RH:
Complete Signed Certificates and Canonical Null Synthesis

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October 29, 2026

Keywords. Riemann hypothesis; Weil positivity; canonical null interpolation; weighted interpolation cost; arithmetic covariance.
MSC 2020. 11M26, 11M06, 47A63.

Abstract

We give quantitative complete signed certificates and describe exact null corrections for a weighted realization of Weil's form. For every fixed $\varepsilon > 0$, the specified inverse-receiving construction has a sufficient trial cutoff $m_M \le M^{1+\varepsilon}$ for all sufficiently large M, retaining the polar constraint, all arithmetic shifts and the complete mixed return. In its guarded input format, construction and verification have polynomial bit cost in the numeric cutoff M. Under RH, the admitted fixed instances have recognizable strict certificates. We also identify the closed canonical derivative carrier with the entire full-multiplicity Mellin zero-jet class in the original reciprocal-theta norm. An equivalent continuous-area norm and uniform estimates on zero-free exterior arms prove strong inward synthesis. Contour moments then give a computable orthogonal projection and optimal exact lifts of any fixed controllable finite head. At exact unit row moment and profile tolerance η, the logarithm of the least genuine-null physical norm has order $\eta^{-4/(8j+13)}$ for a fixed row of highest index j; simultaneous lifts admit linear-order exponential budgets and prescribed orders $J = o(\log\log M)$. A complete evaluated certificate and paid localization give the whole-domain lower bound -3×10^{-8} in the original physical norm. The covariance estimates and critical signed criteria connect these constructive and geometric results to arithmetic Weil positivity.

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1 Complete certificates and canonical null synthesis

Weil positivity turns a zero-location problem into a signed inequality on an infinite-dimensional space. Two difficulties remain when that inequality is approached through finite calculations: a finite matrix must control every omitted direction, and a null correction must remain legal in the original weighted norm. This paper gives quantitative receiving resources for the first difficulty and a complete description of the canonical full-jet carrier for the second. Their consequences include computable null projection, sharp interpolation costs and a whole-domain lower bound from an evaluated certificate.

1.1 The original object and the finite verification problem

Let $s^2 = \psi/\cosh(u/2)$ be the even theta weight defined in Section 2. A physical vector h has profile $f = sh$. The original Hilbert space $\mathcal{K}$ consists of odd $L^2(\mathbb{R})$ vectors with $\langle h, s\sinh(u/2)\rangle = 0$; V is the minimal form domain and H its associated weighted Weil operator. Every norm, constraint and finite-to-infinite return below refers to this realization. The odd Weil criterion is proved in Appendix A.

A positive compression alone controls its finite trial space. Our supported construction instead retains a positive receiving complement and bounds the entire off-diagonal return. Its trial functions approximate a specified inverse in energy norm, and a directed finite-matrix test pays that approximation before giving a lower bound for every supported polar vector. Section 3 states this complete criterion; Section 6 proves its foundations and quantitative approximation theorem.

The principal resource improvement is the following fixed-exponent law: for every fixed $\varepsilon > 0$, the specified receiving construction has

$$m_M \le M^{1+\varepsilon}$$

for all sufficiently large M. Its constants and onset may depend on ε. The core dimension and the receiving trial cutoff count different parts of the certificate. The guarded construction result also specifies outward acquisition, inverse margins and finite acceptance; its polynomial bit bound is in the numeric parameter M. These statements retain the same polar constraint, complete arithmetic shifts and mixed receiving terms as the signed criterion.

Under RH, each fixed admitted instance has a recognizable strict certificate. Conversely, accepted complete certificates along the specified unbounded support schedule return positivity to the original domain by Theorem 2.2. The constructive resource law and the arithmetic acceptance condition have distinct roles in this implication. The evaluated M25 instance, together with the localization proved below, gives

$$q_H[h] \geq -3 \times 10^{-8} \|h\|_2^2 \qquad (h \in V).$$

The proof uses the original physical norm and pays the nonlocal arithmetic boundary terms and polar repair.

1.2 The actual weighted carrier and its constructive consequence

Put $q_0 = (\partial_u^2 - 1/4)\psi$ and $c_j = q_0^{(2j+1)}$. The physical canonical rows are $z_j = c_j/s$. Let $X = L^2(\mathbb{R}, s^{-2}du)$ and let N be the X-closed canonical carrier. With

$$A(z) = -z(z^2 - 1/4)\xi(1/2 + z), \qquad \mathcal{M}f(z) = \int_{\mathbb{R}} f(u)e^{zu}\, du,$$

let J consist of the legal odd polar profiles for which $\mathcal{M}f = AE$ with E entire and even. This is the full-multiplicity zero-jet condition. We prove $N = J$ in the original X topology.

The essential step is strong inward synthesis: $\mathcal{M}f_\rho(z) = A(z)E(\rho z)$ converges to $\mathcal{M}f$ in the transform norm corresponding to X. An explicit continuous gamma weight gives an equivalent area norm. Uniform estimates on zero-free exterior arms control dilation, while odd tilted Plancherel norms return the central strip to those arms. Tight tails pay the passage from local entire convergence to the actual weighted norm. This supplies the metric control needed in addition to exponential type and strict inward carrier membership.

Write $N_H = \{f/s : f \in N\}$ in physical coordinates. The orthogonal projector P_N onto N_H is computable on computable physical L^2 vectors. Certified contour moments detect all principal parts of $\mathcal{M}f/A$, including repeated zeros, without requiring a supplied zero or multiplicity list. The canonical vectors and the Riesz vectors of those constraints give two orthogonal effectively generated dense sides. A residual search then computes the orthogonal decomposition. For a computable finite isometry U whose carrier observation $U^*|_{N_H}$ is onto, put $F = U^*P_NU$. The optimal carrier lift is

$$V_{\mathrm{opt}} = P_N U F^{-1}, \qquad V_{\mathrm{opt}}^* V_{\mathrm{opt}} = F^{-1}.$$

Finite canonical exact lifts approach this optimum with certified whole-map error. The full-jet condition in this statement fixes the multiplicity scope of the projector.

The canonical family has prior same-weight ancestry. Van Hemert's repaired family $h_j = \psi^{(2j+1)} - 4^{-j}\psi'$ satisfies $c_k = h_{k+1} - h_k/4$ and has the same finite spans after the stated weighted identification [13]. The contribution here is exhaustion of the compatible full-jet class in the actual norm and its constructive projection. Laplace–area representations, gamma-integral asymptotics and least-norm lift algebra are used in their classical roles.

1.3 Sharp costs and covariance transfers

The projection result and the interpolation estimates answer different quantitative questions. For a fixed nonzero canonical row combination $z = \sum_{i=0}^{j} \alpha_i z_i$, $\alpha_j \neq 0$, let $C_z^H(\epsilon)$ be the least physical norm of an operator-null vector k with $\langle z, k\rangle = 1$ and $\|sk\|_2 \leq \epsilon$. Corollary 10.2 proves

$$\log C_z^H(\epsilon) = \Theta_z(\epsilon^{-4/(8j+13)}). \tag{1}$$

The upper construction lies in the admitted genuine-null carrier; the lower estimate applies on the original physical space. The simultaneous lift has squared-norm budget $\exp(D_J e^{2L})$ with $D_J \le \exp(C(J+1))$ and onset at most $C(J+1)$ for each fixed set of auxiliary tolerances. This admits every prescribed $J = o(\log\log M)$ in the corresponding cost-paid covariance transfer. The exact polar leading-cost refinement has its own specified comparison class; it is distinguished from the sharp logarithmic order.

For the complete averaged negative source $E_{M,K}$, positive form $P_{M,K}$ and original-norm ridge κ, write

$$\gamma_M(K;\kappa) = \|E_{M,K}^*(P_{M,K}+\kappa I)^{-1}E_{M,K}\|.$$

All source directions are retained. The unconditional covariance theorem has, for fixed $b, \epsilon > 0$, the estimate

$$\sup_K |\gamma_M(K;M^{-b}) - 1| \le C_{b,\epsilon} M^{-1/2}\sqrt{\log M}\exp[(8/\sqrt{\pi}+\epsilon)\sqrt{b\log M}], \tag{2}$$

uniformly over finite genuine-null restrictions, including moving ones. This estimate and the critical signed criteria have separate proof chains. On the original polar recipient,

$$\text{RH} \iff \liminf_{M\to\infty} \sqrt{M}(\gamma_M(\{0\};M^{-1}) - 1)_+ < \infty. \tag{3}$$

On arbitrary finite genuine-null recipients K_M, the corresponding criterion is

$$\text{RH} \iff \liminf_{M\to\infty} \sqrt{M}(\gamma_M(K_M;M^{-1}) - 1)_+ = 0. \tag{4}$$

For restrictions in one fixed canonical span, the bounded version also holds by the independent qualitative interpolation proof in Appendix C. That argument takes the arithmetic limit for each fixed test before taking the small-profile limit, and does not require the quantitative lift.

An eventual all-integer bound $\sqrt{M}(\gamma_M - 1)_+ \le C(\log M)^A$, $A \ge 0$, at a fixed polynomial ridge gives $\Re\rho \le 1/2 + A$ on the specified cost-admitted recipients. This is a conditional zero-location transfer. For comparison, the merged all-moduli Dirichlet endpoint in the Quasi-Riemann tracker is $87338196661320751993/10^{20}$ [36]; applying the transfer below that boundary requires the correspondingly smaller arithmetic exponent, together with any stated moving-recipient cost.

1.4 Increment and organization

The public predecessor of this paper, edition 10, establishes complete coefficient-ball covariance control, the original critical criteria and complete supported signed tests. Its prescribed receiving bounds are exponential in a positive power of M. The present fixed-exponent receiving law strengthens that constructive resource, while weighted carrier exhaustion and projection add a distinct geometric capability. Sharp null cost, linear-order lift dependence and the evaluated global floor describe further consequences in this same realization. The existing strip/divisor closure in *Polynomial cores, inertia and Carleman moment bounds*, edition 3, and *Polynomial synthesis for zero ideals of the Riemann xi function*, edition 2, concerns fixed-strip graph and divisor topologies [25, 31]. Those estimates do not by themselves pay the infinite factorial-line or continuous-area tails required by actual X; that return is proved here before using carrier closure.

The main text places the complete receiving theorem and its proof before carrier classification and sharp interpolation costs. The technical supplement gives the complete covariance proofs and groups alternative receivers, worked data and auxiliary source interfaces by verification role. Shared Green, polar and domain arguments are stated once and cited where used.

2 The original form, norm and domain

The Riemann hypothesis has a positivity endpoint in a fixed odd Weil realization. We retain this realization throughout the complete covariance estimates and the constructive signed certificates. We first specify the original norm and domain; the later reductions return to precisely these objects.

Use $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)e^{-iu\xi}\,\mathrm{d}u$ and Plancherel measure $\mathrm{d}\xi/(2\pi)$. For $u \geq 0$ define

$$\psi(u) = e^{u/2}\sum_{n\geq 1}(4r_n^2 - 6r_n)e^{-r_n}, \quad r_n = \pi n^2 e^{2u}, \qquad s^2 = \psi/\cosh(u/2), \tag{5}$$

with the even continuation given by theta inversion. Let $\mathcal{K}$ be the odd subspace of $L^2(\mathbb{R})$ satisfying

$$\langle h, s\sinh(u/2)\rangle = 0.$$

Write $\Psi = \Gamma'/\Gamma$ and

$$a(\xi) = \Re\Psi(1/4 + i\xi/2) - \Psi(1/4) \geq 0, \qquad m_0 = \Psi(1/4) - \log\pi.$$

The domain V is the completion of the smooth compact odd polar-free core for

$$\|h\|_V^2 = \|h\|_2^2 + \int_{\mathbb{R}} a(\xi)|\widehat{sh}(\xi)|^2\frac{\mathrm{d}\xi}{2\pi}. \tag{6}$$

The complete form on this domain is the closure of

$$\begin{aligned} q_H[h] = &\int_{\mathbb{R}}(m_0 + a)|\widehat{sh}|^2\frac{\mathrm{d}\xi}{2\pi} \\ &- \sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\langle sh, (S_{\log n} + S_{-\log n})sh\rangle, \qquad S_b f(u) = f(u+b). \end{aligned} \tag{7}$$

Inner products are conjugate-linear first. For $f = sh$ the original physical norm is

$$\|h\|_2^2 = \int_{\mathbb{R}}|f(u)|^2\frac{\cosh(u/2)}{\psi(u)}\,\mathrm{d}u.$$

These are the conventions of the weighted-Weil and complementary-receiver predecessors [28, 26, 24].

Notation and proof dependencies. The ambient Hilbert norm is always $\|h\|_2$, and $f = sh$ is its physical profile. Q_b removes the polar row; a further projection removes only a specified finite genuine-null space. The symbols $p_{M,k}, n_{M,k}$ refer to arithmetic averaging, while the explicitly declared symbols in Section S2 and the technical supplement refer to hard cutoffs. After averaging order one is fixed, P_M, N_M, E_M and $R_M(\kappa)$ denote its positive form, negative form, complete source, and actual positive inverse. Scalar hard-cutoff budgets have only their local declared meaning. The unrestricted averaged forms use the original minimal domain V; finite genuine-null constraints use its same-form restriction $V \cap K^{\perp}$. The arguments required here are included; predecessor citations identify the development of the constructions.

Lemma 2.1 (Complete elementary envelopes)**.** *For $r = \pi e^{2|u|}$,*

$$2r^2e^{-r} \leq s(u)^2 \leq 169e^{4|u| - \pi e^{2|u|}}. \tag{8}$$

The arithmetic sum in (7) *converges in original operator norm. Moreover $0 \leq a(\xi) \leq 17\xi^2$.*

Proof. Each theta atom is positive. The first atom, $4r^2 - 6r \geq 2r^2$, and $e^{|u|/2}/\cosh(u/2) \geq 1$ give the lower bound. For the upper, $e^{|u|/2}/\cosh(u/2) \leq 2$ and $\sum_{n\geq 1} n^4 e^{-(n^2-1)r} < 2$: the consecutive ratios are bounded by $16e^{-3r} < 1/2$. Hence $s^2 < 16r^2e^{-r} < 169e^{4|u|-r}$. For $b = \log n$, put $x = |u| + |u+b| \geq b$. The upper envelope gives

$$s(u)s(u+b) \leq 169e^{2x-\pi e^x} \leq 169n^2e^{-\pi n}.$$

Thus the norm of the full omitted arithmetic sum is bounded by

$$\delta_M = 338 \sum_{n>M} n^{3/2} \log n \, e^{-\pi n} \longrightarrow 0. \tag{9}$$

This sums all integers and therefore every omitted prime power. In particular, for every integer $M \geq 2$,

$$\delta_M < 451(M+1)^2 e^{-\pi(M+1)}. \tag{10}$$

Indeed $\log n \leq \sqrt{n}$, and the consecutive ratios of $n^2e^{-\pi n}$ are at most $4e^{-\pi} < 1/4$. Summing the resulting geometric majorant gives the coefficient $1352/3 < 451$. Boundedness of s and this convergence make (6) the original semibounded form topology. Finally the digamma series is

$$a(\xi) = \sum_{n\geq 0} \frac{(\xi/2)^2}{(n+1/4)((n+1/4)^2 + (\xi/2)^2)}.$$

The $n = 0$ coefficient is at most $16\xi^2$, and the others total at most $\xi^2 \sum_{n\geq 1} n^{-3}/4 \leq 3\xi^2/8$. □

Theorem 2.2 (Cofinal complete certificates)**.** *For each j let Π_j be a finite orthogonal projection onto genuine null vectors of the complete associated operator H, and put $P_j = I - \Pi_j$. Suppose a complete signed certificate on the original domain gives*

$$q_H[P_j h] \geq -\epsilon_j \|P_j h\|_2^2 \quad (h \in V), \qquad 0 \leq \epsilon_j \longrightarrow 0 \tag{11}$$

along an unbounded sequence of indices. Then $q_H \geq 0$ on V, and the Riemann hypothesis holds. Each Π_j may use its own finite genuine-null space.

Proof. For a genuine associated-operator null vector $z \in \operatorname{Dom} H$, $q_H[z, h] = 0$ for every $h \in V$. Consequently finite null projections preserve V, $q_H[P_j h] = q_H[h]$ and $\|P_j h\|_2 \leq \|h\|_2$. For each fixed h, (11) gives $q_H[h] \geq -\epsilon_j \|h\|_2^2$; taking the limit proves positivity. Lemma A.1 supplies the endpoint implication. □

The following arithmetic realization and covariance estimates return to precisely this norm and domain. The finite-coordinate constructions in the technical supplement provide alternative implementations of the same endpoint. Related compact-window reductions retain a Fourier envelope and leading–tail coupling [39], or source-exact positive directions in a block–Schur certificate [16]. Our main estimates act on the complete odd weighted realization; the compact-window and whole-line recipients are distinct.

3 Finite-verifiable certificates for the Riemann hypothesis

We first give an explicit certificate criterion for original arithmetic Weil positivity. Every candidate uses finitely many specified functions and finite matrices, with a proved price for the whole receiving complement. The specified energy-return producer is Theorem 3.3; Corollary 3.4 pays its guarded finite verification cost. The complete-residual criterion below remains an independently paid alternative, rather than a residual interpretation of an energy error. The original odd Weil equivalence is proved in Appendix A; the complete construction and supporting proofs are in Section 6.

Original and profile coordinates. The original odd test is h, with ambient norm $\|h\|_2$, and its logarithmic profile is $f = sh$. Here $s^2 = \psi/\cosh(u/2)$, where, for $u \geq 0$,

$$\psi(u) = e^{u/2} \sum_{n\geq 1} (4r_n^2 - 6r_n) e^{-r_n}, \qquad r_n = \pi n^2 e^{2u},$$

with even continuation. Thus $\|h\|_2^2 = \int_{\mathbb{R}} |f(u)|^2 s(u)^{-2}\, du$. The odd-half unitary sends f to $g = \sqrt{2} f|_{(0,L)}$, so $\|g\|_2 = \|f\|_2$. In profile coordinates the original ridge is the multiplier $M^{-1}W$, $W = s^{-2}$; in the original h coordinates it is $M^{-1}I$. The supported polar condition is $g \perp \sinh(x/2)$. Section 2 gives the full form and minimal-domain definition; the supported realization used here is proved in Section 5.

On the odd-half unitary profile space $L^2(0, L)$, put

$$p(x) = \sinh(x/2), \quad \mathcal{F}_L = p^{\perp}, \quad e_q(x) = \sqrt{2/L}\sin(k_q x), \quad k_q = (q + \tfrac{1}{2})\pi/L.$$

The original supported form has the exact realization

$$H_f = D_L + R_L + V_L, \qquad D_L e_q = a(k_q) e_q, \qquad W = s^{-2}.$$

Here $a(k) = \Re\Psi(1/4 + ik/2) - \Psi(1/4)$, $m_0 = \Psi(1/4) - \log\pi$, and $R_L = \sum_{j\geq 0} r_j \otimes r_j \succeq 0$ is the complete Green response, with $c_j = 2j + \frac{1}{2}$ and

$$r_j(x) = \frac{e^{-c_j(L-x)} - e^{-c_j(L+x)}}{\sqrt{1 + e^{-2c_j L}}}, \qquad \|R_L\| \leq \pi/2.$$

The support-uniform norm bound is proved in Lemma 3.1. The bounded $V_L = m_0 - C_L^{\mathrm{ar}}$ contains every active prime power:

$$C_L^{\mathrm{ar}} = \mathcal{U}_L \left[1_{(-L,L)} \sum_{2\leq n<e^{2L}} \frac{\Lambda(n)}{\sqrt{n}} (S_{\log n} + S_{-\log n}) 1_{(-L,L)} \right] \mathcal{U}_L^*.$$

Thus both shift orientations and the reflected channel are retained. The exact physical coordinate is $f = sh$, with the odd-half unitary map understood; the ridge $M^{-1}W$ corresponds to $M^{-1}I$ in the original physical norm.

Lemma 3.1 (A support-uniform Green norm)**.** *For every $L > 0$ and every finite Green prefix,*

$$0 \preceq R_{L,J} \preceq \frac{\pi}{2} I, \qquad 0 \preceq R_L \preceq \frac{\pi}{2} I.$$

The bounds use the complete reflected response normalization of Section 3.

Proof. Set $c_j = 2j + 1/2$ and change to the boundary coordinate $s = L - x$. The exact response profile satisfies

$$r_j(L - s) = \frac{e^{-c_j s} - e^{-c_j(2L-s)}}{\sqrt{1 + e^{-2c_j L}}}, \qquad 0 \leq r_j(L-s) \leq e^{-c_j s}, \quad 0 < s < L.$$

The finite feature Gram $G_{ij} = \langle r_i, r_j \rangle$ therefore has nonnegative entries and obeys the entrywise bound

$$G_{ij} \leq \int_0^L e^{-(c_i + c_j)s}\, \mathrm{d}s \leq \frac{1}{2(i + j + 1/2)} =: \mathsf{H}_{ij}.$$

Put $w_j = 4^{-j}\binom{2j}{j}$. Its generating function and Tonelli's theorem give, for every $i \ge 0$,

$$\begin{aligned}\sum_{j\ge 0} \mathsf{H}_{ij} w_j &= \frac{1}{2}\int_0^1 t^{i-1/2}(1-t)^{-1/2}\,\mathrm{d}t \\ &= \int_0^{\pi/2} \sin^{2i}\theta\,\mathrm{d}\theta = \frac{\pi}{2} w_i.\end{aligned}$$

The last identity follows from integration by parts, starting with $i = 0$. For a finite complex coefficient vector z, use symmetry and

$$2|z_i||z_j| \le |z_i|^2 \frac{w_j}{w_i} + |z_j|^2 \frac{w_i}{w_j}.$$

It follows that

$$z^* G z \le \sum_{i,j<J} \mathsf{H}_{ij}|z_i||z_j| \le \sum_{i<J} \frac{|z_i|^2}{w_i} \sum_{j<J} \mathsf{H}_{ij} w_j \le \frac{\pi}{2}\|z\|^2.$$

If $T_J z = \sum_{j<J} z_j r_j$, then $G = T_J^* T_J$ and $R_{L,J} = T_J T_J^*$ have the same operator norm. This proves the finite-prefix bound. The positive prefixes increase to the original complete response; their uniform bound gives a strong monotone limit and the same inequality for R_L. □

The explicit lower form. For every integer $M \ge 2^{20}$, set

$$\begin{aligned} &U_M = \log M, \quad && \lambda_M = \log U_M, && \rho_M = U_M + 8\lambda_M, \\ &\eta_M = \lambda_M^{-1}, && L_M = \tfrac{1}{2}\log(\rho_M/\pi) + \eta_M, && J_M \in \mathbb{N}, \quad \rho_M\lambda_M/4 \le J_M \le \rho_M\lambda_M/4 + 2. \end{aligned} \tag{12}$$

Write $R_{L,J} = \sum_{j<J} r_j \otimes r_j$. An exactly specified positive dyadic step multiplier W_{low} is constructed below so that

$$\tfrac{1}{2} W \le W_{\mathrm{low}} \le W.$$

Throughout this section every block is built from the same lower form

$$S_M = D_{L_M} + R_{L_M,J_M} + V_{L_M} + M^{-1} W_{\mathrm{low}} \quad \text{on } \mathcal{F}_{L_M}. \tag{13}$$

An elementary full arithmetic budget is $B_M = 6 + 8(\log 2)e^{L_M}$. Fix $\beta = 1$, and choose a certified integer N_M with

$$L_M e^{B_M+\beta+1}/\pi \le N_M \le L_M e^{B_M+\beta+1}/\pi + 2.$$

Such an outward choice is computable by interval refinement. Let U_M be an exact profile isometry onto

$$\mathcal{V}_M = \operatorname{span}(e_0, \dots, e_{N_M-1}) \cap p^\perp, \qquad \dim \mathcal{V}_M = N_M - 1.$$

Define $F_M = [p/\|p\|, \mathsf{U}_M]$, $Q_M = I - F_M F_M^*$, and

$$\begin{aligned} d_M &= \beta + M^{-1} W_{\mathrm{low}}, \\ Z_M &= d_M^{-1} - d_M^{-1} F_M (F_M^* d_M^{-1} F_M)^{-1} F_M^* d_M^{-1}. \end{aligned} \tag{14}$$

Only a finite constraint Gram is inverted in this formula.

Theorem 3.2 (Finite-verifiable RH criterion)**.** *The complete receiving compression* $T_M = Q_M \widehat{S}_M Q_M$*, with domain* $\operatorname{Dom} D_{L_M} \cap \operatorname{ran} Q_M$ *in* $\operatorname{ran} Q_M$*, satisfies*

$$T_M \succeq (d_M)_{Q_M} \succeq \beta I, \qquad T_M^{-1} \preceq Z_M, \qquad \|Z_M\| \le \beta^{-1},$$

where Z_M *is extended by zero on* $\operatorname{ran} F_M$*. For any legal finite-DN trial map* Y*,* $Q_M Y = Y$*, put*

$$\begin{aligned} A_Y &= \widehat{S}_M(\mathsf{U}_M - Y), \\ \Gamma_M(Y) &= (\mathsf{U}_M - Y)^* \widehat{S}_M(\mathsf{U}_M - Y) - A_Y^* Z_M A_Y. \end{aligned} \tag{15}$$

Here $\widehat{S}_M$ *is the full ambient expression in* (13)*; all output functions are retained. Any certified Hermitian lower enclosure* $\underline{\Gamma}_M(Y) \preceq \Gamma_M(Y)$ *with* $\underline{\Gamma}_M(Y) \succ 0$ *proves*

$$q_H[h] \ge -M^{-1}\|h\|_2^2$$

for every original polar-free test supported in $(-L_M, L_M)$*.*

The Riemann hypothesis is equivalent to the existence of such strictly positive finite certificates along an unbounded sequence of integers $M \ge 2^{20}$*. The trial family can be fixed computably: take dyadic finite-DN proposals* w *and apply, columnwise,*

$$w \longmapsto (I - \mathsf{U}_M \mathsf{U}_M^*) \left(w - e_0 \frac{\langle p, w\rangle}{\langle p, e_0\rangle} \right). \tag{16}$$

Under RH, every admitted M *has a strict certificate from this family. Still under RH, if* $r_Y = Q_M A_Y$ *and*

$$\|r_Y\|^2 \le \frac{\beta}{72M},$$

then $\Gamma_M(Y) \succeq I/(72M)$*. A two-sided enclosure* $\|\widehat{\Gamma} - \Gamma_M(Y)\| \le e$*,* $e \le 1/(288M)$*, therefore gives the strict lower certificate* $\widehat{\Gamma} - eI \succeq I/(144M)$*.*

Proof. Section 6 proves the original-domain realization, step floor, receiving bound, and full residual identity

$$0 \preceq K_M - \Gamma_M(Y) = r_Y^*(Z_M - T_M^{-1}) r_Y \preceq \beta^{-1} r_Y^* r_Y,$$

where K_M is the exact Schur matrix of S_M. Thus a passing lower enclosure proves $S_M \succeq 0$. Complete positive order gives $S_M \preceq H_f + M^{-1}W$, yielding the asserted physical bound. Since $L_M \to \infty$, an unbounded sequence admits every compact original polar-free test. Taking the limit and the original minimal closure gives Weil positivity and hence RH.

Conversely, under RH the complete physical Green-tail bound proved below and the step floor give $S_{M,h} \succeq I/(4M)$, or $S_M \succeq W/(4M) \succeq I/(36M)$. Its exact Schur matrix has the same profile scalar lower bound. The legal computable trial family is graph-dense, so its complete residuals tend to zero at each fixed M. The residual inequality gives the stated $I/(72M)$ margin. Every entry of (15) is computable by finitely many explicit panel integrals and a finite matrix inverse. Finite directed refinement therefore supplies a strict outward certificate. For the printed radius guard, $\widehat{\Gamma} - eI \succeq \Gamma_M(Y) - 2eI$, which proves the final bound. □

3.1 A fixed-exponent receiving construction

Fix $0 < \sigma < \tau < \eta < 1/2$ before varying M, and put $\theta = \sigma/\tau$. Use the actual nonnegative cell potential $V = M^{-1}W_{\mathrm{low}} = \sum_{i=1}^n v_i \chi_i$, the complete bounded part $\mathcal{K} = R_{L,J} + m_0 - C_L^{\mathrm{ar}}$, and the explicit $\mathfrak{k}$ bounding $\mathcal{K}$ on both L^2 and X^τ in (30). Thus $\widehat{S}_M = D + V + \mathcal{K}$. Define

$$v_{\max} = \max_i v_i, \quad \mathfrak{h} = \mathfrak{k} + v_{\max}, \quad \alpha = a(k_{N_M}), \quad R_0 = \|p\|^2/|\langle p, e_0\rangle|^2, \quad P_* = \alpha R_0 + \mathfrak{h}(1 + R_0).$$

The constructive bound H in (35) uses these same cells, heights and full polar/core constraints. Set

$$G = \max\{\mathrm{e}, 144MH^2(6 + P_* + \sigma^{-1})\}, \qquad Z = (G \log G)^{1/(2\sigma)}. \tag{17}$$

Choose an outward integer m_M within two above $\max(N_M, L_M Z/\pi)$. All intermediate integer choices in the definition of H use the same outward guard; dependent budgets are recomputed at the chosen integers, not at smaller ideal thresholds. For $m \geq N_M$, let V_m be an exact isometry onto

$$\mathcal{W}_m = \operatorname{span}(e_0, \dots, e_{m-1}) \cap p^{\perp} \cap \mathcal{V}_M^{\perp}.$$

Define the finite matrices

$$H_c = \mathsf{U}_M^* \widehat{S}_M \mathsf{U}_M, \qquad T_m = V_m^* \widehat{S}_M V_m, \qquad B_m = V_m^* \widehat{S}_M \mathsf{U}_M,$$
$$X_m = B_m^* T_m^{-1} B_m, \qquad \epsilon_m^{\mathrm{frac}} = H^2(6 + P_* + \log\langle k_m\rangle)\langle k_m\rangle^{-2\sigma}, \qquad \mathcal{L}_{M,m} = H_c - X_m - \epsilon_m^{\mathrm{frac}} I.$$

At an empty trial space the inverse term is zero.

Theorem 3.3 (Specified finite-matrix criterion)**.** *For $\langle k_m\rangle \geq \exp(1/(2\sigma))$, these matrices satisfy*

$$\mathcal{L}_{M,m} \preceq K_M \preceq H_c - X_m, \qquad T_m \succeq \beta I.$$

RH is equivalent to the existence, along an unbounded sequence of admitted M, of strictly positive finite outward lower enclosures of the specified matrices $\mathcal{L}_{M,m_M}$. Under RH, every admitted M satisfies $\mathcal{L}_{M,m_M} \succeq I/(48M)$, and a two-sided operator center-radius guard $1/(288M)$, followed by subtraction of that radius, gives a lower enclosure at least $I/(72M)$. The trial dimension is $m_M - N_M$, with

$$\log m_M = \frac{\log M}{2\sigma} + o(\log M).$$

Consequently, for every fixed $\varepsilon > 0$, choosing fixed $1/(2 + 2\varepsilon) < \sigma < \tau < \eta < 1/2$ gives $m_M \leq M^{1+\varepsilon}$ for all sufficiently large admitted M. Precisely, for each such fixed choice and ε there is $M_0 = M_0(\varepsilon, \sigma, \tau, \eta, \text{chosen bounds})$ such that the last inequality holds for every admitted $M \geq M_0$. No numerical common onset is claimed. This is a sufficient count, not a minimum, a uniform variable-exponent $M^{1+o(1)}$ prescription, or a runtime estimate.

Proof. The complete constrained energy-approximation proof in Section 6.9 gives, for $X = C^* T_M^{-1} C$,

$$0 \preceq X - X_m \preceq \epsilon_m^{\mathrm{frac}} I.$$

The prescribed m_M makes this scalar at most $1/(144M)$. Under RH, $K_M \succeq I/(36M)$, proving the asserted strict margin. Conversely any passing lower matrix proves the whole $S_M \succeq 0$; the original order and exhaustion in Theorem 3.2 apply. Section 6.9 proves the actual inverse range, the complete 2σ energy return, and the selected growth law. Section 4 pays the finite computation. □

The matrices $\mathcal{L}_{M,m}$ use an energy-return error payment; $\Gamma_M(Y)$ uses the complete step-priced L^2 residual. These are two independently paid finite certificates for the same complete Schur matrix.

