geq 1$, the ratio is bounded below by $A_\phi e^{-r/2}(1 - e^{-1})/\|\phi\|_2$. The two arms are disjoint. Translation invariance and the triangle inequality give $\mathcal{E}(u_L) \leq 4\mathcal{E}(\phi)$ and $\|a(D)u_L\|_2 \leq 2\|a(D)\phi\|_2$; smoothness makes both finite. Take $\alpha = \langle p_L, v\rangle/\langle p_L, u_L\rangle$. The angle bound controls $|\alpha|$ by $C\|v\|_2$; the energy triangle inequality proves the assertion. The seed is smooth in the genuine supported operator domain, and the polar functional is continuous in L^2, so the split is valid throughout the closed local domain. This is geometric conditioning only, with no signed form assertion. □

Use $\rho = 1/4$, $r = 1$ from now on. Let $J_L : L^2(0, L) \longrightarrow L^2(\mathbb{R})$ be the odd isometry given by $J_L v(x) = v(x)/\sqrt{2}$ on $(0, L)$, its odd reflection on $(-L, 0)$, and zero elsewhere. Apply the frame lemma to the polar-free part of $J_L v$ and keep the single exact seed coordinate. This covers the full ordinary DN core, rather than its polar-free intersection. Indeed that intersection alone cannot approximate the whole core at fixed support: if P_N is the ordinary DN projection and $p = \sinh(x/2)$ on $(0, L)$,

$$\operatorname{dist}\left(\frac{P_N p}{\|P_N p\|}, p^\perp\right) = \frac{\|P_N p\|}{\|p\|} \longrightarrow 1.$$

Proposition S47.3 (Exact host and complete energy control)**.** *Fix the host buffer $h_* = 9$. On all full-line profiles supported in $(-L - h_*, L + h_*)$, not merely odd individual atoms, define*

$$\widehat{q}_L = D_{\text{full}} + (m_0 + 2)I - C^{\text{ar}}_{L+h_*} - J_L R_L J_L^*.$$

Here J_L^* includes restriction and odd projection, and R_L is the old *complete Green response. Then*

$$\widehat{q}_L[J_L v, J_L w] = \langle v, \mathcal{A}_{L,2} w\rangle \tag{S182}$$

as closed forms, and

$$|\widehat{q}_L[f, g]| \leq F_L \mathcal{E}(f)^{1/2}\mathcal{E}(g)^{1/2}, \qquad F_L = 12 + 12e^{L+h_*}.$$

The complete frame with its seed converges in this same host form. For the paired arithmetic operator on this host, define $\operatorname{Err}(f, g) = \langle f, (C^{\text{ar}}_{L+h_*} - C^{\text{main}}_{L+h_*})g\rangle$, where $C^{\text{main}}_{L+h_*}$ has kernel $e^{|x-y|/2}$ restricted to the same host. Thus the discrepancy includes both shift orientations and the complete continuous main term. Under the strip premise of Lemma S44.1, it has the bound

$$|\operatorname{Err}(f, g)| \leq Ce^{2dL}\mathcal{E}(f)^{1/2}\mathcal{E}(g)^{1/2}, \qquad d = b - 1/2 + \eta, \tag{S183}$$

for represented old profiles. The factor e^{2dL} records the support dependence of this arithmetic error bound.

Proof. Nonzero cells satisfy $|j| < L+2$, and their atoms have support within four of their cell centers. Thus all partial syntheses lie in the stated fixed host. Infinite synthesis equals the original profile, although finite partials can occupy its collar. On old profiles every additional host arithmetic shift

has zero overlap. The exact original realization gives $D_{\mathrm{full}}|_{J_L L^2(0,L)} = J_L(D_L + R_L)J_L^*$ as forms. Subtracting the transported old response therefore proves (S182). Replacing it by the host response would change the target.

The full paired arithmetic norm is bounded by $2\sum_{n<e^{2(L+h_*)}} \Lambda(n)/\sqrt{n} < 12e^{L+h_*}$, using Lemma S21.1. Also $|m_0 + 2| < 8$ and $\|J_L R_L J_L^*\| \le \pi/2$ by Lemma 3.1. Cauchy–Schwarz for the nonnegative multiplier D_{full} and $\|f\|_2 \le \mathcal{E}(f)^{1/2}$ prove the stated F_L bound. It includes all mixed, collar and parity terms. Energy convergence thus implies convergence in the host form.

Here are the additional uniformities needed for the last assertion. Factoring out $e^{i\omega_k u}$ in g_k leaves

$$\frac{(\omega_k^2 + 1/4)\theta - \theta'' - 2i\omega_k\theta'}{1 + \omega_k^2},$$

whose fixed-order smooth seminorms are bounded uniformly for every $k \in \mathbb{Z}$, including zero. In a pairing of cells j, l, integrate by parts in the internal variable as in the proof of Proposition S44.2. For every fixed m this bounds each window seminorm by $C_m(1+|\omega_k-\omega_{k'}|)^{-m}$. Lemma S44.1 then bounds the corresponding entry by

$$C_m e^{d|j-l|}\ell_k(1 + |\omega_k - \omega_{k'}|)^{-m}.$$

The continuous term here is the actual positive-shift integral over $y \ge 1$. When the translated window meets the origin, replacing its full Mellin main by this truncated main costs only a fixed compact-range term with the same smooth window bound. A window strictly to the negative side has zero actual sum and integral. The finitely wide origin range is bounded directly, retaining its carrier phases. The opposite shift orientation is paid in the same manner. Thus no fictitious primes below one, discarded orientation or unpriced negative window occurs. The logarithmic ratio of ℓ_k and $\ell_{k'}$ is absorbed by one extra inverse separation power. Since the full signed lattice has fixed spacing, the two-sided Schur test yields a weighted coefficient norm bound $Ce^{d|j-l|}$ for each cell pair. The seed adds two translates of one fixed smooth mother at carrier zero; its pairings satisfy the same argument with bounded multiplicity. It need not have zero polar moment. Finally

$$\sup_{|j|<L+2} \sum_{|l|<L+2} e^{d|j-l|} \le C_d e^{2dL}.$$

The same estimate holds for the two additional seed centers after changing the fixed constant. Schur's test on cells and the complete bounded analysis map prove (S183) for finite syntheses. Pass to the energy limit in the bounded host arithmetic form. All color, low-frequency, seed and spatial cross terms have been retained. This is not a lower Riesz assertion for the redundant dictionary. □

