

Relative tail payment and spectral enclosures

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Abstract

We sharpen the relative tail-payment estimate for a self-adjoint diagonal operator perturbed by a bounded self-adjoint operator with bounded first commutator. A smooth spectral cutoff replaces the growing-band argument and gives complete relative tail payment at the threshold $A \geq \max\{16P, (1152K^2/\delta)^{1/3}\}$, where P bounds the perturbation, K bounds its first commutator, and δ is the positive finite-head margin. The estimate retains the original finite head and the complete receiving tail. A shifted version yields indexed lower enclosures for every low Ritz eigenvalue, including an indefinite finite head. A fixed rank-two perturbation of a compact-resolvent diagonal operator shows that the cubic truncation power is optimal in this first-commutator class. Jacobi examples illustrate the resulting enclosures with explicit commutator budgets. This supplement develops the relative tail-payment method of the arithmetic matrix criteria predecessor.

Keywords. Bounded commutator, spectral truncation, Schur complement, Ritz eigenvalue, relative tail estimate.

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1 Introduction

Finite spectral tests for a perturbed diagonal operator require control of the coupling between the tested modes and the complete tail. The relative tail-payment lemma in [1] obtains that control from a positive finite head, a bound on the perturbation, and a bound on its first commutator with the diagonal operator. Its growing-band argument uses

$$A \geq \max \left\{ 8, 16P, \left[\frac{1024(n+1)^2(K^2 + P^2)}{\delta} \right]^{1/3} \right\}, \quad n \geq 1 \text{ minimal with } 4^n \geq 1 + P.$$

Here we replace the spectral ladder by one smooth cutoff. The resulting threshold is

$$A \geq \max \left\{ 16P, \left(\frac{1152K^2}{\delta} \right)^{1/3} \right\}. \tag{1}$$

This removes the growing-band factor and the additive P^2/δ contribution from the cubic term. The gain is particularly useful when the perturbation nearly commutes with the diagonal operator.

The proof uses classical functional-calculus commutator estimates and separated-spectrum Sylvester inversion. The smooth-commutator estimate appears, for example, in Zhang [4, Lemma 5.5]; separated-spectrum inversion is discussed in [5]. Our additional estimate closes the

finite-source Sylvester inequality with its own leakage term and then pays the remaining high head by its energy. The numerical constants are explicit throughout.

The indexed spectral consequence complements the residual and invariant-subspace viewpoint of Knyazev [2] and Nakatsukasa [3]. Its input consists of the diagonal cutoff, the finite Ritz matrix, P , and K . A fixed infinite example establishes the optimal cubic power for this information class. Energy-space localization under stronger nested-commutator assumptions has a different quantitative regime; see Yu and Wei [6]. These comparisons place the present estimate within the established commutator and spectral-approximation framework.

2 Setting and smooth spectral localization

Let $\mathcal{H}$ be a complex Hilbert space, with inner product conjugate-linear in its first argument. Let D be self-adjoint and bounded below, with an increasing eigenbasis and compact resolvent. Its eigenvalues are repeated with multiplicity. Let $U : \mathbb{C}^N \rightarrow \mathcal{H}$ be the isometry selecting the first N eigenvectors, and put

$$Q = I - UU^*, \quad D_Q = D|_{\text{ran } Q}, \quad D_Q \geq AI.$$

Let V be bounded and self-adjoint, preserve $\text{Dom } D$, and satisfy

$$\|V\| \leq P, \quad P \geq 1, \quad \|[D, V]\| \leq K. \quad (2)$$

The last norm is that of the bounded extension of the commutator initially defined on $\text{Dom } D$. Set

$$T = D + V, \quad H = U^*TU, \quad V_N = U^*VU, \quad C = QVU, \quad B = Q[D, V]U. \quad (3)$$

The diagonal spectral projection reduces D . Therefore

$$[H, V_N] = U^*[D, V]U, \quad D_Q C - CH = B - CV_N. \quad (4)$$

Domain invariance implies $C\mathbb{C}^N \subset \text{Dom } D_Q$. The complete receiving block is the self-adjoint operator $QTQ = D_Q + QVQ$ on $\text{Dom } D_Q$; all subsequent block inequalities use its full form domain.