Corollary 3.4 (Guarded polynomial-numeric-M verification)**.** *Fix rational $0 < \sigma < \tau < \eta < 1/2$ and effective outward Hardy, reflection, Green and commutator constants as constructed below. For the constructed floor and exact symbolic coordinates specified in Section 4, a deterministic producer*

and verifier form a rational Hermitian center and a certified operator-norm radius at most $1/(288M)$ *for*

$$H_c - B_m^* T_m^{-1} B_m - \frac{I}{144M}.$$

They terminate in at most M^C *bit operations and use at most* M^C *output bits, for a finite constant* C *depending on the fixed exponents and constructive constants. A passing strictly positive lower matrix is sound without RH; RH guarantees PASS at every admitted index. Otherwise the algorithm may return NOT_CERTIFIED, which is not a certified negative direction. The cost concerns this prescribed format and is not independent of the length of arbitrary external input. An already frozen floor or basis retains its identity and requires its own paid validity/evaluation certificate; description length alone does not establish the same bound for arbitrary permitted inputs.*

Proof. The complete scalar, coordinate, inverse and matrix-error construction is given in Section 4. Its fixed subtraction is legal because (17) pays the entire return by $1/(144M)$. Receiving inversion uses the unconditional floor one; only the positive acceptance margin uses RH. The exponent C is not asserted to be $1+\varepsilon$. Polynomial in numeric M is generally exponential in its binary input length $\lceil \log_2 M \rceil$. □

Proposition 3.5 (Whole physical response and representation bounds)**.** *For* (12)*, the entire omitted Green response obeys*

$$\|s(R_{L_M} - R_{L_M,J_M})s\| \le \frac{1024 \log\log M}{M(\log M)^6} < \frac{1}{4M}, \qquad M \ge 2^{20}. \tag{18}$$

The number of retained Green responses and the sufficient number of dyadic potential cells are $O(\log M \log\log M)$*;* $N_M = M^{o(1)}$*. The generating finite-DN intersection list has profile Gram condition below* $37L_M$*. For any genuine profile isometry* E*, the physical Gram* $G = E^*WE$ *has* $\operatorname{cond} G \le M^{1+o(1)}$*.*

The proof is given in Section 6. These counts describe the core, response and potential representations. The finite trial map and directed precision are additional parts of each accepted certificate. The complete covariance estimate in Corollary S6.2 supplies a complementary quantitative description of the same arithmetic Weil positivity endpoint.

4 Quantitative approximation and verification resources

Theorem 3.3 pays the whole receiving return by one fixed-exponent construction. Corollary 3.4 pays a guarded producer and verifier for its specified finite format. These are sufficient counts, not minimum dimensions. Constants and onsets depend on the fixed exponent triple; no uniform $M^{1+o(1)}$ schedule is asserted. Polynomial in numeric M is generally exponential in binary input length. Since $e^{2L_M} \sim \log M/\pi$, the sufficient count has $\log m_M = (\pi/(2\sigma) + o(1))e^{2L_M}$ on the support scale. Neither count nor cost theorem establishes practical execution, an arithmetic sign estimate, or a passing cofinal family.

Effective inputs and guards. Fix rational exponents and compute Hardy/reflection/Fourier constants at the actual η, τ in Section 6.9. The Fourier difference integral has effective small-argument quadratic and large-argument integrable power bounds, with directed quadrature on the remaining compact interval. Absorbing denominators are explicit. Form $C_\eta, A, G_\tau, \Gamma_c, \gamma_c, M_0, r_\tau, C_0$

Table 1: Distinct sufficient resources for the guarded construction.

Resource	Guarantee
Core and representations	$N_M = M^{o(1)}$; Green-prefix length and potential cell count are $O(\log M \log\log M)$.
Receiving dimension	Fixed $0 < \sigma < \tau < \eta < 1/2$ gives $\log m_M = \log M/(2\sigma) + o(\log M)$; dimension $m_M - N_M$. Each fixed $\varepsilon > 0$ admits $m_M \le M^{1+\varepsilon}$ eventually.
Acceptance precision	Whole receiving fee $1/(144M)$ and two-sided operator radius at most $1/(288M)$.
Constructed-format cost	M^C bit operations and output bits for fixed rational exponents and effective fixed constants; C is not the receiving exponent.
Arithmetic-only alternative	The separate first-moment test in Theorem B.6 has $\log N = O(e^L + \log L)$; it is not the inverse producer.

from those bounds, not from an unchecked numerical constant for another exponent. They may be large near $1/2$ but are fixed before M varies.

For every threshold use a rational enclosure of width below $1/4$ and the ceiling of its rational upper endpoint, selecting an integer between the threshold and two above it without deciding integrality. Apply this in dependency order to J, N, n, h, m. After selecting the actual auxiliary h, recompute b_{lo}, H, G, Z before selecting m. The auxiliary h need not be materialized as an additional trial matrix. Consistent fixed-factor outward bounds retain $H, P_* = M^{o(1)}$ and $m = M^{1/(2\sigma)+o(1)}$.

The potential uses the indexed cells and coarsest dyadic rounding in Section 6. Its strict-slack receipts certify $W/2 \le W_{\mathrm{low}} \le W$ on each ENTIRE cell. Elementary theta brackets give height/denominator lengths $O(\log M)$ with harmless guard bits. The heights d_j/M are rational, not necessarily dyadic for arbitrary M. This specifies one allowed construction. A frozen floor or basis retains its identity and needs its own effective validity/evaluation certificate with paid complexity. Short descriptions alone do not provide polynomial tests of arbitrary weak transcendental inequalities. The cost is for this bounded format, not independent of the length of arbitrarily padded external input.

Exact coordinates and conditioning. Let $v_q = \langle e_q, p\rangle$. Integration gives

$$v_q = \sqrt{2/L}(-1)^q \frac{\cosh(L/2)}{2(1/4 + k_q^2)}.$$

Core columns $e_j - (v_j/v_0)e_0$, $1 \le j < N$, have Gram $I + uu^*$ with condition at most $R_0 < 16L$. In the receiver put $p_N = P_N p$, $s_N = \|p_N\|^2$ and use

$$g_q = e_q - (v_q/s_N)p_N, \qquad N \le q < m.$$

Their high coordinates are independent; they impose the EXACT original p row and all core constraints. Their Gram is $I + tt^*/s_N$, $t = (v_N, \dots, v_{m-1})$, with condition $\le 1/(1-\delta_N) < 2$ by the polar leakage bound. Both have the continuous inverse-square-root formula

$$(I + uu^*)^{-1/2} = I - \frac{uu^*}{\sqrt{1+\|u\|^2}(1+\sqrt{1+\|u\|^2})}.$$

It needs no division by $\|u\|^2$ or rank oracle. It defines exact symbolic isometries U_M, V_m; numerical acquisitions have separate error bounds. The non-DN column in F remains the FULL polar profile, not p_N. The residual alternative also has $(1+v_{\max})^{-1} I \preceq F^*(1+V)^{-1}F \preceq I$. The primary energy producer does not invert that step Gram.

Complete entries without activation decisions. First-m form entries consist of the diagonal $a(k_q)$, exact finite Green coefficient products, potential sine-product integrals, m_0, and every killed prime-power pairing. The normalized Green coefficients are given in Section 6.9. Enumerate integers up to an outward bound $K \geq e^{2L}$, $K = O(\log M)$, and determine prime powers by a sieve or trial division. Extra channels have zero killed overlap; equality with e^{2L} need not be decided. For $a = \log n$ and $\mathrm{clip}(t) = \min(L, \max(0, t))$, forward, backward and reflected intervals are

$$[0, \mathrm{clip}(L-a)], \quad [\mathrm{clip}(a), L], \quad [\mathrm{clip}(a-L), \mathrm{clip}(a)].$$

The reflected integrand retains its minus sign and argument $a-x$. These formulas include $a = L, 2L$ and vanish for $a > 2L$. Min/max have Lipschitz interval extensions. An uncertain endpoint sliver of width w costs at most $2w/L$ for normalized sine products, or $4w/L$ for two endpoints. Phase derivatives cost at most the actual frequencies. Summing all channels with their true weights has polynomial sensitivity in m, L and the feature/height budgets. Each term is integrated on its own clipped panel: no transcendental sorting or endpoint-separation assumption is needed.

The elementary directed scalar algorithms can be chosen as follows.

1. Use $a(k) = \sum_{j \geq 0} 2k^2/[c_j(c_j^2 + k^2)]$ with tail $2k^2(c_R^{-3} + c_R^{-2}/4)$ after $j \geq R$. An outward $R \geq 4(1+|k|)/\sqrt{\delta}+1$ suffices for error $0 < \delta \leq 1$. Its length is polynomial in M for $k = O(m)$ and inverse-polynomial δ. For m_0 use $\Psi(1/4) = -\gamma - \pi/2 - 3\log 2$ and $H_n - \log n - 1/n \leq \gamma \leq H_n - \log n$, taking $n \geq 2/\delta$ and directed logarithms. No fast special-function oracle is required.

2. Logarithms use binary scaling and the atanh series with geometric tail; square roots use rational bisection, and π uses alternating Machin arctangent series. Taylor evaluation after power-of-two argument reduction, with repeated squaring/doubling, supplies exponentials and trigonometric functions. Oscillatory factors have modulus one, not $e^{|\Im z|}$. Real exponential logarithmic magnitudes and required guard lengths are bounded by a fixed power of $\log M$.

3. Use the ENTIRE panel primitive, including $z = 0$,

$$\int_a^{a+\Delta} e^{zx}\, dx = e^{za}\Delta \sum_{j \geq 0} \frac{(z\Delta)^j}{(j+1)!}.$$

   Subdivide into at least $\max(1, 2|z|\Delta)$ symbolic pieces. Each local argument is at most $1/2$, and its tail after t terms is at most $2(1/2)^t/(t+1)!$ times the explicit prefactor. Here $|\Im z| = O(m)$, $|\Re z| = O(J+1)$; the piece count is polynomial in M. Directed summation includes endpoint uncertainties.

These recipes use inverse-polynomial-in-M TRUNCATION tolerances. Additional $(\log M)^{O(1)}$ rounding guard bits do not demand $2^{-(\log M)^2}$ truncation from the slow symbol or Euler series. This distinction keeps their term counts polynomial in M. Rounding individual terms to a sufficient dyadic grid prevents unnecessary denominator growth.

A whole-matrix inverse and rounding budget. Let $r = N-1$, $d = m-N$, and take a rational bound $A \geq \max(1, 6 + \log\langle k_m \rangle + \mathfrak{h})$. It bounds the actual first-m form blocks and full action and is $M^{o(1)}$ on the schedule. Empty receivers use zero return. Otherwise $T_m \succeq I$ unconditionally by (25), not by RH.

Compute H_0, B_0, T_0 with complete operator errors at most δ, including basis acquisition, and explicitly symmetrize Hermitian blocks with paid errors. Frobenius estimates suffice: a square

size-m matrix needs entry radius δ/m, not a dimension-free radius; rectangular blocks use the square root of their entry count. The rank-one basis formulas have polynomial sensitivity bounds, so inverse-polynomial scalar tolerances suffice. For $\delta \le 1/4$, $T_0 \succeq 3I/4$.

Choose an outward a_0 within two above $2(A+1)$. Then $J_0 = I - T_0/a_0$ is positive and $\|J_0\| \le 1 - 3/(4a_0)$. The inverse tail of $a_0^{-1}\sum_{j<s} J_0^j$ is at most $(4/3)\exp[-3s/(4a_0)]$. Select s to pay half the requested inverse error and the other half for directed matrix rounding. Contractivity gives polynomial s, m, a_0 amplification of rounding errors; $s = M^{o(1)}\log M$ suffices. The resolvent identity gives $\|T_m^{-1} - T_0^{-1}\| \le 2\delta$. The verifier may instead check the COMPLETE residual $\|I - T_m W\|$, which bounds $\|T_m^{-1} - W\|$ because $T_m \succeq I$.

For total inverse error η_{inv}, the common center satisfies

$$\|H_c - B_m^* T_m^{-1} B_m - (H_0 - B_0^* W B_0)\| \le \delta + (2A+\delta)\delta + (A+\delta)^2\eta_{\mathrm{inv}}.$$

Set $e = 1/(288M)$ and require, for example, $\delta \le e/[64(A+1)^2]$ and $\eta_{\mathrm{inv}} \le e/[16(A+1)^2]$. This contribution is below $e/4$. Separate $e/4$ budgets for product rounding and symmetrization leave room for every scalar/fee error. Subtract the EXACT rational fee $1/(144M)$ after certifying the analytic cutoff, avoiding numerical tail evaluation. All radii are two-sided OPERATOR bounds; every mixed matrix term is included.

Test the rational Hermitian lower center $\widehat{L} - eI$ by exact rational LDL or leading principal minors. Strictly positive pivots give PASS; zero/nonpositive pivots give NOT_CERTIFIED. The rational test terminates without a spectral separation assumption at zero and does not identify failure with an original negative direction. Fraction-free elimination has polynomial bit cost: after clearing dyadic denominators, determinant bit lengths are $O(r(p + \log r + \log(A+1)))$ at entry precision p, and polynomially many integer operations compute the required minors.

Cost and conditional success. For fixed rational exponents, $m = M^{1/(2\sigma)+o(1)}$, other norm budgets are $M^{o(1)}$, and all scalar loop counts, dimensions, operation counts and bit lengths above are bounded by M^C for some fixed finite C. The effective recipes also handle finitely many smaller admitted indices, absorbed by enlarging C. Certificates may store or regenerate the exact symbolic coordinates, integer/floor guards, scalar truncation bounds, block centers/radii, inverse residual, fee, final center and rational LDL witness. They need not store every summand or iteration. Guards are inequalities with explicit remainders, not unverified flags. This proves Corollary 3.4 for the stated format.

RH supplies the inherited exact Schur floor $1/(36M)$; the return fee leaves $1/(48M)$ and the two-sided guard followed by radius subtraction leaves $1/(72M)$. Hence RH guarantees PASS at each admitted index. Evaluation, termination, receiving inversion and soundness do not use RH. The energy cutoff alone does not certify the unbounded-action residual of $\Gamma_M(Y)$; that alternative retains its separately paid complete-residual criterion.

The arithmetic-only test has different elementary budgets: at $L = 2$ its printed conditions alone give a sufficient $N > 10^{671}$, not a performance estimate for the present receiver. Worked finite examples in Appendix D retain their original schedules and receipts and are not executions of this asymptotic producer.

5 The supported Green realization

The supported Green realization used in Theorem 3.2 follows the positive exponential construction of Li [15]. Its domain identification is included here. The complete physical-tail estimate and finite

certificate construction are proved in Appendix 6. All arithmetic sums below include every active prime power. Optional critical carriers are empty, and the original polar restriction is retained.

5.1 Exact realization and domain

On the odd-half unitary space $L^2(0,L)$ put

$$c_j = 2j + \tfrac{1}{2}, \quad k_q = (q+\tfrac{1}{2})\pi/L, \quad e_q(x) = \sqrt{2/L}\sin(k_q x), \quad D_L e_q = a(k_q)e_q.$$

Lemma 5.1 (Complete Green realization)**.** *The minimal zero-extended supported archimedean form equals $D_L + R_L$, where*

$$R_L = \sum_{j\ge 0} r_j \otimes r_j, \qquad r_j(x) = \frac{2\sinh(c_j x)}{\sqrt{e^{2c_j L}+1}}, \tag{19}$$

$$0 \preceq R_L, \qquad \|R_L\| \le C_L := L + \pi/2. \tag{20}$$

The associated operator domain is $\operatorname{Dom} D_L$*; finite DN spans are an operator graph core.*

Proof. The positive digamma difference expansion is

$$a(\xi) = \sum_{j\ge 0}\left(\frac{2}{c_j} - \frac{2c_j}{c_j^2+\xi^2}\right).$$

For the Dirichlet half-line and Dirichlet–Neumann interval Green functions,

$$G_{\infty,c}(x,y) = \frac{e^{-c|x-y|} - e^{-c(x+y)}}{2c},$$
$$G_{\mathrm{DN},c}(x,y) = \frac{\sinh(c\min(x,y))\cosh(c(L-\max(x,y)))}{c\cosh(cL)},$$

direct subtraction gives $2c(G_{\mathrm{DN},c} - G_{\infty,c}) = r_c \otimes r_c$. For $z = 2L - x - y > 0$, the sum kernel is bounded by

$$\sum_{j\ge 0} e^{-c_j z} = \frac{e^{-z/2}}{1-e^{-2z}} \le 1 + \frac{1}{2z}.$$

The constant kernel has norm L; the Carleman kernel $[2(u+v)]^{-1}$, $u = L - x$, has Schur bound $\pi/2$ with weight $u^{-1/2}$. All positive partial sums therefore converge strongly to the bounded R_L in (20).

The identity first holds on smooth compact odd tests. To identify its closure, extend a finite DN combination oddly across zero and evenly across L, with period $4L$. Cut it off in endpoint layers of width ε and mollify preserving parity. The reflected error has H^t norm $O(\varepsilon^{1/2-t})$ for every fixed $0 < t < 1/2$: rescale each bounded smooth boundary bump in the fractional Sobolev norm. The bound $a(k) \le C_t(1+k^2)^t$, from the logarithmic symbol estimate, shows convergence in the diagonal form norm. Conversely smooth compact tests have rapidly decreasing DN coefficients. Finite sequence truncation is a form core of D_L; hence the original minimal closure is precisely $D_L + R_L$. Boundedness of R_L then gives its operator domain and graph core. This argument retains the supported realization, rather than silently imposing a different endpoint domain. □

At fixed L, the original positive smooth s has positive minimum. Thus $s^{\pm 1}$ are bounded on this interval. The original physical congruence is exactly $S_h = sS_f s$, with physical norm $\|h\|_2$. The polar condition becomes $f \perp p$, $p(x) = \sinh(x/2)$, in profile coordinates. Restrict the closed forms to this subspace, retaining their minimal supported closure; one can enforce the constraint on the smooth approximants by subtracting a fixed compact test with nonzero polar pairing. Finite bounded arithmetic and ridge terms preserve these domain statements. No condition number is assumed uniform as L grows.

6 Proofs for the finite-verifiable criterion

All forms use the original supported minimal closure and its exact polar restriction. Lemma 5.1 proves $H_f = D_L + R_L + V_L$, the domain $\operatorname{Dom} D_L$, and the finite-DN graph core. The arithmetic bound follows from the complete weighted Chebyshev estimate in Section S2. The construction below combines that realization with positive form order, constrained inversion, and a complete residual identity.

6.1 Uniform theta brackets and the dyadic floor

For $x \geq 0$, put $r = \pi e^{2x}$, $t = (1 + e^{-x})/2$, and

$$\Theta(x) = \sum_{n \geq 1} (4r^2 n^4 - 6rn^2) e^{-rn^2}, \qquad W = t/\Theta.$$

Every summand is positive. Write $q_1 = (4r^2 - 6r)e^{-r}$ and

$$W_1 = t/q_1 = \frac{(1 + e^{-x})e^r}{4r(2r - 3)}.$$

Lemma 6.1 (Whole-theta brackets and variation)**.** *On $x \geq 0$,*

$$q_1 \leq \Theta \leq (129/128)q_1, \qquad (128/129)W_1 \leq W \leq W_1, \qquad |(\log W)'| < 21\pi e^{2x}. \tag{21}$$

Proof. Since $r > 3$, $q_1 \geq 2r^2 e^{-r}$. For $n = 2$, $4 \log 2 < 3$; for $n \geq 3$, $4 \log n \leq 4(n - 1) \leq n^2 - 1$. Thus

$$\frac{\Theta - q_1}{q_1} \leq 2 \sum_{n \geq 2} n^4 e^{-r(n^2 - 1)} < 2 \sum_{n \geq 2} e^{-2(n^2 - 1)} \leq \frac{2e^{-6}}{1 - e^{-10}} < \frac{5971968}{1073682775} < \frac{1}{128}.$$

The penultimate inequality uses $e > 8/3$. This proves the first two bounds for the entire series.

Termwise differentiation, justified by uniform Gaussian tails on finite intervals, gives

$$\Theta'(x) = \sum_{n \geq 1} (-8y^3 + 28y^2 - 12y) e^{-y}, \qquad y = rn^2.$$

For $y \geq 3$ the absolute polynomial is at most $20y^3$. Also $\sum_{n \geq 1} n^6 e^{-r(n^2 - 1)} < 2$: for $n \geq 2$, $6 \log n \leq 2n^2$, and the remaining tail is bounded by

$$\sum_{n \geq 2} e^{3 - n^2} \leq \frac{e^{-1}}{1 - e^{-5}} < 16/31 < 1.$$

Consequently $|\Theta'/\Theta| < 20r$. Finally $|t'/t| \leq 1/2$, proving the logarithmic derivative bound. □

Let $K_L = 21\pi e^{2L}$. Choose an outward integer n within two above $2K_L L/\log(4/3)$, and use the exact symbolic nodes $x_j = jL/n$. The width $h = L/n$ is at most $\log(4/3)/(2K_L)$. At each left node use directed evaluation $0 < a \le W_1 \le b$, $b/a \le 17/16$. The complete-theta node enclosure

$$a_j = (128/129)a \le W \le b = b_j, \qquad b_j/a_j \le 2193/2048 < 4/3$$

then needs only the elementary expression W_1. For a definite computational choice put $E_j = a_j e^{-K_L h}$, obtain rational bounds $0 < E_- \le E_j \le E_+$ with $E_-/E_+ \ge 99/100$, and choose the COARSEST power-of-two mesh $s_j \le E_-/200$. Round $(19/20)E_-$ down to that mesh to obtain an exactly represented positive dyadic d_j. This gives

$$(9/10)a_j e^{-K_L h} \le d_j \le a_j e^{-K_L h},$$

because $E_-/400 < s_j \le E_-/200$, $d_j \ge .945E_- \ge .93555E_j > .9E_j$, and $d_j \le .95E_- \le E_j$. Put $W_{\mathrm{low}} = d_j$ on that cell. These mesh comparisons are rational; the strict-slack enclosures avoid transcendental equality decisions. An unspecified arbitrarily fine mesh would not provide the claimed denominator-length bound. Logarithmic variation gives

$$d_j \le W(x), \qquad \frac{d_j}{W(x)} \ge (9/10)(a_j/b_j)e^{-2K_L h} \ge 81/160 > 1/2.$$

Thus $W/2 \le W_{\mathrm{low}} \le W$ as multiplication operators. The guarded cell count is at most $2K_L L/\log(4/3) + 2$. At each fixed L, both potentials are bounded. Their jumps change the operator by a bounded multiplier and preserve the supported form and operator domains.

6.2 The whole physical Green response

Lemma 6.2 (Complete physical sandwich)**.** *For $E_J = R_L - R_{L,J}$, $0 < \eta < L$, and $\sigma_\eta^2 = \operatorname{ess\,sup}_{L-\eta<x<L} s(x)^2$,*

$$0 \preceq sE_J s \preceq \left\{ \sqrt{(L+\pi/2)\sigma_\eta^2} + 3\sqrt{\frac{e^{-2c_J\eta}}{2c_J(1-e^{-4\eta})}} \right\}^2 I. \tag{22}$$

Proof. Use the ambient L^2 edge and interior projections $P_e = 1_{(L-\eta,L)}$, $P_i = I - P_e$. Since $0 \preceq E_J \preceq R_L$,

$$\|E_J^{1/2} P_e s\| \le \sqrt{L+\pi/2}\,\sigma_\eta.$$

On the interior $0 \le r_j(x) \le e^{-c_j(L-x)}$, whence Tonelli and the complete positive feature sum give

$$\operatorname{Tr}(P_i E_J P_i) \le \sum_{j\ge J} \frac{e^{-2c_j\eta}}{2c_j} \le \frac{e^{-2c_J\eta}}{2c_J(1-e^{-4\eta})}.$$

Use $\|s\|_\infty < 3$ for the interior factor, and apply the operator triangle inequality before squaring $\|E_J^{1/2}s\|^2 = \|sE_J s\|$. All cross terms are included. Positive partial sums pass monotonically to the bounded response. The characteristic functions act only in L^2, so no multiplication of a form-domain vector by a discontinuous cutoff is required. Restriction then preserves every polar constraint. □

All response indices are chosen outward as in (12); finite interval refinement supplies such an integer without deciding an exact-ceiling equality. Here are the explicit admissions. Write $U = \log M$, $\lambda = \log U$, $\rho = U + 8\lambda$. For $M \geq 2^{20}$, $13 < U$, $2 < \lambda$, and $\log U \leq U/4$. The last inequality follows at 13 from $\log 13 < 3 < 13/4$, and then by differentiation. Thus $\rho \leq 3U$, $\eta < 1/2$, and

$$C_L = L + \pi/2 < \tfrac{1}{2}\lambda + \tfrac{5}{2} < 2\lambda.$$

The width satisfies $L > 1$. Indeed its derivative has the sign of $U(\lambda^2 - 2) + 8\lambda(\lambda - 2) > 0$. At $U_0 = 20 \log 2$, $\rho/\pi > (40/3 + 16)/4 = 22/3 > 4$, $\lambda_0 < 3$, and hence $L(U_0) > \log 2 + 1/3 > 1$.

At the edge start $\pi e^{2(L-\eta)} = \rho$. The complete theta envelope $s(x)^2 \leq 169 e^{4x - \pi e^{2x}}$, decreasing for $x \geq 0$, therefore gives

$$\sigma_\eta^2 \leq 169(\rho/\pi)^2 e^{-\rho} \leq \frac{169}{MU^6}, \qquad C_L \sigma_\eta^2 \leq \frac{338\lambda}{MU^6}.$$

Moreover $4J\eta \geq \rho$, $2c_J \geq \rho/\eta$, and $1 - e^{-t} \geq t/(1+t)$ imply

$$2c_J(1 - e^{-4\eta}) \geq \frac{4\rho}{1 + 4\eta} > \frac{4}{3}U, \qquad \tau_J := \frac{e^{-2c_J\eta}}{2c_J(1 - e^{-4\eta})} \leq \frac{3}{4MU^9}.$$

Applying $(a+b)^2 \leq 2a^2 + 2b^2$ in (22) proves

$$\Delta_J \leq \frac{676\lambda}{MU^6} + \frac{27}{2MU^9} \leq \frac{1024\lambda}{MU^6}, \qquad M\Delta_J \leq \frac{1024}{U^5} < \frac{1024}{13^5} < \frac{1}{4}.$$

These elementary inequalities prove (18) for every admitted integer, not just asymptotically.

The complete lower form satisfies $S_M \preceq H_f + M^{-1}W$. Under original Weil positivity, its exact physical congruence is

$$S_{M,h} = H_h - sE_J s + M^{-1} s W_{\mathrm{low}} s \succeq \left(\frac{1}{2M} - \Delta_J\right) I \succeq I/(4M).$$

Equivalently $S_M \succeq W/(4M) \succeq I/(36M)$. The bounded response and potential changes preserve the same original minimal supported closure. Compact polar-free exhaustion uses the exact physical row $s(x)\sinh(x/2)$, as in the proof of the odd criterion.

6.3 The small-support intersection core and receiving floor

The scalar symbol bounds needed below follow directly from

$$a(k) = \sum_{j \geq 0} \frac{2k^2}{c_j(c_j^2 + k^2)}.$$

For $k \geq 1$, comparison with the decreasing integral gives $a(k) \geq \log k$; separating $j = 0$ and bounding the remaining sum by its integral gives

$$a(k) \leq 5 + \log(1 + k), \qquad k \geq 0. \tag{23}$$

For the lower bound, the integral over $c \in [1/2, \infty)$, divided by the spacing 2, is $\frac{1}{2}\log(1+4k^2) \geq \log k$. For the upper, the first summand is at most 4; the remaining integral is at most $\frac{1}{2}\log(1 + 4k^2) \leq \log 2 + \log(1 + k)$.

Let $\bar{p} = p/\|p\|$, P_n be the first n DN projection, and $\delta_n = \|(I - P_n)\bar{p}\|^2$. Integration by parts gives the exact coefficients

$$|\langle \bar{p}, e_q \rangle|^2 = \frac{\cosh(L/2)^2}{L(\sinh L - L)(1/4 + k_q^2)^2}. \tag{24}$$

For $L \geq 1$, the ratio $\cosh(L/2)^2/(\sinh L - L)$ decreases: its derivative is $-L \sinh L/[2(\sinh L - L)^2]$. Its value at 1 is below 9, since $\cosh 1 < 2$ and $\sinh 1 - 1 > 1/6$. The first-term-plus-integral estimate of the remaining fourth-power series therefore yields, for $n \geq 1$,

$$\delta_n \leq \frac{9L^3}{\pi^4 (n + 1/2)^3} = \frac{9}{\pi k_n^3}.$$

If $f \perp \mathcal{V}_M$ and $f \perp p$, its low-frequency component is proportional to $P_{N_M} p$. The exact polar row and Cauchy–Schwarz give $\|P_{N_M} f\|^2 \leq \delta_{N_M} \|f\|^2$, hence

$$D_L[f] \geq \{a(k_{N_M}) - \delta_{N_M}[a(k_{N_M}) - a(k_0)]\}\|f\|^2.$$

Put $X = B_M + \beta + 1 > 2$. Since $k_{N_M} \geq e^X$, the decreasing function $[5 + \log(1 + k)]/k^3$ and (23) give

$$\delta_{N_M}[a(k_{N_M}) - a(k_0)] \leq \frac{9}{\pi}(X + 6)e^{-3X} < 3(X + 6)e^{-3X} \leq 24e^{-6} < 3/8.$$

The deliberately generous $X + 6$ bound follows from $\log(1 + e^X) < X + 1$. Thus D_L on the whole receiving complement exceeds $B_M + \beta + 5/8$. Dropping only the positive retained responses and using $V_L \succeq -B_M I$ proves

$$T_M \succeq \beta I + M^{-1} Q_M W_{\text{low}} Q_M. \tag{25}$$

This proof is safe for every larger outward N_M.

For completeness write $v_q = \langle e_q, p \rangle$, $v_0 > 0$. The finite generating list $w_j = v_0 e_j - v_j e_0$, $1 \leq j < N_M$, has Gram $v_0^2 I + v_{\text{tail}} v_{\text{tail}}^*$. By (24) and $\cosh(L/2)^2/(\sinh L - L) > 1/2$,

$$\operatorname{cond} G_w \leq \frac{\|p\|^2}{v_0^2} < 2L(1/4 + \pi^2/(4L^2))^2 < 37L, \qquad L \geq 1.$$

Its inverse square root defines a computable exact profile isometry. All inputs remain finite DN combinations.

6.4 Constrained step inversion and the full residual

Let $F = F_M$, $Q = Q_M$, $d = d_M$, and $G_d = F^* d^{-1} F$. Since $F^* F = I$ and d is a finite positive step multiplier,

$$d_{\max}^{-1} I \preceq G_d \preceq d_{\min}^{-1} I.$$

Thus the finite inverse is both well-defined and susceptible to directed verification with an explicit lower bound. For $g \in \operatorname{ran} Q$, the function

$$z = d^{-1} g - d^{-1} F G_d^{-1} F^* d^{-1} g$$

satisfies $F^* z = 0$ and $Q dz = g$. It is the solution for the restricted multiplier form d_Q. The same formula annihilates F, so it defines $Z = (d_Q)^{-1}$ extended by zero. From (25), inverse form order gives

$$0 \preceq T_M^{-1} \preceq Z, \qquad \|Z\| \leq \beta^{-1}.$$

Write $H_c = \mathsf{U}_M^* S_M \mathsf{U}_M$, $C = Q\widehat{S}_M \mathsf{U}_M$, and $K = H_c - C^* T_M^{-1} C$. For a legal operator-domain Y, $QY = Y$, the complete residual identity is

$$K = (\mathsf{U}_M - Y)^* \widehat{S}_M (\mathsf{U}_M - Y) - (C - T_M Y)^* T_M^{-1} (C - T_M Y).$$

This follows by expanding both squares and cancelling all linear and quadratic Y terms. As $r_Y = C - T_M Y = Q\widehat{S}_M(\mathsf{U}_M - Y)$ and $ZF = 0$, replacing T_M^{-1} by Z yields exactly

$$K - \Gamma_M(Y) = r_Y^*(Z - T_M^{-1}) r_Y, \qquad 0 \preceq K - \Gamma_M(Y) \preceq \beta^{-1} r_Y^* r_Y.$$

The positive receiving block makes this Schur identity valid even when the core block is indefinite. Acceptance retains the complete ambient action covariance, not a square of a finite output compression.

6.5 Computable trials, complete entries, and strict acceptance

Finite DN sums form a graph core for D_L. Formula (24) also shows $p \in \operatorname{Dom} D_L$, because $a(k_q) = O(\log(2+q))$ and the coefficient squares are $O(q^{-4})$. Restriction to $p^\perp$, and then to the finite-DN core complement, has the expected operator domain: the off-diagonal rows against these finitely many domain vectors are bounded. More explicitly, $\langle p, D_L f\rangle = \langle D_L p, f\rangle$, and the same identity holds for the finite core columns. Thus their removed row components are bounded in $\|f\|_2$, making the ambient and restricted graph norms equivalent on the constrained domain. All other terms in $\widehat{S}_M - D_L$ are bounded at fixed M.

For a true receiving-domain f, DN truncation converges in this graph norm. Exact pole correction using e_0, followed by projection off the finite core, preserves graph convergence and finite-DN structure. The corrections in (16) are graph-continuous and act identically on true receiving vectors. Dyadic finite-DN proposals therefore give a computable graph-dense trial family. Each column of $T_M^{-1}C$ belongs to the receiving operator domain. Approximating these finitely many columns by that family makes $\|C - T_M Y\| \to 0$.

We spell out finite evaluability. The function d^{-1} is a finite step function. For finite-DN inputs, diagonal actions are finite sine sums; each retained response is an explicit exponential; each killed prime translation is a finite sine expression on its overlap panels. The odd-half identification supplies the reflected pieces as well. The potential action is another finite piecewise sine sum. Hence every entry in $A_Y^* d^{-1} A_Y$, $F^* d^{-1} A_Y$, G_d, and the form matrix uses finitely many sine, cosine and exponential integrals over a common finite partition. For example, the entire expression

$$\int_a^b e^{zx}\,dx = e^{za}(b-a) \sum_{m\ge 0} \frac{[z(b-a)]^m}{(m+1)!}$$

has an explicit exponential-series remainder and includes $z = 0$. It avoids an equality decision at a vanishing denominator. Outward panel endpoints and continuous zero-overlap limits similarly handle coincident shift endpoints.