Proposition S47.4 (Quantitative finite-core pullback)**.** *Let $U_N : \mathbb{C}^N \to L^2(0,L)$ be the ordinary first-N DN isometry, $k_{\max} = k_{N-1}$, and*

$$M_N^2 = 1 + a(k_{\max}) + \pi/2.$$

Retain every active cell, every local frequency $|\omega_k| \le \Omega$, and the exact seed. The resulting synthesis map S_Ω from $\mathbb{C}^N$ to the common host satisfies, for $\Omega \ge 2$,

$$\|S_\Omega - J_L U_N\|^2_{\mathbb{C}^N \to \mathcal{E}} \le C_{\mathrm{cov}}\Omega^{-1/4}N(1 + k_{\max})^{1/2}. \tag{S184}$$

Its pullback $H_\Omega = S_\Omega^* \widehat{q}_L S_\Omega$ approximates the same *core matrix $H_N = U_N^* \mathcal{A}_{L,2} U_N$ with*

$$\|H_\Omega - H_N\| \le \eta_\Omega := F_L\epsilon(2M_N + \epsilon) \tag{S185}$$

whenever the left map norm in (S184) *is at most ϵ. The fixed C_{cov} is independent of L, N, Ω; a numerical use must certify it or directly certify that map error.*

Proof. A normalized zero-extended DN sine has, for $s = 1/4$,

$$\|J_L e_q\|^2_{H^s(\mathbb{R})} \le C_s(1 + k_q)^{2s}, \qquad L \ge 4.$$

This follows by expressing its two arms as interval indicators times modulations. The squared fractional seminorm of a normalized interval indicator is $C_s L^{-2s}$; modulation costs at most $C_s(1 + k_q)^{2s}$. The triangle and Cauchy–Schwarz inequalities give

$$\|J_L U_N z\|^2_{H^s} \le C_s N(1 + k_{\max})^{1/2}\|z\|^2.$$

The polar-seed subtraction preserves this bound because its coefficient is bounded by $C\|z\|$ and its translated smooth H^s norm is fixed.

The analysis proof in Lemma S47.1 works with H^s in place of logarithmic energy: replace $J(t)$ by $|t|^{-1-2s}$ in the difference estimates. Its small-distance error integral converges for $s < 1$, and its exterior/periodic image integrals converge for $s > 0$. The multiplier P^{-1} gains two derivatives in this norm as well. Periodic Parseval therefore gives

$$\sum_{j,k}(1 + \omega_k^2)^s |d_{jk}|^2 \le C_s\|v_0\|^2_{H^s}.$$

For $|\omega_k| > \Omega \ge 2$, $\ell_k/(1 + \omega_k^2)^{1/4} \le C\Omega^{-1/4}$. Apply bounded energy synthesis to precisely these discarded coefficients to prove (S184). Finite DN zero extensions belong to the required minimal energy closure: cut off their boundary strips and mollify. For any $0 < t < 1/2$ the strips of a fixed piecewise smooth vector have H^t norm tending to zero, hence energy norm tending to zero. Correcting the one polar moment by the fixed smooth seed preserves this limit. This verifies the actual input domain rather than assuming smooth DN endpoint values.

By the original Green identity and Lemma 3.1, $\|J_L U_N\|_{\mathbb{C}^N \to \mathcal{E}} \le M_N$. Write $S_\Omega = J_L U_N + E_\Omega$ and expand the full form pullback into two mixed terms and one error square. The host continuity bound gives $2F_L M_N \epsilon + F_L \epsilon^2$, proving (S185) as an induced matrix estimate, without an additional entrywise dimension fee. □

Corollary S47.5 (Certified acceptance on the original core range)**.** *Write the finite synthesis as a dictionary map times its actual coefficient map, $S_\Omega = \mathcal{D}_\Omega A_\Omega$, retaining the seed row. Let G_Ω be the complete host-form dictionary Gram, so $H_\Omega = A_\Omega^* G_\Omega A_\Omega$. If a Hermitian center C_Ω has certified induced error r_Ω to this exact pullback and*

$$C_\Omega - (r_\Omega + \eta_\Omega)I \succeq I/5,$$

then $H_N \succeq I/5$. At every cutoff admitted by Theorem B.6, it follows that $\mathcal{A}_{L,2} \succeq 52I/405$. Under RH, each fixed admitted L, N can be recognized in this representation by refinement.

Proof. The first assertion is (S185) and the directed center error. Apply Theorem B.6 only at its prescribed cutoff and with its actual domain and commutator hypotheses. Under RH that theorem gives $H_N \succ 2I/5$. Taking $\eta_\Omega, r_\Omega \le 1/40$ makes the displayed lower exceed $(2/5 - 2\eta_\Omega - 2r_\Omega)I \succeq 3I/10$. For example $\epsilon \le \min\{1, [80F_L(M_N + 1)]^{-1}\}$ suffices for $\eta_\Omega < 1/40$. The two comparison losses of each error are retained. Increasing Ω and the numerical precision supplies conditional fixed-instance recognition, not a successful cofinal sign.

All finite coefficients are effectively enclosable. The specified smooth bumps and their normalized integrals have effective derivative and support bounds. Analysis coefficients use their integrals and the explicit compact P^{-1} kernel. Arithmetic entries involve only the finite active prime-power

list and smooth overlap integrals. For D_{full} use Fourier integrals; integration by parts bounds the omitted Fourier tails by any chosen inverse power, while the symbol grows logarithmically. The old response entries can also be enclosed without a false norm limit: for a bounded old half-profile w, $|\langle r_j, w\rangle| \le \|w\|_\infty / c_j$, so its paired scalar series tail is bounded by the product of the two suprema times $c_J^{-2} + (2c_J)^{-1}$. Finite directed quadrature and all coefficient, coordinate and Gram errors must then be included in r_Ω. A numerical receipt cannot replace C_{cov} by an unspecified existence constant. □

This criterion tests the pullback on the actual original-core input range. Positivity of G_Ω on arbitrary redundant dictionary coefficients is neither assumed nor required. A sufficient cutoff is $\Omega \ge C_{\mathrm{cov}}^4 \epsilon^{-8} N^4 (1+k_{\max})^2$. There are $O((L+1)(1+\Omega))$ dictionary atoms and one seed coordinate. The bound is polynomial in N, e^L, L and the inverse fixed margin; on the elementary near-threshold CO schedule it is $\exp(O(e^L)) = \exp(O(\sqrt{\log M}))$. It counts dictionary size, not arithmetic operations or bit complexity, and native H_N evaluation can be smaller. The representation transfers a certified finite sign through the displayed complete error budget; the arithmetic discrepancy retains its explicit support-dependent factor.