Lemma 2.1 (Explicit smooth commutator). *For $h > 0$, put $b_h = (2h)^{-1}\mathbf{1}_{[-h, h]}$, and define*

$$f = \mathbf{1}_{[-3/8, 3/8]} * b_{1/16} * b_{1/16}.$$

Then $0 \leq f \leq 1$, $f = 1$ on $[-1/4, 1/4]$, and $f = 0$ outside $[-1/2, 1/2]$. For self-adjoint finite matrices L, W and $a > 0$,

$$\|[W, f(L/a)]\| \leq \frac{21}{a} \| [W, L] \|. \quad (5)$$

Proof. The support and plateau follow directly from convolution with the two normalized boxes. With Fourier convention $\widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-itx} dx$,

$$\widehat{f}(t) = \frac{2 \sin(3t/8)}{t} \left(\frac{\sin(t/16)}{t/16} \right)^2.$$

Consequently $\widehat{f}$ and $t\widehat{f}(t)$ are integrable, and

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{R}} |t\widehat{f}(t)| dt &\leq \frac{1}{\pi} \left(\int_0^{16} 2 dt + \int_{16}^{\infty} \frac{512}{t^2} dt \right) \\ &= \frac{64}{\pi} < 21. \end{aligned}$$

Duhamel's formula gives $\| [W, e^{itL/a}] \| \leq |t| \| [W, L] \| / a$. Integrating this bound in the Fourier inversion formula proves (5). This is a concrete Fourier proof of the standard smooth-functional-calculus commutator estimate. $\square$

Lemma 2.2 (Shifted low-head leakage). *Let $\lambda \in \mathbb{R}$, $a = A - \lambda \geq 16P$, and $L = H - \lambda I$. Then*

$$\| Cf(L/a) \| \leq \frac{37K}{7a} < \frac{6K}{a} \quad \text{if } K > 0, \quad (6)$$

and the left side is zero when $K = 0$.

Proof. Put $F = f(L/a)$, $E = \mathbf{1}_{(-\infty, a/2]}(L)$, and $X = CF = XE$. Lemma 2.1 and (4) give $\| [V_N, F] \| \leq 21K/a$. On the source $\text{ran } E$, the exact shifted identity is

$$(D_Q - \lambda)X - XL_E = BF - XV_NE - C[V_N, F]E, \quad L_E = L|_{\text{ran } E}. \quad (7)$$

Indeed $V_N F = F V_N + [V_N, F]$. The source restriction in (7) fixes its upper spectral endpoint at $a/2$, whereas $D_Q - \lambda \geq aI$.

If the right side is Y , differentiation of $e^{-t(D_Q - \lambda)} X e^{tL_E}$ and integration give

$$X = \int_0^\infty e^{-t(D_Q - \lambda)} Y e^{tL_E} dt, \quad \|X\| \leq \frac{2}{a} \|Y\|.$$

The domain statement following (4) justifies the derivative; the terminal term vanishes at least as $e^{-at/2}$. Hence

$$\|X\| \leq \frac{2}{a} \left(K + P \|X\| + \frac{21PK}{a} \right).$$

Since $P/a \leq 1/16$, absorption proves

$$\|X\| \leq \frac{2 + 42P/a}{1 - 2P/a} \frac{K}{a} \leq \frac{37}{7} \frac{K}{a}.$$

The same inequality gives $X = 0$ when $K = 0$. $\square$

3 Complete relative tail payment

Theorem 3.1 (Relative tail payment). *In the setting (2)–(3), suppose $H \geq \delta I$, $\delta > 0$, and (1) holds. For every $\tau \in [A/2, A/2 + 1]$,*

$$QTQ \geq \tau I, \quad \tau^{-1} C^* C \leq \frac{3}{16} H, \quad H - \tau^{-1} C^* C \geq \frac{13}{16} H. \quad (8)$$

In particular T is positive on its complete form domain.

Proof. Let $E_0 = \mathbf{1}_{(-\infty, A/4]}(H)$. Positivity of H and the plateau of f imply

$$CE_0 = Cf(H/A)E_0, \quad \|CE_0\| \leq 6K/A$$

by Lemma 2.2. On $\text{ran } E_0^\perp$, $H > A/4$. Thus

$$\|CH^{-1/2}\| \leq \frac{6K}{A\sqrt{\delta}} + \frac{2P}{\sqrt{A}}. \quad (9)$$

The triangle inequality in (9) retains the coupling between both head parts. Using $(x + y)^2 \leq 2x^2 + 2y^2$ and $\tau \geq A/2$,

$$\frac{\|CH^{-1/2}\|^2}{\tau} \leq \frac{144K^2}{\delta A^3} + \frac{16P^2}{A^2} \leq \frac{1}{8} + \frac{1}{16} = \frac{3}{16}.$$

Congruence by $H^{1/2}$ proves the two finite-head inequalities. The complete receiver satisfies $QTQ \geq (A - P)I \geq \tau I$, because $A \geq 16P$ and $P \geq 1$. For head coordinate u and tail vector v , completing the square gives

$$\langle u, Hu \rangle + 2\Re\langle v, Cu \rangle + \tau \|v\|^2 = \langle u, (H - \tau^{-1}C^*C)u \rangle + \tau \|v + \tau^{-1}Cu\|^2 \geq 0.$$

This proves positivity on the original full form domain. $\square$

Remark 3.2 (Dependence on the commutator). At $K = 0$, condition (1) reduces to $A \geq 16P$ for every positive head margin. More generally the cubic term depends on K^2/δ . The explicit constant 1152 is a sufficient constant; the next two sections identify the resulting spectral scale and its optimal power.