All scalar coefficients are computable with directed errors. The displayed positive series for $a(k)$ has tail at $j \ge R$ at most $2k^2[c_R^{-3} + c_R^{-2}/4]$, by its first term and integral. Logarithms, exponentials, trigonometric functions and square roots have their usual convergent rational enclosures. The Euler constant entering m_0 may be enclosed between $H_n - \log n - 1/n$ and $H_n - \log n$. Finite-dimensional positive square roots and inverses can be enclosed using the explicit Gram lower bounds. The outward arithmetic index may be chosen within two of e^{2L}; any additional killed shift has zero overlap on the actual support. Consequently a finite certificate contains directed bounds for every scalar, node, height, action, constraint inverse, and final matrix.

Under RH the complete lower form is at least $I/(36M)$ in profile norm. Minimization over the entire receiving space with fixed core coefficient therefore gives $K \succeq I/(36M)$. The residual tolerance $\|r_Y\|^2 \le \beta/(72M)$ proves $\Gamma_M(Y) \succeq I/(72M)$. The computable graph core attains this tolerance at each fixed M. Directed refinement of the finitely many entries can attain any positive error radius. With $\|\widehat{\Gamma} - \Gamma_M(Y)\| \le e \le 1/(288M)$, the lower matrix obeys $\widehat{\Gamma} - eI \succeq I/(144M)$. This proves conditional fixed-instance completeness and finite recognition by a strict positive matrix test. Conversely, every accepted certificate gives a complete original-norm allowance M^{-1}; cofinal exhaustion proves Theorem 3.2.

6.6 The constrained energy identity and legal first-m repair

Write $\widehat{S}_M = D_L + \mathcal{B}$, where $\mathcal{B} = V + \mathcal{K}$ and $\|\mathcal{B}\| \le \mathfrak{h} = v_{\max} + \mathfrak{k}$. The sharper X^τ estimate for the actual Green prefix is derived in Section 6.9; no order inference from a positive response bound is required.

For the normalized polar row, cosine Parseval gives

$$\sum_{q\ge 0} k_q^2 |\langle p, e_q\rangle|^2 = \|p'\|^2, \qquad \frac{\|p'\|^2}{\|p\|^2} = \frac{\sinh L + L}{4(\sinh L - L)} < 4, \quad L \ge 1.$$

The ratio decreases, and the inequality follows at one from $\sinh 1 > 7/6 > 17/15$. Thus $\|\mathcal{J}\bar{p}\| \le \sqrt{17}$. The symbol bound also gives $a(k) \le 4 + \frac{1}{2}\log(1+4k^2) \le 4 + 2k$, hence $a(k)^2 \le 32 + 8k^2$ and $\|D_L \bar{p}\| \le 8$. For $F = [\bar{p}, \mathsf{U}_M]$, on the whole coefficient space,

$$\|D_L F\| \le \sqrt{64 + \alpha^2}.$$

The constrained graph argument above identifies $\operatorname{Dom} T_M = \operatorname{Dom} D_L \cap \operatorname{ran} Q_M$; it does not turn Q_M into a suffix projection.

Let P_m be the first-m DN projection, $m \ge N_M$, and put $z_0 = (I - \mathsf{U}_M \mathsf{U}_M^*)e_0$. Then z_0 lies in the first N_M modes, has norm at most one, $D_L[z_0] \le \alpha$, and $\langle p, z_0\rangle = \langle p, e_0\rangle \ne 0$. For $y = T_M^{-1} C v$, define

$$w_m = P_m y - z_0 \frac{\langle p, P_m y\rangle}{\langle p, e_0\rangle} \in \mathcal{W}_m, \qquad g = (I - P_m)y.$$

Truncation preserves core orthogonality because the core is first-N_M; the displayed repair imposes the EXACT original polar row. Writing $\zeta = \langle p, P_m y\rangle/\langle p, e_0\rangle = -\langle p, g\rangle/\langle p, e_0\rangle$, we have $y - w_m = g + z_0\zeta$. Only these high/low pieces are orthogonal in L^2 and D_L; the bounded perturbation is paid on their SUM:

$$\|y - w_m\|^2 \le (1 + R_0)\|g\|^2, \quad D_L[y - w_m] \le D_L[g] + \alpha R_0 \|g\|^2, \quad T_M[y - w_m] \le D_L[g] + P_* \|g\|^2.$$

This retains every potential/Green/arithmetic mixed energy. The pole formula yields

$$R_0 = \frac{L(\sinh L - L)}{\cosh(L/2)^2}\left(\frac{1}{4} + \frac{\pi^2}{4L^2}\right)^2 < 16L :$$

the hyperbolic ratio is below two and $2(1/4 + 10/4)^2 < 16$. For $Y = T_M^{-1} C$ and $Y_m = V_m T_m^{-1} B_m$, the positive receiving form gives the exact Galerkin identity and best-energy approximation [5]:

$$0 \preceq X - X_m = (Y - Y_m)^* T_M (Y - Y_m), \qquad T_M[(Y - Y_m)v] \le T_M[Yv - w_m].$$

It prices the WHOLE receiving complement on each common complex core vector. No compressed action square replaces a complete Gram. The remaining fixed-exponent proof supplies its explicit energy tail.

6.7 Actual arithmetic commutators and the sharper return bound

We establish the support-domain and arithmetic-commutator foundations used by the receiving construction. The guarded cost proof is in Section 4. On the full line let $A = a(D)$ and use the bounded nonnegative partial multipliers

$$A_n = a_n(D), \qquad a_n(\xi) = \sum_{j<n} \{2/c_j - 2c_j/(c_j^2 + \xi^2)\}.$$

Their off-diagonal kernel is $-K_n(|x-y|)$, with

$$0 \le K_n(d) \le \frac{e^{-d/2}}{1-e^{-2d}} \le e^{-d/2} + \frac{1}{2d}.$$

Across a half-line cut, write the distances as $u, v > 0$. The exponential cross kernel has rank one and norm 1; the Carleman kernel $1/[2(u+v)]$ has norm at most $\pi/2$. The commutator consists of a cross block and its negative adjoint, so its norm is at most $1 + \pi/2$. For the interval projection $P = 1_{(-L,L)}$, subtraction of the two half-line projections gives uniformly in n, L

$$\|[A_n, P]\| \le c_* := 2 + \pi. \tag{26}$$

The diagonal constants cancel before this estimate.

For $f \in \operatorname{Dom} A$, $A_n f \to Af$, and (26) bounds $\|A_n P f\|$ uniformly. Monotone convergence of $a_n(\xi)^2 |\widehat{Pf}(\xi)|^2$ proves $Pf \in \operatorname{Dom} A$. Strong limits then give the bounded commutator $[A, P]$. The off-diagonal blocks $PA(I-P)$ and $(I-P)AP$ are bounded by c_*. Removing their self-adjoint sum from A yields a self-adjoint block-diagonal operator on $\operatorname{Dom} A$. Its interval part A_I has domain $\operatorname{Dom} A \cap \operatorname{ran} P$, and represents the restricted full-line closed form.

This is the original minimal supported realization. Indeed the block argument also bounds every interval projection uniformly in the norm $\|f\|^2 + \|A^{1/2} f\|^2$. Interior interval cutoffs converge in that norm: prove it first on the global Schwartz core, where a boundary strip of width ε has H^t norm $O(\varepsilon^{1/2-t})$, $0 < t < 1/2$, and then use the uniform projection bound and density. The logarithmic multiplier is bounded by every positive Sobolev power. An interior-supported vector can then be mollified, with convergence by Fourier dominated convergence and with support retained. Even mollifiers preserve odd parity. Thus compact smooth odd tests are a form core of the odd restricted form. Lemma 5.1 identifies this same closure with $D_L + R_L$.

Let τ_a be full-line translation and $T_a = P\tau_a P$ the actual killed translation on $\operatorname{ran} P$. Both P and τ_a preserve $\operatorname{Dom} A$; translation commutes with A. On the compressed domain the exact identity is

$$[A_I, T_a] = -PA(I-P)\tau_a P + P\tau_a (I-P)AP.$$

It proves $\|[A_I, T_a]\| \le 2c_*$ for every real a. Only the pair $T_a + T_{-a}$ is restricted to the odd sector. It has norm at most 2, preserves the odd supported domain, and the half-unitary identity $D_L = A_I|_{\mathrm{odd}} - R_L$ gives

$$\|[D_L, (T_a + T_{-a})|_{\mathrm{odd}}]\| \le 4(2+\pi) + 4C_L. \tag{27}$$

There is no shift-size singularity. Triangle inequality in this commutator estimate retains every active prime power, while the arithmetic operator in the form keeps its actual signed coefficients.

We also pay the other components. For a step b on $(0, L)$, each jump at t has

$$\|[D_L, 1_{(t,L)}]\| \le 2C_L, \qquad \|[D_L, b]\| \le 2C_L \ \mathrm{TV}(b).$$

To prove the first estimate, use the finite digamma identity $D_n = A_{I,n}|_{\mathrm{odd}} - R_{L,n}$. In half-interval coordinates the opposite-side kernel of $A_{I,n}$ is bounded by $1+1/[2(u+v)]$, giving norm at most C_L. The $R_{L,n}$ commutator costs at most C_L. Uniformity in n, followed by the same diagonal monotone convergence argument, proves both preservation of $\operatorname{Dom} D_L$ and the stated bound. Sum the jumps to obtain the variation estimate.

For the finite Green terms,

$$\|r_j\|^2 \le (2c_j)^{-1}, \qquad \|r_j'\|^2 \le 2c_j, \qquad \|D_L r_j\|^2 \le 16/c_j + 16c_j.$$

These follow from the explicit exponential profiles and the same DN cosine Parseval estimate used for p. Hence $\|D_L r_j\|\|r_j\| \le \sqrt{8/c_j^2 + 8} < 7$. The rank-one commutator identity then gives $\|[D_L, R_{L,J}]\| \le 14J$, with domain preservation. If n_{cell} is the potential cell count, an explicit bound for the whole perturbation $\mathcal{B}$ is therefore

$$\begin{aligned} K_1 = 14J + [4(2+\pi) + 4C_L] \sum_{2 \le n < e^{2L}} \frac{\Lambda(n)}{\sqrt{n}} \\ + 4C_L M^{-1} n_{\mathrm{cell}} \|W_{\mathrm{low}}\|_\infty \ \ge \ \|[D_L, \mathcal{B}]\|. \end{aligned} \tag{28}$$

Here $\mathrm{TV}(W_{\mathrm{low}}) \le 2n_{\mathrm{cell}}\|W_{\mathrm{low}}\|_\infty$; monotonicity of the heights is unnecessary. Every component preserves $\operatorname{Dom} D_L$.

6.8 Representation growth

For (12),

$$e^{2L} = \rho e^{2/\lambda}/\pi = (1/\pi + o(1)) \log M, \quad J = O(\log M \log\log M), \quad L = O(\log\log M).$$

The elementary arithmetic budget gives $B = O(\sqrt{\log M})$, so $\log N = O(\sqrt{\log M})$ and $N = M^{o(1)}$. Also $K_L = 21\pi e^{2L} = O(\log M)$, proving the cell count in Proposition 3.5. The first theta term gives $s(x)^2 = \Theta/t \ge 2r^2 e^{-r}$; this last function decreases for $r \ge \pi$. With $r_L = \pi e^{2L} = (1+o(1)) \log M$, any profile isometry E therefore has

$$I/9 \preceq E^* W E \preceq \frac{e^{r_L}}{2r_L^2} I, \qquad \mathrm{cond}(E^* W E) \le \frac{9e^{r_L}}{2r_L^2} = M^{1+o(1)}.$$

These are representation and metric bounds for the stated core and profile isometries. The explicit residual tolerance governs trial enrichment and finite precision in each certificate.

For the quantitative energy-return construction, the exact height bound gives $\log(1+v_{\max}) = O(\log M/\log\log M)$, while $\alpha = O(B_M + 1)$ and $R_0 = O(L_M)$. These are the actual height/core inputs to the fixed-exponent argument below. Its selected receiving cutoff replaces the graph prescriptions; entry acquisition, inverse and complete rounding costs are proved in Section 4.

6.9 Fixed-exponent regularity of the complete receiving inverse

We prove Theorem 3.3 for fixed $0 < \sigma < \tau < \eta < 1/2$, $\theta = \sigma/\tau$. Throughout $D = D_L$, $F = [\bar{p}, \mathsf{U}_M]$, $Q = I - FF^*$, and X^s is the DN scale

$$\mathcal{J} e_q = \langle k_q \rangle e_q, \qquad X^s = \operatorname{Dom} \mathcal{J}^s \quad (s \ge 0),$$

with completion for negative exponents. The auxiliary scale does not replace the original supported domain. The following Hardy argument is used at any fixed $0 < s < 1/2$, not only one numerical exponent.

Fractional Hardy and cut estimates. For an interval I, use the double-integral convention

$$[v]_{s,I}^2 = \int_I \int_I \frac{|v(x) - v(y)|^2}{|x-y|^{1+2s}}\, dy\, dx.$$

Put

$$c_s = \frac{3^{2s} - 2^{2s}}{2s} < 1, \quad e = \frac{1 - c_s}{2c_s}, \quad d_s^{\mathrm{H}} = 1 - (1+e)c_s = \frac{1 - c_s}{2} > 0, \quad h_s = \frac{(1 + e^{-1})2^{1+2s}}{d_s^{\mathrm{H}}}.$$

For smooth compactly supported v on the closed positive half-line, average

$$|v(x)|^2 \le (1+e)|v(y)|^2 + (1+e^{-1})|v(x) - v(y)|^2$$

over $2x < y < 3x$ with measure dy/x, then integrate with weight x^{-2s}. Fubini evaluates the first averaged integral as

$$\int_0^\infty x^{-1-2s} \int_{2x}^{3x} |v(y)|^2\, dy\, dx = c_s \int_0^\infty y^{-2s} |v(y)|^2\, dy.$$

Since $|x - y| \le 2x$ on this wedge, the difference integral is at most $2^{1+2s}[v]_{s,(0,\infty)}^2$. Absorption gives

$$\int_0^\infty x^{-2s} |v(x)|^2\, dx \le h_s [v]_{s,(0,\infty)}^2.$$

On $(0, L)$, apply the averaging only for $x < L/3$; its first term is at most $(1+e)c_s$ times the entire weighted integral. The remainder at $x \ge L/3$ is bounded by $(3/L)^{2s}\|v\|^2$. Consequently

$$\int_0^L x^{-2s} |v(x)|^2\, dx \le h_s [v]_{s,(0,L)}^2 + \frac{(3/L)^{2s}}{d_s^{\mathrm{H}}} \|v\|^2. \tag{29}$$

Reflection gives the same bound at L. Smooth-up-to-boundary density follows from even reflection and mollification: the reflected difference kernel is bounded by the original one. Fatou then extends (29) to the intrinsic fractional space. Thus this argument does not presuppose the zero-extension estimate it will prove.

Zero extension from $(0, L)$ to the line adds exactly

$$\frac{1}{s} \int_0^L \{x^{-2s} + (L - x)^{-2s}\} |v(x)|^2\, dx$$

to the seminorm. On the line, a half-line cut is bounded by the half-line Hardy inequality applied on the retained side; two such cuts bound multiplication by any interval indicator, independently of its width. Translations are isometries and restriction decreases the intrinsic norm.

For completeness, the DN scale is the same fractional space at this exponent. Extend oddly across 0 and evenly across L, with period $4L$. At even joins the difference kernel is directly bounded by the original kernel. At odd joins the extra sums are bounded by the two Hardy integrals in (29). Remote period interactions cost at most a fixed multiple of $L^{-2s}\|v\|^2$. For the periodic kernel $\sum_{n \in \mathbb{Z}} |x - y + 4Ln|^{-1-2s}$, Fourier expansion gives the multiplier

$$2 \int_{\mathbb{R}} (1 - \cos z) |z|^{-1-2s}\, dz\ |k|^{2s}.$$

The integral is finite and positive. The odd/even symmetries select exactly $k_q = (q + 1/2)\pi/L$. Restriction gives the reverse norm estimate, and comparison of $1 + |k|^{2s}$ with $(1 + k^2)^s$ proves

equivalence with $\|\mathcal{J}^s v\|$. For $L \geq 1$ these constants may be uniform, or enlarged polynomially in $1 + L$. No endpoint trace is imposed below exponent $1/2$. The Fourier and reflection arguments have the classical background in [20]; the estimates needed here have been derived explicitly.

Choose $A_s \geq 1$ to dominate the resulting norm constants for interval multiplication and the complete extension, translation, restriction and parity compositions just described. This means products of their factor norms, not just the largest separate factor norm. The number of factors in each composition is fixed. The constants are computable from the displayed Hardy constants, the convergent Fourier integral and the reflection estimates, with at most polynomial slack in $1 + L$; they are independent of cut positions and minimum cell width. In particular, for a shift $a > 0$, the actual killed pair on the odd half-profile is

$$v(x + a)1_{x+a<L} + v(x - a)1_{x>a} - v(a - x)1_{x<a,\ a-x<L}.$$

Odd zero extension, both translations and restriction give norm at most $2A_s$. This retains the reflected arm as well as both shift orientations.

Fixed scale constants and the complete Green prefix. Choose fixed effective constants $C_\eta \geq 1$ comparing X^η and the intrinsic inhomogeneous Gagliardo norm in both directions. For $L \geq 1$ the Hardy remainder L^{-2s} is at most one, so the reflection argument gives constants uniform in L. Let $A \geq 1$ bound interval cuts on every $|s| \leq \eta$ and the fixed-length odd-extension, paired-translation and restriction compositions at τ. Obtain the endpoint constants from the preceding formulas, use duality at negative exponents and interpolate between endpoints. Products of factor norms, rather than their maximum, are included in A.

The actual Green profiles have the exact DN coefficients

$$\langle e_q, r_j \rangle = \sqrt{2/L}(-1)^q \frac{c_j \sqrt{1 + e^{-2c_j L}}}{c_j^2 + k_q^2}.$$

For $x, y > 0$, $f(c) = c^2/[(c^2 + x^2)(c^2 + y^2)]$ is unimodal, has supremum $(x + y)^{-2}$, and $\int_0^\infty f(c)\, dc = \pi/[2(x + y)]$, including the continuous $x = y$ limit. Splitting the spacing-two sum at its mode gives $\sum_j f(c_j) \leq \frac{1}{2}\int_{1/2}^\infty f(c)\, dc + 2 \sup f$. For $x + y \geq 1$ this is at most $(\pi/4 + 2)/(x + y)$; for $x + y \leq 1$, use $\sum c_j^{-2} < 5 \leq 5/(x + y)$. Thus, for every actual prefix,

$$|(R_{L,J})_{qr}| \leq \frac{20}{L(k_q + k_r)} \leq \frac{7}{q + r + 1}.$$

This is an absolute coefficient estimate, not an inference of Sobolev order from $0 \preceq R_{L,J} \preceq R_L$. Writing $x = q + 1/2, y = r + 1/2$, use $(\langle k_q \rangle / \langle k_r \rangle)^\tau \leq 1 + (x/y)^\tau$. Schur weights $x^{-1/2}$ and the decreasing-integrand comparison

$$\sum_{r \geq 0} \frac{x^\nu y^{-\nu}}{x + y} \leq 2 + \frac{\pi}{\sin(\pi\nu)}, \qquad 0 < \nu < 1,$$

give both row and column bounds at $\nu = 1/2, 1/2 \pm \tau$. The integral part is $\pi/\sin(\pi\nu)$; the first term is at most two. Hence set

$$G_\tau = 7\{4 + \pi + \pi/\cos(\pi\tau)\}, \qquad \mathfrak{k} = |m_0| + \max(G_\tau, \pi/2) + 2A \sum_{2 \leq n < e^{2L}} \frac{\Lambda(n)}{\sqrt{n}}. \tag{30}$$

It bounds $\mathcal{K}$ on X^0, X^τ, hence on X^σ. Each finite r_j is in the DN X^1 form domain; the bounded conjugate matrix therefore extends that same actual finite-rank map. There is no Neumann derivative trace at this form-domain endpoint. The arithmetic composition used here is exactly

$$v(x+a)1_{x+a<L} + v(x-a)1_{x>a} - v(a-x)1_{x<a,\ a-x<L},$$

so both orientations and the reflected arm are retained. The complete weighted-prime bound gives $\mathfrak{k} = O(e^L) = O(\sqrt{\log M})$, independently of the Green-prefix length. Replacing this by the rough $2J$ bound would not support the full exponent range proved below.

A logarithmic commutator row. Let $A_{\mathrm{cut}}(u_i) = \sum_i \chi_i u_i$ and $B_{\mathrm{cut}} u = (\chi_i u)_i$, with the exact direct-sum DN scales. Disjoint cells give norm one for both maps at zero. For $v_i = \chi_i u_i$, the Gagliardo cross-cell inequality $|v_i(x) - v_j(y)|^2 \le 2|v_i(x)|^2 + 2|v_j(y)|^2$ proves

$$\Big\| \sum_i v_i \Big\|_{Z_\eta}^2 \le 2 \sum_i \|v_i\|_{Z_\eta}^2.$$

Within-cell differences and L^2 parts are accounted for by the same component norms. Therefore, putting

$$\Gamma_c = \max(1, \sqrt{2} C_\eta^2 A), \quad \gamma_c = \Gamma_c^{\tau/\eta}, \quad d = 1 + \log(n+2),$$

we have

$$\|A_{\mathrm{cut}}\|_\eta \le \Gamma_c, \quad \|B_{\mathrm{cut}}\|_\eta \le A\sqrt{n}, \quad \|A_{\mathrm{cut}}\|_s \le \Gamma_c^{s/\eta}\ (0 \le s \le \eta).$$

Duality gives $\|A_{\mathrm{cut}}\|_{-s} \le (A\sqrt{n})^{s/\eta}$. Interpolation of the exact spectral scales also yields

$$\|A_{\mathrm{cut}}\|_\sigma \le \gamma_c^\theta, \qquad \|B_{\mathrm{cut}}\|_\sigma \le A^\theta n^{\theta/2}.$$

Here is the analytic-family and domain justification. Between finite spectral test vectors, form $\mathcal{F}(z) = \mathcal{J}^z A_{\mathrm{cut}} \mathcal{J}_{\mathrm{vec}}^{-z}$. Its paired coefficients are finite entire exponential sums. Finite spectral compressions have locally uniform operator bounds on $-\eta < \Re z < \eta$, from the preceding scale estimates and unitary imaginary powers. The coefficients stabilize; density extends them to all vectors, with the same local bounds. Scalar Cauchy integrals define bounded operator derivatives, and the second-order Cauchy remainder proves norm holomorphy, rather than formal unbounded conjugation. The bounds are

$$\|\mathcal{F}(x+it)\| \le \Gamma_c^{x/\eta} \quad (x \ge 0), \qquad \|\mathcal{F}(x+it)\| \le (A\sqrt{n})^{-x/\eta} \quad (x \le 0).$$

On the circle of radius $\eta/(2d)$ at zero these norms are at most $M_0 = \max(\Gamma_c, Ae^{1/4})$; at τ use $r_\tau = \min(\tau, \eta - \tau)/2$, whose circle lies entirely in the positive strip. Thus

$$\|\mathcal{F}'(0)\| \le 2M_0 d/\eta, \qquad \|\mathcal{F}'(\tau)\| \le \Gamma_c / r_\tau.$$

On finite spectral input the derivative is $(\log \mathcal{J}) A_{\mathrm{cut}} - A_{\mathrm{cut}} \log \mathcal{J}_{\mathrm{vec}}$. Finite inputs have cut outputs in X^η; $\log \mathcal{J} : X^\eta \to X^\tau$ is bounded because $\eta > \tau$. Spectral graph approximation and closedness of the diagonal logarithm then prove actual graph-domain preservation and the commutator identity at zero and at τ. The two bounded extensions agree on their common core. Since $D = \log \mathcal{J} + B_d$, $\|B_d\| \le 6$, its row remainder costs 12 at zero and $12\gamma_c$ at τ, without summing n individual norms. For the compatible row $\mathcal{C}_{\mathrm{row}} = DA_{\mathrm{cut}} - A_{\mathrm{cut}} D_{\mathrm{vec}}$, take

$$C_0 = \max(1, 2M_0/\eta + 12, \Gamma_c/r_\tau + 12\gamma_c), \qquad \|\mathcal{C}_{\mathrm{row}}\|_0 \le C_0 d, \quad \|\mathcal{C}_{\mathrm{row}}\|_\tau \le C_0. \tag{31}$$

In particular each cell preserves $\mathrm{Dom}\, D$. All constants here are effective at the chosen fixed exponents and independent of the partition size, widths, heights and $L \ge 1$.

The same parametrix error before inversion. With the actual unchanged heights define

$$\ell = 1+4C_0[d+A^\theta d^{1-\theta}n^{\theta/2}], \quad r_i = (D+v_i+\ell)^{-1}, \quad R = A_{\mathrm{cut}}\,\mathrm{diag}(r_i)B_{\mathrm{cut}}, \quad E = \mathcal{C}_{\mathrm{row}}\,\mathrm{diag}(r_i)B_{\mathrm{cut}}. \tag{32}$$

The diagonal resolvents have norm at most $1/\ell$ on every scale. The proved cut-domain identity makes R domain-valued on L^2, and $V\chi_i = v_i\chi_i$ gives

$$(D+V+\ell)R = I+E, \qquad \|E\|_0 \le C_0 d/\ell, \quad \|E\|_\tau \le C_0 A\sqrt{n}/\ell.$$

Interpolate this SAME error, at this SAME ℓ and these SAME heights, before inversion:

$$\|E\|_\sigma \le C_0 A^\theta d^{1-\theta}n^{\theta/2}/\ell \le 1/4.$$

The separate d term also makes $\|E\|_0 \le 1/4$. Thus $W_E = (I+E)^{-1}$ has compatible norms at most two on zero and σ. No inverse on X^τ is asserted. The positive self-adjoint reference has floor ℓ; its domain-valued right inverse RW_E is its actual L^2 inverse. Writing $f_n = (\gamma_c A)^\theta n^{\theta/2}$, its positive-scale bound is

$$\|RW_E\|_\sigma \le r_\sigma = 2f_n/\ell.$$

Choose an outward integer h within two above

$$\max(1, L\exp(\Omega+1)/\pi), \qquad \Omega = \max(1, 8f_n(\ell+\mathfrak{k})).$$

Let P_h be the first-h DN projection and split $RW_E = R_{\mathrm{lo}}+R_{\mathrm{hi}}$ by placing $P_h, I-P_h$ IMMEDIATELY after each r_i, before $B_{\mathrm{cut}}W_E$. Then $a(k_q) \ge \Omega$ for $q \ge h$, and

$$\|R_{\mathrm{hi}}\|_0 \le 2/(\Omega+\ell), \quad \|R_{\mathrm{hi}}\|_\sigma \le 2f_n/(\Omega+\ell), \quad \|R_{\mathrm{lo}}\|_{0\to\sigma} \le b_{\mathrm{lo}} = 2\gamma_c^\theta\langle k_h\rangle^\sigma/\ell.$$

Consequently $R_{\mathrm{hi}}(\ell-\mathcal{K})$ contracts by at most $1/4$ on both spaces. The right factor W_E is not moved through a cell, and is used only on L^2 for the low smoothing bound. The remaining $n^{\theta/2}$ factor in Ω is essential.

Full constraints and actual inverse range. For $F = [\bar{p}, \mathsf{U}_M]$, define the whole-map bounds

$$w_N = \langle k_{N_M}\rangle^\sigma, \quad p_\sigma = \sqrt{17}+w_N, \quad d_0 = \sqrt{64+\alpha^2}+v_{\max}+\mathfrak{k}. \tag{33}$$

The polar Parseval estimate proved above gives $\|\mathcal{J}\bar{p}\| \le \sqrt{17}$, $\|DF\| \le \sqrt{64+\alpha^2}$. Since U_M is a first-N_M isometry, $\|\mathcal{J}^\sigma F\| \le p_\sigma$ without a $\sqrt{N_M}$ sum over columns. Cuts and all arithmetic channels give

$$b_\sigma = \mathfrak{k}+nAv_{\max}, \qquad c_\sigma^* = (1+p_\sigma)(\alpha+b_\sigma)w_N \ge \|\mathcal{J}^\sigma C\|, \quad C = Q\widehat{S}_M\mathsf{U}_M. \tag{34}$$

Here $\|\mathcal{J}^\sigma Q\mathcal{J}^{-\sigma}\| \le 1+p_\sigma$; no commutation with Q is used.

Lemma 6.3 (Fixed-exponent receiving inverse range)**.** *On the original receiving domain,*

$$\|\mathcal{J}^\sigma T_M^{-1}C\| \le H, \qquad H = 2c_\sigma^*[b_{\mathrm{lo}}(\ell+\mathfrak{k})/\beta + r_\sigma(1+p_\sigma d_0/\beta)]. \tag{35}$$

The bound holds on the entire complex core coefficient ball.

Proof. For $f \in QX \cap X^\sigma$, first use only the original L^2 receiving inverse: $y = T_M^{-1} f \in \operatorname{Dom} D \cap QX$, $\|y\| \le \beta^{-1}\|f\|$. Its exact ambient equation is

$$\widehat{S}_M y = f + F\mu, \quad \mu = (DF)^* y + F^*(V + \mathcal{K})y, \quad \|\mu\| \le d_0 \beta^{-1}\|f\|.$$

All polar and core rows are retained. Applying the positive-reference inverse gives in L^2

$$y = R_{\mathrm{hi}}(\ell - \mathcal{K})y + R_{\mathrm{lo}}(\ell - \mathcal{K})y + RW_E(f + F\mu).$$

The last two terms belong to X^σ, with bound

$$b_{\mathrm{lo}}(\ell + \mathfrak{k})\beta^{-1}\|f\| + r_\sigma(\|f\|_{X^\sigma} + p_\sigma d_0 \beta^{-1}\|f\|).$$

Solve using the common high-block contraction in X^σ. Its Neumann series also converges on L^2, with the same sum on the intersection; L^2 uniqueness identifies the constructed solution with the original y. This proves range admission BEFORE applying an unbounded weight to y. Taking $f = Cv$ proves the displayed whole-map bound. Heights remain paid in d_0, c_σ^*. □

The Hilbert-scale interpolation and Cauchy principles are classical; the fixed-exponent construction above is proved directly. The small-exponent extrapolation alternative is not needed here.

The full 2σ energy tail. Use the actual legal repair and Galerkin identity proved above. For $y = T_M^{-1} Cv$, $g = (I - P_m)y$, high/low orthogonality is used only for L^2 and D; the entire $V + \mathcal{K}$ norm pays the mixed energy:

$$T_M[y - w_m] \le D[g] + P_*\|g\|^2.$$

For $x \ge \exp(1/(2\sigma))$, $(6 + P_* + \log x)x^{-2\sigma}$ decreases, since its derivative has sign $1 - 2\sigma(6 + P_* + \log x)$. Thus, retaining the COMPLETE return, not a finite output compression,

$$0 \preceq X - X_m = (Y - Y_m)^* T_M (Y - Y_m) \preceq H^2(6 + P_* + \log x_m)x_m^{-2\sigma} I, \quad x_m = \langle k_m \rangle. \tag{36}$$

The Loewner bound holds for every common complex coefficient vector.

At (17), $x_{m_M} \ge Z \ge \exp(1/(2\sigma))$. Since $\log G \ge 1$ and $\log\log G \le \log G$,

$$H^2(6 + P_* + \log Z)Z^{-2\sigma} \le H^2(6 + P_* + \sigma^{-1})/G \le 1/(144M).$$

This also covers $G = \mathrm{e}$ and a dominating core cutoff. The $\log G$ factor pays the logarithmic symbol and is not dropped. The unconditional receiving floor proves finite inversion. Under RH the unchanged complete physical sandwich gives exact Schur floor $1/(36M)$; subtracting the return fee leaves $1/(48M)$. A two-sided operator radius at most $1/(288M)$, then subtraction of that radius, leaves $1/(72M)$. A passing lower matrix implies positivity on the FULL supported polar-free space. The original order $S_M \preceq H_f + M^{-1}W$, the unitary half-profile map and $f = sh$ return $q_H[h] \ge -M^{-1}\|h\|_2^2$ in the original minimal domain. The complete Green tail and original cofinal exhaustion are unchanged.

Fixed-exponent growth and effective guards. Put $U_0 = \log M, \lambda = \log U_0$. The selected construction has

$$n, J = O(U_0\lambda), \quad L = O(\lambda), \quad \log N_M = O(\sqrt{U_0}), \quad \alpha, \mathfrak{k} = O(\sqrt{U_0}), \quad \log(1 + v_{\max}) = O(U_0/\lambda).$$

In particular $r_L = (U_0 + 8\lambda)e^{2/\lambda}$, so the last bound does not treat the actual theta height as fixed. The high/low cutoff pays all three terms

$$\Omega = O(U_0^{\theta/2}\lambda^{1+\theta/2} + U_0^{\theta}\lambda + U_0^{(1+\theta)/2}\lambda^{\theta/2}) = o(U_0).$$

Each power of U_0 is strictly below one because $\theta < 1$. A guarded near-threshold h gives $\log\langle k_h\rangle \le \Omega + O(1) = o(U_0)$. The source/constraint budgets have logarithms $O(U_0/\lambda + \sqrt{U_0} + \log(n+2)) = o(U_0)$; therefore $\log(H+2) = o(U_0)$. For the selected growth equality also note $\ell \le C(n+2)^{\theta/2}d$, $r_\sigma = 2f_n/\ell \ge c/d$, and $c_\sigma^* \ge 1$ eventually. Thus $H \ge 2r_\sigma$, so $\log H = o(U_0)$ on both sides. It follows that

$$\log G = U_0 + o(U_0), \qquad \log Z = U_0/(2\sigma) + o(U_0), \qquad \log m_M = U_0/(2\sigma) + o(U_0).$$

The actual dimension $m_M - N_M$ has the same selected leading law. For fixed $\varepsilon > 0$, the strict gap $1/(2\sigma) < 1 + \varepsilon$ gives an exponent-dependent onset. Neither an endpoint choice $\sigma = 1/2$ nor a uniform variable-exponent schedule is asserted.