S47.2 Conditional completion of the polar direction

The preceding polar-free constructions admit an exact one-dimensional completion on the original odd supported space. The estimate below retains the raw prime operator and expresses the additional debit in the same unweighted profile norm. Two arithmetic hypotheses give alternative bounds for that debit.

Supported operators and a smooth polar seed. Let $\mathcal{H}_L = L^2_{\mathrm{odd}}(-L, L)$, with profiles extended by zero outside $(-L, L)$, and let

$$(J_L g)(x) = 2^{-1/2} \operatorname{sgn}(x) g(|x|)$$

be the unitary map from the odd-half coordinates to $\mathcal{H}_L$. Write $\mathcal{D}_L$ for the operator associated with the supported closed form of the full-line multiplier $a(D)$ on its minimal odd domain $\mathcal{V}_L$. The complete supported arithmetic action is

$$(\mathcal{C}_L f)(x) = \sum_{2 \le n < e^{2L}} \frac{\Lambda(n)}{\sqrt{n}} [f(x + \log n) + f(x - \log n)], \qquad |x| < L.$$

Both channels use the zero extension of the same odd profile. Define

$$\widetilde{B}_L = \mathcal{D}_L + m_0 I - \mathcal{C}_L, \qquad \mathcal{R}_L = J_L R_L J_L^*, \qquad \widetilde{\mathcal{A}}_{L,c} = J_L \mathcal{A}_{L,c} J_L^* = \widetilde{B}_L - \mathcal{R}_L + cI,$$

where c is fixed. Here $0 \preceq \mathcal{R}_L \preceq (\pi/2) I$ by Lemma 3.1, and $\mathcal{D}_L = J_L (D_L + R_L) J_L^*$ as forms. These raw operators have the common form domain $\mathcal{V}_L$.

Fix $0 < \rho < r$ and a nonzero, nonnegative, real even $\phi \in C_c^\infty(-\rho, \rho)$. For $L \ge r + \rho + 1$ put

$$a_L = L - r, \qquad p_L(x) = \sinh(x/2), \qquad u_L(x) = \phi(x - a_L) - \phi(x + a_L), \qquad m_L = \langle p_L, u_L \rangle.$$

Lemma S47.2 gives $m_L \asymp e^{L/2}$, a uniform moment angle, and the uniform seed-action bound $\|a(D) u_L\|_2 \le 2 \|a(D)\phi\|_2$. In particular u_L belongs to the operator domains of both raw supported operators. For every $f \in \mathcal{V}_L$,

$$\alpha = \frac{\langle p_L, f \rangle}{m_L}, \qquad f_0 = f - \alpha u_L \perp p_L, \qquad \|f_0\|_2 + |\alpha| \le C_\phi \|f\|_2. \tag{S186}$$

The split preserves the closed form domain, since the local polar functional is L^2-continuous and the seed is smooth and compact.

The raw main kernel and scalar completion. Let $\mathcal{C}_L^{\mathrm{main}}$ be the supported integral operator with kernel $e^{|x-y|/2}$, the continuous $y \geq 1$ counterpart of the paired arithmetic shifts. On the full odd space,

$$\mathcal{C}_L^{\mathrm{main}} = -2p_L \otimes p_L - \mathcal{K}_L, \qquad 0 \preceq \mathcal{K}_L \preceq 4I, \tag{S187}$$

where $\mathcal{K}_L$ has kernel $e^{-|x-y|/2}$ and $(p_L \otimes p_L)f = p_L\langle p_L, f\rangle$. Indeed $e^{|t|/2} = 2\cosh(t/2) - e^{-|t|/2}$, and the even $\cosh(x/2)\cosh(y/2)$ channel vanishes on odd inputs. Positivity and the bound for $\mathcal{K}_L$ follow from its full-line Fourier multiplier $(\xi^2 + 1/4)^{-1}$. The signed-pole geometric counterpart is $\widetilde{B}_L - 2p_L \otimes p_L$; its restriction to $p_L^{\perp}$ agrees with the raw form.

Lemma S47.6 (Completion from a supported seed residual)**.** *For $\mathcal{T}_L = \widetilde{B}_L$ or $\mathcal{T}_L = \widetilde{\mathcal{A}}_{L,c}$, suppose*

$$\|(\mathcal{C}_L - \mathcal{C}_L^{\mathrm{main}})u_L\|_{L^2(-L,L)} \leq \varepsilon(L), \qquad \varepsilon(L) \geq 1, \qquad \varepsilon(L) = o(e^L).$$

Then, for all sufficiently large L, every $f = f_0 + \alpha u_L$ as in (S186) *satisfies*

$$q_{\mathcal{T}_L}[f] \geq q_{\mathcal{T}_L}[f_0] - Ce^{-L}\varepsilon(L)^2\|f_0\|_2^2.$$

In particular, if $b_0(L) \geq 0$ and $q_{\mathcal{T}_L}[g] \geq -b_0(L)\|g\|_2^2$ for all $g \in \mathcal{V}_L \cap p_L^{\perp}$, then

$$q_{\mathcal{T}_L}[f] \geq -C[b_0(L) + e^{-L}\varepsilon(L)^2]\|f\|_2^2, \qquad f \in \mathcal{V}_L. \tag{S188}$$

All constants may depend on the fixed seed and shift, but not on L.

Proof. Put $e_L = (\mathcal{C}_L - \mathcal{C}_L^{\mathrm{main}})u_L$. Equation (S187) gives

$$\widetilde{B}_L u_L = 2m_L p_L + (\mathcal{D}_L + m_0 I + \mathcal{K}_L)u_L - e_L.$$

The non-rank-one remainder has norm at most $C\varepsilon(L)$. The same assertion holds for $\widetilde{\mathcal{A}}_{L,c}$ by the uniform response bound. Therefore

$$\langle u_L, \mathcal{T}_L u_L\rangle = 2m_L^2 + O(\varepsilon(L)) \geq c_0 e^L, \qquad |\langle f_0, \mathcal{T}_L u_L\rangle| \leq C\varepsilon(L)\|f_0\|_2.$$

The first inequality holds eventually because $\varepsilon(L) = o(e^L)$; the second uses $f_0 \perp p_L$. Completing the scalar square in $q_{\mathcal{T}_L}[f_0 + \alpha u_L]$ gives the first conclusion. The nonnegative base debit and the uniform conditioning in (S186) give (S188). The operator-domain seed justifies the mixed pairing throughout $\mathcal{V}_L$. □

Proposition S47.7 (Completion under a zero-strip hypothesis)**.** *Assume every nontrivial zeta zero has real part at most b, where $1/2 \leq b < 1$. Fix $0 < \eta < 1 - b$ and put $d = b - 1/2 + \eta$. If a nonnegative $b_0(L)$ gives the polar-free lower bound in Lemma S47.6, then*

$$q_{\mathcal{T}_L}[f] \geq -C[b_0(L) + e^{(4b-3+4\eta)L}]\|f\|_2^2 \qquad (f \in \mathcal{V}_L) \tag{S189}$$

for $\mathcal{T}_L = \widetilde{B}_L$ and $\widetilde{\mathcal{A}}_{L,c}$, respectively, using each operator's own polar-free lower bound.