4 Indexed eigenvalue enclosures

Let $\theta_1 \leq \dots \leq \theta_N$ be the eigenvalues of H , and let $\lambda_1(T) \leq \lambda_2(T) \leq \dots$ be those of T , each repeated with multiplicity. Bounded perturbation preserves compact resolvent and semiboundedness, so these sequences and the minmax characterization apply.

Theorem 4.1 (All low Ritz indices). *Fix $j \leq N$ and $\lambda < \theta_j$. If*

$$a = A - \lambda \geq 16P, \quad (\theta_j - \lambda)(A - \lambda)^3 \geq 1152K^2, \quad (10)$$

then $\lambda_j(T) \geq \lambda$. In particular, if $g_j = A - \theta_j \geq 16P$, then

$$\theta_j - \frac{1152K^2}{g_j^3} \leq \lambda_j(T) \leq \theta_j. \quad (11)$$

Proof. Write $L = H - \lambda I$, and let E_j select an orthonormal list of head eigenvectors numbered $j, \dots, N$; an arbitrary ordering within a repeated eigenspace suffices. This projection reduces L , and

$$L_j = L|_{\text{ran } E_j} \geq \epsilon I, \quad \epsilon = \theta_j - \lambda > 0.$$

Lemma 2.2 applies to the entire finite head, including its negative shifted eigenvalues. On the restricted positive source $\text{ran } E_j$, split at $L_j = a/4$ to obtain

$$\|CE_jL_j^{-1/2}\| \leq \frac{6K}{a\sqrt{\epsilon}} + \frac{2P}{\sqrt{a}}.$$

For $\tau = a/2$, condition (10) gives

$$\tau^{-1} \|CE_jL_j^{-1/2}\|^2 \leq \frac{144K^2}{\epsilon a^3} + \frac{16P^2}{a^2} \leq \frac{3}{16}.$$

The complete shifted receiver is at least $(a - P)I \geq \tau I$. Completing the square as in Theorem 3.1 proves $T - \lambda \geq 0$ on the form subspace consisting of $U \text{ran } E_j$ and the complete tail. That subspace has codimension $j - 1$. Minmax yields $\lambda_j(T) \geq \lambda$.

For $K > 0$, choose $\lambda = \theta_j - 1152K^2/g_j^3$; then $a \geq g_j$, and (10) holds. For $K = 0$, apply the same implication to every $\lambda < \theta_j$ and pass to the limit. The Ritz variational principle gives the upper bound in (11). $\square$

Corollary 4.2 (Implicit refinement). *If $K > 0$ and $g_j > 0$, let $\epsilon_j > 0$ be the unique root*

$$\epsilon_j(g_j + \epsilon_j)^3 = 1152K^2.$$

Whenever $g_j + \epsilon_j \geq 16P$, one has $\lambda_j(T) \geq \theta_j - \epsilon_j$.

Proof. The left side is strictly increasing from zero to infinity on positive ϵ_j . Substitute $\lambda = \theta_j - \epsilon_j$ in (10). $\square$

5 Sharpness of the cubic power

Proposition 5.1 (Two-dimensional margin scale). *For $D = \text{diag}(0, A)$ and $V = \begin{pmatrix} \delta & b \\ b & 0 \end{pmatrix}$, with $A, b, \delta > 0$, the one-dimensional head is $H = \delta$, $\|[D, V]\| = Ab = K$, and positivity requires $\delta A^3 \geq K^2$.*

Proof. The determinant of $D + V$ is $\delta A - b^2$. Its nonnegativity is necessary for positivity. Substitute $b = K/A$. The choice $\delta = b^2/(2A)$ gives a negative determinant with a positive head; taking A sufficiently large relative to b also gives $A \geq 16 \max\{1, \|V\|\}$. $\square$

Theorem 5.2 (Fixed-operator sharp cubic power). *On $\ell^2(\{0, 1, \dots\})$, define*

$$De_0 = 0, \quad De_n = 2^n e_n \ (n \geq 1), \quad a_0 = 0, \quad a_n = \frac{1}{n2^n},$$

and

$$V = -|e_0\rangle\langle e_0| + |e_0\rangle\langle a| + |a\rangle\langle e_0|.$$