For every computable real threshold t, obtain a rational enclosure $[a, b]$ of width less than $1/4$ and choose $j = \lceil b\rceil$. Then $t \le j < t + 2$, without deciding whether t is integral. Apply this to J, N, n, h, m in their forward dependency order, and recompute b_{lo}, H, G, Z after choosing the actual h. Fixed factor outward bounds suffice if used consistently. For fixed rational exponents every constant above is obtained from the displayed Hardy/Fourier/Schur recipes at the chosen η, τ; a former numerical C_τ is not automatically a C_η. This completes the analytic proof of Theorem 3.3.

7 A whole-domain physical lower bound

A supported certificate can control every vector of the original form domain, including vectors whose support extends beyond the certified window. We prove this by nonlocal localization and an explicit repair of the polar moment. Applied to the moving-window instance $M = 2^{25}$, the result is

$$q_H[h] \ge -\frac{3}{10^8}\|h\|_2^2, \qquad h \in V. \tag{37}$$

The norm and minimal domain are exactly those of Section 2. This is a single whole-domain lower bound; the vanishing-allowance hypothesis of Theorem 2.2 is a separate condition.

7.1 Ambient form, complete tails and a compact moment repair

Let $\mathcal{O}$ be odd physical $L^2(\mathbb{R})$, put $b = s\sinh(u/2)$, and let $\widetilde{V}$ be the minimal completion of all smooth compact odd functions for the norm (6), without imposing the polar moment. Write $\mathcal{A}[f] = \int a(\xi)|\widehat{f}(\xi)|^2\,\mathrm{d}\xi/(2\pi)$. The same arithmetic expression (7) defines a closed semibounded auxiliary form $\widetilde{q}$ on $\widetilde{V}$: the nonnegative form $\mathcal{A}[sh]$ is closable, and its remaining terms are bounded physical operators by Lemma 2.1. Indeed $a^{1/2}(D)M_s$ is a closed operator on its maximal domain, so its restriction to the compact core is closable. Only its minimal closure is used here. A fixed smooth compact odd p with $\langle b, p\rangle = 1$ gives

$$V = \widetilde{V} \cap b^{\perp}, \qquad q_H = \widetilde{q}|_V. \tag{38}$$

To prove the nontrivial inclusion, approximate $h \in \widetilde{V} \cap b^{\perp}$ by ambient core vectors h_j. The corrected vectors $h_j - \langle b, h_j\rangle p$ converge in the same form norm and belong to the original core. The converse follows from continuity of the physical polar functional. Thus the auxiliary form does not change the realization of H.

For $R \geq 0$ and $r_R = \pi e^{2R}$, define

$$B_R = 13e^{2R}e^{-r_R/2}, \qquad t_R = \sqrt{5}\, r_R^{3/4} e^{-r_R/2}, \qquad d_K = 451(K+1)^2 e^{-\pi(K+1)}. \tag{39}$$

The elementary envelopes imply

$$\|s\|_\infty < 3, \qquad \sup_{|u|\geq R} s(u) \leq B_R, \qquad \|1_{\{|u|\geq R\}} b\|_2 \leq t_R. \tag{40}$$

For the first assertion, $e^{4u-\pi e^{2u}}$ decreases for $u \geq 0$ and $169e^{-\pi} < 9$. For the last, both arms and $\sinh(u/2)^2 \leq e^{|u|}/4$ give

$$\|1_{\{|u|\geq R\}} b\|_2^2 \leq \frac{169}{4\pi^{5/2}} \int_{r_R}^\infty x^{3/2} e^{-x}\, \mathrm{d}x \leq \frac{169}{2\pi^{5/2}} r_R^{3/2} e^{-r_R} < 5 r_R^{3/2} e^{-r_R}.$$

Here the logarithmic derivative of $x^{3/2}e^{-x}$, and $r_R \geq \pi > 3$, bound the integral by $r_R^{3/2}e^{-r_R}/(1 - 3/(2r_R))$. By (10), the complete physical prime operator omitted after the integer $K \geq 2$ has norm at most d_K. That estimate includes both orientations of every omitted prime power and uses no polar cancellation, so it applies to $\widetilde{q}$.

An explicit repair will also control the mixed form term. Set

$$\phi_0(u) = u(1-4u^2)^3\, 1_{\{|u|<1/2\}}, \qquad m = \int_{\mathbb{R}} \phi_0(u) \sinh(u/2)\, \mathrm{d}u, \qquad \phi = \phi_0/m, \quad g = \phi/s.$$

The zero extension of ϕ_0 belongs to $H^2(\mathbb{R})$, since its value and first derivative vanish at the endpoints. Exact integration and $\sinh(u/2) \geq u/2$ for $u \geq 0$ give

$$m \geq \int_0^{1/2} u^2(1-4u^2)^3\, \mathrm{d}u = \frac{2}{315},$$
$$\|\phi_0\|_2^2 = \frac{256}{45045}, \qquad \|\phi_0''\|_2^2 = \frac{2048}{77}.$$

Consequently $\|\phi\|_2 < 12$ and $\|\phi''\|_2 < 813$. On the repair support, $\pi \leq r \leq \pi e < 9$ and the lower theta envelope gives $s^2 \geq 162e^{-9} > 1/64$. For example, $\pi < 22/7$, $e < 11/4$ prove both strict scalar comparisons. Thus

$$\langle b, g\rangle = 1, \qquad \|g\|_2 \leq G_0 = 96, \qquad \|\widetilde{H} g\|_2 \leq G_1 = 43000. \tag{41}$$

Here $\widetilde{H}$ is the operator associated with the ambient form. To verify the last assertion, first approximate the compact H^2 profile ϕ smoothly. The inequality $a(\xi) \leq 17\xi^2$ then represents its form action by

$$\widetilde{H} g = s\, a(D)\phi + m_0 s\phi - P_{\mathrm{ar}} g,$$

where P_{ar} is the complete paired physical prime operator. This proves $g \in \operatorname{Dom} \widetilde{H}$. The digamma value at $1/4$ gives

$$|m_0| = \gamma + \pi/2 + 3\log 2 + \log \pi < 6.$$

For instance $\gamma < 1$, $\pi/2 < 11/7$, $3\log 2 < 21/10$, and $\log \pi < 6/5$ suffice. Moreover

$$\|P_{\mathrm{ar}}\| \leq 338 \sum_{n\geq 2} n^2 e^{-\pi n} < \frac{1352/400}{1 - 9/80} < 4.$$

The first term uses $e^3 > 20$, and successive ratios are at most $(3/2)^2 e^{-\pi} < 9/80$. Therefore

$$\|\widetilde{H} g\|_2 < 3 \cdot 17 \cdot 813 + 6 \cdot 3 \cdot 12 + 4 \cdot 96 = 42063 < 43000, \qquad |\widetilde{q}[g]| \leq G_0 G_1.$$

The repair is intentionally nonpolar; all its uses are in the ambient form.

7.2 Localization with all nonlocal mixed terms

The digamma series in Lemma 2.1, with $c_j = 2j + 1/2$, has the positive representation

$$\begin{aligned} a(\xi) &= \sum_{j\ge 0} \frac{2\xi^2}{c_j(c_j^2+\xi^2)} = \int_{\mathbb{R}} (1-\cos(\xi t))\nu(t)\,\mathrm{d}t, \\ \nu(t) &= \frac{e^{-|t|/2}}{1-e^{-2|t|}}, \qquad \mathcal{A}[f] = \frac{1}{2}\int_{\mathbb{R}} \nu(t)\|S_t f - f\|_2^2\,\mathrm{d}t. \end{aligned} \tag{42}$$

Indeed integration of each $e^{-c_j|t|}$ gives its displayed summand; positivity permits summation. This representation retains the nonlocal nature of the archimedean form.

Choose $R \ge 1/2$, $0 < w \le 1$, $L = R + w$, and the even Lipschitz functions

$$\theta(u) = \frac{\pi}{2}\min\{1, \max\{0, (|u|-R)/w\}\}, \qquad \chi = \cos\theta, \quad \eta = \sin\theta.$$

Then $\chi^2 + \eta^2 = 1$, $\chi = 1$ on $[-R, R]$, and $\chi = 0$ outside $[-L, L]$. Put

$$\Delta(u,v) = (\chi(u)-\chi(v))^2 + (\eta(u)-\eta(v))^2 \le \min\{\pi^2|u-v|^2/w^2, 4\}.$$

If $\Delta \ne 0$, at least one endpoint is outside $[-R, R]$; hence $s(u)s(v) \le 3B_R$. Expanding (42) gives the exact identity

$$\begin{aligned} &\mathcal{A}[\chi f] + \mathcal{A}[\eta f] - \mathcal{A}[f] \\ &\quad = \frac{1}{2}\int_{\mathbb{R}} \nu(t) \int_{\mathbb{R}} \Delta(u, u+t)\Re(\overline{f(u)}f(u+t))\,\mathrm{d}u\,\mathrm{d}t. \end{aligned} \tag{43}$$

For $f = sh$, its modulus is at most $(3B_R/2)J(w)\|h\|_2^2$, where

$$J(w) = \int_{\mathbb{R}} \nu(t)\min\{\pi^2 t^2/w^2, 4\}\,\mathrm{d}t \le 18 + 10\log(2/w). \tag{44}$$

For $0 < t \le 1$, concavity of $1 - e^{-2t}$ gives $\nu(t) \le 5/(4t)$; for $t \ge 1$, $\nu(t) \le (5/4)e^{-t/2}$. Split the even integral at $2w/\pi < 1$ and at 1. The three bounds are 5, $10\log(\pi/(2w))$, and $20e^{-1/2} < 13$, proving (44).

Set $S_K = \sum_{2\le n\le K} \Lambda(n)/\sqrt{n}$, or replace it by a certified upper bound. A finite paired prime term is $-2a_n\Re\int \overline{f}(u)f(u+\log n)\,\mathrm{d}u$, where $a_n = \Lambda(n)/\sqrt{n}$. Its localization difference is exactly

$$a_n \int_{\mathbb{R}} \Delta(u, u+\log n)\Re(\overline{f}(u)f(u+\log n))\,\mathrm{d}u.$$

Thus its modulus is at most $12a_nB_R\|h\|_2^2$. This computation includes both shift orientations and, on odd functions, the reflected channel. The omitted physical operator costs $2d_K$: the map $h \mapsto (\chi h, \eta h)$ is an isometry, so its diagonal compression has norm at most d_K, and subtracting the unlocalized tail costs one more d_K. The multiplication term $m_0 s^2$ localizes exactly. We have proved

$$|\widetilde{q}[\chi h] + \widetilde{q}[\eta h] - \widetilde{q}[h]| \le \{B_R[27 + 15\log(2/w) + 12S_K] + 2d_K\}\,\|h\|_2^2. \tag{45}$$

No local differential commutator formula is used.

The exterior satisfies a second complete bound:

$$\widetilde{q}[\eta h] \ge -\{(6 + 2S_K)B_R^2 + d_K\}\|h\|_2^2. \tag{46}$$

Discard the nonnegative archimedean energy. The potential costs $6B_R^2$; each surviving finite prime pair has both η-weighted endpoints outside R, costing $2a_nB_R^2$. The omitted tail costs d_K. No polar condition on ηh is needed. For a compact smooth h, all localized profiles are compact H^1; the bound $a(\xi) \le 17\xi^2$ justifies their form membership and the identities by smooth approximation.

7.3 The supported-to-global theorem

Theorem 7.1 (Explicit physical global return)**.** *Let V_L be the original minimal closure of the compact odd polar-free core supported in $(-L, L)$. Suppose a complete supported certificate proves*

$$q_H[v] \geq -\varepsilon\|v\|_2^2, \qquad v \in V_L, \quad \varepsilon \geq 0.$$

For any R, w, K as above with $R + w = L$,

$$q_H[h] \geq -\mathcal{F}(\varepsilon; R, w, K)\|h\|_2^2 \quad (h \in V), \tag{47}$$

where every fee is explicit:

$$\begin{aligned}\mathcal{F} = {} & \varepsilon(1 + 96t_R)^2 + 86000t_R(1 + 96t_R) + 4128000t_R^2 \\ & + B_R[27 + 15\log(2/w) + 12S_K] + (6 + 2S_K)B_R^2 + 3d_K.\end{aligned} \tag{48}$$

Proof. First let h belong to the original compact polar-free core and put $\ell = \langle b, \chi h\rangle$. Since $\langle b, h\rangle = 0$, (40) gives

$$|\ell| = |\langle b, (1 - \chi)h\rangle| \leq t_R\|h\|_2.$$

The repaired vector $v = \chi h - \ell g$ is odd, polar-free, supported in $[-L, L]$, and $\|v\|_2 \leq (1+96t_R)\|h\|_2$. It lies in the precise supported minimal domain V_L: its profile $sv = \chi sh - \ell\phi$ belongs to $H_0^1(-L, L)$. Approximate it by smooth odd profiles compactly supported in the open interval and remove their residual polar moments with a fixed smooth interior unit-moment correction. These corrections tend to zero; positivity and smoothness of s on this interval pay division by s, while $a \leq 17\xi^2$ pays convergence in the supported form norm. This also verifies the endpoint condition of the supported certificate.

Since $\chi h = v + \ell g$ and $g \in \operatorname{Dom}\widetilde{H}$, the exact form expansion and (41) imply

$$\widetilde{q}[\chi h] \geq -\{\varepsilon(1 + 96t_R)^2 + 86000t_R(1 + 96t_R) + 4128000t_R^2\}\|h\|_2^2.$$

Here the cross term is bounded by $2|\ell|\|v\|_2\|\widetilde{H}g\|_2$, and the last term uses $|\widetilde{q}[g]| \leq 96 \cdot 43000$. Combine this with (45) and (46); the omitted-tail charges are $2d_K + d_K = 3d_K$. This proves the bound for every compact polar h, irrespective of its support radius. Finally an original core sequence converges in the norm (6); continuity of the closed form and physical norm extends the same uniform inequality to every $h \in V$. □

7.4 The complete $M = 2^{25}$ supported instance

We now specify the numerical premise of Theorem 7.1. Use the common supported construction of Section S13, (S51), and the exact realization in Section 5, with

$$M = 2^{25}, \quad \kappa = 2^{-25}, \quad L = \tfrac{1}{2}\log(4\log M/\pi) + \tfrac{1}{10}, \quad J = 91, \quad N = 1046. \tag{49}$$

Here $26 < e^{2L} < 27$, so the complete active prime powers are

$$2, 4, 8, 16, 3, 9, 5, 25, 7, 11, 13, 17, 19, 23,$$

with weights $\Lambda(n)/\sqrt{n}$ and both killed shift orientations. Larger supported overlaps vanish. The full lower form is

$$S = D_L + R_{L,91} + m_0 - C_L^{\mathrm{ar}} + \kappa W_{\mathrm{low}} \quad \text{on } p^{\perp}, \qquad p(x) = \sinh(x/2).$$

Its complete theta-floor and reference construction is the one in Section S45: 8152 radial cells and all arithmetic split points give 8166 common cells and 8167 nodes. In particular $W/2 \le W_{\mathrm{low}} \le W$, and

$$\alpha \le a(k_N) - \frac{9L^3}{\pi^4(N+1/2)^3}\{a(k_N) - a(k_0)\} + m_0.$$

One-sided arithmetic row prices on every cell and the positive Green remainder give, on the entire receiving complement,

$$T = S_Q \succeq d_Q, \quad d = \kappa W_{\mathrm{low}} + \alpha - r_{\mathrm{up}} > 0, \quad T \succ 0.0100175\, I \succ I/100. \tag{50}$$

All $N = 1046$ columns of $F = [p, C]$ are retained, where

$$C = (e_j - (p_j/p_0)e_0)_{1\le j<N}, \qquad Q = \mathrm{proj}_{\ker F^*}, \qquad G = F^* d^{-1} F.$$

The exact upper-triangular dyadic conditioner R_G, with denominator 2^{256} and positive diagonal, satisfies the full 1046-dimensional directed check

$$\|R_G^* G R_G - I\| < 9.5514\, 10^{-75} < 10^{-65} < \delta := 10^{-60}, \qquad G^{-1} \succeq \frac{R_G R_G^*}{1+\delta}. \tag{51}$$

The last inequality follows by inverting $R_G^* G R_G \preceq (1+\delta)I$; its direction is important because the constraint inverse is subtracted from the response.

Put $E = \widehat{S} - d$. All complete mixed output moments are kept:

$$B_E = C^* E C, \quad K_E = (EC)^* d^{-1} EC, \quad H_{\mathrm{act}} = F^*C + F^* d^{-1} EC, \quad A = \frac{H_{\mathrm{act}}^* R_G R_G^* H_{\mathrm{act}}}{1+\delta} - B_E - K_E.$$

The ridge cancels symbolically in E. The data include the full pole/action row, all sine/cosine products and all Green response cross terms; the endpoint primitives are those in Section S45. This is a zero-receiving-trial case with 1046 primitive inputs and the entire 1045-dimensional pole-free core, not a selected finite compression.

For clarity, the constrained multiplication inverse is

$$Z = d^{-1} - d^{-1} F G^{-1} F^* d^{-1}, \qquad T^{-1} \preceq Z.$$

The complete Schur comparison (S51), proved in Section 6, therefore bounds the actual core Schur matrix below by A. Positivity of A is the remaining finite sign condition.

Let $v_j = p_j/p_0$, $G_C = I + vv^*$, and let R_{uu} be the core block of R_G. The analytic conditioner

$$P_C = G_C^{-1} R_{uu}^{-*}, \qquad G_C^{-1} = I - \frac{vv^*}{1+v^*v}$$

is invertible. The primary interval block elimination returned `NOT_CERTIFIED`; it is not the sign certificate. A separate complete verifier reconstructs this analytic P_C and all $1045^2 = 1{,}092{,}025$ entries of the lower matrix. It uses an upper-triangular proposal R_w whose stored float64 entries are interpreted as their *exact dyadic rational values*, with positive diagonal. For this exact invertible R_w, the full directed residual proves

$$\|R_w^*(P_C^* A P_C) R_w - I\|_\infty < 2.5751\, 10^{-9} < 1. \tag{52}$$

For the underlying Hermitian matrix the absolute-row-sum norm bounds the spectral norm. Thus the whitened matrix is strictly positive, and invertibility of both conditioners gives $A \succ 0$. The midpoint proposal itself is not an acceptance test, and no claim of 1045 positive interval elimination pivots is needed. Completion of squares against the whole positive receiver proves $S \succeq 0$. Finally $R_L - R_{L,91} \succeq 0$, $W - W_{\mathrm{low}} \ge 0$, and $sWs = I$ return to the physical norm and yield

$$q_H[h] \ge -2^{-25}\|h\|_2^2, \qquad h \in V_L. \tag{53}$$

Identifiable numerical inputs. The case identifier is `SD_M2pow25_N8_beta0.01`; the `N8` component is a seed label, whereas the complete reference and constraint dimension are $N = 1046$. The scientific input basenames are `JointreceiverSD_M2pow25_N8_beta0.01stepreferen ce.json`, `Moving-windowweightedconstraintconditionerSD_M2pow25_N8_beta0.01.j son`, `MovingmixedSD25_b0p01N1046.json`, `MovingsignedSD25_b0p01N1046.json`, and `M25independentexactdyadicconditioner.npy`. The weighted and signed check records are respectively `Moving-windowM25weightedreferenceindependentverification.json` and `Moving-windowM25signedprocessorindependentverification.json`. The latter records `PASS_INDEPENDENT_COMPLETE_SIGNED_LOWER` separately from the producer's unresolved block test. The mixed and signed input SHA-256 identities are, respectively,

`776dffad9818110e4423b838665f617d30908d8a4b2253bbb593502c37d3b701`
`cf964c4d1b94c9c1da711b2a22ab30f71f136d7c9e30436fd95075ef8ff03dca`.

The programs `verify-M25-weighted-reference.py`, `verify-M25-signed-output.py` and `exact_congruence.py` specify the checks. The mixed acquisition uses 768-bit interval arithmetic, the signed assembly 1536 bits, the separate signed congruence 2048 bits, and the weighted check 4096 bits, with Python/python-flint (Arb) for the directed computations; the dyadic proposal uses NumPy. Checking the saved full matrices and regenerating their elementary mixed moments are distinct reproducibility operations.

7.5 Exact scalar specialization and the global bound

For the exact support (49), set

$$r_L = 4(\log M)e^{1/5}, \qquad w = r_L^{-2}, \qquad R = L - w, \qquad K = 32, \qquad S_K \le 2\sqrt{32}\log 32. \tag{54}$$

The last estimate follows from $\Lambda(n) \le \log K$ and $\sum_{n=2}^{K} n^{-1/2} \le 2\sqrt{K}$. Here $r_R = r_L e^{-2w}$, $0 < w < 1$ and $r_R > \pi e$, so the localization hypotheses hold. Rational outward evaluation gives

$$84.63756753435591 < r_R < 84.63756753435592, \quad B_R < 1.464005\,10^{-16}, \quad t_R < 2.608214\,10^{-17}.$$

The complete scalar calculation, including all terms of (48), gives the exact upper bound

$$\begin{aligned}\mathcal{F}(2^{-25}; R, w, 32) &\le \frac{4567550748298320362134980099223475601077908326 7}{1532495540865888858358347027150309183618739122183602176} \\ &< \frac{3}{10^8}.\end{aligned} \tag{55}$$

Its value is approximately $2.98046593056810 34\,10^{-8}$; the strict comparison uses the exact fraction, not this decimal. For scale, the four outward component bounds are

repair, including the supported allowance	$2.980456546\,10^{-8}$
archimedean localization	$2.496961\,10^{-14}$
arithmetic localization, including $2d_{32}$	$6.888491\,10^{-14}$
exterior, including d_{32}	$1.809396\,10^{-30}$.

These rounded component bounds illustrate the sizes; the fraction in (55) comes from the unrounded complete evaluation.

The scalar specification is `check_M25_global_fee.py` with `M25conditionalexactglobalf ee.json`. It uses exact rational intervals rounded outward on the 2^{-180} grid. Machin's identity uses 160 alternating arctangent terms with the next term as remainder. Logarithms are reduced

to $1 \le x \le 2$ and use $z = (x-1)/(x+1)$ with the 160-term positive atanh expansion and tail $2z^{321}/[321(1-z^2)]$. For $0 \le x < 200$, the exponential sum through $x^{512}/512!$ has remainder at most

$$\frac{x^{513}}{513!}\frac{1}{1-x/514};$$

negative arguments are inverted. Integer square roots of the rational endpoints, rounded down and one grid unit up, enclose all roots. These formulas give complete, rationally bounded remainders for every elementary operation used in (55). The scalar record retains its conditional name: it evaluates the return fee, while (53) supplies the supported premise. Combining them in Theorem 7.1 proves (37) on the whole original V.

8 Genuine nulls and canonical row geometry

We first construct the genuine null vectors used for interpolation and compute the distinct exponential tails of their canonical rows. These two facts identify both the legal correction space and the finite-row detector needed for the quantitative theorem.

Retain $\mathcal{K}, V, s, H$ from Section 2, and set

$$q_0 = (\partial_u^2 - 1/4)\psi, \qquad z_j = q_0^{(2j+1)}/s \quad (j \ge 0), \qquad K_0 = \operatorname{span}\{z_0, \dots, z_J\}, \quad J < \infty.$$

Lemma 8.1 (Genuine derivative nulls)**.** *For every $j \ge 0$, $z_j = (\psi^{(2j+3)} - \psi^{(2j+1)}/4)/s$ belongs to $\operatorname{Dom} H$ and satisfies $Hz_j = 0$. Every finite initial family is linearly independent.*

Proof. The theta Mellin identity of Appendix A gives

$$\mathcal{M}(sz_j)(z) = (z^2 - 1/4)(-z)^{2j+1}\xi(1/2+z).$$

It vanishes at both polar points and at every centered nontrivial zero. To verify the original domain, use compact even cutoffs χ_R and the polar-preserving approximants $(\partial_u^2 - 1/4)(\chi_R\psi^{(2j+1)})/s$. The complete derivative tails, also quantified in Proposition S18.2, give convergence in L^2 after division by s and convergence of the weighted physical derivative in L^2. The bound $a(\xi) \le 17\xi^2$ proves convergence in V. The same complete tails and integration by parts give arbitrary vertical inverse powers for the Mellin transforms on $|\Re z| \le 1/2$. The zero-count bound therefore passes the polarized explicit formula through these approximants as in Appendix A. It yields $q_H[z_j, h] = 0$ first on the compact core and then on all V. Form representation gives the asserted operator nullity. Finally a vanishing finite combination would make a nonzero polynomial times $(z^2 - 1/4)\xi(1/2+z)$ identically zero. Analyticity excludes this. □

These are the genuine operator nulls just constructed. All orthogonal complements below are taken in the original physical Hilbert space $\mathcal{K}$. Profile norms are unweighted $L^2(\mathbb{R})$ norms of sh. Define $\tau_t q(u) = q(u-t)$ and fix

$$0 < a < \tfrac{1}{2}\log 2, \qquad r_t = \tfrac{1}{2}(\tau_t q_0 - \tau_{-t} q_0), \quad |t| < a, \qquad \mathcal{D}_a = \operatorname{span}_{\mathbb{C}}\{r_t : |t| < a\}. \tag{56}$$

The span consists of finite sums.

Lemma 8.2 (Small translates in the original domain)**.** *For every $r \in \mathcal{D}_a$, $r/s \in \operatorname{Dom} H$ and $H(r/s) = 0$. Moreover $\mathcal{D}_a$ is dense in odd unweighted $L^2(\mathbb{R})$.*

Proof. The Mellin convention is $Mf(z) = \int_{\mathbb{R}} f(u)e^{zu}\,\mathrm{d}u$. The theta identity in Appendix A gives

$$Mq_0(z) = (z^2 - 1/4)\xi(1/2 + z), \qquad Mr_t(z) = \sinh(tz)Mq_0(z). \tag{57}$$

Thus both polar moments vanish. Lemma B.1 supplies strong mixed nullity for the complete raw form $\mathcal{Q}$. We first verify admission to the weighted minimal domain.

At each fixed derivative order the theta series gives, for real t,

$$|q_0(u \pm t)| + |q_0'(u \pm t)| \le C_t e^{C_t|u|} \exp[-\pi e^{2(|u|-|t|)}].$$

Indeed differentiation introduces only polynomial factors in $\pi n^2 e^{2u}$ on the positive tail, and evenness supplies the negative tail. Dividing the square by the lower envelope in (8) yields

$$|r_t(u)/s(u)|^2 \le C_t e^{C_t|u|} \exp[-\pi(2e^{-2|t|} - 1)e^{2|u|}]. \tag{58}$$

The coefficient $2e^{-2|t|} - 1$ is strictly positive for the translations in (56). Hence $h_t = r_t/s \in L^2$, and $r_t \in H^1$. The vanished moments put h_t in $\mathcal{K}$.

Take smooth even cutoffs χ_R, equal to one on $[-R, R]$, with uniformly bounded first derivatives and support in $[-R - 1, R + 1]$. Choose a fixed compact smooth odd profile p with $\int p(u)\sinh(u/2)\,\mathrm{d}u = 1$. Set

$$c_R = \int \chi_R(u) r_t(u) \sinh(u/2)\,\mathrm{d}u, \qquad f_R = \chi_R r_t - c_R p.$$

Full theta tails and the zero moment give $c_R \to 0$. Each f_R/s is in the compact odd polar-free core. The same tails give

$$\|f_R/s - h_t\|_2 \to 0, \qquad \|f_R - r_t\|_{H^1} \to 0.$$

By $a(\xi) \le 17\xi^2$, these imply convergence in the exact norm (6); thus $h_t \in V$. They also give $\mathcal{S}_1(f_R - r_t) \to 0$. For a compact core vector v, mixed continuity in Lemma B.1 and raw mixed nullity give

$$q_H[h_t, v] = \lim_R \mathcal{Q}(f_R, sv) = \mathcal{Q}(r_t, sv) = 0.$$

Continuity of the closed form in V extends this identity to every $v \in V$. The associated-operator representation theorem gives $h_t \in \operatorname{Dom} H$, $Hh_t = 0$. Finite sums retain this property.

For density, we need the analyticity of the translated kernel. The theta series and each fixed derivative converge normally in

$$\mathfrak{S} = \{u \in \mathbb{C} : |\operatorname{Im} u| < \pi/4\},$$

since $\operatorname{Re} e^{2u} > 0$ there. The evenness identity on the real line extends throughout this strip. On every smaller closed strip, the positive-tail series bounds, followed by evenness, give uniform double-exponential decay at both real tails. Consequently, for odd $g \in L^2$,

$$F_g(t) = \int_{\mathbb{R}} \overline{g(u)} q_0(u - t)\,\mathrm{d}u$$

is holomorphic for $|\operatorname{Im} t| < \pi/4$. Locally uniform L^2 bounds on the kernel and its derivatives justify differentiation. For real t, parity gives $F_g(-t) = -F_g(t)$, so $\langle g, r_t\rangle = F_g(t)$. If g annihilates $\mathcal{D}_a$, the identity theorem therefore gives $F_g(t) = 0$ for every real t. The correlation is the Fourier transform of an L^1 product of two L^2 Fourier transforms. Fourier uniqueness gives $\widehat{\overline{g}}\,\widehat{q}_0 = 0$ almost everywhere. By (57),

$$\widehat{q}_0(\xi) = (-\xi^2 - 1/4)\xi(1/2 - i\xi)$$

is nonzero almost everywhere: the xi function is nontrivial entire and its real-frequency zeros are isolated. Thus $g = 0$, proving density. □

Lemma 8.3 (Every canonical row combination is profile-unbounded)**.** *If $0 \neq z \in K_0$, the functional*

$$T_z(r) = \langle z, r/s \rangle \qquad (r \in \mathcal{D}_a)$$

is unbounded for the unweighted norm $\|r\|_2$.

Proof. We compute both the row growth and its convolution coefficient. For the first theta atom write its mth derivative as $e^{u/2}P_m(r)e^{-r}$, where $r = \pi e^{2u}$. Then

$$P_0(r) = 4r^2 - 6r, \qquad P_{m+1}(r) = (1/2 - 2r)P_m(r) + 2rP_m'(r).$$

Thus P_m has degree $m+2$ and leading coefficient $4(-2)^m$. The positive-tail weight satisfies $s^2 \sim 8r^2e^{-r}$. At $m = 2j+3$ the derivative of order $m-2$ in $q_0^{(2j+1)}$ cannot change its leading term. It follows that

$$\frac{z_j(u)}{s(u)} \sim \frac{(-2\pi)^{2j+3}}{2} e^{\lambda_j u}, \qquad \lambda_j = 4j + 13/2 \quad (u \to +\infty). \tag{59}$$

At any fixed order the higher theta atoms are smaller by a polynomial factor times e^{-3r}, and all lower polynomial powers give smaller exponents. These facts justify the asymptotic and the corresponding exponential upper bound. For a nonzero finite row combination, choose the largest index with nonzero coefficient. Distinct exponents give

$$w(u) := \overline{z(u)}/s(u) \sim ce^{\lambda u}, \quad c \neq 0, \qquad |w(u)| \leq Ce^{\lambda|u|}, \tag{60}$$

where $\lambda = \lambda_j$ for that index. Oddness supplies the negative tail and smooth positivity of s pays compact intervals.

For clarity, the Mellin normalization can be checked directly. Put $A = (z+1/2)/2$. For real $z > 1/2$, substitution $r = \pi n^2 e^{2u}$ in a single atom gives

$$\tfrac{1}{2}(\pi n^2)^{-A}[4\Gamma(A+2) - 6\Gamma(A+1)].$$

The sum of absolute atom integrals is finite because $2A > 1$. Summation gives

$$M\psi(z) = \tfrac{1}{2}(z^2 - 1/4)\pi^{-A}\Gamma(A)\zeta(2A) = \xi(1/2+z).$$

Both sides extend entirely; integration by parts twice gives (57). In particular

$$Mq_0(\lambda_j) = (\lambda_j^2 - 1/4)\xi(4j+7) > 0, \tag{61}$$

since every factor in the defining xi product is positive at the real argument $4j+7 > 1$.

Using w only on the real axis, define

$$F_w(t) = \int_{\mathbb{R}} w(u)q_0(u-t)\,\mathrm{d}u.$$

This is holomorphic on $|\operatorname{Im} t| < \pi/4$: on each compact parameter set the double-exponential kernel estimates from Lemma 8.2 dominate the exponential bound in (60), also after differentiation. Only the kernel is continued to complex arguments. Changing variables $u = t+v$, dominated convergence gives

$$e^{-\lambda t}F_w(t) \longrightarrow cMq_0(\lambda) \neq 0 \qquad (t \to +\infty). \tag{62}$$

Indeed for $t \geq 0$, $e^{-\lambda t}|w(t+v)| \leq Ce^{\lambda|v|}$, and this majorant times $|q_0(v)|$ is integrable.

