

Relative finite optimization of arithmetic approximation distances

Scott Schubr
Luminary Oddities

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Abstract

We give finite rational approximations to the complete least-squares distance from the constant sequence to fractional-part sequences $\{n/k\}$ in the norm $\sum_{n \geq 1} |f(n)|^2 / [n(n+1)]$. The exact periodic mean is a positive gcd covariance form. Montgomery–Vaughan’s circular Hilbert inequality yields a coefficient-uniform complete-tail error with constant $D = N(N-1)$. This gives rational lower and upper optimization problems and a midpoint optimizer whose actual squared error is at most $(1-\beta^2)^{-1}$ times the complete optimum, where $\beta = 2D(2D+1)/[(R+1)(R+2)] < 1$. The latter is a Kantorovich application and gives a sufficient cutoff $R = O(N^2 \varepsilon^{-1/4})$ for relative $1 + \varepsilon$ quality of the optimizer. The computed optimum-value brackets are separate and retain their explicitly stated widths. An adjacent arithmetic wavelet shows that no bounded uniform prefix extension is possible at $R \sim N \log N$; at $R \sim N^{1+\alpha}$ its full-to-prefix ratio tends to $1/\alpha$. The mean, Hilbert inequality, wavelet head identity and quadratic optimization tools are classical or previously established; their combination gives explicit native-norm truncation guarantees. A standard-library exact rational implementation is included.

1 The finite optimization problem

What the optimization returns The finite procedure returns three objects: a midpoint candidate, a bracket for the optimum value, and a bound for the candidate’s complete norm. The relative 1.0001 guarantee concerns the midpoint candidate, separately from the absolute brackets. Coefficient-uniform periodic-mean correction permits optimization after truncation. The sufficient cutoff and the short-prefix obstruction retain their different quantifiers.

Let $N \geq 2$ and write

$$\gamma_k(n) = \{n/k\}, \quad f_a(n) = a_0 + \sum_{k=2}^N a_k \gamma_k(n), \quad n \geq 1.$$

The coordinates of a are indexed by $0, 2, \dots, N$. Our complete and finite squared norms are

$$\mathcal{F}(a) = \sum_{n \geq 1} \frac{|f_a(n)|^2}{n(n+1)}, \quad P_R(a) = \sum_{n=1}^R \frac{|f_a(n)|^2}{n(n+1)}. \quad (1)$$

Here $R \geq 0$ is an integer and $P_0 = 0$. The arithmetic distance is

$$d_N^2 = \min_{a_0=1} \mathcal{F}(a). \quad (2)$$

Complex coefficients are allowed. A real target has real minimizers, since the imaginary part contributes a separate nonnegative form. The sequence model is the unit-cell step space

$L^2((1, \infty), x^{-2} dx)$: the mass of $[n, n+1]$ is $1/[n(n+1)]$. Moreover

$$\lfloor x/k \rfloor - \frac{\lfloor x \rfloor}{k} = -\gamma_k(\lfloor x \rfloor).$$

Thus (2) is the discrete arithmetic approximation problem in the Nyman–Beurling–Báez-Duarte criterion [1, 2].

Put $D = N(N-1)$ and let $\mathcal{S}(a)$ be the exact average of $|f_a(n)|^2$ over any common integer period. The principal finite objective is the midpoint form

$$G_R(a) = P_R(a) + \frac{\mathcal{S}(a)}{R+1}. \quad (3)$$

The following theorem distinguishes the quality of an optimizer from the width of a computed enclosure of the optimum value.

Theorem 1.1 (Relative finite optimization). *Let $N \geq 2$ and $R \geq 0$ be integers. Define*

$$t = \frac{D}{(R+1)(R+2)}, \quad \delta = \frac{D}{R+2}, \quad \beta = \frac{2D(2D+1)}{(R+1)(R+2)}.$$

The forms $\mathcal{F}$ and $\mathcal{S}$ are positive definite and

$$|\mathcal{F}(a) - G_R(a)| \leq t\mathcal{S}(a), \quad \mathcal{F}(a) \geq \frac{\mathcal{S}(a)}{2(2D+1)}. \quad (4)$$

All entries of G_R and $\mathcal{S}$ are rational. Let a_G be the unique minimizer of G_R on $a_0 = 1$. If $\beta < 1$, then

$$d_N^2 \leq \mathcal{F}(a_G) \leq \frac{d_N^2}{1 - \beta^2}. \quad (5)$$

In particular $\beta^2 \leq \varepsilon/(1 + \varepsilon)$ is sufficient for $\mathcal{F}(a_G) \leq (1 + \varepsilon)d_N^2$. It can be enforced at $R = O(N^2\varepsilon^{-1/4})$ for $0 < \varepsilon \leq 1$.