Proof. For real T, define the exact truncated-main discrepancy

$$F_\phi(T) = \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}}\phi(\log n - T) - \int_1^\infty y^{-1/2}\phi(\log y - T)\,dy.$$

For $T \geq \rho$ the main integral is the full Mellin residue, so Lemma S44.1 at $\omega = 0$ gives $|F_\phi(T)| \leq Ce^{dT}$. For $0 \leq T \leq \rho$, the difference between the full residue and the truncated main integral is bounded

on a fixed compact interval. For $-\rho \le T < 0$, both terms use only $0 \le \log y \le 2\rho$, a fixed finite arithmetic list and a bounded main integral. For $T < -\rho$ both terms vanish. Enlarging one constant therefore gives, for every real T,

$$|F_\phi(T)| \le Ce^{d\max(T,0)}\mathbf{1}_{\{T\ge-\rho\}}.$$

Evenness of ϕ gives the exact four-window identity

$$e_L(x) = F_\phi(a_L - x) - F_\phi(-a_L - x) + F_\phi(x - a_L) - F_\phi(x + a_L), \qquad |x| < L.$$

This retains both translation directions and all active prime powers; the compact seed makes every shift beyond the supported cutoff vanish. In particular

$$|e_L(x)| \le C\big[e^{d|x-a_L|} + e^{d|x+a_L|}\big].$$

The complete spatial norm is paid by

$$\int_{-L}^{L} e^{2d|x-a_L|}\,dx = \frac{e^{2d(L+a_L)} + e^{2d(L-a_L)} - 2}{2d} \le C_{d,r}e^{4dL},$$

and the same bound holds for $x + a_L$. Hence $\|e_L\|_2 \le Ce^{2dL}$, with no factor depending on the receiving profile or on the interval length outside this exponential. Since $0 < 2d < 1$, this is an admissible $\varepsilon(L)$ in Lemma S47.6. Its squared cost divided by the raw seed reserve e^L gives (S189). □

Proposition S47.8 (Completion under a prime-primitive bound)**.** *Let $\psi_{\mathrm{C}}(y) = \sum_{n\le y}\Lambda(n)$ be the actual Chebyshev function. Suppose it obeys*

$$|\psi_{\mathrm{C}}(y) - (y-1)| \le C_\sigma y^\sigma \quad (y \ge 1), \qquad \frac{1}{2} < \sigma < 1, \qquad \psi_{\mathrm{C}}(1) = 0.$$

For either raw supported operator, a nonnegative polar-free debit $b_0(L)$ then gives

$$q_{\mathcal{T}_L}[f] \ge -C\big[b_0(L) + e^{(4\sigma-3)L}\big]\|f\|_2^2, \qquad f \in \mathcal{V}_L, \tag{S190}$$

for all sufficiently large L.

Proof. Set $E(y) = \psi_{\mathrm{C}}(y) - (y-1)$ and, for each fixed $|x| < L$,

$$G_x(y) = y^{-1/2}\big[u_L(x + \log y) + u_L(x - \log y)\big].$$

Stieltjes integration by parts gives exactly

$$e_L(x) = \int_{[1,\infty)} G_x(y)\,dE(y) = -\int_1^\infty E(y)G_x'(y)\,dy.$$

Here $E(1) = 0$ removes the lower boundary, and compact support removes the upper boundary. Only the fixed smooth seed is differentiated: $|G_x'(y)| \le C_\phi y^{-3/2}$ on at most four fixed-log-width bands, whose nonnegative centers lie near $|x - a_L|$ and $|x + a_L|$. Bands meeting $\log y = 0$ contribute a bounded constant. With $\gamma = \sigma - 1/2 > 0$, integration on these bands gives

$$|e_L(x)| \le C\big[e^{\gamma|x-a_L|} + e^{\gamma|x+a_L|}\big].$$

The spatial integral used above, with d replaced by γ, yields $\|e_L\|_2 \le Ce^{(2\sigma-1)L}$. Since $\sigma < 1$, the shared scalar completion applies and proves (S190). □

The primitive hypothesis with $\sigma < 3/4$ gives a vanishing additional debit, and at $\sigma = 3/4$ a bounded additional debit. Under the zero-strip hypothesis, $b < 3/4$ permits $\eta < 3/4 - b$ and again a vanishing debit; at $b = 3/4$, each fixed $\eta > 0$ gives the explicit factor $e^{4\eta L}$. The constants may depend on the chosen positive η, so this endpoint statement retains that factor. For every allowed $\sigma < 1$ or $b+\eta < 1$, the additional debit is $o(e^L)$. These completions transport a separately supplied polar-free lower bound to the full raw supported operator, retaining its original unweighted norm and the exact polar main term. The unitary identity $\|J_L g\|_2 = \|g\|_2$ transports the same debit to $\mathcal{A}_{L,c}$ in its odd-half norm. In the theta-congruent coordinates $f = sh$, the norm factor is exactly $\|sh\|_2^2$; the bound $\|s\|_\infty < 3$ of the main paper further gives $\|sh\|_2^2 \le 9\|h\|_2^2$ whenever those original supported coordinates are used.

S47.3 Actual-prime thin collars and local support extension

We work in the ordinary odd-half profile space. Here $0 < \ell < U$, $J = (\ell, U)$ and $\delta = U - \ell$. The support hypotheses in this subsection are independent of those in the full-core construction. The physical old-interval/collar decomposition also occurs in the support-variation analysis of [4, Section 4]. The following estimate uses the integer spacing of the actual prime channels for the arithmetic-only recipient of this paper.