Suppose $|T_z(r)| \leq B\|r\|_2$ on $\mathcal{D}_a$. Hahn–Banach and Riesz provide odd $g \in L^2$ with $T_z(r) = \langle g, r\rangle$ there. Since w is odd and q_0 even, $\int wr_t = F_w(t)$, the factor $1/2$ being cancelled by $F_w(-t) = -F_w(t)$. Thus $F_w(t) = F_g(t)$ for $|t| < a$. Strip analyticity extends equality to every real t. But

$$|F_g(t)| \leq \|g\|_2\|q_0\|_2 \qquad (t \in \mathbb{R}),$$

contradicting (62). The large real parameters in this contradiction occur only in the auxiliary convolutions; the physical nulls remain the small translates in (56). □

9 Full-jet synthesis in the original theta norm

We determine the closed space generated by the canonical nulls in the *physical* norm. The answer is the entire-divisibility class with all polar and xi multiplicities retained. A continuous Mellin-area representation makes the missing norm estimate accessible: two zero-free arms control the whole norm of an odd profile, and the exact gamma factor gives a uniform dilation estimate on those arms. We then use the classification to compute the carrier projection and the optimal null lift for an admitted finite head.

Retain $s, \psi, q_0, z_j, \mathcal{K}, V, H$ from Section 8. Work also in the odd ambient physical space $\mathcal{H}_o = L^2_{\mathrm{odd}}(\mathbb{R},\, \mathrm{d}u)$, with conjugate-linear first inner product. Put

$$X = L^2(\mathbb{R}, s^{-2}\,\mathrm{d}u), \qquad Mf(z) = \int_{\mathbb{R}} f(u)e^{zu}\,\mathrm{d}u,$$

$$Q(z) = (z^2 - 1/4)\xi(1/2+z) = Mq_0(z), \qquad A(z) = -zQ(z) = M(q_0')(z), \tag{63}$$

$$N_H = \overline{\mathrm{span}_{\mathbb{C}}\{z_j : j \ge 0\}}^{\mathcal{H}_o}, \qquad \mathscr{N} = sN_H = \overline{\mathrm{span}_{\mathbb{C}}\{q_0^{(2j+1)} : j \ge 0\}}^{X}. \tag{64}$$

Multiplication by s is an isometry from physical L^2 to X. Every fixed Mellin derivative evaluation is X-continuous, since $u^m s(u)e^{xu} \in L^2$ for every fixed m, x. Define

$$\mathscr{J} = \{f \in X : f \text{ odd},\ Mf/A \text{ entire}\}, \qquad J_H = \{f/s : f \in \mathscr{J}\}. \tag{65}$$

This is a closed subspace. The zero at 0 is automatic for odd input; the simple polar zeros $\pm 1/2$ and every centered nontrivial xi zero are imposed with their full multiplicities. Equivalently, $Mf = AE$ with E entire and even. These are divisibility conditions, rather than merely values at the distinct xi zeros.

The canonical family and its predecessors. The finite derivative family already occurs, in the same theta weight, in van Hemert's parity construction [13]. In the present notation set

$$h_j^- = \psi^{(2j+1)} - 4^{-j}\psi', \quad j \ge 1, \qquad h_0^- = 0, \qquad c_k = q_0^{(2k+1)}.$$

Then exactly

$$c_k = h_{k+1}^- - \tfrac{1}{4}h_k^-, \qquad h_j^- = \sum_{k=0}^{j-1} 4^{-(j-1-k)}c_k. \tag{66}$$

The weighted realization agrees as well: with $\mathrm{d}\nu = 2\cosh(u/2)\psi(u)\,\mathrm{d}u$, $\|h_j^-/\psi\|^2_{L^2(\nu)} = 2\|h_j^-\|_X^2$. Thus these are identical finite algebraic spans and identical closed weighted carriers after the indicated change of realization. The theorem below identifies that carrier with the *whole* compatible class (65) and proves approximation in its original norm. Mellin–Laplace area representations are classical [12]; factorial line norms have close half-plane counterparts [23], and the continuous gamma integral used below belongs to Volterra–Ramanujan theory [10]. Full-divisor synthesis through entire division and safe lines also has co-Poisson antecedents [4, §§6.24–6.28]. Here the decisive estimate is for the reciprocal-theta norm in (65).

Theorem 9.1 (Weighted carrier synthesis)**.** *The spaces in* (64)–(65) *satisfy*

$$\mathscr{N} = \mathscr{J}, \qquad N_H = J_H \subset \mathrm{Dom}\, H \cap \ker H.$$

For $f \in \mathscr{J}$, write $Mf = AE$. For every $0 < \rho < 1$ there is a unique $D_\rho f \in \mathscr{N}$ such that

$$M(D_\rho f)(z) = A(z)E(\rho z). \tag{67}$$

Each fixed $D_\rho : \mathscr{J} \to X$ is bounded, and

$$\sup_{3/4\le\rho<1} \|D_\rho\|_{\mathscr{J}\to X} < \infty, \qquad \lim_{\rho\uparrow 1} \|D_\rho f - f\|_X = 0 \quad (f \in \mathscr{J}). \tag{68}$$

Consequently $A(z)p(z^2)$, with p polynomial, is dense in the Mellin image of $\mathscr{J}$ in the transported physical norm.

We prove the strict inward assertion before estimating its boundary limit. This keeps existence of the approximants separate from their uniform norm control.

9.1 Strict inward synthesis in the physical space

Set

$$\Omega = \{a + ib : |b| < \pi/4, \quad 2|a| - \log\cos(2b) < \log 2\}, \qquad K = \overline{\Omega}. \tag{69}$$

The central-strip qualifier is part of this definition. The set is bounded, convex, symmetric about both axes, and contains 0 in its interior. For a compact set of parameters in Ω, the translated complete theta series and every fixed derivative have an integrable X majorant. Indeed, on the two physical tails their squared numerators divided by s^2 are bounded by a fixed power of $r = \pi e^{2|u|}$, times a fixed exponential in $|u|$, times

$$\exp\{-[2e^{2a}\cos(2b) - 1]r\}, \qquad \exp\{-[2e^{-2a}\cos(2b) - 1]r\},$$

respectively. Both bracketed coefficients have a strictly positive minimum on the compact parameter set. Higher theta atoms are absorbed by the normally convergent sums of a fixed power of n times $\exp[-c(n^2-1)r]$. Thus dominated difference quotients and Cauchy integrals prove X-valued holomorphy of the translated kernels.

In particular the odd, even-in-t family

$$T(t)(u) = \tfrac{1}{2}[q_0'(u-t) + q_0'(u+t)], \qquad t \in \Omega, \qquad MT(t)(z) = A(z)\cosh(tz) \tag{70}$$

is X-holomorphic. The transform identity follows by shifting the scalar contour within $|\operatorname{Im} u| < \pi/4$; the complete double-exponential tails make its closing sides vanish for each fixed z. Near $t = 0$ its X-Taylor series is

$$T(t) = \sum_{j\ge 0} \frac{t^{2j}}{(2j)!} q_0^{(2j+1)}.$$

It therefore takes values in $\mathscr{N}$ near zero. The bounded quotient map $X \to X/\mathscr{N}$ and the identity theorem on the connected set Ω give $T(t) \in \mathscr{N}$ throughout Ω. The same argument for $r_t = (q_0(\cdot - t) - q_0(\cdot + t))/2$, and real finite differences at zero, shows

$$N_H = \overline{\{r/s : r \in \mathcal{D}_a\}}^{L^2} = \mathcal{N}_a \qquad (0 < a < \tfrac{1}{2}\log 2). \tag{71}$$

This agrees with the physical space used in the quantitative interpolation theorem. Lemma 8.1 and closedness of H give $N_H \subset \operatorname{Dom} H \cap \ker H$. Continuity of every divisor jet also gives $\mathscr{N} \subset \mathscr{J}$.

The following envelope puts the quotient's Borel singularities in exactly the closed admissible lens.

Lemma 9.2 (Uniform quotient envelope)**.** *For $f \in \mathscr{J}$, $Mf = AE$, there is a fixed C such that*

$$|E(z)| \le C\|f\|_X(1 + |z|)^{-4}\exp(h_K(z)), \qquad h_K(z) = \sup_{t\in K} \operatorname{Re}(tz). \tag{72}$$

In particular E has finite exponential type.

Proof. The complete real theta bound $s^2 \le Cr^2e^{-r}$ gives, for $x \ge 1$,

$$|Mf(x+iy)| \le C\|f\|_X \pi^{-x/2}\Gamma(x+2)^{1/2}. \tag{73}$$

The substitution $r = \pi e^{2|u|}$ pays both tails; the negative arm is bounded by the same enlarged right side. The exact divisor is

$$A(z) = -\tfrac{1}{2}z(z^2-1/4)^2\pi^{-z/2-1/4}\Gamma(z/2+1/4)\zeta(z+1/2). \tag{74}$$

For $x \ge 1$ the inverse-zeta Dirichlet series is absolutely convergent and $|1/\zeta(z+1/2)| \le \zeta(3/2)$. Uniform sector Stirling [21, §5.11], with $R = |z| \ge 2$, $\theta = \arg z$, gives

$$\begin{aligned}\log|\Gamma(z/2+1/4)| &= (x/2-1/4)\log(R/2) - (y/2)\theta - x/2 + O(1),\\ \tfrac{1}{2}\log\Gamma(x+2) &= (x/2+3/4)\log x - x/2 + O(1).\end{aligned}$$

The degree-five prefactor in (74) leaves $R^{-5}R^{1/4}x^{3/4} \le R^{-4}$. Hence

$$|E(z)| \le C\|f\|_X R^{-4}e^{RF(\theta)}, \qquad F(\theta) = \tfrac{1}{2}[\cos\theta\log(2\cos\theta) + \theta\sin\theta], \quad x \ge 1. \tag{75}$$

To justify the intervening sector argument without presuming finite type, first normalize $\|f\|_X = 1$. The theta moment estimate on disks gives $\log^+ \max_{|z|=R}|Mf(z)| = O(R\log(2+R))$. The fixed entire function $Q = Mq_0$ has the same upper; its X norm is finite by the complete theta bounds. Let $H_0 = Mf/z$ after removal at zero. The quotient $H_0/Q = -E$ is entire and $Q(0) \ne 0$. For the circle mean $m(R,F) = (2\pi)^{-1}\int \log^+|F(Re^{it})|\,\mathrm{d}t$, Jensen's formula yields

$$m(R,1/Q) \le m(R,Q) + |\log|Q(0)||, \qquad m(R,E) = O(R\log(2+R)).$$

Generic circles avoiding the discrete zeros suffice. The Poisson upper for $\log|E|$ on a disk of radius $2R$ then proves

$$|E(z)| \le C\|f\|_X \exp[C(1+|z|)\log(2+|z|)], \tag{76}$$

after rescaling. This also supplies a uniform bound on each fixed disk. The nonvanishing $Q(0)$ follows from $\xi(1/2) \ne 0$; for example the alternating eta series is positive at $1/2$, whereas $1-2^{1-1/2} < 0$.

Put $c = \pi/6$. On the rays $\theta = \pi/3, 2\pi/3$, (75) and evenness give $|z^4E(z)e^{icz}| \le C\|f\|_X$. In their intervening sector $\operatorname{Re} z^2 \le -|z|^2/2$. Apply the maximum principle to $z^4E(z)e^{icz+\epsilon z^2}$ on finite sectors. Equation (76) makes the outer arcs vanish for each $\epsilon > 0$. Let the radius tend to infinity and then $\epsilon \downarrow 0$. Thus on the upper sector $|E(z)| \le C\|f\|_X|z|^{-4}e^{c\operatorname{Im} z}$ for $|z| \ge 1$; evenness gives the lower sector. Remaining angles use (75) and its even reflection.

Finally the support function of K is, in $0 \le \theta \le \pi/2$,

$$h_K(Re^{i\theta})/R = \begin{cases} F(\theta), & 0 \le \theta \le \pi/3,\\ c\sin\theta, & \pi/3 \le \theta \le \pi/2.\end{cases}$$

Indeed on its right arc $a = \frac{1}{2}\log(2\cos(2b))$; maximizing $a\cos\theta + |b|\sin\theta$ gives $|b| = \theta/2$ until the corner $|b| = \pi/6$, and then that corner. Reflection supplies the other quadrants. The compact-disk bound replaces $|z|^{-4}$ by $(1+|z|)^{-4}$, proving (72). □

Lemma 9.3 (Strict inward return)**.** *For each $0 < \rho < 1$, equation* (67) *defines a bounded linear map $D_\rho : \mathscr{J} \to \mathscr{N}$.*

Proof. For E of finite exponential type its Borel germ at infinity is $\beta_E(t) = \sum_{n\ge 0} E^{(n)}(0)t^{-n-1}$. Equation (72) continues it outside K by

$$\beta_\theta(t) = e^{i\theta}\int_0^\infty E(re^{i\theta})e^{-tre^{i\theta}}\,\mathrm{d}r, \qquad \mathrm{Re}(te^{i\theta}) > h_K(e^{i\theta}).$$

These integrals converge normally and agree with the germ when the real projection of t is sufficiently large. On each nonempty overlap the two half-planes have a common unbounded cone where both agree with that germ, so analytic uniqueness glues them. Opposite supporting half-planes have empty overlap. Convex separation shows their union is $\mathbb{C}\setminus K$. The integrable factor $(1+r)^{-4}$ also bounds each such integral by $C\|f\|_X$.

The germ for $E(\rho z)$ is $\rho^{-1}\beta_E(t/\rho)$, holomorphic outside ρK. Since $0\in\Omega$ and K is convex, ρK is compactly contained in Ω. Choose a rectifiable Jordan contour Γ_ρ enclosing ρK and contained in $\Omega\setminus\rho K$, with its interior also in Ω. Define the X-valued contour integral

$$f_\rho = \frac{1}{2\pi i}\int_{\Gamma_\rho} \rho^{-1}\beta_E(t/\rho)T(t)\,\mathrm{d}t \;\in \mathscr{N}. \tag{77}$$

It exists by norm-continuity on the compact contour and is bounded by $C_\rho\|f\|_X$. Borel inversion follows first by coefficient integration on a large circle and then by contour deformation to Γ_ρ. Since E is even, replacing e^{tz} by $\cosh(tz)$ in that inversion still gives $E(\rho z)$. Bounded Mellin evaluation commutes with (77), so (70) gives $Mf_\rho = AE(\rho\,\cdot)$. Fourier uniqueness on the imaginary axis proves uniqueness and identifies $f_\rho = D_\rho f$. □

9.2 Continuous-area calibration of the exact physical norm

Define the positive density and its bilateral Laplace transform by

$$W(x) = \frac{\pi^{|x|}\exp(e^{-|x|}/3)}{\Gamma(|x|+3)}, \qquad K_W(u) = \int_{\mathbb{R}} W(x)e^{2xu}\,\mathrm{d}x, \qquad \mathrm{d}\mu(z) = \frac{\mathrm{d}x\,\mathrm{d}y}{2\pi}. \tag{78}$$

The correction in W makes $\phi = -\log W$ convex also at the origin. Indeed, for $t\ge 0$ put $b(t) = \log\Gamma(t+3) - t\log\pi - e^{-t}/3$. Then

$$b'(0) = 11/6 - \gamma - \log\pi > 0, \qquad b''(t) = \psi_1(t+3) - e^{-t}/3 > \frac{1}{t+3} - \frac{e^{-t}}{3} \ge 0.$$

Here $\psi_1 = (\Gamma'/\Gamma)'$ is the trigamma function and γ is Euler's constant. The last inequality follows from $3e^t \ge t+3$, and the strict trigamma bound follows by integral comparison of its positive series. Thus $\phi(x+iy) = b(|x|)$ is subharmonic; its cusp contributes the positive measure $2b'(0)\delta_{x=0}\,\mathrm{d}y$.

Lemma 9.4 (Actual norm calibration)**.** *For every $f\in X$, $G = Mf$,*

$$\int_{\mathbb{C}} |G(z)|^2 W(\mathrm{Re}\,z)\,\mathrm{d}\mu(z) = \int_{\mathbb{R}} |f(u)|^2 K_W(u)\,\mathrm{d}u. \tag{79}$$

With $r(u) = \pi e^{2|u|}$,

$$K_W(u) \sim r(u)^{-2}e^{r(u)}, \qquad K_W(u)s(u)^2 \longrightarrow 8 \quad (|u|\to\infty). \tag{80}$$

There are fixed constants $0 < m \le M < \infty$ such that the area norm $\|G\|_{\mathcal{A}}^2$ on the left of (79) satisfies

$$m\|f\|_X^2 \le \|Mf\|_{\mathcal{A}}^2 \le M\|f\|_X^2. \tag{81}$$

Proof. Every exponential tilt $e^{xu}f$ is in $L^1 \cap L^2$. Plancherel on the vertical line gives

$$\int_{\mathbb{R}} |G(x+iy)|^2 \frac{\mathrm{d}y}{2\pi} = \int_{\mathbb{R}} |f(u)|^2 e^{2xu}\,\mathrm{d}u. \tag{82}$$

Tonelli proves (79) with its exact normalization.

For completeness, the classical continuous gamma asymptotic needed for this calibration can be proved with all tails retained. Put $V(r) = \int_2^\infty r^t/\Gamma(t+1)\,\mathrm{d}t$. Uniform real Stirling gives

$$\frac{r^t}{\Gamma(t+1)} = (2\pi t)^{-1/2} e^{H_r(t)}(1+O(t^{-1})), \quad H_r(t) = t[1+\log(r/t)], \quad H_r''(t) = -1/t.$$

On $t = r + \sqrt{r}\,v$, $|v| \le r^{1/12}$, $H_r(t) = r - v^2/2 + O(r^{-1/4})$. The normalized integral over this window tends to $(2\pi)^{-1/2}\int_{\mathbb{R}} e^{-v^2/2}\,\mathrm{d}v = 1$. On the rest of $[r/2, 2r]$, $H_r(t) \le r - (t-r)^2/(4r)$, giving a vanishing Gaussian tail. On $[2, r/2]$ the maximum is $(1+\log 2)r/2 < r$ and the interval has length at most r. On $[2r, \infty)$, $H_r(t) \le 2(1-\log 2)r - (t-2r)\log 2$. Both latter integrals are $o(e^r)$. Hence $V(r) \sim e^r$.

For $u \to +\infty$, set $r = \pi e^{2u}$ and $t = x + 2$. The positive arm of K_W is exactly

$$r^{-2}\int_2^\infty \frac{r^t}{\Gamma(t+1)} \exp(e^{2-t}/3)\,\mathrm{d}t.$$

The extra factor lies in $[1, e^{1/3}]$ and tends uniformly to one on the central saddle window; the preceding tail bounds apply. The negative arm is at most $\int_0^\infty W(x)\,\mathrm{d}x$, so is negligible. Reflection proves the first asymptotic in (80). The complete first-atom expansion is $s^2 = 8r^2e^{-r}(1+O(r^{-1/2}))$, with higher atoms exponentially smaller, proving the second. Stirling also gives compact-u domination for K_W. Its positive continuous product with s^2 therefore has a positive infimum and finite supremum on $\mathbb{R}$; these give m, M in (81). □

This is also an exact realization of the ambient Mellin image: every entire G of finite $\mathcal{A}$ norm is Mf for some $f \in X$. To see the converse, W is bounded below on each compact x-interval, so Fourier transformation in y is valid in local $L^2_x(L^2_y)$. If $F_x(u)$ is the inverse transform with factor $1/(2\pi)$, the Cauchy–Riemann equation gives $\partial_x F_x = uF_x$ in distributions. On compact u-intervals this is a bounded-multiplier Hilbert-space equation, so $F_x(u) = e^{xu}f(u)$. The solutions agree on overlaps. Fubini and Plancherel give $\int |f|^2 K_W < \infty$; hence $f \in X$. Its entire Mellin transform agrees almost everywhere with G, and therefore everywhere. This is the standard Laplace–area mechanism [12, Theorem 2.1], here calibrated to the actual weight.

9.3 Safe-arm control and the strong boundary return

For odd f, the tilted norm in (82), denoted by $L_f(x)$, is even and nondecreasing for $x \ge 0$, since

$$L_f(x) = \int_{\mathbb{R}} |f(u)|^2 \cosh(2xu)\,\mathrm{d}u.$$

Let $\mathcal{U} = \{z : |\operatorname{Re} z| \ge 1\}$ and put

$$C_{\mathrm{arm}} = 1 + \frac{\int_{-1}^1 W(x)\,\mathrm{d}x}{2\int_1^2 W(x)\,\mathrm{d}x}. \tag{83}$$

The middle strip contributes at most $L_f(1)\int_{-1}^1 W$, whereas $1 \le |x| \le 2$ contributes at least $2L_f(1)\int_1^2 W$. Consequently

$$\|Mf\|^2_{\mathcal{A},\mathcal{U}} \le \|Mf\|^2_{\mathcal{A}} \le C_{\mathrm{arm}}\|Mf\|^2_{\mathcal{A},\mathcal{U}} \qquad (f \in X \text{ odd}). \tag{84}$$

Lemma 9.5 (Uniform divided weight on the safe arms)**.** *Fix $\alpha = 3/4$, and define $\omega(z) = |A(z)|^2 W(\operatorname{Re} z)$. For $z = x + iy$, $x \geq \alpha$, $R = |z|$, $\theta = \arg z$, put*

$$P(z) = x \log(2x/R) + y\theta.$$

There are constants $0 < c_\alpha \leq C_\alpha < \infty$, uniform in $x \geq \alpha$ and every real y, such that

$$c_\alpha R^{19/2} x^{-5/2} e^{-P(z)} \leq \omega(z) \leq C_\alpha R^{19/2} x^{-5/2} e^{-P(z)}. \tag{85}$$

In particular, for $t \geq 1$,

$$\omega(tz) \leq K_\alpha t^7 \omega(z), \qquad |\operatorname{Re} z| \geq \alpha, \qquad K_\alpha = C_\alpha / c_\alpha. \tag{86}$$

Proof. On $x \geq 3/4$, the zeta factor in (74) satisfies

$$\zeta(5/4)^{-1} \leq |\zeta(z + 1/2)| \leq \zeta(5/4).$$

Both estimates use absolute convergence in the zero-free half-plane. Also $|z|^2 |z^2 - 1/4|^4 \asymp R^{10}$. Uniform sector Stirling gives

$$|\Gamma(z/2 + 1/4)|^2 \asymp (R/2)^{x-1/2} e^{-y\theta - x}, \qquad \Gamma(x + 3) \asymp x^{x+5/2} e^{-x}.$$

The constants are uniform on the entire stated sector, including $x/R \to 0$; compact radii are filled by continuity away from the polar zeros. Multiplying these formulas cancels the powers π^x and π^{-x} and proves (85). The bounded correction $\exp(e^{-x}/3)$ is absorbed in its constants.

For $v = y/x$,

$$P(z) = x\{\log 2 + v \arctan v - \tfrac{1}{2} \log(1 + v^2)\} \geq x \log 2 > 0.$$

The expression after $\log 2$ is even, has derivative $\arctan v$, and vanishes at zero. Since $P(tz) = tP(z)$, the ratio in (85) is at most $K_\alpha t^{19/2 - 5/2} e^{-(t-1)P(z)} \leq K_\alpha t^7$. Reflection, using $A(-z) = -A(z)$, proves the left-arm statement. □

Proof of Theorem 9.1. Strict inward membership was proved in Lemma 9.3. For $\alpha \leq \rho < 1$, change variables $w = \rho z$ in the safe-arm norm and use (86):

$$\begin{aligned} \|M(D_\rho f)\|_{\mathcal{A},\mathcal{U}}^2 &= \rho^{-2} \int_{|\operatorname{Re} w| \geq \rho} |E(w)|^2 \omega(w/\rho) \, \mathrm{d}\mu(w) \\ &\leq K_\alpha \rho^{-9} \int_{|\operatorname{Re} w| \geq \rho} |E(w)|^2 \omega(w) \, \mathrm{d}\mu(w) \leq K_\alpha \alpha^{-9} \|Mf\|_{\mathcal{A}}^2. \end{aligned} \tag{87}$$

The moving band $\rho \leq |\operatorname{Re} w| \leq 1$ is included: the comparison is valid down to α, not just on $\mathcal{U}$. Equations (81) and (84), applied to the already admitted odd $D_\rho f$, give

$$\|D_\rho f\|_X^2 \leq (M/m) C_{\mathrm{arm}} K_\alpha \alpha^{-9} \|f\|_X^2.$$

For strong convergence retain the tails in the same change of variables:

$$\int_{\mathcal{U} \cap \{|z| > T\}} |E(\rho z)|^2 \omega(z) \, \mathrm{d}\mu(z) \leq K_\alpha \alpha^{-9} \int_{|w| > \alpha T} |E(w)|^2 \omega(w) \, \mathrm{d}\mu(w). \tag{88}$$

The right side tends to zero uniformly in $\rho \in [\alpha, 1)$. On each compact subset, $E(\rho z) \to E(z)$ uniformly. Using twice the individual squared tails for the squared difference proves $\|A(E(\rho \cdot) - E)\|_{\mathcal{A},\mathcal{U}} \to 0$. The difference $D_\rho f - f$ is an odd X vector, so the odd strip inequality returns this to the full area norm, and then to X. Closedness of $\mathscr{N}$ gives $f \in \mathscr{N}$. The reverse inclusion was already proved. Operator nullity follows from (71) and closedness of the original operator. □

The theorem identifies the closed canonical space of genuine operator-null vectors by the full divisibility conditions in (65). Each zero multiplicity is part of that defining condition and is retained in the synthesis and projection below.

9.4 Computing the carrier without labeling the zeros

Use the standard computable L^2 representation: a vector is given by rational step approximants with certified L^2 errors tending to zero. Uniform computability means one algorithm for the input name, an index when present, and a positive rational requested tolerance. The contour construction below makes this effective structure explicit for the actual functions, including all multiplicities.

Lemma 9.6 (Effective theta data)**.** *The vectors $z_j = q_0^{(2j+1)}/s$ are uniformly computable in j. The odd evaluation vectors $v_z(u) = s(u)\overline{\sinh(zu)}$ are computable uniformly on compact parameter sets, with effective norm moduli. The entire function A and its derivatives are computable uniformly on compact disks.*

Proof. For $u \geq 0$, $r = \pi e^{2u}$, differentiation of the complete theta series gives

$$q_0^{(k)}(u) = e^{u/2} \sum_{n \geq 1} Q_k(n^2 r) e^{-n^2 r}.$$

The exact polynomial recurrence is

$$Q_0(r) = 16r^4 - 112r^3 + 164r^2 - 36r, \qquad Q_{k+1} = (1/2 - 2r)Q_k + 2rQ_k'.$$

For each indexed k, the coefficients are obtained by exact polynomial operations. Finitely many terms and an eventual geometric ratio less than $1/2$ give an effective uniform bound for $\sum n^{2k+8} e^{-(n^2-1)r}$, $r \geq \pi$. The positive lower theta envelope then bounds $|q_0^{(k)}/s|^2$ by $C_k r^{2k+13/2} e^{-r}$. After $\mathrm{d}u = \mathrm{d}r/(2r)$ this has an effective integrable tail. On compact real intervals the series is effectively convergent and s is positive with a computable positive lower bound. Rational step approximation on the compact interval and these tails give the asserted L^2 names, uniformly in k and the tolerance.

For contour evaluation put $E_m(R) = \| |u|^m s(u) e^{R|u|} \|_2$. The same theta bounds give effective finite uppers for every $E_m(R)$; for example

$$E_0(R)^2 \leq 16\pi^{-R}\Gamma(R+2), \qquad E_m(R)^2 \leq (2m)!\, E_0(R + 1/2)^2 \quad (R \geq 0).$$

The second bound uses $t^{2m} \leq (2m)! e^t$. These envelopes give norm-certified evaluation-vector approximation and all needed parameter derivative bounds. Integrating $q_0' e^{zu}$ with the analogous complete tails computes A and its derivatives. All estimates retain the two physical tails. □

Theorem 9.7 (Computable physical carrier projection)**.** *The orthogonal projection $P_N : \mathcal{H}_o \to N_H$ is computable in the vectorwise norm-certified sense, uniformly from a computable odd input and a positive rational tolerance. Its complement and the real-valued distance to N_H are computable as well.*

Proof. Certified contours and full principal parts. For each integer $n \geq 1$, search fairly over rational radii $R \in (n+1, n+2)$, meshes and precision levels until a certificate

$$\min_{|z|=R} |A(z)| \geq a_R > 0$$

is found. Effective compact bounds on A' make a fine mesh of certified A values a certificate on the entire circle. Zeros are discrete and only finitely many zero moduli meet this compact annulus, so a valid rational radius exists and its positive lower bound is eventually certified. Fair interleaving

avoids waiting forever at an invalid radius. The resulting circles Γ_R exhaust the plane. Compute the integer

$$n_R = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{A'(z)}{A(z)}\,\mathrm{d}z$$

by contour quadrature with certified error less than $1/4$ and integer rounding. This counts multiplicities without labeling roots. For $0 \le k < n_R$ define

$$\ell_{R,k}(h) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{z^k G_h(z)}{A(z)}\,\mathrm{d}z, \qquad G_h = M(sh). \tag{89}$$

When $n_R = 0$ the circle contributes no constraints. Exactly,

$$J_H = \bigcap_{R,k} \ker \ell_{R,k}. \tag{90}$$

One implication is Cauchy's theorem. For the converse, the meromorphic quotient on a fixed circle has principal parts at finitely many zeros of total possible pole order n_R. Its residue pairing against polynomials is determined by their jets at those zeros. Hermite interpolation of degree less than n_R realizes every such jet list, so the first n_R moments detect every principal-part coefficient. If they all vanish, there is no pole inside that circle. Exhaustion proves (90). Hermite interpolation is used only to prove completeness of these constraints; the algorithm never needs root coordinates or an individual multiplicity oracle.

Computable Riesz vectors. Set

$$H_{R,k}(u) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{z^k \sinh(zu)}{A(z)}\,\mathrm{d}z, \qquad w_{R,k}(u) = s(u)\overline{H_{R,k}(u)}. \tag{91}$$

Oddness permits replacing e^{zu} by $\sinh(zu)$ in G_h. Norm-dominated Fubini and the conjugate-linear first convention give $\langle w_{R,k}, h\rangle = \ell_{R,k}(h)$, with the conjugation in (91) applying to the *whole* contour integral. In particular

$$\|w_{R,k}\|_2 \le R^{k+1}E_0(R)/a_R. \tag{92}$$

This is a computable Hilbert-valued contour integral. More explicitly, if $M_1 \ge \max_{\Gamma_R}|A'|$, the norm of the parameter derivative of its integrand on $z = Re^{i\theta}$, including $1/(2\pi i)$, is bounded by

$$\frac{1}{2\pi}\left\{\left[\frac{(k+1)R^{k+1}}{a_R} + \frac{R^{k+2}M_1}{a_R^2}\right]E_0(R) + \frac{R^{k+2}}{a_R}E_1(R)\right\}. \tag{93}$$

Lemma 9.6, Riemann sums and this modulus therefore compute $w_{R,k}$ to any prescribed L^2 tolerance.

Let $C_H = \overline{\operatorname{span}\{w_{R,k}\}}^{L^2}$. Equation (90) says $C_H = J_H^{\perp}$. Every z_j is exactly orthogonal to every $w_{R,k}$, since $G_{z_j} = A(z)z^{2j}$ makes every contour residue zero. Theorem 9.1 now gives

$$\mathcal{H}_o = N_H \oplus^{\perp} C_H.$$

A terminating two-sided search. Enumerate all pairs (n, w) of finite rational-complex combinations of z_j and $w_{R,k}$. Given a name for h and rational $\eta > 0$, fairly refine candidate residual norms until an outward upper proves $\|h - n - w\|_2 < \eta/2$. The strict inequality is semidecidable. The two dense generating spans guarantee a pair with residual less than $\eta/4$, so the search terminates. Exact orthogonality gives

$$\|P_N h - n\|_2 < \eta/2, \qquad \|(I - P_N)h - w\|_2 < \eta/2.$$

Approximate the selected exact computable finite sum by rational steps with error less than $\eta/2$ to obtain the requested output name. Taking its norm also computes $\|(I - P_N)h\|_2$. □

The algorithm supplies a strong computable projection with $\|P_N\| \le 1$. Its stopping rule uses both orthogonal sides rather than a prescribed rate for canonical prefixes. Distance is computed as a real number with arbitrary error tolerance, which is distinct from deciding exact membership. The construction takes place in $\mathcal{H}_o$: finite complementary sums need not be polar. On a polar input the exact projected components are both polar, since $N_H \subset \mathcal{K}$.

9.5 Optimal genuine-null lifts for a fixed finite head

Theorem 9.8 (Computable optimal carrier lift)**.** *Let $r \ge 1$ and let $U : \mathbb{C}^r \to \mathcal{K}$ be a supplied computable exact physical isometry satisfying the finite-head controllability condition*

$$U^*|_{N_H} : N_H \longrightarrow \mathbb{C}^r \quad \text{is onto.} \tag{94}$$

Then $F = U^* P_N U$ is computable and positive definite. The map

$$V_{\mathrm{opt}} = P_N U F^{-1} \tag{95}$$

is a computable genuine original-operator-null lift, and

$$U^* V_{\mathrm{opt}} = I_r, \qquad V_{\mathrm{opt}}^* V_{\mathrm{opt}} = F^{-1}. \tag{96}$$

For every other carrier lift V with $U^*V = I_r$,

$$V^*V = F^{-1} + Z^*Z \succeq F^{-1}, \qquad Z = V - V_{\mathrm{opt}}. \tag{97}$$

Thus the minimum squared operator-norm amplification is $\lambda_{\min}(F)^{-1}$. Finite canonical lifts with exact head constraints can be computed within any positive tolerance of this optimum.