For an exact value enclosure, suppose additionally that $\delta < 1$. Set

$$Q_{\pm}(a) = G_R(a) \pm t\mathcal{S}(a), \quad q_{\pm} = \min_{a_0=1} Q_{\pm}(a).$$

These minima and their minimizing coefficients are rational, and

$$0 < q_- \leq d_N^2 \leq q_+ \leq \frac{1 + \delta}{1 - \delta} q_-. \quad (6)$$

The actual midpoint candidate has the computable complete-norm bounds

$$Q_-(a_G) \leq \mathcal{F}(a_G) \leq Q_+(a_G). \quad (7)$$

The quarter-power cutoff above concerns (5), not the relative width of (6).

For example, a sufficient integer choice for the optimizer guarantee is

$$R \geq \sqrt{2D(2D+1)} \left(\frac{1 + \varepsilon}{\varepsilon} \right)^{1/4}. \quad (8)$$

One can instead test the exact rational inequality on β^2 ; neither the algorithm nor its acceptance test needs a floating root.

Relationship to earlier formulas. Balazard[2] describes the native step space, its arithmetic inversion and Vasyunin biorthogonal system. The periodic mean and an Abel tail enclosure with pair-period fluctuation bounds occur in[5, Theorem 3]. The all-band head identity in[6, Theorem 2(c)] supplies the wavelet mechanism used in Section 4. Our uniform bound uses Montgomery–Vaughan’s circle Hilbert inequality[3]; the midpoint improvement uses the classical Kantorovich inequality, whose elementary form is proved below; see also[4]. The resulting application gives a complete-native relative optimization guarantee with an explicit cutoff and exact finite data. Related complete arithmetic head/tail optimization in[7] uses a translated Poisson recipient and prescribed coefficient nulls. Those different norms and constraints are retained in that work.

Known pairwise Gram methods are a serious alternative. In particular, the cross term for indices j, k has period $\text{lcm}(j, k) \leq N^2$; a common period for all generators need not be enumerated. Pair-period special-function formulas and certified Gram calculations already occur in[5]. The present method gives a simple uniform relative truncation rule and exact rational construction. Its rational numerators can be large.

2 Exact means and the complete tail

Lemma 2.1 (Arithmetic covariance). *Let $b_k = (k-1)/(2k)$ and let $J_2(d) = d^2 \prod_{p|d} (1-p^{-2})$. Then*

$$\mathcal{S}(a) = \left| a_0 + \sum_{k=2}^N b_k a_k \right|^2 + \frac{1}{12} \sum_{d=2}^N J_2(d) \left| \sum_{\substack{k \leq N \\ d|k}} \frac{a_k}{k} \right|^2. \quad (9)$$

This is a positive definite rational Hermitian form.

Proof. For a uniformly chosen common-period residue, γ_k has mean b_k . Let $g = \text{gcd}(j, k)$ and condition on the residue u modulo g . By the Chinese remainder theorem the remaining residues modulo j and k are conditionally independent. Their conditional means are $(u + (j-g)/2)/j$ and $(u + (k-g)/2)/k$. Since the variance of the uniform residue $u \in \{0, \dots, g-1\}$ is $(g^2-1)/12$, we obtain

$$\text{Cov}(\gamma_j, \gamma_k) = \frac{\text{gcd}(j, k)^2 - 1}{12jk}.$$

Expanding the full complex coefficient quadratic form and using $g^2 - 1 = \sum_{d|g, d \geq 2} J_2(d)$ proves (9). The divisibility map $a \mapsto (\sum_{d|k} a_k/k)_{2 \leq d \leq N}$ is triangular with nonzero diagonal. Its vanishing forces every a_k , $k \geq 2$, to vanish; the first term then determines a_0 . Hence the form is positive definite. This derivation retains all mixed terms. $\square$