Lemma S47.9 (Complete thin-collar arithmetic payment)**.** *If $\delta e^{2U} \le 1/4$, the complete paired arithmetic action satisfies*

$$\|1_{(0,\ell)} C_U^{\mathrm{ar}} 1_J\|^2 \le 2 \sum_{n<e^{2U}} \frac{\Lambda(n)^2}{n} \le 24U^2 + 12U, \qquad \|1_J C_U^{\mathrm{ar}} 1_J\| \le 2Ue^{-\ell}.$$

Proof. Put $t_n = \log n$ and $w_n = \Lambda(n)/\sqrt{n}$. On a collar input u, the old-output block is exactly

$$\sum_{n<e^{2U}} w_n \big[u(x+t_n)1_J(x+t_n) - u(t_n-x)1_J(t_n-x)\big], \qquad 0 < x < \ell.$$

The ordinary $x - t_n$ arm cannot reach J. For fixed x, the two displayed channels require integers in $(e^{\ell-x}, e^{U-x})$ and $(e^{\ell+x}, e^{U+x})$, respectively. Each interval has length less than one, since its length is at most $\delta e^{2U} \le 1/4$. Thus at most two scalar summands occur. Pointwise Cauchy–Schwarz pays their entire cross term by twice the sum of squares. For each n, the two changes of variables give the disjoint source pieces $J \cap (t_n, t_n + \ell)$ and $J \cap (t_n - \ell, t_n)$. Their squared input integrals sum to at most $\|u\|^2$. Summation proves the first inequality with every prime power retained.

The proof of Lemma S21.1 gives $\Psi(X) < 3X$ for $X > 1$. Partial summation therefore gives

$$\sum_{n\le X} \frac{\Lambda(n)}{n} \le 3(1 + \log X), \qquad \sum_{n\le X} \frac{\Lambda(n)^2}{n} \le 3\log X(1 + \log X).$$

Take $X = e^{2U}$. For the collar diagonal, $\delta < \log 2$ excludes ordinary shifts between collar points. A reflected channel requires $n \in (e^{2\ell}, e^{2U})$, an interval of length at most $2\delta e^{2U} \le 1/2$. There is at most one such integer. Its coefficient is at most $2Ue^{-\ell}$, and its compressed reflection is a self-adjoint contraction. This proves the remaining bound, including activation endpoints, whose overlaps have measure zero. □

Proposition S47.10 (Paid local support extension)**.** *Let $c \ge 0$ be fixed and suppose the entire arithmetic-only operator $\mathcal{A}_{\ell,c} \succeq \mu I$, $\mu > 0$. Set $\overline{U} = \ell + 1$ and*

$$B_{\mathrm{mix}} = 2 + 3\pi/2 + \sqrt{24\overline{U}^2 + 12\overline{U}}, \qquad w_* = 2\overline{U}e^{-\ell},$$

$$\tau = \max(1, 2B_{\mathrm{mix}}^2/\mu), \qquad T_* = \tau + |m_0| + w_* + 1.$$

Choose any positive δ with

$$\delta \le \min\{\ell/16, 1, e^{-2\overline{U}}/4, (\pi/9)e^{-2T_*}\}.$$

For every $\ell \le L \le \ell + \delta$, on the complete original odd-half domain,

$$\mathcal{A}_{L,c} \succeq \frac{\min(\mu/2, \tau)}{(1 + B_{\mathrm{mix}}/\tau)^2} I \succ 0.$$

For $\ell \ge 1$ and $0 < \mu \le 1$, a safe dyadic choice may be made with $\log(1/\delta) = O((\ell+1)^2/\mu + \ell)$, with fixed symbol constants.

Proof. The full raw form is compatible under physical zero extension. The true Green features in (19) give, on the old interval,

$$R_\ell - R_L|_{(0,\ell)} = \sum_{j \ge 0} 4 \left(\frac{1}{e^{2c_j \ell} + 1} - \frac{1}{e^{2c_j L} + 1} \right) \sinh(c_j \cdot) \otimes \sinh(c_j \cdot) \succeq 0.$$

The order holds first for positive partial sums and then for the bounded strong limits. Extra arithmetic shifts have zero overlap on the old support. Consequently the old compression of $\mathcal{A}_{L,c}$ is at least $\mathcal{A}_{\ell,c} \succeq \mu I$.

The spatial-cut argument at (26) preserves the original logarithmic domain and bounds the complete archimedean cross block by $2 + \pi$. Subtracting R_L, whose norm is at most $\pi/2$, gives a DN cross bound $2 + 3\pi/2$. Lemma S47.9 then bounds the complete old/collar cross block by B_{mix}, uniformly for all the stated L. The scalar shift has no cross block.

To bound the whole collar, choose an integer $n \in [\ell/(8\delta), \ell/(4\delta)]$. Its existence follows from $\delta \le \ell/16$. On a vector f supported in (ℓ, L), the first n DN modes satisfy

$$\|P_n f\|^2 \le \frac{2n(L-\ell)}{L} \|f\|^2 \le \tfrac{1}{2}\|f\|^2.$$

The positive symbol is increasing. Its defining digamma sum, compared with the integral of the decreasing function $2k^2/[t(t^2+k^2)]$ on the mesh $t = 1/2 + 2j$, gives $a(k) \ge \frac{1}{2}\log(1 + 4k^2) \ge \log k$. Also $L \le 17\ell/16$ and $k_n(L) \ge n\pi/L > \pi/(9\delta) > 1$. Hence the entire collar has

$$D_L[f] \ge \tfrac{1}{2}\log(\pi/(9\delta))\|f\|^2.$$

Its arithmetic compression costs at most w_* by the same lemma. The chosen δ and $c \ge 0$ therefore make the whole collar block at least τI.

The old/collar split is valid on the same closed form domain by the spatial-cut proof just cited; the bounded Green and arithmetic terms preserve it. Write its blocks as A_{00}, C, T. Then $T \succeq \tau I$ and $A_{00} - CT^{-1}C^* \succeq (\mu - B_{\mathrm{mix}}^2/\tau) I \succeq \mu I/2$. Complete square completion uses a triangular map whose inverse norm is at most $1 + B_{\mathrm{mix}}/\tau$. This proves the displayed whole-operator bound, retaining all new directions and mixed terms. For the final assertion, $B_{\mathrm{mix}} = O(\ell + 1)$, $w_* = O(1)$ and $\tau = O((\ell+1)^2/\mu)$; the other logarithmic radius prices are $O(\ell)$ or smaller. A dyadic below the explicit minimum changes its logarithm by at most a fixed additive constant. □

Corollary S47.11 (Local extension of the shift-two example)**.** *Let $\ell = \frac{1}{2}\log(96 \log 2/\pi) + 1/10$ be the support of Theorem B.8. Throughout $\ell \le L \le \ell + 2^{-1300000}$, $\mathcal{A}_{L,2} \succeq I/1281$.*