Proof. Put $X_U = P_N U$. Then $F = X_U^* X_U$, so $0 \preceq F \preceq I_r$; its kernel is trivial by (94). Theorem 9.7, applied to its r columns with Euclidean aggregation of the error tolerances, computes F in matrix norm. Refining Hermitian eigenvalue enclosures eventually certifies a positive rational

$$0 < f_0 \le \lambda_{\min}(F).$$

For example finite nets of the unit sphere give certified enclosures from the Rayleigh quotients and $\|F\| \le 1$. The positivity promise ensures termination; it need not provide a conditioning constant in advance. Matrix inversion is effective once f_0 is certified.

Direct multiplication gives (96). Every column of X_U is in the closed original-operator kernel. For another lift, $U^*Z = 0$ and $P_N Z = Z$, so $V_{\mathrm{opt}}^* Z = F^{-1} U^* P_N Z = 0$. This proves the whole-matrix equality (97), including every mixed coefficient term, and the norm optimum.

To obtain finite canonical lifts, the projection algorithm computes finite canonical maps X_ℓ with certified $\|X_\ell - X_U\| \le \epsilon_\ell \to 0$. Set $B_\ell = U^* X_\ell$; it need not be Hermitian. For $\epsilon_\ell < f_0$ it is invertible and

$$\|B_\ell^{-1}\| \le (f_0 - \epsilon_\ell)^{-1}, \qquad \|B_\ell^{-1} - F^{-1}\| \le \frac{\epsilon_\ell}{f_0(f_0 - \epsilon_\ell)}.$$

Define $V_\ell = X_\ell B_\ell^{-1}$. Its canonical range is finite and $U^* V_\ell = I_r$ exactly. Moreover

$$\begin{aligned} \|V_\ell - V_{\mathrm{opt}}\| &\le d_\ell := \frac{\epsilon_\ell}{f_0 - \epsilon_\ell}(1 + f_0^{-1}), \\ \|V_\ell^* V_\ell - F^{-1}\| &\le (2 f_0^{-1/2} + d_\ell) d_\ell. \end{aligned} \tag{98}$$

Choose a sufficiently small rational ϵ_ℓ for the requested Gram or amplification tolerance. These are exact computable complex coefficients after inversion, with norm-certified numerical names; rounding the coefficients alone is not the definition of the exact head constraint. The bounds in (98) pay the common-map acquisition and inverse errors. □

The controllability premise is satisfied by the actual normalized positive-width raw polynomial heads. To verify this particular case, let R_0c be a compact odd polynomial profile on $[-L, L]$, $L > 0$, with independent columns, and normalize $U = Q_b M_s R_0 C$ by its positive physical Gram, where C is invertible. If Uc annihilates every admitted r_t/s, its polar component drops out and

$$0 = \langle Q_b s R_0 C c, r_t/s \rangle = \langle R_0 C c, r_t \rangle \qquad (|t| < a).$$

The unweighted profile density in Lemma 8.2 implies $R_0 C c = 0$, hence $c = 0$. Finite-dimensional duality proves (94). The Gram normalization is legitimate: a compact polynomial profile cannot give a nonzero multiple of the full polar row outside its support. Its positive finite Gram and inverse square root are computable. Other supplied heads use the explicit premise (94). The fixed-head algorithm obtains a conditioning certificate by refinement; its theorem asserts termination and certified tolerances rather than a uniform rank or running-time law for changing heads.

Corollary 9.9 (Optimal fixed-head signed return)**.** *Suppose the original form has a complete split on V,*

$$q_H[h] = \ell_+[h] - \|Y^* h\|^2 + \langle h, Eh \rangle, \qquad \ell_+ \geq 0, \quad E = E^*, \quad \|E\| \leq e,$$

where $e \geq 0$ is a supplied computable error bound, $\operatorname{ran} Y \subset \operatorname{ran} U$*, and U satisfies Theorem 9.8 with columns in V. Then*

$$q_H[h] \geq -\frac{e}{\lambda_{\min}(U^* P_N U)} \|h\|_2^2 \qquad (h \in V). \tag{99}$$

The scalar lift cost in this bound is computable at each admitted finite head, and finite canonical lifts approach it with certified whole-Gram error.

Proof. For any genuine carrier lift V_0 with $U^* V_0 = I$, write $V_0 = U + Z$, $U^* Z = 0$, and $h = Uc + x$, $U^* x = 0$. The vector $w = h - V_0 c = x - Zc$ is in the original V and $Y^* w = 0$. Exact operator-nullity of the columns gives $q_H[h] = q_H[w] \geq -e\|w\|^2$; this cancels the mixed terms of the *complete* original form before estimation. The remainder map $(c, x) \mapsto x - Zc$ has squared norm $1 + \|Z\|^2 = \|V_0\|^2$. Apply (96)–(97) to obtain (99); finite canonical near-optimality follows from (98). □

The optimization is over carrier lifts for the specified head and complete split budget. It provides a norm-certified fixed-head quantity for the signed criteria, while the behavior of that quantity along a varying arithmetic family remains the hypothesis of the corresponding criterion.

10 Quantitative canonical null interpolation

We now impose exact canonical moments while controlling both the profile and physical norms. The proof starts with a bounded regularized dual splitting; theta-strip continuation then gives an anisotropic Gram lower bound before surjectivity or a constrained minimizer is used. The row list is finite at each use; all order dependence is quantified uniformly below. The resulting endpoint cost order also holds in the full kernel of the original physical operator, as shown in Corollary 10.2. The classical exponential loss for continuation along a bounded half-strip is quantified by Trefethen [35, Theorem 4.1]. Our self-contained proof combines this continuation mechanism with theta-specific boundary bounds, anisotropic row detection and harmonic barriers uniform along the widening-strip schedule.

Retain $a, \mathcal{D}_a, q_0, z_j$ from (56), and define the physical closed span

$$\mathcal{N}_a = \overline{\{r/s : r \in \mathcal{D}_a\}}^{L^2}.$$

Lemma 8.2 and closedness of H give $\mathcal{N}_a \subset \operatorname{Dom} H \cap \ker H \subset V$: if $k_l \to k$ physically and $Hk_l = 0$, closedness applies to $(k_l, Hk_l) \to (k, 0)$. Thus limits in this space remain genuine nulls on the original polar recipient.

Fix $J < \infty$ and put $K_J = \operatorname{span}\{z_0, \dots, z_J\}$, $\lambda_j = 4j + 13/2$. On $\mathcal{N}_a$ define bounded maps

$$Bk = sk, \qquad Tk = (\langle z_j, k\rangle)_{j=0}^{J}.$$

The same formula defines the bounded physical moment map on $\mathcal{K}$; when its argument is outside $\mathcal{N}_a$, T denotes that extension. For $\tau > 0$ set

$$\mathsf{A}_\tau = B^*B + \tau I_{\mathcal{N}_a}, \qquad \mathsf{S}_\tau = T\mathsf{A}_\tau^{-1}T^*, \qquad D_L = \operatorname{diag}(e^{\lambda_0 L}, \dots, e^{\lambda_J L}). \tag{100}$$

Here D_L is a finite diagonal matrix, distinct from the supported archimedean operators used elsewhere. Coercivity makes A_τ invertible and S_τ nonnegative. The anisotropic lower bound below will prove its positive definiteness and surjectivity of T.

Theorem 10.1 (Uniform endpoint quantitative canonical lift)**.** *Fix $0 < a < \frac{1}{2}\log 2$ and put $n = J + 1$. For every $\theta > 0$ there is $C_{a,\theta} \geq 1$, independent of $J \geq 0$, such that one may choose*

$$D_J = \exp(C_{a,\theta} n), \qquad L_J = C_{a,\theta} n. \tag{101}$$

Every smooth compact odd polar-free profile f supported in $[-L, L]$, $L \geq L_J$, admits a correction $k \in \mathcal{N}_a$ with

$$v = f/s - k \in V \cap K_J^{\perp}, \qquad q_H[v] = \mathcal{Q}[f], \tag{102}$$

$$\|sk\|_2^2 + \exp[-D_J e^{2L}]\|k\|_2^2 \leq \theta^2 \|f\|_2^2. \tag{103}$$

For each J, L this correction is linear in f. There is also the fixed-Λ Gram version: for each $\Lambda > 0$ there is $C_{a,\Lambda} \geq 1$ such that, for every $J \geq 0$,

$$\mathsf{S}_{\tau_L} \succ \Lambda D_L^2, \qquad \tau_L = \exp[-D_J e^{2L}], \qquad D_J = \exp(C_{a,\Lambda} n), \quad L \geq C_{a,\Lambda} n. \tag{104}$$

The constants may depend on the indicated fixed tolerance or Gram threshold, but not on J, L or f; no uniform assertion for parameters varying with J is made. Strict order means positive definite difference. In particular each fixed finite list has an endpoint budget $\exp(D_J e^{2L})$ for the physical squared norm, without a factor $\log(2 + L)$ in its exponent.

Proof.

The bounded physical energy dual. Set $a_j = (-2\pi)^{2j+3}/2$, $d_j = a_j(\lambda_j^2 - 1/4)\xi(4j + 7) \neq 0$, and $\Delta_n = \operatorname{diag}(d_j)_{j<n}$. These scalars are real. Normalize the rows by $\widehat{T} = \Delta_n^{-1} T$ and $\widehat{\mathsf{S}}_\tau = \widehat{T}\mathsf{A}_\tau^{-1}\widehat{T}^*$. Thus $\mathsf{S}_\tau = \Delta_n \widehat{\mathsf{S}}_\tau \Delta_n$; normalization changes neither the physical energy nor the exact target constraints. For any coefficient vector c, put $p = \mathsf{A}_\tau^{-1}\widehat{T}^* c$, $g = Bp$ and $h = \sqrt{\tau} p$. The following exact dual splitting holds on $\mathcal{N}_a$:

$$\langle c, \widehat{T}k\rangle = \langle g, Bk\rangle + \sqrt{\tau}\langle h, k\rangle, \qquad \|g\|_2^2 + \|h\|_2^2 = c^* \widehat{\mathsf{S}}_\tau c. \tag{105}$$

Both functions are odd. This is a dual representation in the sum of the profile and physical energies; its first term alone need not represent a profile-bounded canonical row.

Explicit boundary growth of the theta kernel. Set $h_* = \pi/4$. The derivative-polynomial recurrence used in (59) gives the exact analytic expansion

$$q_0(u) = e^{u/2} \sum_{n\geq 1} (16r_n^4 - 112r_n^3 + 164r_n^2 - 36r_n)e^{-r_n}, \qquad r_n = \pi n^2 e^{2u}. \tag{106}$$

Normal convergence on compact substrips justifies the differentiation. For $x \geq 0$, $|y| \leq h_0 < h_*$, put $\rho = \cos(2h_0) > 0$ and $R_n = \pi n^2 e^{2x}$. The polynomial in (106) has modulus at most $328R_n^4$, since $R_n \geq \pi > 1$. Moreover

$$R^4 e^{-\rho R} \leq 4096\rho^{-4} e^{-\rho R/2}, \qquad \sum_{n\geq 1} e^{-\rho R_n/2} \leq e^{-\rho\pi e^{2x}/4} \sum_{n\geq 1} e^{-\rho\pi n^2 e^{2x}/4} \leq \rho^{-1/2} e^{-x} e^{-\rho\pi e^{2x}/4}.$$

The first inequality follows by maximizing $t^4 e^{-t/2}$; the last compares a decreasing Gaussian sum with its integral from zero to infinity. With $A_* = 328 \cdot 4096$, evenness therefore gives on both arms

$$|q_0(x+iy)| \leq A_* \rho^{-9/2} e^{-|x|/2} e^{-\rho\pi e^{2|x|}/4}. \tag{107}$$

In particular, for $\lambda > 1/2$ and $\alpha = (\lambda - 1/2)/2$,

$$\sup_{|y|\leq h_0} \|q_0(\,\cdot - iy)\|_2 \leq \sqrt{2} A_* \rho^{-9/2}, \tag{108}$$

$$\sup_{|y|\leq h_0} \int_{\mathbb{R}} e^{\lambda|u|} |q_0(u - iy)| \, \mathrm{d}u \leq A_* (4/\pi)^\alpha \Gamma(\alpha) \rho^{-(\lambda/2+17/4)}. \tag{109}$$

For the first bound drop the last exponential in (107) and integrate its square. For the second integrate the bound over $x \geq 0$, multiply by two, substitute $v = \rho\pi e^{2x}/4$, and enlarge the incomplete gamma integral to the complete one. These estimates track every dependence on strip width.

Relative complete-theta estimates and Mellin normalization. Put $w_j = z_j/s = q_0^{(2j+1)}/s^2$, the real canonical profile row. Here C denotes an absolute constant that may increase; C_a allows only the fixed slit geometry. Put $k = 2j+1$, $p_j = \lambda_j = 2k+9/2$. The derivative polynomials satisfy

$$Q_{k+1} = (1/2 - 2r)Q_k + 2rQ_k', \qquad Q_0 = 16r^4 - 112r^3 + 164r^2 - 36r.$$

Starting from a monomial $q_d r^d$, a path of k recurrence steps with l diagonal steps has degree $d+k-l$ and total absolute coefficient at most $|q_d| 2^k \binom{k}{l} (k+5)^l$. Each raising step costs 2; each diagonal step costs at most $2k + 17/2 < 2(k+5)$. The unique all-raising path from $d = 4$ gives $16(-2)^k r^{k+4}$. Summing the other paths bounds the relative polynomial error by

$$(1 + (k+5)/r)^k - 1 + \sum_{d=1}^{3} \frac{|q_d|}{16r^{4-d}} (1 + (k+5)/r)^k.$$

Consequently, including the complete theta sum,

$$\left| \frac{q_0^{(k)}(u)}{16(-2)^k e^{u/2} r^{k+4} e^{-r}} - 1 \right| \leq C(k+1)^2/r, \qquad r \geq C(k+1)^2. \tag{110}$$

Indeed every higher atom has relative majorant $C\ell^{2k+8} e^{-(\ell^2-1)r}$; half its exponential absorbs the power in this range and the remaining square gaps give a uniformly summable Ce^{-cr} tail. The

positive real denominator obeys $\psi = 4e^{u/2}r^2e^{-r}(1+\eta)$, $|\eta| \le C/r$, $1+\eta \ge 1/2$. Thus division and the exact cosh factor give

$$\left|\frac{w_j(u)}{a_j e^{p_j u}} - 1\right| \le C\{(k+1)^2 e^{-2u} + e^{-u}\}, \qquad u \ge \log[C(k+1)]. \tag{111}$$

The e^{-u} term is essential. This is a positive-arm asymptotic; oddness, rather than that model exponential, controls the other arm.

Write $\mathfrak{M}(p) = \int e^{pv} q_0(v)\,\mathrm{d}v = (p^2 - 1/4)\xi(p+1/2) > 0$ for $p \ge 13/2$. The exact normalization is

$$|d_j| = 2^k \pi^{-1/2}(p_j^2 - 1/4)^2 \Gamma(k+5/2)\zeta(2k+5) \asymp 2^k \Gamma(k+13/2).$$

The comparison follows from the four-step gamma ratio; all constants are absolute, and $1 \le \zeta(2k+5) \le \zeta(7)$. In particular $|d_j|^{-1} \le e^{C(k+1)}$. Absolute kernel moments have the uniform comparison

$$\mathfrak{M}_{\mathrm{abs}}(p) := \int e^{p|v|}|q_0(v)|\,\mathrm{d}v \le C\mathfrak{M}(p), \qquad p \ge 13/2. \tag{112}$$

To check it, bound the fixed polynomial $Q_0(x)$ by Cx^4 for $x \ge \pi$, substitute $x = \ell^2\pi e^{2|v|}$ in every absolute atom, and enlarge the resulting positive integral to $(0,\infty)$. The upper is $C\pi^{-p/2-1/4}\Gamma(p/2+17/4)\zeta(p+1/2)$. In the exact positive $\mathfrak{M}(p)$ the gamma is $\Gamma(p/2+1/4)$ and the prefactor is $(p^2-1/4)^2/2$. Their degree-four ratio is bounded. This pays signed normalization without assuming pointwise positivity of q_0.

The same recurrence paths give the global atom bound

$$\sum_{\ell \ge 1} |Q_k(\ell^2 r)| e^{-(\ell^2-1)r} \le C^{k+1} 2^k (r+k+5)^{k+4}, \qquad r \ge \pi.$$

For completeness, apply $|Q_k(y)| \le C2^k(y+k+5)^{k+4}$ and $(A+z)^N \le 2^N A^N e^{z/2}$ with $A = r+k+5 > N = k+4$, $z = (\ell^2-1)r$; the remaining Gaussian sum is absolute. Divide by the positive theta denominator and by d_j, then split $(r+k+5)^{k+4}$ into its two endpoint powers. Elementary Stirling bounds absorb $(k+5)^{k+4}/\Gamma(k+13/2)$ into $e^{C(k+1)}$. This yields the useful global estimate

$$|w_j(u)/d_j| \le e^{C(k+1)} \left\{ \frac{e^{p_j|u|}}{\Gamma(k+13/2)} + e^{|u|/2} \right\} \le e^{C(k+1)} e^{p_j|u|}.$$

In particular the normalized supported targets satisfy, uniformly for $L \ge 0$,

$$\|D_L^{-1}\widehat{T}(f/s)\|^2 \le H_n \|f\|_2^2, \qquad H_n \le e^{Cn}.$$

Here Cauchy–Schwarz uses $e^{-2p_jL}\int_{-L}^{L} e^{2p_j|u|}\,\mathrm{d}u \le 1/p_j$, and then sums the n rows. There is no uncontrolled factor L.

Both real convolution arms and the complex strip. Define

$$\widehat{F}_j(t) = d_j^{-1}\int w_j(u) q_0(u-t)\,\mathrm{d}u, \quad F_g(t) = \int \overline{g(u)} q_0(u-t)\,\mathrm{d}u, \quad \widehat{F}_c = \sum_{j<n} \overline{c_j}\widehat{F}_j.$$

Only the kernel is continued to complex arguments. Subtract the full-line exponential convolution before splitting at $u = t/2$. For $t \ge Cn$ the high part of $e^{-p_j t}\widehat{F}_j(t) - 1$ is at most $C(k+1)^2 e^{-t/2}\mathfrak{M}_{\mathrm{abs}}(p_j)/\mathfrak{M}(p_j)$. For the other part, the full kernel envelope $|q_0(v)| \le Ce^{C|v|}e^{-ce^{2|v|}}$

on $v = u - t < -t/2$ gives $\exp[C(k+1)\log(k+2) - c'e^t]$. The latter bound uses the rough absolute moment of one half of the double-exponential kernel and the global normalized row bound; the other half pays $e^{-c'e^t}$. It includes all negative u and is at most e^{-t} at $t \ge Cn$. Therefore

$$|e^{-p_j t}\widehat{F}_j(t) - 1| \le e^{Cn}e^{-t/2}, \qquad t \ge Cn, \quad j < n. \tag{113}$$

For all $t \ge 0$, the global two-term row estimate and (112) also give $|e^{-p_j t}\widehat{F}_j(t)| \le e^{Cn}$.

The two-term bound retains the inverse gamma needed for complex correlations. From (107), with $\rho = \cos(2y)$, substitution on both arms gives

$$\int e^{p_j|v|}|q_0(v - iy)|\,\mathrm{d}v \le C^{k+1}\Gamma(k+2)\rho^{-(k+13/2)}, \qquad \int e^{|v|/2}|q_0(v - iy)|\,\mathrm{d}v \le C\rho^{-19/4}.$$

For the second estimate it suffices to use $e^{|v|/2} \le e^{|v|}$. The ratio $\Gamma(k+2)/\Gamma(k+13/2)$ is bounded. Thus

$$|\widehat{F}_j(x + iy)| \le e^{Cn}e^{p_j|x|}\rho^{-(k+13/2)}, \qquad |F_g(x + iy)| \le C\rho^{-9/2}\|g\|_2.$$

These absolute bounds on slightly larger compact substrips justify holomorphy. Since $\rho \ge 4(h_* - |y|)/\pi$, one common boundary exponent is $P = 2n + 11/2$; all conversion constants cost only e^{Cn}. No complex lower bound for the theta denominator is used.

Normalized growing damping and its complete anchor. Use the anchored convolution primitive of the weighted-localization predecessor [30], with rate $b = n$ and $m = 2n + 6$,

$$(\mathcal{R}_n F)(t) = \frac{b^m}{(m-1)!}\int_0^t (t - v)^{m-1}e^{-b(t-v)}F(v)\,\mathrm{d}v.$$

Its holomorphic path is immaterial. Put $G_j = \mathcal{R}_n\widehat{F}_j$, $G_g = \mathcal{R}_n F_g$, $G_c = \sum \overline{c_j}G_j$, $G = G_c - G_g$. The positive-real kernel has mass at most one, so $|G_g(t)| \le \|q_0\|_2\|g\|_2$ for $t \ge 0$. The exponential gain and exact anchored formula are

$$d'_{j,n} = \left(\frac{b}{b + p_j}\right)^m, \qquad \mathcal{R}_n(e^{p_j\cdot})(t) = d'_{j,n}e^{p_j t}\left[1 - e^{-(b+p_j)t}\sum_{v=0}^{m-1}\frac{((b + p_j)t)^v}{v!}\right].$$

Here $p_j/n \le 13/2$, $m \le 8n$, hence $e^{-Cn} \le d'_{j,n} \le 1$. Exponential Markov for the shape-m, rate-$(b+p_j)$ gamma tail bounds the anchor remainder by $e^{-t}[(b+p_j)/(b+p_j-1)]^m \le Ce^{-t}$. For the error integral split at $v = t/2$. Its late half uses (113); its early half is bounded by $e^{Cn}d'_{j,n}2^m e^{-(b+p_j)t/4}$. Consequently

$$e^{-Cn} \le d'_{j,n} \le 1, \qquad |e^{-p_j t}G_j(t) - d'_{j,n}| \le e^{Cn}e^{-t/4}, \quad t \ge Cn. \tag{114}$$

In particular no late asymptotic is used on the early primitive segment.

The bent real-then-vertical contour removes the strip boundary blowup: $m - P = 1/2$. The normalized horizontal multiplier is at most e^{bh_*}; the vertical multiplier is at most $b^m h_*^{m-P}/[(m-1)!(m-P)] \le e^{Cn}$. On a fixed negative real arm the additional multiplier is at most $b^m e^{b\beta}(\beta + h_*)^{m-1}\beta/(m-1)! \le e^{C_a n}$. These inequalities follow by enlarging the horizontal gamma integral and using $(Y - v)^{m-1}(h_* - v)^{-P} \le (h_* - v)^{m-P-1}$ vertically; Stirling's factorial lower bound pays b^m. Thus throughout the open strip

$$|G_j(x + iy)| \le e^{Cn}e^{p_j x}, \qquad |G_g(x + iy)| \le e^{Cn}\|g\|_2 \quad (x \ge 0), \tag{115}$$

with $e^{C_a n}$ bounds on each fixed negative arm. Fix $0 < a_0 < a$, for example $a_0 = a/2$. On the real slit $I = [-a_0, a_0]$, anchoring costs at most $e^{ba_0}(ba_0)^m/m! \le e^{C_a n}$; the negative half is included.

Compressed cardinal inversion with factorial separation. Take $u_i = i/n$, $x_j = e^{p_j/n}$, $0 \le i, j < n$, and $W_{ij} = x_j^i$. The jth inverse row consists of the coefficients of its cardinal polynomial, so

$$\|W^{-1}\|_2 \le \sqrt{n}\max_j \prod_{l\ne j} \frac{1+x_l}{|x_j - x_l|} \le \sqrt{n}\,(1+e^{13/2})^{n-1}\frac{(n/4)^{n-1}2^{n-1}}{(n-1)!} \le e^{Cn}.$$

Indeed $|x_j - x_l| \ge 4|j-l|/n$ retains the denominator $j!(n-1-j)!$; replacing all separations by the minimum loses this factorial. The empty product covers $n = 1$. This is the classical cardinal-coefficient inverse argument [11, Section 3]. Let $V_{ij} = d'_{j,n}e^{p_j u_i}$, so $\sigma_{\min}(V) \ge e^{-Cn}$. Writing $t_L(c) = \|D_L c\|$ and $a_L = (\overline{c_j}e^{p_j L})_j$, the samples factor exactly as

$$(G_c(L + C_0 + u_i))_i = (V + E)\,\mathrm{diag}(e^{p_j C_0})a_L, \qquad \|E\|_2 \le e^{Cn}e^{-(L+C_0)/4}.$$

For $L \ge Cn$ this error is at most half the smallest singular value, uniformly for every $C_0 \ge 0$. No diagonal is commuted past E. Hence

$$\max_i |G_c(L + C_0 + u_i)| \ge e^{-Cn}e^{p_0 C_0}t_L(c). \tag{116}$$

An exponential dual budget, before the fixed parameters. Fix $q \ge 0$ and any prescribed $1 \le Q_n \le e^{qn}$. Suppose a nonzero normalized-row direction satisfies

$$c^*\widehat{\mathsf{S}}_\tau c \le Q_n t_L(c)^2. \tag{117}$$

The exact dual gives $\|g\|_2, \|h\|_2 \le \sqrt{Q_n}t_L(c)$. Choose $C_0 = C'_{a,q}n$ large enough that the real detector beats $\|q_0\|_2\sqrt{Q_n}t_L(c)$ by at least $t_L(c)$. Then some $t_i = L + C_0 + u_i$ satisfies

$$|G(t_i)| \ge t_L(c). \tag{118}$$

Fix $\beta > a_0$, $x_0 > a_0 + 2h_*$, and $B_0 = C_0 + B_a$, with fixed $B_a > \max\{2, x_0 + 2h_* + 2\}$. On the outer boundary of

$$\Omega_L = \{-\beta < \mathrm{Re}\, t < L + B_0, |\,\mathrm{Im}\, t| < h_0\} \setminus I$$

all samples stay at right-edge distance greater than one, and

$$|G| \le \mathcal{B}_n t_L(c), \qquad \mathcal{B}_n \ge e, \qquad \log \mathcal{B}_n \le C_{a,q}n^2. \tag{119}$$

The positive-arm normalization pays $e^{p_j B_0}$, the fixed negative arm is covered by the primitive bound, and the nuisance pays $e^{Cn}\sqrt{Q_n}$. Thus the largest logarithmic loss is $p_{n-1}B_0 = O_{a,q}(n^2)$, with no $\log L$ loss. On I the dual applied to legal real r_t/s and oddness give

$$|\widehat{F}_c(t) - F_g(t)| \le C_I\sqrt{\tau Q_n}\,t_L(c), \qquad C_I = \sup_{t\in I}\|r_t/s\|_2 < \infty. \tag{120}$$

The strict common theta-tail majorant proves $C_I < \infty$; no large or complex physical translate is admitted. The slit primitive gives

$$|G(t)| \le \mathcal{A}_n\sqrt{\tau}\,t_L(c), \qquad 1 \le \mathcal{A}_n \le e^{C_{a,q}n}, \quad t \in I. \tag{121}$$

A fixed Dirichlet constant and a matched cosine barrier. Take $3h_*/4 \le h_1 < h_0 < h_*$ and put $\nu = \pi/(2h_1)$, so $2 < \nu \le \nu_{\max} = 8/3$. Let ω_L be harmonic measure of the slit banks in Ω_L, with value one on the slit and zero on the outer boundary. Consider the fixed slit rectangle

$$D_* = \{-\beta < \operatorname{Re} t < x_0 + 2h_*, |\operatorname{Im} t| < h_*/2\} \setminus I$$

with slit harmonic measure ω_*. Its compact disk $|t - x_0| \le h_*/4$ lies in its interior; hence

$$m_* = \min_{|t-x_0|\le h_*/4} \omega_*(t) > 0.$$

This is a single fixed Dirichlet-problem constant. For large L, $D_* \subset \Omega_L$ and boundary comparison gives $\omega_L \ge \omega_*$ on D_*.

The disk $|t-x_0| < h_0$ avoids the slit and lies inside the uncut rectangle. On $h_0/4 < |t-x_0| < h_0$, compare ω_L with

$$\frac{m_*}{\log 4} \log \frac{h_0}{|t - x_0|}.$$

The inner boundary has $\omega_L \ge m_*$ and the outer boundary has $\omega_L \ge 0$, so the maximum principle applies. Tangency at the horizontal rectangle edges is handled by exhaustion. Using $\log(1/r) \ge 1-r$ on the annulus and the inner-disk bound m_* gives for all $|y| < h_0$

$$\omega_L(x_0 + iy) \ge \frac{m_*}{h_* \log 4}(h_0 - |y|).$$

For $|y| \le h_1$ we have $\cos(\nu y) = \sin(\nu(h_1 - |y|)) \le \nu_{\max}(h_0 - |y|)$. Thus the fixed constant $\mu = m_*/(h_* \log 4\, \nu_{\max})$ satisfies $\omega_L(x_0 + iy) \ge \mu \cos(\nu y)$. Comparison on $(x_0, L + B_0) \times (-h_1, h_1)$ with

$$H_L(x, y) = \mu \frac{\sinh(\nu(L + B_0 - x))}{\sinh(\nu(L + B_0 - x_0))} \cos(\nu y)$$

is now valid on every edge. At the real samples their right-edge distance exceeds one. Since $2 < \nu \le \nu_{\max}$,

$$\omega_L(t_i) \ge c_* e^{-\nu L}, \qquad c_* = 2\mu \sinh(2) e^{-\nu_{\max}(B_0 - x_0)} > 0. \tag{122}$$

All constants are independent of h_0, h_1 in the stated range.

A single widening-strip schedule. For $L \ge 0$ choose

$$d_L = \frac{h_*}{16(1 + L)}, \quad h_0 = h_* - d_L, \quad h_1 = h_* - 2d_L, \quad \nu_L = \frac{16(1 + L)}{7 + 8L}. \tag{123}$$

These widths satisfy the preceding hypotheses, and

$$(\nu_L - 2)L = \frac{2L}{7 + 8L} < \frac{1}{4}, \qquad \rho_L = \cos(2h_0) = \sin(2d_L) \ge \frac{4d_L}{\pi} = \frac{1}{16(1 + L)}.$$

The sine inequality is its chord bound on $[0, \pi/2]$. Consequently (122) gives $\omega_L(t_i) \ge c_h e^{-2L}$ with $c_h = c_* e^{-1/4}$, and the outer bound for G remains $\mathcal{B}_n t_L(c)$, independent of these widths. Since $B_0 = C_0 + B_a = O_{a,q}(n)$, we may decrease c_h to a number $c_n \le 1$ with $c_n \ge e^{-C_{a,q} n}$. The fixed Dirichlet minimum m_* and μ depend only on the fixed left-hand geometry, not on n.

For an analytic function bounded by $\mathcal{B}$ on the outer boundary and by $\epsilon_0 \le \mathcal{B}$ on both slit banks, the subharmonic maximum principle gives, applied here to G,

$$\log |G(t_i)| \le \log \mathcal{B} - \omega_L(t_i) \log(\mathcal{B}/\epsilon_0). \tag{124}$$

Zeros cause no difficulty: use $\frac{1}{2}\log(|G|^2+\eta^2)$ and its harmonic boundary majorant, then let $\eta \downarrow 0$. The domains are bounded and G is continuous across the slit before removal. Normalize $t_L(c)=1$ and use $\epsilon_0 = \mathcal{A}_n\sqrt{\tau}$, $\mathcal{B}=\mathcal{B}_n$. The detector and harmonic lower imply

$$c_n e^{-2L}\left[\tfrac{1}{2}\log(1/\tau) - \log \mathcal{A}_n\right] \le \log \mathcal{B}_n$$

whenever the bracket is nonnegative and $\epsilon_0 \le \mathcal{B}_n$. Set

$$\widehat{D}_n = \frac{4(\log \mathcal{B}_n + \log \mathcal{A}_n + 1)}{c_n}, \qquad \tau = \exp[-\widehat{D}_n e^{2L}].$$

For $L \ge 0$, this makes the bracket positive, ensures the required slit/outer ordering, and makes the left side strictly larger than $\log \mathcal{B}_n$, a contradiction. The bounds above give $\widehat{D}_n \le e^{C_{a,q}n}$ and onset at most $C_{a,q}n$. Only $\log \mathcal{B}_n = O_{a,q}(n^2)$ enters this choice. Choose the fixed geometry, primitive constants and perturbation onset first, then q, the detecting offset, outer/slit bounds and harmonic prefactor, and finally one common $C_{a,q}$. Decreasing τ to $\exp[-e^{C_{a,q}n}e^{2L}]$ increases the normalized Gram in positive order. We have proved the exponential-budget statement

$$\widehat{\mathsf{S}}_{\tau_L} \succ Q_n D_L^2, \qquad 1 \le Q_n \le e^{qn}, \quad \tau_L = \exp[-e^{C_{a,q}n}e^{2L}], \quad L \ge C_{a,q}n.$$

This argument establishes a strict normalized lower before any surjectivity or minimizer is invoked.

The normalized lower and invertibility of Δ_n make S_{τ_L} positive definite. It also proves that T is onto: a vector orthogonal to its range has $T^*c = 0$, hence $c^*\mathsf{S}_{\tau_L}c = 0$ and therefore $c = 0$. The range is a finite-dimensional linear subspace, hence all of $\mathbb{C}^{J+1}$. It follows that S_τ is positive definite for every $\tau > 0$. No surjectivity was used before this point.