Lemma 2.2 (Uniform interval discrepancy). *For every interval I of consecutive integers,*

$$\left| \sum_{n \in I} |f_a(n)|^2 - |I| \mathcal{S}(a) \right| \leq D \mathcal{S}(a). \quad (10)$$

Proof. Finite discrete Fourier expansion gives $f_a(n) = \sum_{\theta \in \Theta} A_\theta \exp(2\pi i n \theta)$, where Θ is contained in the distinct reduced rational frequencies of denominator at most N , including zero. Their circular separation is at least $1/[N(N-1)]$. Indeed, two different denominators have product at most $N(N-1)$, and equal denominators give separation at least $1/N$; the same argument applies across zero modulo one. Parseval on a common period gives $\sum_\theta |A_\theta|^2 = \mathcal{S}(a)$.

Theorem 1(1.2) of Montgomery–Vaughan[3] states, for frequencies of circular separation δ_Θ ,

$$\left| \sum_{\theta \neq \phi} u_\theta \overline{u_\phi} \csc(\pi(\theta - \phi)) \right| \leq \delta_\Theta^{-1} \sum_{\theta} |u_\theta|^2. \quad (11)$$

For $I = \{A + 1, \dots, A + L\}$, the off-diagonal interval kernel is

$$\exp(2\pi i(A + (L + 1)/2)(\theta - \phi)) \frac{\sin(\pi L(\theta - \phi))}{\sin(\pi(\theta - \phi))}.$$

Split the sine into two exponentials. The discrepancy is $(H(u_+) - H(u_-))/(2i)$, where H is the cosecant form in (11) and

$$(u_+)_\theta = A_\theta \exp(2\pi i(A + L + 1/2)\theta), \quad (u_-)_\theta = A_\theta \exp(2\pi i(A + 1/2)\theta).$$

Both vectors have squared norm $\mathcal{S}(a)$. Equation (11) and $\delta_\Theta^{-1} \leq D$ prove (10). $\square$

Proposition 2.3 (Complete tail and coercivity). *For every integer $R \geq 0$,*

$$\left| \sum_{n > R} \frac{|f_a(n)|^2}{n(n+1)} - \frac{\mathcal{S}(a)}{R+1} \right| \leq \frac{D\mathcal{S}(a)}{(R+1)(R+2)}. \quad (12)$$

Also $\mathcal{F}(a) \geq \mathcal{S}(a)/[2(2D+1)]$.

Proof. Put $w_n = 1/[n(n+1)]$ and $E_t = \sum_{j=1}^t (|f_a(R+j)|^2 - \mathcal{S}(a))$. Lemma 2.2 gives $|E_t| \leq D\mathcal{S}(a)$. Discrete Abel summation, with its infinite boundary tending to zero, gives

$$\sum_{n > R} w_n (|f_a(n)|^2 - \mathcal{S}(a)) = \sum_{t \geq 1} E_t (w_{R+t} - w_{R+t+1}).$$

The positive weight differences sum to w_{R+1} , while $\sum_{n > R} w_n = 1/(R+1)$. This proves the entire-tail bound. Finally the first $2D$ unweighted cells have energy at least $D\mathcal{S}(a)$ by (10). Each of their weights is at least $1/[2D(2D+1)]$. Therefore

$$\mathcal{F}(a) \geq P_{2D}(a) \geq \frac{\mathcal{S}(a)}{2(2D+1)}.$$

$\square$

3 Rational optimization and midpoint stability

Write a positive definite form H in block coordinates (a_0, c) as

$$H(a_0, c) = h_{00}|a_0|^2 + 2\operatorname{Re}(\overline{a_0}h_{0c}c) + c^*H_{cc}c.$$

Its unique minimum at $a_0 = 1$ is attained at $c = -H_{cc}^{-1}h_{0c}^*$, and has value $h_{00} - h_{0c}H_{cc}^{-1}h_{0c}^*$. For rational real entries these quantities are rational. This is ordinary finite quadratic minimization; positive definiteness is available analytically, rather than being inferred from a floating matrix factorization.