Proof. The retained reference enclosures of the shift-two example give $\|C_\ell^{\mathrm{ar}}\| < 7$, receiving floor greater than 1.182, and complete core Schur lower $I/10$. The whole triangular payment is therefore at most $(1+7)^2$, giving $\mathcal{A}_{\ell,2} \succeq I/640$. The exact support expression and $2/3 < \log 2 < 7/10$, $3 < \pi < 4$ give $1 < \ell < 37/20$. Since $\delta < 1/8 < 2-\ell$, the proposed interval has $L < 2$, and one may sharpen the common budgets to $B_{\mathrm{mix}} < 18$, $w_* \le 4$ and $|m_0| < 6$. Take $\tau = 420000 > 2 \cdot 18^2 \cdot 640 = 414720$. For $\delta = 2^{-1300000}$, the thinness and $\ell/16$ guards hold, and

$$\tfrac{1}{2}\log(\pi/(9\delta)) - 10 > 1299967/3 > 420000,$$

using $\log 2 > 2/3$ and $\log(\pi/9) > -2$. Thus the same proof applies with these sharper budgets. Its lower is

$$\frac{1/1280}{(1+18/420000)^2} = \frac{3828125}{4900420009} > \frac{1}{1281}.$$

This uses the complete previously certified starting example and exact scalar comparisons. □

S48 The earlier complete supported certificate

The finite criterion can be evaluated on a complete receiving space. The following worked example uses an alternative fixed support and receiving reference in the same original-form normalization. Put

$$M = 2^{20}, \qquad \kappa = 2^{-20}, \qquad L = \tfrac{1}{2}\log\Big(4\log(2^{20})/\pi\Big) + \tfrac{1}{10}, \qquad J = 74, \qquad N = 329.$$

Use the exact original supported realization and normalization defined above. Let W_{low} be the dyadic theta floor specified by the accompanying certificate data, with $W/2 \le W_{\mathrm{low}} \le W$, and put

$$S_\star = D_L + R_{L,74} + m_0 - C_L^{\mathrm{ar}} + \kappa W_{\mathrm{low}}.$$

Every prime power $n < e^{2L}$ and both killed-shift orientations occur in C_L^{ar}.

Theorem S48.1 (Evaluated complete supported certificate)**.** *On the original minimal closed odd supported profile domain, subject to $\langle p, f\rangle = 0$, the form $S_\star$ is nonnegative. Consequently every original odd polar-free test supported in $(-L, L)$ satisfies*

$$q_H[h] \ge -2^{-20}\|h\|_2^2.$$

Proof. Apply the common complete receiver construction of main-paper Section S13 with $N = 329$, $J = 74$ and the support specified above. Thus C has 328 polar-free columns, $F = [p, C]$, $G_C = C^*C$ and $Q = \mathrm{proj}_{\ker F^*}$. The reference-cell verification in the supplementary section *Verification of the complete supported example* certifies the complete arithmetic row upper r_{up}, the leakage-paid diagonal lower α and

$$(S_\star)_Q \succeq d_Q, \qquad d = \kappa W_{\mathrm{low}} + \alpha - r_{\mathrm{up}} > 1/200.$$

The exact invertible dyadic conditioner R_G retains all 329 constraints and satisfies

$$\|R_G^*(F^* d^{-1} F)R_G - I\| < \delta, \qquad \delta = 4 \times 10^{-75}.$$

Form B_E, K_E, H from the complete outputs as in main-paper (S51). The directed lower matrix is

$$A_\star = \frac{H^* R_G R_G^* H}{1+\delta} - B_E - K_E.$$

All 328 directed pivots of $P_C^* A_\star P_C$ are positive, where $P_C = G_C^{-1} R_{uu}^{-*}$ and R_{uu} is the 328×328 core block of R_G. The supplementary verification gives the exact panels, elementary primitives, interval data and replay. The common full receiving comparison therefore proves $S_\star \succeq 0$, and its physical congruence proves the original-norm lower bound. □

The zero receiving trial suffices. A separately verified 64-trial extension is supplementary reproducibility evidence for the same instance. The evaluated instance supplies a concrete realization of the complete finite-certificate mechanism at the stated support and cutoff.

S49 Complete fixed-support ridge continuation

The supported examples can also be studied as parameter families. The following comparison retains the full constrained reference inverse and all mixed columns. Support, constraints, Green model and legal trials remain fixed throughout each comparison.

Write $S_\kappa = S_0 + \kappa W_0$, $d_\kappa = d_0 + \kappa W_0$, where W_0 is a bounded positive multiplier on the fixed support. Let Q be the complete receiving projection and assume $QS_\kappa Q \succeq Qd_\kappa Q$, with d_κ uniformly positive on the parameter interval considered. Extend $Z_\kappa = (Qd_\kappa Q)^{-1}$ by zero off $\operatorname{ran} Q$. For a fixed legal finite input map V, set

$$\Gamma_\kappa = V^* S_\kappa V - (S_\kappa V)^* Z_\kappa (S_\kappa V), \qquad X_\kappa = V - Z_\kappa S_\kappa V.$$

These are sufficient reference-inverse lower matrices. The complete Schur comparison retains the corresponding actual receiving operator.

Proposition S49.1 (Complete derivative and downward loss)**.** *On the common domain,*

$$\Gamma_\kappa' = X_\kappa^* W_0 X_\kappa \succeq 0.$$

At an admitted anchor κ_0, put $D_0 = X_{\kappa_0}^* W_0 X_{\kappa_0}$ and $a = \operatorname{ess\,sup} W_0/d_{\kappa_0}$. For $0 \le t \le h$ and $ha < 1$,

$$\Gamma_{\kappa_0 - t} \succeq \Gamma_{\kappa_0} - \frac{t}{1-ta} D_0, \qquad \Gamma'_{\kappa_0 - t} \preceq (1-ta)^{-2} D_0.$$

Assume $\kappa_0 > 0$ and $0 < h < \kappa_0$. If a positive accepted anchor lower $L_0 \preceq \Gamma_{\kappa_0}$ satisfies $\kappa_0 D_0 \preceq cL_0$, and $b = 1 - ha > 0$, $g = 1 - ch/(\kappa_0 b) > 0$, then throughout that band

$$\kappa \Gamma_\kappa' \preceq \frac{c}{b^2 g} \Gamma_\kappa.$$

Proof. The constrained resolvent identity gives $Z_\kappa' = -Z_\kappa W_0 Z_\kappa$. Differentiating the entire quadratic expression, including both mixed terms, gives the derivative square. Put $A = W_0^{1/2} Z_{\kappa_0} W_0^{1/2}$. The multiplier-minus-finite-rank formula implies $0 \preceq A \preceq (W_0/d_{\kappa_0})I$, so $\|A\| \le a$. Subtracting the constrained equations for X yields

$$W_0^{1/2} X_{\kappa_0 - t} = (I - tA)^{-1} W_0^{1/2} X_{\kappa_0}.$$

Spectral calculus proves the derivative bound; integration gives the loss $t/(1-ta)$. The reference remains positive because $d_{\kappa_0 - t} \ge (1-ta) d_{\kappa_0}$. Finally $D_0 \preceq c\Gamma_{\kappa_0}/\kappa_0$ and the loss bound give $\Gamma_\kappa \succeq g\Gamma_{\kappa_0}$. Combining these matrix orders proves the relative estimate without simultaneous diagonalization. □