Then V has rank at most two, preserves $\text{Dom } D$, and

$$\|V\| < 2, \quad \|[D, V]\| = \frac{\pi}{\sqrt{6}}.$$

Let λ be the ground eigenvalue of $T = D + V$. For $N \geq 1$, let θ_N be its ground Ritz value on $\text{span}\{e_0, \dots, e_N\}$, and put $A_N = 2^{N+1}$. Then

$$\theta_N - \lambda \geq \frac{1}{4(N+1)^2 A_N^3}. \quad (12)$$

For every $\alpha > 0$, this fixed operator's errors fail to be $O(A_N^{-3-\alpha})$.

Proof. The vector a belongs to $\text{Dom } D$ because $\|Da\|^2 = \sum_{n \geq 1} n^{-2} = \pi^2/6$. Each vector in the range of V lies in $\text{span}\{e_0, a\} \subset \text{Dom } D$, which proves domain invariance. On that two-dimensional range the eigenvalues of V give

$$\|V\| = \frac{1 + \sqrt{1 + 4\|a\|^2}}{2} < 2, \quad \|a\|^2 \leq \sum_{n \geq 1} 4^{-n} = \frac{1}{3}.$$

The commutator has off-diagonal column Da , orthogonal to e_0 , so its norm is $\|Da\|$.

For $x < 0$, write $d_n = 2^n$ and

$$F_N(x) = x + 1 + \sum_{n=1}^N \frac{a_n^2}{d_n - x}, \quad F_\infty(x) = x + 1 + \sum_{n \geq 1} \frac{a_n^2}{d_n - x}.$$

The series and its derivative converge uniformly on compact subintervals of $(-\infty, 0)$. Scalar Schur complementation against the positive diagonal block shows that their unique roots below -1 are θ_N and λ , respectively. The tail has codimension one, so minmax allows at most one negative eigenvalue; the root gives exactly one. Since $D \geq 0$ and $\|V\| < 2$, both roots lie in $[-2, -1)$.

At the full root,

$$F_N(\lambda) = - \sum_{n > N} \frac{a_n^2}{d_n - \lambda}.$$

The mean value theorem therefore gives, for some $\xi_N \in (\lambda, \theta_N)$,

$$\theta_N - \lambda = \frac{\sum_{n > N} a_n^2 / (d_n - \lambda)}{F'_N(\xi_N)}.$$

On $[-2, -1)$,

$$1 \leq F'_N(x) = 1 + \sum_{n \leq N} \frac{a_n^2}{(d_n - x)^2} < 2.$$

For the first omitted index, $d_{N+1} - \lambda \leq d_{N+1} + 2 \leq 2d_{N+1}$. That summand alone is at least $1/[2(N+1)^2 A_N^3]$. Division by a derivative less than two proves (12). Finally $A_N^\alpha / (N+1)^2 \rightarrow \infty$, which proves the assertion for every $\alpha > 0$. The gap $A_N - \theta_N$ lies between $A_N + 1$ and $A_N + 2$, so the same optimal power holds in the gap variable of (11). $\square$

6 Jacobi applications

Let S be the unilateral shift on $\ell^2(\mathbb{N})$, indexed from one, and let W be a bounded real diagonal operator. Take

$$De_n = ne_n, \quad V = t(S + S^*) + W, \quad t \in \mathbb{R}.$$

The shifts preserve $\text{Dom } D$, as does bounded diagonal multiplication. Since $[D, S] = S$ and $[D, S^*] = -S^*$, valid budgets are

$$P = \max\{1, 2|t| + \|W\|\}, \quad K = 2|t|.$$

For the first N coordinates, $A = N + 1$. Every finite-head eigenvalue satisfying $N + 1 - \theta_j \geq 16P$ consequently obeys

$$\theta_j - \frac{4608t^2}{(N + 1 - \theta_j)^3} \leq \lambda_j(T) \leq \theta_j.$$

The diagonal term enters the perturbation budget while its commutator vanishes.

For the slower diagonal $De_n = \log(n+1)e_n$, adjacent diagonal differences are at most $\log 2$. The weighted shifts preserve $\text{Dom } D$, and

$$\|[D, V]\| \leq 2|t|\log 2, \quad A = \log(N + 2).$$

Theorem 4.1 applies with these budgets and the corresponding finite-head eigenvalues. These examples express the enclosure using diagonal, nearest-neighbor, and finite-head data; the fixed rank-two example in Theorem 5.2 supplies the sharp-power statement for the general first-commutator class.

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