Lemma 3.1 (Kantorovich stability for one affine constraint). *Let F, G be positive definite Hermitian forms on a finite-dimensional complex vector space, and suppose $(1 - \beta)F \preceq G \preceq (1 + \beta)F$, $0 \leq \beta < 1$. For any nonzero complex linear functional ℓ , let x_F, x_G minimize the respective forms subject to $\ell(x) = 1$. Then*

$$F(x_G) \leq \frac{F(x_F)}{1 - \beta^2}. \quad (13)$$

Proof. Whiten F and write $A = F^{-1/2}GF^{-1/2}$. Its eigenvalues lie in $[1 - \beta, 1 + \beta]$. Represent the transformed functional by a vector of norm $\nu > 0$, and let u be its unit direction. The two constrained minimizers give

$$\frac{F(x_G)}{F(x_F)} = \frac{u^* A^{-2} u}{(u^* A^{-1} u)^2}.$$

Let $l = (1 + \beta)^{-1}$ and $h = (1 - \beta)^{-1}$. The spectral values t of A^{-1} belong to $[l, h]$. For the probability weights induced by u , put $\mu = \mathbb{E}t$. The scalar inequality $(t - l)(h - t) \geq 0$ yields

$$\frac{\mathbb{E}t^2}{\mu^2} \leq \frac{(l + h)\mu - lh}{\mu^2} \leq \frac{(l + h)^2}{4lh} = \frac{1}{1 - \beta^2}.$$

The middle maximum follows by differentiating a scalar rational function on $[l, h]$, or by completing a square. This is the classical Kantorovich bound in the normalization needed here; compare[4]. $\square$

Proof of Theorem 1.1. Lemma 2.1 and Proposition 2.3 give (4), positive definiteness and rationality. In particular

$$|G_R(a) - \mathcal{F}(a)| \leq t\mathcal{S}(a) \leq \beta\mathcal{F}(a).$$

Apply Lemma 3.1 with $\ell(a) = a_0$ to prove (5). Condition (8) implies $(R + 1)(R + 2) \geq 2D(2D + 1)\sqrt{(1 + \varepsilon)}/\varepsilon$, which proves the sufficient precision law.

If $\delta < 1$, the two forms

$$Q_{\pm} = P_R + \frac{1 \pm \delta}{R + 1} \mathcal{S}$$

are positive definite. The exact tail bound gives $Q_- \preceq \mathcal{F} \preceq Q_+$, and the PSD identity

$$\frac{1 + \delta}{1 - \delta} Q_- - Q_+ = \left(\frac{1 + \delta}{1 - \delta} - 1 \right) P_R$$

gives their relative comparison. Minimize on $a_0 = 1$ to obtain (6), and evaluate at a_G for (7). The block formula above proves rational computability. These value brackets and the midpoint quality estimate are separate consequences of the same complete-tail control. $\square$

Corollary 3.2 (A posteriori projection error). *Let a_* minimize $\mathcal{F}$ on $a_0 = 1$. For any normalized candidate a ,*

$$\mathcal{F}(a - a_*) = \mathcal{F}(a) - d_N^2.$$

When $\delta < 1$, the computable quantity $Q_+(a) - q_-$ bounds this nonnegative error from above. If $a = a_G$ and $\beta < 1$, the actual error $\mathcal{F}(a_G - a_*)$ is at most $\beta^2 d_N^2 / (1 - \beta^2)$.

Proof. The difference $a - a_*$ has zero constant coordinate and belongs to the tangent space of the constraint. The normal equations for a_* make its $\mathcal{F}$ inner product with a_* zero. Expanding the squared norm proves the identity, and the two bounds follow from Theorem 1.1. $\square$

4 A short-prefix obstruction

The preceding sufficient cutoff does not settle which smaller cutoffs are possible. The following explicit native vector prevents a bounded uniform extension at a natural logarithmic cutoff. Let

$$v_m(x) = m\lfloor x/m \rfloor - (m+1)\lfloor x/(m+1) \rfloor, \quad m \geq 2.$$

On integer cells it is $(m+1)\gamma_{m+1} - m\gamma_m$. Its residue formula shows $|v_m| \leq m$ everywhere and its period is $m(m+1)$.

Lemma 4.1 (Exact adjacent-wavelet head). *For integers $1 \leq h \leq m$ and real $h(m+1) \leq X \leq (h+1)m$,*

$$\int_1^X \frac{|v_m(x)|^2}{x^2} dx = H_h - \frac{h^2}{X}, \quad H_h = \sum_{j=1}^h \frac{1}{j}. \quad (14)$$

This is the adjacent-wavelet specialization of the all-band identity in [6, Theorem 2(c)].