Exact constrained energy after the Gram lower. For $d \in \mathbb{C}^{J+1}$ the unique minimum-energy vector with $Tk = d$ is

$$k_\tau(d) = \mathsf{A}_\tau^{-1}T^*\mathsf{S}_\tau^{-1}d, \qquad \min_{Tk=d}\{\|Bk\|_2^2 + \tau\|k\|_2^2\} = d^*\mathsf{S}_\tau^{-1}d. \tag{125}$$

This is the constrained Hilbert energy minimization underlying regularization [34]. The displayed vector has target d; for $e \in \ker T$, $\langle e, \mathsf{A}_\tau k_\tau(d)\rangle = \langle Te, \mathsf{S}_\tau^{-1}d\rangle = 0$. Expansion of the positive energy around it proves the identity and uniqueness. For a fixed $\theta > 0$, choose $Q_n = 1 + H_n/\theta^2$. Since $H_n \le e^{hn}$ for fixed h, one has $Q_n \le e^{q_\theta n}$ for a fixed q_θ; enlarge h if needed. Apply the normalized budget lower to the exact normalized target $\widehat{d} = \Delta_n^{-1}T(f/s)$. Inverse positive order, without a commutation assumption, yields

$$\widehat{d}^*\widehat{\mathsf{S}}_{\tau_L}^{-1}\widehat{d} \le Q_n^{-1}\|D_L^{-1}\widehat{d}\|^2 \le \theta^2\|f\|_2^2.$$

The original and normalized minimizers coincide because their target constraints are equivalent. This proves every fixed-tolerance assertion; scaling a tolerance-one correction would not preserve its target.

For a fixed $\Lambda > 0$, set $R_n = \max_{j<n}|d_j|^{-2} \le e^{cn}$ and choose $Q_n = 1 + \Lambda R_n \le e^{q_\Lambda n}$ with fixed q_Λ. The normalized budget lower and the original congruence give

$$\mathsf{S}_{\tau_L} \succ Q_n \Delta_n D_L^2 \Delta_n = Q_n \Delta_n^2 D_L^2 \succeq \Lambda D_L^2.$$

Only the explicitly diagonal matrices commute here; the Gram itself need not commute with the normalization. This proves the ORIGINAL-Gram statement (104), with its own $C_{a,\Lambda}$. For $f = 0$ use zero. Finally $Tk = T(f/s)$ and $k \in \mathcal{N}_a \subset \operatorname{Dom} H \cap \ker H$ give (102), including every mixed form term. This proves the theorem. □

Corollary 10.2 (Sharp logarithmic cost order in the full original physical kernel)**.** *Let* $z = \sum_{i=0}^{j} \alpha_i z_i$ *with* $\alpha_j \neq 0$ *and* $\lambda = 4j + 13/2$*. Distinguish the ambient original odd polar Hilbert space* $\mathcal{K}$*, the admitted physical closed null span* $\mathcal{N}_a$*, and the full original operator kernel*

$$\mathcal{Z}_H = \{k \in \operatorname{Dom} H : Hk = 0\}, \qquad \mathcal{N}_a \subset \mathcal{Z}_H \subset \mathcal{K}.$$

For $X \in \{\mathcal{K}, \mathcal{Z}_H, \mathcal{N}_a\}$ *define*

$$C_z^X(\varepsilon) = \inf\{\|k\|_2 : k \in X,\ \langle z, k\rangle = 1,\ \|sk\|_2 \leq \varepsilon\}.$$

The ambient cost imposes no form-domain or nullity condition; the full-kernel cost uses precisely the original physical operator. Write $C_z = C_z^{\mathcal{N}_a}$ *for the cost in the constructed null span. These are infima; no attainment under the two constraints is asserted. Every one of these costs is finite for each* $\varepsilon > 0$*, tends to infinity as* $\varepsilon \downarrow 0$*, and satisfies*

$$\lim_{\varepsilon \downarrow 0} \frac{\log \log C_z^X(\varepsilon)}{\log(1/\varepsilon)} = \frac{2}{\lambda} = \frac{4}{8j + 13}. \tag{126}$$

With a fixed throughout, more precisely, uniformly across these three classes for small ε*, fixed positive constants depending on a and the row give*

$$\begin{gathered} c_z \varepsilon^{-2/\lambda} - O_z(\log(1/\varepsilon)) \leq \log C_z^{\mathcal{K}}(\varepsilon), \\ \log C_z^{\mathcal{K}}(\varepsilon) \leq \log C_z^{\mathcal{Z}_H}(\varepsilon) \leq \log C_z^{\mathcal{N}_a}(\varepsilon), \\ \log C_z^{\mathcal{N}_a}(\varepsilon) \leq C_z^{+} \varepsilon^{-2/\lambda}. \end{gathered} \tag{127}$$

Thus $\log C_z^X(\varepsilon) = \Theta_z(\varepsilon^{-2/\lambda})$ *for each of these classes. Moreover, with* $a_z = \alpha_j(-2\pi)^{2j+3}/2$*, the polar leading-constant lower in Proposition 10.3 gives*

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{2/\lambda} \log C_z^X(\varepsilon) \geq d_z^{\mathrm{pol}} = \frac{\pi}{2}(\lambda/|a_z|^2)^{1/\lambda} \left(\frac{\lambda + 1/2}{\lambda - 1/2}\right)^{2/\lambda} > 0.$$

No null leading-constant equality or limit is asserted. In particular, the full original physical kernel has the sharp interpolation-cost power $4/13$ *for the first canonical row.*

Proof. The lower bound holds even for arbitrary physical vectors. From (60) take $|z(u)/s(u)| \leq Ce^{\lambda|u|}$ and put $D_z = C/\sqrt{\lambda}$. Splitting the moment at R gives, for every feasible k,

$$1 \leq \varepsilon A_z(R) + \|k\|_2 B_z(R), \quad A_z(R) = \|1_{|u| \leq R} z/s\|_2 \leq D_z e^{\lambda R}, \quad B_z(R) = \|1_{|u| > R} z\|_2.$$

For $r = \pi e^{2R}$ the upper theta envelope gives, including both tails and the substitution $\mathrm{d}u = \mathrm{d}x/(2x)$,

$$B_z(R)^2 \leq 169 C^2 \pi^{-(\lambda+2)} \int_r^\infty x^{\lambda+1} e^{-x}\, \mathrm{d}x \leq 338 C^2 \pi^{-(\lambda+2)} r^{\lambda+1} e^{-r}$$

when $r \geq 2(\lambda + 1)$. The last inequality follows by writing $x = r + v$ and bounding $(1 + v/r)^{\lambda+1} \leq e^{v/2}$. For small ε choose $R = \lambda^{-1} \log(1/(2D_z\varepsilon))$. The interior contribution is at most $1/2$, so $\|k\|_2 \geq 1/(2B_z(R))$. Taking logarithms gives

$$\log \|k\|_2 \geq \frac{\pi}{2}(2D_z\varepsilon)^{-2/\lambda} - \frac{\lambda + 1}{\lambda} \log(1/\varepsilon) - C_z'.$$

This uniform inequality passes to the ambient infimum and hence to the costs of both null classes.

For the upper use the full list ending at j and prescribe $d_i = 0$ for $i < j$, $d_j = 1/\overline{\alpha_j}$. Then $Tk = d$ implies $\langle z, k\rangle = 1$ under the conjugate-linear-first convention. Fix $\Lambda = 1$ once. Equations (104) and (125) give

$$\|sk\|_2^2 + \exp[-De^{2L}]\|k\|_2^2 \le |d_j|^2 e^{-2\lambda L}.$$

Take a fixed $C_0 \ge \max\{1, |d_j|\}$ and $L = \lambda^{-1}\log(C_0/\varepsilon)$. For small ε this exceeds the fixed onset, and hence

$$\|sk\|_2 \le \varepsilon, \qquad \|k\|_2 \le \varepsilon \exp\left[\tfrac{D}{2}(C_0/\varepsilon)^{2/\lambda}\right].$$

The constants and target have not been varied with ε. This proves the claimed upper. It also gives nonemptiness for every positive tolerance by using a smaller tolerance if necessary. Set inclusion gives $C_z^{\mathcal{K}} \le C_z^{\mathcal{Z}_H} \le C_z^{\mathcal{N}_a}$. The universal lower and the upper constructed inside $\mathcal{N}_a$ therefore prove (127). Each cost exceeds one eventually, so its iterated logarithm is defined. For each class the liminf ratio is at least $2/\lambda$ and the limsup is at most $2/\lambda$, by the two-sided constant-order bounds just proved. This proves (126), without identifying $\mathcal{N}_a$ with the full kernel. The sharper lower d_z^{pol} follows by inclusion from the exact polar ambient coefficient in (129); that ambient upper is not used as a null upper. The unrestricted odd coefficient remains exact only for the larger space without the polar constraint.

The same logarithmic order also holds for the finite span $\mathcal{F} = \{r/s : r \in \mathcal{D}_a\}$, with cost defined by the same infimum. Indeed $T(\mathcal{F})$ is dense in the onto finite-dimensional image $T(\mathcal{N}_a)$ and is a linear subspace, hence is onto. Choose once a physical bounded right inverse $R : \mathbb{C}^{j+1} \to \mathcal{F}$. Construct k above at tolerance $\varepsilon/2$ and approximate it physically by $u_l \in \mathcal{F}$. Then $u_l + R(d - Tu_l)$ has the target exactly and converges physically to k. Boundedness of B gives profile convergence. Eventually its profile norm is at most ε and its physical norm at most $2\|k\|_2$. The fixed factor from using $\varepsilon/2$ changes only the upper constant. This proves an infimum bound, not attainment or a uniform translate-count estimate. □

Fixed-row asymptotics. Constants and onsets depend on the full row coefficients and highest index. For arbitrary combinations they are not uniform when a highest coefficient tends to zero. The separate full-list theorem above does quantify growth in the canonical list length; it does not make these normalized single-combination asymptotics uniform in its coefficients. For example, $z_0 + \delta z_1$ has exponent $4/21$ for each fixed $\delta \ne 0$, whereas z_0 has exponent $4/13$. The leading row tails balance when $4\pi^2|\delta|e^{4u} \simeq 1$; this is a tail comparison, not a certified tolerance threshold. A tiny highest-row perturbation can change the eventual exponent while delaying its onset. The fixed-row limit alone does not establish a joint increasing-index asymptotic for such degenerating combinations.

10.1 Exact ambient and polar leading constants

The genuine-null upper can be compared with a sharp leading cost that already holds in the larger odd physical space. This separates the weight-induced cost from the additional null constraint. Here the ambient class is all of $L^2_{\mathrm{odd}}(\mathbb{R})$, without the polar constraint; it is larger than the polar physical space $\mathcal{K}$. The coefficient below is exact for this ambient class. Imposing the original polar constraint strictly increases it; that sharper coefficient gives a universal lower for genuine physical operator nulls.

Proposition 10.3 (Ambient interpolation cost and a universal null lower)**.** *Let z be a fixed nonzero finite canonical row combination, with highest derivative index j. Put $\lambda = 4j + 13/2$ and define*

$a \neq 0$ *by* $z(u)/s(u) \sim ae^{\lambda u}$ *as* $u \to +\infty$. *For*

$$C_z^{\mathrm{amb}}(\epsilon) = \inf\{\|k\|_2 : k \in L^2_{\mathrm{odd}}(\mathbb{R}),\ \langle z, k\rangle = 1,\ \|sk\|_2 \le \epsilon\},$$

one has

$$\lim_{\epsilon\downarrow 0} \epsilon^{2/\lambda} \log C_z^{\mathrm{amb}}(\epsilon) = d_z := \frac{\pi}{2}\left(\frac{\lambda}{|a|^2}\right)^{1/\lambda}. \tag{128}$$

For the same infimum restricted to the odd polar Hilbert space $\mathcal{K}$, *one has the exact coefficient*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2/\lambda} \log C_z^{\mathcal{K}}(\epsilon) = d_z^{\mathrm{pol}} := d_z\left(\frac{\lambda + 1/2}{\lambda - 1/2}\right)^{2/\lambda} > d_z. \tag{129}$$

For genuine physical operator nulls or the physical small-translate closed span, the corresponding liminf is at least d_z^{pol}. *No exact leading coefficient for either null class is asserted.*

Proof. Set

$$S(\tau) = \int_{\mathbb{R}} \frac{|z(u)|^2}{s(u)^2 + \tau}\,\mathrm{d}u, \qquad V = \log(1/\tau), \qquad L_V = \tfrac{1}{2}\log(V/\pi).$$

The fixed-row asymptotic, oddness and the actual theta tail give

$$S(e^{-V}) \sim \frac{|a|^2}{\lambda}(V/\pi)^{\lambda} \qquad (V \to \infty). \tag{130}$$

Here is the two-sided estimate needed for its leading constant. For each fixed $\delta > 0$, the theta lower envelope implies $e^{-V}/s(u)^2 \to 0$ uniformly on $|u| \le L_V - \delta$. Indeed its supremum is at most a fixed constant times $V^{-2}\exp[-V(1 - e^{-2\delta})]$ for large V. Also

$$\int_{|u|\le R} |z(u)/s(u)|^2\,\mathrm{d}u \sim \frac{|a|^2}{\lambda} e^{2\lambda R}.$$

This gives the required lower with factor $e^{-2\lambda\delta}$. For the upper, split at $L_V + \delta$, use $s^2/(s^2+\tau) \le 1$ inside, and use $s^2/(s^2+\tau) \le s^2/\tau$ outside. The global fixed-row bound $|z/s| \le Ce^{\lambda|u|}$ and theta upper imply

$$\int_{|u|>R} |z(u)|^2\,\mathrm{d}u \le C(\pi e^{2R})^{\lambda+1} e^{-\pi e^{2R}}$$

for large R. To see this, substitute $r = \pi e^{2|u|}$ on both tails and bound the resulting incomplete gamma integral by twice its first power-exponential term once $r \ge 2(\lambda + 1)$. After multiplication by e^V, the exterior at $L_V + \delta$ is $O(V^{\lambda+1}e^{-V(e^{2\delta}-1)}) = o(V^{\lambda})$. The inner contribution gives factor $e^{2\lambda\delta}$. Let $\delta \downarrow 0$ to prove (130).

For every unit-moment physical vector, weighted Cauchy–Schwarz gives

$$1 \le S(\tau)(\|sk\|_2^2 + \tau\|k\|_2^2).$$

Put $V_\epsilon = 2d_z\epsilon^{-2/\lambda}$. For fixed $0 < \eta < 1$ take $V = (1-\eta)V_\epsilon$. Equation (130) gives $S(e^{-V})\epsilon^2 \to (1-\eta)^{\lambda} < 1$, so

$$\|k\|_2^2 \ge c_\eta \epsilon^2 e^V$$

for every admitted vector at small ϵ. This yields a liminf at least $(1-\eta)d_z$ in (128). It also proves the same lower for every smaller correction class.

For the ambient upper fix $\eta > 0$, set $V = (1+\eta)V_\epsilon$, $\tau = e^{-V}$, and take

$$k_\tau(u) = \frac{z(u)}{(s(u)^2+\tau)S(\tau)}.$$

This is an odd physical L^2 vector with exact unit moment and

$$\|sk_\tau\|_2^2 + \tau\|k_\tau\|_2^2 = S(\tau)^{-1}.$$

Now $S(\tau)\epsilon^2 \to (1+\eta)^\lambda > 1$, making it admissible and giving $\|k_\tau\|_2 \le \epsilon e^{V/2}$. The limsup is at most $(1+\eta)d_z$. Let $\eta \downarrow 0$ in both bounds. The lower also shows that the costs tend to infinity, so the logarithms are valid eventually.

To impose the original polar constraint put $b = s\sinh(u/2)$ and $A_\tau = M_{s^2} + \tau I$ on odd physical L^2. Define

$$S_{bb} = \langle b, A_\tau^{-1} b\rangle, \qquad S_{bz} = \langle b, A_\tau^{-1} z\rangle, \qquad S_{\mathrm{pol}} = S(\tau) - |S_{bz}|^2/S_{bb}.$$

The two physical rows are independent: their profile tail exponents are $1/2$ and $\lambda > 1/2$. Their positive energy Gram is therefore invertible. The exact constrained energy representer is

$$v_\tau = A_\tau^{-1}\left(z - b\frac{S_{bz}}{S_{bb}}\right), \qquad \langle b, v_\tau\rangle = 0, \quad \langle z, v_\tau\rangle = S_{\mathrm{pol}}, \quad \|sv_\tau\|^2 + \tau\|v_\tau\|^2 = S_{\mathrm{pol}}.$$

The same transition estimate used in (130) pays the mixed entries as well. Indeed $w_0 = z/s \sim ae^{\lambda u}$ and $w_1 = b/s = \sinh(u/2) \sim \frac{1}{2}e^{u/2}$ on the positive arm; both are odd on the full line. Put $\nu_0 = \lambda$, $\nu_1 = 1/2$, $a_0 = a$, $a_1 = 1/2$, and $\chi_\tau = s^2/(s^2+\tau)$. For each of the four entries,

$$\frac{\int \chi_{e^{-V}}\,\overline{w_i}w_j\,\mathrm{d}u}{\overline{a_i}a_j e^{(\nu_i+\nu_j)L_V}} \longrightarrow \frac{2}{\nu_i+\nu_j}.$$

For precision, inside $[-L_V+\delta, L_V-\delta]$ the multiplier χ tends uniformly to one. The normalized integral of its complement up to $L_V+\delta$ is $O(\delta)$, using the fixed exponential envelopes of the rows. Beyond $L_V+\delta$ the bound $\chi \le e^V s^2$ makes the normalized absolute mixed integral exponentially small, exactly as in the scalar tail estimate above. Compact intervals and the lower-order row terms vanish after normalization. First let $V \to \infty$, then $\delta \downarrow 0$. This proves entrywise complex Gram convergence, retaining the phase of a; diagonal modulus bounds alone would not suffice.

With $E_V = e^{L_V}$ the limits give

$$S_{bb} \sim \tfrac{1}{2}E_V, \qquad S_{bz} \sim \frac{a}{\lambda+1/2}E_V^{\lambda+1/2}, \qquad S_{\mathrm{pol}} \sim \frac{|a|^2}{\lambda}\left(\frac{\lambda-1/2}{\lambda+1/2}\right)^2 (V/\pi)^\lambda.$$

For a polar unit-moment vector, energy Cauchy–Schwarz now reads $1 \le S_{\mathrm{pol}}(\|sk\|^2 + \tau\|k\|^2)$, since its moment against b is zero. Repeat the two energy tests with $V_\epsilon = 2d_z^{\mathrm{pol}}\epsilon^{-2/\lambda}$ and $V = (1\mp\eta)V_\epsilon$. The lower test is uniform over the polar class; the upper uses $k_\tau = v_\tau/S_{\mathrm{pol}}$, with the polar constraint and unit target exact. Its energy is S_{pol}^{-1}. The same norm-versus-squared-norm factor $1/2$ as above proves (129). Set inclusion then gives the sharper null liminf, without using this polar ambient upper as a null upper. □

For the first row, $a = -4\pi^3$ and $\lambda = 13/2$, so $d_{z_0} = (\pi/2)[13/(32\pi^6)]^{2/13}$. The polar coefficient is $d_{z_0}^{\mathrm{pol}} = d_{z_0}(49/36)^{2/13}$. The ambient upper uses the explicitly defined k_τ; the genuine-null upper is supplied separately by the canonical interpolation theorem.

10.2 The qualitative consequence of the quantitative estimate

The quantitative Gram bound also implies exact finite-sum interpolation with arbitrarily small profile norm. The same conclusion has an independent proof in Lemma C.1, Appendix C, which is sufficient for the fixed-span bounded criterion without the estimates of this section.

To see the quantitative deduction, extend the desired moments on any independent subset of the canonical rows to one fixed full-list target $d' \in \mathbb{C}^{J+1}$. The minimizer (125), with $\Lambda = 1$ in (104), satisfies

$$Tk_L = d', \qquad \|Bk_L\|_2^2 \le (d')^* \mathsf{S}_{\tau_L}^{-1} d' \le \|D_L^{-1} d'\|^2 \longrightarrow 0.$$

For finite sums, the physically dense subspace $\mathcal{F} = \{r/s : r \in \mathcal{D}_a\}$ has onto moment image: T is bounded and onto on $\mathcal{N}_a$, so $T(\mathcal{F})$ is a dense linear subspace of a finite-dimensional space. Choose a bounded physical right inverse $R : \mathbb{C}^{J+1} \to \mathcal{F}$. For $u_n \in \mathcal{F}$ converging physically to a chosen k_L, the vectors $u_n + R(d' - Tu_n)$ have the moments exactly and converge physically to k_L. Boundedness of B gives profile convergence. Choose L first and n second to achieve any positive profile tolerance. This supplies a second route to the interpolation conclusion; the independent appendix proof is the route used in the bounded criterion.

11 Consequences of the complete constructions

The fixed-exponent receiving theorem quantifies the inverse approximation inside a complete signed certificate. Its guarded arithmetic model specifies how a finite lower enclosure is formed and accepted. The weighted carrier theorem supplies a different capability: full-jet compatibility is equivalent to actual-norm canonical synthesis, and the resulting projector computes optimal lifts of each admitted finite head. Sharp interpolation cost and its order-uniform budget pay the physical norm when those corrections vary with the support.

These conclusions return to the same form and original constrained domain. The evaluated M25 lower bound uses full receiving and mixed matrices followed by the paid localization. The covariance criteria retain their complete-source hypotheses and their recipient-dependent normalization. Alternative arithmetic-only, augmented-source and high-frequency certificates in the technical supplement provide additional specified ways to supply those hypotheses.

A The odd original-domain criterion

We give the needed realization-specific reduction to the classical Weil criterion [37, 1]. For $F(z) = \int f(u) e^{zu}\,\mathrm{d}u$, the complete explicit formula has zero-side form

$$W[f, g] = \sum_{\rho} \overline{F(1/2 - \overline{\rho})} G(\rho - 1/2). \tag{131}$$

The sum counts multiplicities and is conjugate-linear in its first argument, consistently with the physical inner product. On odd real tests $W[f] = -\sum_\rho F(\rho - 1/2)^2$. The physical form in (7) equals $W[f] + 2|F(1/2)|^2$ on odd tests; the polar term vanishes on $\mathcal{K}$.

Lemma A.1 (Odd original-domain criterion)**.** *Nonnegativity of q_H on V is equivalent to RH.*

Proof. The theta Mellin identity, in the normalization (5), is

$$M\psi(z) = \xi(1/2 + z), \qquad M\psi'(z) = -z\xi(1/2 + z), \qquad M\psi'(1/2) = -1/4.$$

Indeed, with $t = e^{2u}$ and $A(t) = \sum_{n\ge1} e^{-\pi n^2 t}$, one has $\psi = e^{u/2}(4t^2A'' + 6tA')$. Two integrations by parts on $0 < t < \infty$, initially for $\Re s > 1$, give

$$M\psi(s - 1/2) = \frac{s(s-1)}{2}\int_0^\infty A(t)t^{s/2-1}\,\mathrm{d}t = \xi(s).$$

Theta inversion gives the entire continuation. For any compact smooth real odd f, put $g = f + 4F(1/2)\psi'$. Its polar moment is zero and its transform agrees with F at every nontrivial zero, so $W[g] = W[f]$.

Here g/s belongs to the original minimal domain. To see this directly, take even smooth cutoffs χ_R and a fixed compact real odd b with $Mb(1/2) = 1$. Set $g_R = \chi_R g - M(\chi_R g)(1/2)b$. Then g_R is in the exact polar compact core. Complete theta derivative envelopes give

$$\|(g_R - g)/s\|_2 \to 0, \qquad \|(g_R - g)'\|_2 \to 0.$$

For example $|\psi'| \le 40e^{|u|/2}r^3e^{-r}$ and the lower bound (8) give an integrable upper $800e^{|u|}r^4e^{-r}$ for $|\psi'/s|^2$. The derivative atom polynomials are $-8r^3 + 30r^2 - 15r$ and $16r^4 - 112r^3 + 165r^2 - 75r/2$. Their absolute coefficient bounds for $r \ge \pi > 3$, and consecutive atom ratios, give $|\psi''| \le 160e^{|u|/2}r^4e^{-r}$. These complete tails pay both limits. Bounded cutoff derivatives and the vanishing compact moment correction preserve them. The bound $a \le 17\xi^2$ now gives convergence in (6).

The same tails give exponential-weighted L^1 convergence of $g_R - g$ and its first derivative. Integration by parts therefore gives

$$\sup_{|\Re z|\le1/2}(1 + |\Im z|)|M(g_R - g)(z)| \longrightarrow 0.$$

The zero count implies $\sum_\rho(1 + |\Im\rho|)^{-2} < \infty$, so the complete quadratic zero sum converges as well. Passing the compact explicit formula to the limit yields $q_H[g/s] = W[g] = W[f]$. Thus positivity on V implies $W[f] \ge 0$ for every real odd compact smooth f.

Suppose an off-line centered zero $\sigma_0 = \delta + i\gamma$ exists, with $\delta, \gamma > 0$ by symmetry. Nontrivial real zeros are absent because the alternating eta series gives $\zeta(s) < 0$ for real $0 < s < 1$. Fix $\Gamma > \gamma + 1$. An even real polynomial P annihilates every zero orbit in $|\Im\sigma| \le \Gamma$ except the quartet of σ_0, and $P(\sigma_0) \ne 0$. Choose real a, b with $iP(\sigma_0)(a + b\sigma_0^2) = 1$, possible because $\Im\sigma_0^2 \ne 0$. Put

$$F_T(z) = iP(z)(a + bz^2)[e^{T(z-i\gamma)^2} - e^{T(z+i\gamma)^2}], \quad T \ge 1.$$

This entire function is odd and real type. It is the bilateral Laplace transform of a real odd Gaussian multiplied by polynomial and trigonometric factors. At each fixed T, even compact cutoffs of that physical function converge in exponential Schwartz seminorms. Repeated integration by parts gives transform convergence with every polynomial vertical decay uniformly on $|\Re z| \le 1/2$. The zero count $N_\zeta(Y) = O(Y\log(Y + 2))$ therefore permits dominated convergence in (131), so $W[f_T] \ge 0$.

At σ_0,

$$e^{-T\delta^2}F_T(\sigma_0) = 1 - e^{-4T\gamma^2+4iT\delta\gamma} \longrightarrow 1.$$

The target quartet, of multiplicity m, contributes $-4m\Re F_T(\sigma_0)^2 \sim -4me^{2T\delta^2}$. All other zeros in the finite window are annihilated. For exterior zeros put $d_\rho = ||\Im\sigma_\rho| - \gamma| > 1$. A fixed polynomial envelope bounds their squared transform by

$$C(1 + |\Im\sigma_\rho|)^{2D}e^{2T(1/4-d_\rho^2)} \le Ce^{-3T/2}(1 + |\Im\sigma_\rho|)^{2D}e^{-(d_\rho^2-1)}.$$

The full remaining zero sum is $O(e^{-3T/2})$ by the same zero-count bound. The negative quartet dominates, a contradiction. This proves RH. Conversely, RH makes (131) a sum of squared absolute transforms on the polar core; original form closure extends positivity to V. □

B A complementary constant-shift certificate criterion

The vanishing-allowance criterion of Theorem 3.2 and the criterion below retain the same original arithmetic Weil form. They use different norms and different finite receiving constructions. In particular, the fixed shift below is in the unweighted logarithmic profile norm, not in the theta-congruent norm. Its use is justified by a uniform-semiboundedness argument, which we include in full. Each criterion specifies its own operator, norm and finite verification test.

B.1 Uniform semiboundedness on the odd compact class

Let $\mathcal{Q}(f,g)$ be the original archimedean-minus-prime form on the logarithmic line, with Fourier measure $d\xi/(2\pi)$. Thus its archimedean multiplier is $m_0 + a(\xi)$ and its arithmetic term retains both translations for every prime power. Put

$$\mathcal{S}_1(f) = \sup_{u\in\mathbb{R}} e^{|u|}\max\{|f(u)|, |f'(u)|\}.$$

The argument below uses the compact and strong test classes throughout.

Lemma B.1 (Mixed continuity and compact null approximation)**.** *There is an explicit finite constant C_* such that*

$$|\mathcal{Q}(f,g)| \le C_*\mathcal{S}_1(f)\mathcal{S}_1(g), \qquad C_* = 17 + |m_0| + 2\sum_{n\ge 2}\frac{\log n(1+\log n)}{n^{3/2}}.$$

The complex span of the odd functions

$$r_t = \tfrac{1}{2}(\tau_t q_0 - \tau_{-t}q_0), \quad q_0 = (\partial_u^2 - \tfrac{1}{4})\psi, \quad t\in\mathbb{R},$$

is dense in odd unweighted $L^2(\mathbb{R})$. Every r_t is a strong two-moment-zero mixed null of $\mathcal{Q}$, and every finite linear combination can be approximated in $\mathcal{S}_1$ by smooth compact odd two-moment-zero functions.

Proof. Since $\int e^{-2|u|}du = 1$, the L^2 norms of f and f' are at most $\mathcal{S}_1(f)$. The digamma series gives $a(\xi) \le (\xi^2/4)\sum_{n\ge 0}(n+1/4)^{-3} \le 17\xi^2$: the sum is at most $64 + (5/4)^{-3} + 1/\{2(5/4)^2\} < 68$. This bound and Plancherel therefore bound the archimedean mixed term by $(17+|m_0|)\mathcal{S}_1(f)\mathcal{S}_1(g)$. For $t \ge 0$,

$$\int_{\mathbb{R}} e^{-|u|}e^{-|u+t|}du = (1+t)e^{-t}.$$

Apply this to the two arithmetic translates at $t = \log n$, use $\Lambda(n) \le \log n$, and sum. The displayed series converges by the integral test. This proves continuity for the complete raw form, without splitting a divergent bare prime diagonal. For completeness, each exponential moment obeys $|Mf(\pm 1/2)| \le (8/3)\mathcal{S}_1(f)$, so adding both explicit-formula pole products costs at most $(128/9)\mathcal{S}_1(f)\mathcal{S}_1(g)$.

The theta Mellin identity gives

$$Mq_0(z) = (z^2 - \tfrac{1}{4})\xi(1/2+z).$$

Translation multiplies this transform by an exponential. It still vanishes at both poles and every centered nontrivial zero. The complete polarized explicit formula on the strong test class thus proves mixed nullity, with zero multiplicities retained. Strong theta derivative decay pays all Mellin and

arithmetic limits. The Fourier transform of q_0 is nonzero almost everywhere, since its entire xi factor has only isolated real-frequency zeros and its polynomial factor has none.

If an odd $h \in L^2$ is orthogonal to all r_t, parity makes its correlation with $\tau_{-t}q_0$ the negative of its correlation with $\tau_t q_0$. Both correlations are therefore zero. Fourier uniqueness, applied to the L^1 product of their two L^2 Fourier transforms, implies $\widehat{h}\,\overline{\widehat{q_0}} = 0$ almost everywhere and hence $h = 0$. This proves the asserted density, not continuity of the Weil form in the unweighted L^2 norm.

Choose even smooth compact cutoffs χ_R with uniformly bounded derivatives through order three. Then $q_R = (\partial_u^2 - 1/4)(\chi_R\psi)$ is even, compact and has both exponential moments exactly zero, by integration by parts. Complete theta derivative decay gives $\mathcal{S}_1(q_R - q_0) \to 0$. Translations satisfy $\mathcal{S}_1(\tau_t f) \le e^{|t|}\mathcal{S}_1(f)$. For each fixed finite sum, replace its finitely many translates of q_0 by translates of q_R, choosing R after the sum and the desired tolerance are fixed. Antisymmetrization preserves oddness and the two zero moments, proving the compact approximation assertion. □

Proposition B.2 (A uniform unweighted floor)**.** *RH is equivalent to the existence of one finite $K \ge 0$ such that*

$$\mathcal{Q}[f] \ge -K\|f\|_2^2$$

for every smooth compact odd two-moment-zero test f, across all supports.

Proof. Fix such an f. By Lemma B.1, choose finite odd mixed-null sums r_n with $\|f - r_n\|_2 \le 1/n$, and compact odd two-moment-zero approximants g_n satisfying

$$\mathcal{S}_1(g_n - r_n) \le \{n[1 + \mathcal{S}_1(f - r_n)]\}^{-1}.$$

Write $e_n = g_n - r_n$ and $v_n = f - g_n$. Mixed nullity gives $\mathcal{Q}[f - r_n] = \mathcal{Q}[f]$, whereas mixed continuity gives

$$|\mathcal{Q}[v_n] - \mathcal{Q}[f]| \le 2C_*/n + C_*/n^2, \qquad \|v_n\|_2 \le 2/n.$$

The same floor applies to every compact admissible v_n, even though their supports grow. It follows that $\mathcal{Q}[f] \ge 0$. The odd original-domain Weil criterion then gives RH. Conversely RH gives positivity and the choice $K = 0$. This proof uses neither a support-dependent floor nor a global L^2-closed extension. □

B.2 The supported constant-shift operator

For $g \in L^2(0, L)$ use the odd-half unitary identification $f(x) = g(x)/\sqrt{2}$ for $x > 0$ and $f(-x) = -g(x)/\sqrt{2}$. Then, with $p(x) = \sinh(x/2)$,

$$Mf(1/2) = \sqrt{2}\int_0^L p(x)g(x)dx, \qquad Mf(-1/2) = -Mf(1/2).$$

The sum of the two explicit-formula pole products is therefore $-4|\langle p, g\rangle|^2$. The complete raw supported odd form is

$$B_L = D_L + R_L + m_0 - C_L^{\mathrm{ar}}$$

on the full odd-half space, without a polar constraint. Its minimal supported closure and operator domain are those in Lemma 5.1, with bounded arithmetic added.