Application to the complete M24 example. With exactly the supported model and input contracts of the M24 example, the complete derivative-weight integrals and their independently reconstructed 659×659 derivative Gram certify a continuous ridge band. Its upper endpoint is $\kappa_0 = 2^{-24}$, and its exact lower endpoint is $\kappa_0 - h$, where

$$h = \frac{1125889145388605}{75557863725914323419136}, \qquad a = \frac{612601868649647177140450428117319 68}{365132188228942762398806 4773}.$$

The lower endpoint is approximately $4.47036260082 \times 10^{-8}$. Complete directed matrix tests establish $\kappa_0 D_0 \preceq (1 + 2^{-20}) L_0$ and the downward-band lower. Substitution in Proposition S49.1 gives

$$\kappa \Gamma_\kappa' \preceq 3\Gamma_\kappa \quad (\kappa_0 - h \le \kappa \le \kappa_0).$$

All 434281 derivative-matrix entries and all 659 directions are retained. This application uses new derivative-weight primitives; the original M24 moment archive alone does not reproduce that additional claim. Its scientific inputs, exact congruences and numerical contracts accompany this extension in the separate ridge-continuation dataset and portable verifier. The interval keeps the same support; a changing support changes the operator and requires a separate comparison.

Proposition S49.2 (Uniform capped return comparison)**.** *Fix an admitted capped model with* $S_w = S_{\mathrm{base}} + w$, $d_w = d_{\mathrm{base}} + w$, $d_{\mathrm{base}} \geq \beta I$, *and* $Q(S_{\mathrm{base}} - d_{\mathrm{base}})Q \succeq 0$. *For any* $0 \leq w_1 \leq w_0 \leq \beta I$, *keep the same constraints and legal input* V. *Put* $\Delta = w_0 - w_1$, $A = \Delta^{1/2} Z_{w_0} \Delta^{1/2}$ *and* $B = \Delta^{1/2} X_{w_0}$. *Then*

$$0 \preceq A \preceq \tfrac{1}{2} I, \qquad \Gamma(w_1) = \Gamma(w_0) - B^*(I - A)^{-1} B.$$

For $M_j = B^* A^j B$ *and every integer* $m \geq 0$,

$$\sum_{j<m} M_j \preceq B^*(I - A)^{-1} B \preceq \sum_{j<m} M_j + 2^{1-m} M_0.$$

This holds also for $w_1 = 0$, *including changes across clipping knots.*

Proof. Inverse order gives $Z_{w_0} \preceq d_{w_0}^{-1}$ and $\Delta/(d_{\mathrm{base}} + w_0) \leq 1/2$. Minimize the complete quadratic functional $V^* S_w V - 2\Re((S_w V)^* y) + y^* d_w y$ over $y \in \operatorname{ran} Q$. Writing $y = Z_{w_0} S_{w_0} V + z$, completion of the square at w_1 subtracts $B^* B + B^* \Delta^{1/2} Z_{w_1} \Delta^{1/2} B$ from $\Gamma(w_0)$. The constrained resolvent identity makes its middle factor $(I - A)^{-1}$. The positive geometric series and $A^m (I - A)^{-1} \preceq 2^{1-m} I$ prove the moment enclosure. No inverse of the possibly singular Δ is used. □

The contraction is uniform under the stated admissions. Certification also includes the size of the complete observation Gram M_0 and a directed signed lower bound after subtraction of the return charge. Every capped-model test uses its own recomputed inputs, including the moment, full-response and inverse errors.

S50 An alternative growing-floor arithmetic matrix criterion

The fixed-shift test has a complementary growing-floor version whose scalar tail payment uses the complete spatial action Gram. Proposition S22.1, with exponent $a = 0$, permits an unweighted negative floor growing subexponentially with support. The construction below composes that implication with the complete arithmetic source Gram. Its matrix contains no Green action or receiving inverse; the complete receiving contribution is paid by an explicit scalar floor.

For each integer $j \geq 4$, set

$$L = j, \qquad \beta_j = e^{\sqrt{j}}, \qquad P_j = 8 j e^j.$$

Choose a certified rational τ_j and an outward integer N_j with

$$4P_j^2/\beta_j \leq \tau_j \leq 4P_j^2/\beta_j + 1,$$
$$X_j = P_j + |m_0| + \tau_j + 1, \qquad \frac{j e^{X_j}}{\pi} \leq N_j \leq \frac{j e^{X_j}}{\pi} + 2. \tag{S191}$$

These choices are effective: approximate the real lower threshold for τ_j to a rational interval of width less than one and use its upper endpoint. An interval of width less than one for $j e^{X_j}/\pi$ supplies an integer in the stated outward range, without an exact-real equality decision. Fix these same chosen values in all subsequent matrices.

Let U_j be the ordinary isometry onto the first N_j DN modes at support j, and $Q_j = I - U_j U_j^*$. Use the complete supported self-adjoint arithmetic operator $C_j = C_j^{\mathrm{ar}}$, with both killed translations and the reflected odd channel for every $n < e^{2j}$. Put

$$\begin{aligned} A_j &= D_j + m_0 - C_j + \beta_j I, & & \\ A_j^{\mathrm{ar}} &= U_j^* C_j U_j, & G_j^{\mathrm{full}} &= (C_j U_j)^* (C_j U_j) = U_j^* C_j^2 U_j, \\ G_j^{\mathrm{rec}} &= G_j^{\mathrm{full}} - (A_j^{\mathrm{ar}})^2, & H_j &= \mathrm{diag}_{q<N_j}\, a(k_q) + (m_0+\beta_j) I - A_j^{\mathrm{ar}}, \\ \Gamma_j &= H_j - \tau_j^{-1} G_j^{\mathrm{rec}}. & & \end{aligned} \tag{S192}$$

The square in $(A_j^{\mathrm{ar}})^2$ is a matrix product. In particular, G_j^{full} integrates the complete spatial arithmetic outputs, not a finite receiving-output projection.