Proof. For $0 \leq j \leq h$, the baseline on $[jm, (j+1)m)$ is $-j$. On $[jm, j(m+1))$ with $1 \leq j \leq h$, it is instead $m+1-j$. These spike intervals are disjoint when $j \leq m$, and all are included under the stated condition on X . The baseline integral through X is $(2h - H_h)/m - h^2/X$. The sum of spike increments is $((m+1)/m)H_h - 2h/m$, obtained by integrating each constant square difference against x^{-2} . Adding the terms gives (14). Endpoints have measure zero. $\square$

Theorem 4.2 (Full-to-prefix ratios). *Let $N = m+1$ and regard v_m as a sequence in (1). Its complete squared norm satisfies*

$$H_m - \frac{m}{m+1} \leq \|v_m\|^2 \leq H_m. \quad (15)$$

If integers $R_m \sim m \log m$, then

$$\frac{\|v_m\|^2}{P_{R_m}(v_m)} \sim \frac{\log m}{\log \log m} \rightarrow \infty. \quad (16)$$

For fixed $0 < \alpha < 1$ and $R_m \sim m^{1+\alpha}$,

$$\frac{\|v_m\|^2}{P_{R_m}(v_m)} \rightarrow \frac{1}{\alpha}. \quad (17)$$

Consequently an arbitrary-coefficient uniform extension constant at the latter scale must be at least $1/\alpha$ asymptotically.

Proof. At $X = m(m+1)$, Lemma 4.1 gives $H_m - m/(m+1)$. The ENTIRE remaining tail is between zero and $m^2/X = m/(m+1)$, proving (15). For any $X \rightarrow \infty$ with $X/m \rightarrow \infty$ and $X/m = o(m)$, choose completed-spike endpoints bracketing X . Their indices h differ by a bounded number and satisfy $h \sim X/m$. Equation (14) and monotonicity then give the head asymptotic $\log(X/m)$ whenever $X \sim m \log m$ or $X \sim m^{1+\alpha}$: the correction $h^2/X \sim h/m$ tends to zero. The ordinary sequence prefix P_R is the integral through $X = R+1$, so the one-cell shift has no effect. Since (15) gives $\|v_m\|^2 \sim \log m$, both ratios follow. $\square$

The logarithmic obstruction also applies to uniform inequalities for all affine residuals $1+u$, $u \in \text{span}\{\gamma_2, \dots, \gamma_N\}$: substitute $u = tv_{N-1}$ and divide by $|t|^2$ as $|t| \rightarrow \infty$. The intermediate upper problem remains distinct from the sufficient rule in Theorem 1.1.

5 Exact finite examples

The accompanying `verify.py` uses only the Python standard library. It constructs the exact rational matrices P_R and $\mathcal{S}$, checks positive definiteness by exact LDL pivots, solves each affine minimization by exact Gaussian elimination, and checks stationarity and the explicit PSD comparison identity. It computes the midpoint coefficients as well as both optimum-value brackets; their exact rational endpoints and coefficients are written to `results.json`. Decimal strings are displays only.

For $R = 21D - 2$, the following outward-rounded displays describe the exact value enclosures. In each case the separately proved midpoint candidate guarantee is $\mathcal{F}(a_G) \leq 1.0001d_N^2$.

N	R	lower d_N^2	upper d_N^2
4	250	0.06590073868	0.06598978357
8	1174	0.02423778448	0.02425072528
12	2770	0.02049527656	0.02050081399

The table displays certified absolute lower and upper brackets. The relative 1.0001 midpoint-candidate guarantee concerns the actual midpoint candidate's quality relative to the unknown complete optimum. The same run also pays complete candidate-norm bounds via (7).

Supplementary exact controls verify the arithmetic covariance on finite periods, interval discrepancy consequences at small N , and the inherited wavelet head identities. The Kantorovich step has exact sharpness controls: for rational $p > q > 0$, take $F = I$, $G = \text{diag}(1 + \beta, 1 - \beta)$, $\beta = (p^2 - q^2)/(p^2 + q^2)$ and constrain $px_1 + qx_2 = 1$. The ratio in Lemma 3.1 is exactly $(1 - \beta^2)^{-1}$. All acceptance checks are explicit exceptions and remain active under Python `-O`. The infinite-tail claims follow from the analytic argument, rather than finite tests of its conclusion.

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