Theorem B.3 (Complementary constant-shift RH criterion)**.** *Fix $\beta > 0$ and any increasing support sequence $L_j \to \infty$. The following are equivalent:*

1. *RH;*

2. $B_L + \beta I \succeq 0$ *for every* $L > 0$*;*

3. $B_{L_j} + \beta I \succeq 0$ *for every member of the sequence.*

Under RH the stronger floor $B_L + \beta I \succeq \beta I$ *holds.*

Proof. Under RH the zero-side form is a sum of squared absolute Mellin transforms. The exact pole calculation makes the raw odd form equal to that nonnegative sum plus $4|\langle p, g\rangle|^2$. Positivity passes from compact tests to the same minimal supported closure, so $B_L \succeq 0$. Conversely every compact odd polar-free test lies in each sufficiently large support. The complete supported forms agree on this test under zero extension. Assertion 3 therefore gives the one uniform unweighted floor $\mathcal{Q}[f] \geq -\beta\|f\|_2^2$. Proposition B.2 proves RH. No nesting of finite Schur matrices or assumption of global closability is used. □

B.3 A complete finite receiving test with fixed margin

Put $A_L = B_L + \beta I = D_L + V_L^{\mathrm{sh}}$ and choose explicit bounds $\|V_L^{\mathrm{sh}}\| \leq P$ and $\|[D_L, V_L^{\mathrm{sh}}]\| \leq K_1$, with domain preservation. The full-response commutator theorem and the complete paired-arithmetic bound supply these hypotheses; the scalar shift has zero commutator. Fix a receiving floor $\tau > 0$, independently of β. Choose a certified finite prefix $U = [e_0, \ldots, e_{N-1}]$ such that

$$a(k_N) + m_0 - \|C_L^{\mathrm{ar}}\| + \beta \geq \tau.$$

With $Q = I - UU^*$, the entire receiving operator $T = QA_LQ$ on $\operatorname{ran} Q$ satisfies $T \succeq \tau I$, because $R_L \succeq 0$. Set $H = U^*A_LU$, $C = QA_LU$, and $\alpha_N = \max_{q<N} a(k_q)$.

Proposition B.4 (No-polar cubic return enclosure)**.** *Define*

$$G_2 = P\alpha_N + (1 + P/\tau)(K_1 + P^2).$$

Let $E_{N,m}$ *be the isometry onto modes* $N, \ldots, m-1$*,* $m > N$*, and put*

$$X_m = C^*E_{N,m}(E_{N,m}^*A_LE_{N,m})^{-1}E_{N,m}^*C.$$

Then

$$0 \preceq C^*T^{-1}C - X_m \preceq \epsilon_m I, \qquad \epsilon_m = G_2^2\{a(k_m)^{-3} + Pa(k_m)^{-4}\}.$$

A strictly positive directed lower enclosure of $H - X_m - \epsilon_m I$ *certifies* $A_L \succeq 0$ *on its entire supported domain. Under RH,* $\epsilon_m \leq \beta/4$ *and a two-sided center radius* $e \leq \beta/8$ *leave an outward lower of at least* $\beta I/2$*.*

Proof. Here Q commutes with D_L. The exact mixed map is $C = QV_L^{\mathrm{sh}}U$, so $\|C\| \leq P$ and $\|D_LC\| \leq P\alpha_N + K_1$. For $y = T^{-1}C$, coercivity and the operator equation give

$$\|y\| \leq P/\tau, \qquad \|D_Ly\| \leq P + P^2/\tau.$$

Domain preservation and the bounded commutator imply that $D_Ly = C - QV_L^{\mathrm{sh}}Qy$ belongs to $\operatorname{Dom} D_L$. Applying D_L gives

$$D_L^2y = D_LC - QV_L^{\mathrm{sh}}QD_Ly - Q[D_L, V_L^{\mathrm{sh}}]Qy, \qquad \|D_L^2y\| \leq G_2.$$

This supplies the required second-graph regularity rather than assuming it for a generic bounded perturbation.

Galerkin energy minimization uses the legal receiving trial obtained by retaining modes below m in y. The omitted vector h obeys

$$\langle h, D_L h\rangle \le a(k_m)^{-3}\|D_L^2 y\|^2, \qquad \|h\|^2 \le a(k_m)^{-4}\|D_L^2 y\|^2.$$

Adding the bounded perturbation fee proves the complete return estimate in finite coefficient order. Schur square completion proves the certificate implication. Under RH, $A_L \succeq \beta I$, so its exact Schur complement is at least βI. Subtracting $\epsilon_m I$ leaves at least $3\beta I/4$. A center enclosure of radius e followed by subtraction of eI can lose $2e$, which leaves at least $\beta I/2$ under the stated guard. □

Every finite coefficient in this test is effectively enclosable: the complete Green coefficient series has its paid tail, every active arithmetic pair is retained, and the receiving finite matrix is coercive. All inverse, coefficient and final matrix errors must be included in the directed lower. This gives the complete finite-recognition test for the stated supported form.

For a sufficient explicit threshold take

$$a(k_m) \ge \max\{1, P, (8G_2^2/\beta)^{1/3}\}.$$

It gives $\epsilon_m \le \beta/4$. For fixed β, choose $\tau = 1$ and a near-threshold outward prefix. The complete bounds yield $P = O(e^L + L)$, $\alpha_N = O(e^L)$ and $K_1 = O(Le^L)$, hence $G_2 = O(e^{3L})$ and a sufficient near-threshold choice has $\log m = O(e^{2L})$ for $L \ge 1$. On $L = \frac{1}{2}\log(10\log M/\pi) + 2$, this is $m = M^{O(1)}$. Here M labels support growth in the fixed-shift realization. This trial-dimension bound has the fixed conditional margin and unweighted norm of Theorem B.3. Arithmetic precision and validation costs enter separately in the directed finite test.

B.4 An arithmetic-only fixed-shift construction

The support-uniform response norm gives a further finite target for the same original positivity question. Keep the unweighted odd-half coordinates and minimal supported domain of Appendix B. For a fixed $\beta > \pi/2$, define

$$\mathcal{A}_{L,\beta} = D_L + m_0 - C_L^{\mathrm{ar}} + \beta I, \qquad \delta_\beta = \beta - \pi/2 > 0.$$

The complete arithmetic operator retains both killed translation orientations and the reflected odd channel for every active prime power.

Corollary B.5 (Arithmetic-only RH criterion)**.** *Fix $\beta > \pi/2$ and any increasing sequence $L_j \to \infty$. The following are equivalent:*

1. *RH;*
2. *$\mathcal{A}_{L,\beta} \succeq 0$ for every $L > 0$;*
3. *$\mathcal{A}_{L_j,\beta} \succeq 0$ for every j.*

Under RH,

$$\mathcal{A}_{L,\beta} \succeq \delta_\beta I \quad \text{uniformly in } L.$$

In particular $\beta = 2$ has conditional margin greater than $2/5$, and $\beta = 3$ has conditional margin greater than $7/5$.

Proof. Write the complete raw odd form as $B_L = D_L + R_L + m_0 - C_L^{\mathrm{ar}}$. In the half-unitary convention, the two explicit-formula pole products are $-4|\langle \sinh(x/2), g\rangle|^2$. Thus under RH the raw form B_L is the nonnegative zero-side form plus the positive term $4|\langle \sinh(x/2), g\rangle|^2$. This proves $B_L \succeq 0$ on compact odd tests and on its same minimal supported closure. Lemma 3.1 now gives

$$\mathcal{A}_{L,\beta} = B_L + \beta I - R_L \succeq (\beta - \pi/2) I.$$

Conversely, cofinal positivity of $\mathcal{A}_{L,\beta}$ gives $B_L = \mathcal{A}_{L,\beta} + R_L - \beta I \succeq -\beta I$ on each accepted support. Every compact odd polar-free test belongs to all sufficiently large supports, and the complete forms agree on it under zero extension. Hence the original raw form has one uniform lower bound $-\beta\|f\|_2^2$ on that compact class. The exact-null density and strong compact-approximation argument of Proposition B.2 converts this bound to original Weil positivity and RH. The stated rational margins follow from $\pi < 16/5$. □

Complete arithmetic and commutator bounds. Let

$$\Sigma_L = \sum_{2 \le n < e^{2L}} \frac{\Lambda(n)}{\sqrt{n}}, \qquad P_\beta = |m_0| + \beta + 2\Sigma_L, \qquad K_\beta = (8 + 6\pi)\Sigma_L.$$

Then $V_{L,\beta} = m_0 - C_L^{\mathrm{ar}} + \beta I$ preserves $\operatorname{Dom} D_L$ and

$$\|V_{L,\beta}\| \le P_\beta, \qquad \|[D_L, V_{L,\beta}]\| \le K_\beta.$$

Indeed each killed symmetric pair has norm at most two. The crosscut domain argument in Appendix 6 gives, before odd restriction, a commutator bound $4(2+\pi)$ for the supported archimedean operator. Since that operator equals $D_L + R_L$ on the odd sector, bounded subtraction and Lemma 3.1 give for the same pair

$$\|[D_L, (T_t + T_{-t})|_{\mathrm{odd}}]\| \le 4(2+\pi) + 4\|R_L\| \le 8 + 6\pi.$$

Bounded subtraction preserves the supported domain, and the killed-pair domain invariance is supplied by the crosscut argument. Summation with the complete arithmetic coefficients proves the two bounds above.

Complementary complete-return implementations. The same domain and commutator bounds allow Proposition B.4 to be applied directly with $V = V_{L,\beta}$, $P = P_\beta$ and $K_1 = K_\beta$. Choose a DN prefix U and $\tau > 0$ with $a(k_N) + m_0 - 2\Sigma_L + \beta \ge \tau$, put $Q = I - UU^*$, and set $H = U^*\mathcal{A}_{L,\beta}U$, $C = -QC_L^{\mathrm{ar}}U$. The proposition gives a complete finite Galerkin return enclosure; its constants and induced error bounds are recomputed for these data.

A scalar receiving payment gives the complementary finite lower

$$\Gamma = H - \tau^{-1}C^*C, \qquad C^*C = U^*(C_L^{\mathrm{ar}})^2U - (U^*C_L^{\mathrm{ar}}U)^2.$$

The first term integrates complete spatial output functions. The identity retains every prime-pair and reflected mixed contribution. A positive directed lower for Γ proves positivity on the entire supported domain by Schur square completion. This is the full-output construction used by the evaluated arithmetic examples below. The following theorem gives an additional first-moment-only test by paying the receiving return analytically at its prescribed cutoff.

B.5 First-moment certificates with analytic tail payment

The arithmetic-only realization admits a finite test involving its first moments alone. A prescribed spectral cutoff and the complete first commutator bound pay the entire receiving return analytically. This provides a simpler verification object alongside the complete-return constructions above.

Fix $\mathcal{A}_{L,2} = D_L + V_L$, where $V_L = m_0 - C_L^{\mathrm{ar}} + 2I$, and set $\delta_0 = 1/5$. Choose rational budgets $P \geq 1$ and $K \geq 0$ satisfying

$$\|V_L\| \leq P, \qquad \|[D_L, V_L]\| \leq K.$$

The arithmetic domain invariance established above is retained. Explicit outward budgets are available from

$$P \geq 8 + 12e^L, \qquad K \geq 192e^L;$$

Lemma S21.1 below proves their elementary prime estimate. The dimension estimate below uses rational budgets within one unit above these two elementary lower bounds. Sharper complete budgets may also be used. Using the smooth-cutoff relative payment of *Relative tail payment and spectral enclosures* [29], choose a rational A between the following maximum and that maximum plus one:

$$A \geq \max\left\{16P, \left(\frac{1152K^2}{\delta_0}\right)^{1/3}\right\}. \tag{132}$$

Choose an outward integer N with $Le^A/\pi \leq N \leq Le^A/\pi + 2$. All subsequent matrices use these same choices. Rational enclosures of width less than one give these choices without deciding equality of computable real numbers.

Theorem B.6 (Core-only arithmetic certificate)**.** *Let $U = [e_0, \ldots, e_{N-1}]$ be the ordinary DN mode isometry and put*

$$H_N = U^*\mathcal{A}_{L,2}U = \mathrm{diag}_{q<N}\, a(k_q) + (m_0 + 2)I - U^*C_L^{\mathrm{ar}}U. \tag{133}$$

If a complete directed lower enclosure certifies $H_N \succeq I/5$, then, on the entire original supported domain,

$$\mathcal{A}_{L,2} \succeq \frac{52}{405}I > \frac{1}{8}I.$$

For any prescribed increasing unbounded support sequence L_j, RH is equivalent to the existence of these directed certificates at every L_j. Under RH, the exact cores satisfy $H_N \succ 2I/5$. For the near-threshold choices above, the sufficient matrix dimension is

$$\log N = O(e^L + \log L), \qquad N = \exp(O(\sqrt{\log M})) \quad \textit{when } e^{2L} = O(\log M),\ L \geq 1.$$

The dimension bound concerns this finite verification object; precision and validation costs enter separately.

Proof. Write $Q = I - UU^*$, $C = QV_LU$ and $T = Q\mathcal{A}_{L,2}Q|_{\mathrm{ran}\,Q}$. The omitted diagonal satisfies $a(k_N) > A$. For any rational $\tau \in [A/2, A/2 + 1]$, $T \succeq (A - P)I \succeq \tau I$. Once the finite premise $H_N \succeq \delta_0 I$ is verified, Lemma S21.2 gives the complete covariance payment

$$\tau^{-1}C^*C \preceq 3H_N/16.$$

Thus the true Schur complement satisfies $H_N - C^*T^{-1}C \succeq 13H_N/16 \succeq 13I/80$. Moreover $\|T^{-1}C\| \leq P/\tau \leq 1/8$ and $\tau \geq 8$. For head u and a tail form vector v, the exact completion is

$$\langle u, (H_N - C^*T^{-1}C)u\rangle + \|T^{1/2}(v + T^{-1}Cu)\|^2.$$

The triangular map $(u, v) \mapsto (u, v + T^{-1}Cu)$ has inverse norm at most $1 + \|T^{-1}C\| \le 9/8$. Thus the completed energy is bounded below by $\min(13/80, 8)$ times the squared transformed norm, and Schur square completion yields the whole bound $\min(13/80, 8)(9/8)^{-2}I = 52I/405$.

Under RH, Corollary B.5 gives $\mathcal{A}_{L,2} \succeq (2 - \pi/2)I \succ 2I/5$, hence the same floor for each compression. Directed recognition has a strict margin: a Hermitian center with induced radius $e \le 1/20$ has outward lower strictly above $3I/10$. The possible center displacement and the subtraction together cost at most $2e$. Entrywise radii are first converted to an induced matrix bound.

Conversely, successful tests on the prescribed cofinal supports give $B_{L_j} = \mathcal{A}_{L_j,2} + R_{L_j} - 2I \succeq -2I$. Every compact odd polar-free test embeds in a sufficiently large support, with its original unweighted norm preserved. The uniform-floor implication in Proposition B.2 proves RH.

Finally $P, K = O(e^L)$ permit $A = O(e^L)$ in (132). The outward choice of N gives the displayed dimension bounds, of the same asymptotic order as before. The sharper cutoff changes a sufficient admission, not the directed head premise or a passed arithmetic certificate. The full proof of the relative payment and its domain requirements is in Appendix S21. □

Complete finite entries. For $t = \log n \in (0, 2L)$ and $w_n = \Lambda(n)/\sqrt{n}$, the arithmetic entry in (133) is

$$(U^* C_L^{\mathrm{ar}} U)_{ij} = \frac{2}{L} \sum_{n<e^{2L}} w_n \Bigg[\int_0^{\max(0,L-t)} \sin(k_i x) \sin(k_j(x+t))\, dx + \int_{\max(0,t-L)}^{L} \sin(k_i x) \sin(k_j(x-t))\, dx \Bigg].$$

The second sine itself implements odd reflection when its argument is negative. Both orientations and every active prime power are retained. The identity $2\sin(k_i x)\sin(k_j(x+\sigma t)) = \cos((k_i - k_j)x - \sigma k_j t) - \cos((k_i + k_j)x + \sigma k_j t)$ gives exact finite primitives. When $i = j$, the zero-frequency primitive is evaluated as a constant integral, with no division by an interval containing zero. The diagonal is also directly enclosable through $a(k_q) + m_0 = \operatorname{Re}\Psi(1/4 + ik_q/2) - \log\pi$. Thus the test uses only signed arithmetic first moments and the known diagonal. Its cutoff supplies the analytic whole-tail guarantee. The two evaluated certificates below retain their original complete Schur proofs and specified cutoffs.

B.6 A complete arithmetic-only supported certificate

The arithmetic-only criterion of Corollary B.5 admits a complete finite calculation at the same support used by the second SD example, now in the unweighted odd-half profile norm. Set

$$L = \frac{1}{2}\log\left(\frac{96\log 2}{\pi}\right) + \frac{1}{10}, \qquad N = 256, \qquad \beta = 3,$$

and use the original minimal supported realization of

$$\mathcal{A}_{L,3} = D_L + m_0 - C_L^{\mathrm{ar}} + 3I, \qquad B_L = \mathcal{A}_{L,3} + R_L - 3I.$$

Here $U = [e_0, \dots, e_{255}]$ is the orthonormal DN core and $Q = I - UU^*$ is its entire receiving complement.

Theorem B.7 (Evaluated arithmetic-only certificate)**.** *At the stated support, $\mathcal{A}_{L,3}$ is nonnegative on its complete odd supported domain. Consequently the complete raw odd form satisfies*

$$B_L \succeq -3I.$$

Proof. The complete active prime powers are

$$\{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25\}.$$

The arithmetic row bound can be verified directly in the unweighted logarithmic norm. On $(-L, L)$ use the positive even Schur weight $h(x) = \cosh(x/2)$. For $x \geq 0$, define

$$\begin{aligned} p_a(x) &= \sum_{n<e^{L+x}} \Lambda(n), \quad h_a(x) = \sum_{n<e^{L+x}} \Lambda(n)/n, \\ p_b(x) &= \sum_{n<e^{L-x}} \Lambda(n), \quad h_b(x) = \sum_{n<e^{L-x}} \Lambda(n)/n, \end{aligned}$$

where all sums run over the displayed active prime powers. The two killed translation orientations give the weighted row price

$$r(x) = \frac{A(x) + e^x B(x)}{1 + e^x}, \qquad A = p_a + h_b, \quad B = p_b + h_a.$$

Indeed their respective weight ratios are $\Lambda(n)(1 + e^x/n)/(1 + e^x)$ and $\Lambda(n)(1/n + e^x)/(1 + e^x)$, with the corresponding support indicators. Symmetry makes r even, and the complete paired operator is symmetric on the full supported interval. Applying

$$2|v(x)v(y)| \leq \frac{h(y)}{h(x)}|v(x)|^2 + \frac{h(x)}{h(y)}|v(y)|^2$$

to every paired shift proves its absolute quadratic-form bound by $\int r(x)|v(x)|^2\,\mathrm{d}x$. Restriction to the invariant odd sector, followed by the unitary odd-half identification, retains the reflected channel and gives the same bound for C_L^{ar}.

The coefficients A, B are constant on each complete reference cell. The derivative of its row price has the sign of $B - A$; both one-sided endpoint bounds therefore control the whole cell. The verifier checks all 7,828 cells and their exact consecutive endpoints. The reference uses the identical expression $(\sqrt{K}A + zB)/(\sqrt{K} + z)$, where $K = e^{2L}$ and $z = e^{L+x}$. Consequently the exact dyadic upper r_{up} gives $\|C_L^{\mathrm{ar}}\| \leq r_{\max} := \max r_{\mathrm{up}}$. Monotonicity of the digamma diagonal gives, on the whole suffix,

$$T = Q\mathcal{A}_{L,3}Q \succeq [\Re\Psi(1/4 + ik_N/2) - \log\pi - r_{\max} + 3]\, I \succ \tau I, \qquad \tau = \frac{71111}{65536}.$$

The directed bracket before I exceeds 1.0850825.

Define the complete finite matrices

$$\mathsf{B} = U^* C_L^{\mathrm{ar}} U, \qquad \mathsf{S} = (C_L^{\mathrm{ar}} U)^* (C_L^{\mathrm{ar}} U), \qquad H = U^* \mathcal{A}_{L,3} U.$$

The diagonal and scalar terms preserve the DN core, so the exact mixed map is $C = Q\mathcal{A}_{L,3}U = -QC_L^{\mathrm{ar}}U$ and

$$C^*C = \mathsf{S} - \mathsf{B}^*\mathsf{B}.$$

The full arithmetic-output Gram S is integrated on $(0, L)$. The 19 arithmetic integration intervals are formed from $0, L$ and the interior points $\log n$, $L - \log n$, $\log n - L$. On each interval the action columns are finite sine/cosine combinations. Their exact elementary antiderivatives, evaluated by directed arithmetic, enclose every entry of S and B. The separate closed-form first-moment formulas check all $65,536$ entries of B.

Inverse order therefore gives the complete Schur lower

$$H - C^*T^{-1}C \succeq \Gamma_{\mathrm{ar}} := H - \tau^{-1}(\mathsf{S} - \mathsf{B}^*\mathsf{B}).$$

The scientific enclosures give 256 strictly positive directed pivots for $\Gamma_{\mathrm{ar}} - I/10$. The portable verifier reconstructs the digamma diagonal, first moments and complete Schur matrix, reverses the DN coordinate order, and checks the factorization independently. Thus $\Gamma_{\mathrm{ar}} \succ I/10$ in the orthonormal core coordinates. Completion of squares against $T \succeq \tau I$ proves $\mathcal{A}_{L,3} \succeq 0$ on its entire supported domain. Finally $B_L = \mathcal{A}_{L,3} + R_L - 3I$ and $R_L \succeq 0$ give the raw-form bound. □

Scientific data and error bounds. The arithmetic-only companion contains the original 768-bit directed enclosures for H, S, B, $\mathsf{S} - \mathsf{B}^*\mathsf{B}$ and Γ_{ar}, together with the exact reference-cell data and rational τ. The source Gram includes the entire arithmetic output before the finite-core subtraction. The manifest identifies each scientific file and the prescribed support. The generator recomputes the continuous elementary integrals; the independent verifier checks the complete reference, recomputes every first moment from its separate closed-form identity and validates all 256 reversed-order pivots at 2,048 bits. All coefficient, integration, subtraction, inverse-reference and factorization errors are included in the interval operations. The margin $1/10$ is used for the finite Schur inequality; whole-domain positivity follows from its stated receiving bound and square completion.

B.7 A stronger arithmetic-only supported certificate

Increasing the DN core at the same support strengthens the supported lower bound in Theorem B.7. Set

$$L = \tfrac{1}{2}\log(96\log 2/\pi) + \tfrac{1}{10}, \qquad N = 768, \qquad \beta = 2.$$

Use $U = [e_0, \dots, e_{767}]$ and its entire orthogonal complement $Q = I - UU^*$ in the unweighted odd-half realization.

Theorem B.8 (Evaluated shift-two arithmetic certificate)**.** *At this support, $\mathcal{A}_{L,2} = D_L + m_0 - C_L^{\mathrm{ar}} + 2I$ is nonnegative on its complete odd supported domain. The complete raw odd form therefore satisfies $B_L \succeq -2I$.*

Proof. The active arithmetic shifts and the weighted-Schur majorant are those of Theorem B.7: all fourteen prime powers are retained. On each of the 7,828 reference cells, the price $(A + e^x B)/(1 + e^x)$ has constant-sign derivative, so checking both endpoints bounds the complete cell. The even weight $\cosh(x/2)$ then gives $\|C_L^{\mathrm{ar}}\| \le r_{\max}$ on the full supported space and its invariant odd sector. Monotonicity of the diagonal symbol on the entire DN suffix gives

$$Q\mathcal{A}_{L,2}Q \succeq t_* Q, \qquad t_* = \Re\Psi\left(\tfrac{1}{4} + \frac{i(768 + 1/2)\pi}{2L}\right) - \log\pi - r_{\max} + 2 > 1.18239457729109417.$$

The rational receiving bound $\tau = 77489/65536$ satisfies $0 < \tau < t_*$.

Define the complete continuous-output moments

$$H = U^*\mathcal{A}_{L,2}U, \quad J = U^*C_L^{\mathrm{ar}}U, \quad S = (C_L^{\mathrm{ar}}U)^*(C_L^{\mathrm{ar}}U).$$

Integration over all nineteen arithmetic panels encloses every entry of J and S, including all reflected and killed-shift terms. The full mixed return is $S - J^*J$. The scientific interval matrices enclose

$$\Gamma = H - \tau^{-1}(S - J^*J).$$

Directed factorization of $\Gamma - \frac{1}{10}I$ gives all 768 strictly positive pivots. Thus the actual Schur complement dominates $\Gamma \succeq \frac{1}{10}I$ on the finite core. Completing the square with the positive entire receiving suffix proves $\mathcal{A}_{L,2} \succeq 0$. Finally $B_L = \mathcal{A}_{L,2} + R_L - 2I$ and $R_L \succeq 0$ give the stated raw-form bound. □

The companion supplies all $768^2 = 589,824$ entries of each continuous moment matrix, the exact receiving bound, and the interval Schur matrix. The independent verifier reconstructs every arithmetic first moment from a separate closed formula, reconstructs all three Schur ingredients, checks every reference cell, and reverses the core order before factorization. The coefficient $1/10$ is the finite Schur-core margin used in the completion-of-squares argument. The result strengthens the supported raw lower bound from -3 to -2 at this fixed support.

Supporting certificate transfers. Section S22 of the technical supplement gives three complementary transfers for this arithmetic-only realization: complete-core representation with certified pullback error, conditional completion of a polar-free raw bound by a smooth edge direction, and local support extension from a positive whole-operator margin. Each retains its stated assumptions, original profile coordinates and complete mixed-error payment.

C Independent qualitative canonical interpolation

The fixed-span bounded-critical criterion needs exact canonical moments and arbitrarily small profiles, but no quantitative physical-norm bound. We prove that qualitative statement directly from the profile-unbounded row functionals of Lemma 8.3. This argument uses neither the quantitative Gram estimate nor Theorem 10.1.

Retain the original physical space, K_0 and the finite translate span $\mathcal{D}_a$ of Section 8. Every correction below is a finite sum of the admitted small translates.

Lemma C.1 (Exact canonical interpolation with small profile)**.** *Let $K \subset K_0$ be any subspace, with independent basis $\zeta_1, \dots, \zeta_k$. For every $d \in \mathbb{C}^k$ and $\varepsilon > 0$ there exists $r \in \mathcal{D}_a$ such that*

$$\langle \zeta_i, r/s \rangle = d_i \quad (1 \le i \le k), \qquad \|r\|_2 < \varepsilon. \tag{134}$$

Consequently, for every $h \in V$ there are genuine operator nulls k_n such that

$$v_n = h - k_n \in V \cap K^{\perp}, \qquad q_H[v_n] = q_H[h], \qquad \|s(v_n - h)\|_2 \longrightarrow 0. \tag{135}$$

If a finite $B \ge 0$ satisfies

$$q_H[v] \ge -B\|sv\|_2^2 \qquad (v \in V \cap K^{\perp}), \tag{136}$$

the same floor holds on all V, and RH holds.

Proof. For $k = 0$ take $r = 0$. Otherwise define the linear map $T : \mathcal{D}_a \to \mathbb{C}^k$ by $(Tr)_i = \langle \zeta_i, r/s \rangle$. Consider its graph in $L^2_{\mathrm{odd}} \times \mathbb{C}^k$. If (g, c) is orthogonal to this graph, then

$$\langle g, r \rangle + \sum_i \overline{c_i}(Tr)_i = 0 \qquad (r \in \mathcal{D}_a).$$

For $c \neq 0$, this would bound the functional of the nonzero row $\sum_i c_i\zeta_i$ by $\|g\|_2\|r\|_2$, contradicting Lemma 8.3. Thus $c = 0$, and density in Lemma 8.2 gives $g = 0$. The graph is dense. In particular there are $r_n \in \mathcal{D}_a$ with $r_n \to 0$ and $Tr_n \to d$.

To make interpolation exact, note that $T(\mathcal{D}_a)$ is a dense linear subspace of the finite-dimensional space $\mathbb{C}^k$, hence equals $\mathbb{C}^k$. Choose preimages of its coordinate basis and let $R : \mathbb{C}^k \to \mathcal{D}_a$ be the resulting linear right inverse. Its finite image gives $\|Re\|_2 \leq C_R\|e\|$. Therefore

$$\widetilde{r}_n = r_n + R(d - Tr_n), \qquad T\widetilde{r}_n = d, \qquad \|\widetilde{r}_n\|_2 \to 0.$$

Choosing a sufficiently large n proves (134). Every correction is still a finite sum of admissible small translates.

For $h \in V$, use $d_i = \langle\zeta_i, h\rangle$ and put $k_n = \widetilde{r}_n/s$. Lemma 8.2 gives genuine nullity, so polarization gives exact cancellation of all mixed terms: $q_H[h - k_n] = q_H[h]$. This proves (135). Apply (136) to v_n and pass only the profile norm to its limit. This proves $q_H[h] \geq -B\|sh\|_2^2$ for every $h \in V$. Every smooth compact odd two-moment-zero profile f has f/s in the original core. Proposition B.2 now proves RH. □

D Guide to the supporting verification constructions

Norms, verification routes and evaluated evidence

Table 2 identifies the principal constructions in the original realization and their distinct verification scopes. Physical, profile and complete-source norms are retained at their stated interfaces.

Table 2: Principal constructions, norms and verification scope.

Construction	Norm and conclusion	Proof or evaluated evidence
Complete receiving certificate	Original physical $\|h\|_2$; supported allowance $M^{-1}\|h\|_2^2$, with the complete complement and mixed return	Theorems 3.2–3.3 and Corollary 3.4; fixed-ε receiving cutoff $m_M \leq M^{1+\varepsilon}$ eventually and guarded cost in numeric M.
Full-jet carrier and projection	Original $X = L^2(s^{-2}du)$ and physical L^2; actual-norm $N = J$, strong inward synthesis and computable carrier projection	Theorems 9.1 and 9.7; full multiplicities and certified contour moments.
Optimal fixed-head lift	Physical carrier norm; minimum whole Gram F^{-1} for a fixed computable controllable head	Theorem 9.8; finite canonical exact lifts approach the carrier optimum with certified whole-map error.
Canonical interpolation	Physical $\|k\|_2$ and profile $\|sk\|_2$; sharp logarithmic cost order and order-uniform exact correction	Corollary 10.2 and Theorem 10.1; linear-order exponential budget and onset, with prescribed $J = o(\log\log M)$.
M25 whole-domain lower bound	Original physical norm and minimal domain; $q_H \geq -3 \times 10^{-8} I$	Section 7; all 1046 constraints, separate complete 1045-core congruence, paid localization and exact rational fee (55). One index.
Complete covariance and transfer	Actual complete source metric; unconditional polynomial-ridge bound and critical signed criteria; cost-paid canonical restrictions	Corollaries S6.2 and S6.3, Theorem S9.1; the sufficient signed arithmetic cap remains a hypothesis.

The independent qualitative interpolation proof remains in Appendix C. Leading-cost refinements retain their stated ambient, polar and genuine-null comparison classes. The interpolation budget is paid at the same arithmetic cutoff as the covariance transfer; computable fixed-head optimality supplies no uniform changing-head conditioning estimate.

The indexed technical supplement preserves the earlier M20/M24 supported certificates, fixed-support ridge continuation, complete arithmetic benchmarks, packet construction and alternative high-frequency and source interfaces. Each retains its own norm, hypotheses and complete return. The package index locates their scientific inputs and programs. For M25, the public download manifest distinguishes acquired numerical values from portable I/O/provenance changes. Saved-matrix verification, fresh moment acquisition, exact scalar-fee evaluation and the analytic localization proof are separate checks.

AI Declaration

OpenAI ChatGPT and Codex assisted with mathematical exploration, proof development, literature comparison, exposition, programming, internal verification and manuscript preparation. Scott Schubr directed and reviewed this work and is responsible for the mathematical arguments, computations and attribution of prior work.

Statements and Declarations

Funding. The research was self-funded. No external grant or sponsorship supported this work.

Competing interests. The author declares no competing interests.

Data and code availability. The editable manuscript, technical supplement, portable scientific code and input manifests are supplied in the source package. The numerical inputs for the two complete shrinking-allowance examples, the two arithmetic-only supported certificates and the ridge-continuation calculation are supplied in separate M20, M24, arithmetic-only and ridge-continuation companion archives, whose exact identities and extraction instructions are given in the package index. The numerical examples apply the analytic criteria using these scientific inputs and their complete error bounds. The complete M25 saved matrices, conditioning witness and portable verification sources are supplied in the M25 download volumes, with lossless reconstruction instructions and explicit public/original identity mapping. The source package also supplies `code/check_M25_global_fee.py` and its exact rational output in `data/M25conditionalexactglobalfee.json`. The scalar fee check is distinct from saved-matrix replay and fresh moment acquisition.

Prior dissemination. Earlier author-posted versions of this work are available in the Luminary Research Archive. The necessary realization-specific proofs are included here; citations to earlier author-posted papers identify their mathematical lineage.

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