Theorem S50.1 (Explicit growing-floor matrix criterion)**.** *Each Γ_j is a finite, effectively enclosable matrix. The following are equivalent:*

1. *RH;*
2. *$\Gamma_j \succ 0$ for every sufficiently large integer j;*
3. *for every sufficiently large integer j, Γ_j admits a strictly positive directed lower enclosure with every coefficient and matrix error paid.*

Under RH, $\Gamma_j \succeq \beta_j I/2$ for every $j \geq 4$. A two-sided matrix-center radius $e_j \leq \beta_j/8$ then leaves an outward lower at least $\beta_j I/4$. The specified sufficient dimension satisfies

$$\log N_j = O(je^j + j^2 e^{2j-\sqrt{j}} + \log j).$$

For $j = \lceil L_0(M) \rceil$, where $L_0(M) = \frac{1}{2}\log(10 \log M/\pi) + 2$, this gives $N_j = M^{o(1)}$. Here M labels support growth, not a shrinking original-norm allowance.

Proof. The elementary bounds $\Lambda(n) \leq \log n \leq 2j$ for $n < e^{2j}$ and $\sum_{2\leq n<K} n^{-1/2} \leq 2\sqrt{K}$ give

$$\sum_{2\leq n<e^{2j}} \frac{\Lambda(n)}{\sqrt{n}} \leq 4je^j, \qquad \|C_j\| \leq P_j.$$

The second estimate retains the entire paired arithmetic operator; odd restriction cannot increase its norm. On the whole DN suffix, $k_{N_j} > e^{X_j} > 1$ and $a(k) \geq \log k$ imply

$$T_j = Q_j A_j Q_j|_{\mathrm{ran}\, Q_j} \succeq [a(k_{N_j}) + m_0 - P_j + \beta_j] I \succeq (\tau_j + 1 + \beta_j) I \succeq \tau_j I.$$

This is the full receiving floor on the same minimal supported domain. The exact mixed map is $C_j^{\mathrm{mix}} = -Q_j C_j U_j$, and therefore

$$(C_j^{\mathrm{mix}})^* C_j^{\mathrm{mix}} = U_j^* C_j Q_j C_j U_j = G_j^{\mathrm{rec}} \succeq 0.$$

Inverse order proves the complete Schur lower

$$H_j - (C_j^{\mathrm{mix}})^* T_j^{-1} C_j^{\mathrm{mix}} \succeq \Gamma_j.$$

Hence a positive directed lower for Γ_j proves $A_j \succeq 0$ on its entire supported domain by Schur square completion.

Every arithmetic input action is a finite collection of killed and reflected sine pieces. A common finite endpoint partition reduces both A_j^{ar} and G_j^{full} to continuous piecewise trigonometric integrals, with all paired cross terms retained. The digamma diagonal also has directed enclosures. This proves finite entry computability. Subtraction in G_j^{rec} must be enclosed in the full matrix order; its exact positivity alone does not price numerical cancellation.

Under RH, the odd pole calculation and Lemma 3.1 give $A_j \succeq (\beta_j - \pi/2)I$. Since $\|C_j^{\mathrm{mix}}\| \le P_j$ and $\tau_j \ge 4P_j^2/\beta_j$,

$$\Gamma_j \succeq (3\beta_j/4 - \pi/2)I \succeq \beta_j I/2.$$

Here $j \ge 4$ gives $\beta_j \ge e^2 > 2\pi$. Subtracting a two-sided center radius loses at most $2e_j$, giving the stated directed margin and effective recognition at each fixed j.

Conversely, suppose the matrices pass for every integer $j \ge J$. The raw supported form $B_j = A_j + R_j - \beta_j I$ then satisfies $B_j \succeq -\beta_j I$. Embed any compact real odd polar-free profile supported in $[-L, L]$ at $j = \lceil L \rceil \ge J$. The complete forms agree on it under zero extension and the half-unitary map preserves its unweighted norm. Thus

$$q_{\mathrm{geo}}[f] \ge -e^{\sqrt{\lceil L \rceil}} \|f\|_2^2.$$

For every $\varepsilon > 0$, $\sqrt{\lceil L \rceil} \le \sqrt{L+1} \le \varepsilon(L+1) + 1/(4\varepsilon)$. This supplies one support-growth constant $C_\varepsilon = e^{\varepsilon + 1/(4\varepsilon)}$ for all admitted profiles and sufficiently large supports. Proposition S22.1 with $a = 0$ proves RH. No arbitrary gapped support subsequence is used in this step.

Finally the upper guards in (S191) give $\log N_j = O(\log j + P_j + P_j^2/\beta_j)$, yielding the stated bound. For $j = \lceil L_0(M) \rceil$, $e^{2j} = O(\log M)$ and $j = O(\log \log M)$; hence

$$\frac{j^2 e^{2j - \sqrt{j}}}{\log M} = O(j^2 e^{-\sqrt{j}}) \longrightarrow 0, \qquad \frac{j e^j + \log j}{\log M} \longrightarrow 0.$$

This proves the sufficient dimension assertion for the prescribed near-threshold choices. Entry count, precision and validation costs are separate; this dimension estimate is not a total runtime bound. □

Corollary S50.2 (A prescribed controlled-sparse support sequence)**.** *Fix a strictly increasing unbounded integer sequence j_k such that $j_{k+1} = o(j_k^2)$. RH is equivalent to existence of strictly positive, fully paid lower enclosures of the matrices* (S192) *at every sufficiently large j_k. In particular the prescribed sequence $j_k = 2^k$ suffices.*

Proof. Under RH the preceding strict margin applies at every support. Conversely take an arbitrary sufficiently large support L and its first containing sequence support. If $j_k < L \le j_{k+1}$, the accepted form gives the common floor $e^{\sqrt{j_{k+1}}}$ for every legal profile at that L. Moreover

$$\frac{\sqrt{j_{k+1}}}{L} \le \frac{\sqrt{j_{k+1}}}{j_k} \longrightarrow 0.$$

Equality $L = j_k$ uses that support directly. Thus for every $\varepsilon > 0$ the floor is at most $e^{\varepsilon L}$ for all sufficiently large real L, uniformly over the profiles. The support-growth theorem proves RH. For $j_k = 2^k$, the next containing support is at most $2L$, giving the explicit envelope $e^{\sqrt{2L}} \le e^{1/(2\varepsilon)} e^{\varepsilon L}$. □

Along a sparse sequence the resource label is its native $M_j = \exp\{(\pi/10)e^{2(j-2)}\}$, for which $L_0(M_j) = j$ and $N_j = M_j^{o(1)}$. Rounding an arbitrary $L_0(M)$ up to its next doubling support can change this comparison and does not establish a uniform $N = M^{o(1)}$ bound in that arbitrary smaller label. The criterion specifies an infinite signed matrix target; finitely many passing tests do not decide its all-needed-support premise.

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