Quantitative Null-Space Compression for a Weighted Weil Operator

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Abstract

We study a fixed minimal weighted realization $H = A + B$ of the constrained Weil form, with nonnegative logarithmic reference A and the complete bounded paired arithmetic perturbation B. For every $0 < \tau \leq 1$, explicit Fourier-band moments cut out a finite-codimensional subspace of the entire operator kernel on which both A and $A^{1/2}$ have norm at most τ. The number $n(\tau)$ of observations satisfies

$$\log n(\tau) = O\Big(\sqrt{1 + \log(1/\tau)}\, \log(3 + \log(1/\tau))\Big).$$

Consequently $A\Pi_Y$ and $A^{1/2}\Pi_Y$ belong to every Schatten class for every closed $Y \subseteq \ker H$. The proof combines two complete phase-space tails, a finite-rank real-phase Taylor approximation, and the exact null equation. We also give norm estimates for finite null deflation, a full residual-Gram receiving certificate, an explicit weighted bound $H \geq -99I/100$, and a multiplicity obstruction to identifying the derivative-null closure with the whole kernel. The observations are explicit; the projection inside the unknown kernel is not computationally constructed.

1 Introduction and principal result

Whole-kernel theorem and finite applications The central theorem controls the entire kernel and its singular values. Finite exact-null deflation, the sufficient residual-Gram certificate and the auxiliary lower floor are applications. Exact null vectors preserve the complete form and its mixed terms. The multiplicity obstruction explains why a fixed finite null prefix has a different scope from whole-kernel compression. Keep the original realization $T = I + H$ distinct from the positive reference operator.

The distinction between a collection of exact null vectors and the entire null space is consequential in a spectral reformulation of the Weil criterion. Estimates on selected vectors need not bound the output on their closed span; even a complete description of one known radical need not describe every operator null. We address a quantitative question that does not require such a description: how much of a positive logarithmic reference can act on an arbitrary actual null vector after finitely many explicit observations have vanished?

Here is the realization and normalization used throughout. Put

$$\psi(u) = 2\sum_{k\geq 1}\Big(2\pi^2 k^4 e^{9u/2} - 3\pi k^2 e^{5u/2}\Big)\, e^{-\pi k^2 e^{2u}}, \tag{1}$$

$$c(u) = \cosh(u/2), \qquad w = \psi c, \qquad s = \sqrt{\psi/c}, \tag{2}$$

$$\mathcal{K} = \{h \in L^2(\mathbb{R}) : \langle \sqrt{w}, h\rangle = \langle \sqrt{w}\tanh(u/2), h\rangle = 0\}. \tag{3}$$

The theta identity makes ψ even and strictly positive. All Hilbert inner products are conjugate-linear in the first argument. Our Fourier convention is $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)e^{-iu\xi}\,du$, with Plancherel measure $d\xi/(2\pi)$. Define

$$a(\xi) = \operatorname{Re}\frac{\Gamma'}{\Gamma}(1/4 + i\xi/2) - \frac{\Gamma'}{\Gamma}(1/4), \tag{4}$$

$$m_0 = \frac{\Gamma'}{\Gamma}(1/4) - \log\pi, c_n = \frac{\Lambda(n)}{\sqrt{n}}, \qquad (S_b f)(u) = f(u+b). \tag{5}$$

The nonnegative form $\mathfrak{a}[h] = \int a(\xi)|\widehat{sh}(\xi)|^2\,d\xi/(2\pi)$ is closed on the closure $\mathcal{V}$ of the constrained C_c^∞ core in its form norm. Its selfadjoint operator is A. The complete perturbation is

$$\begin{aligned} B &= P_{\mathcal{K}}\left[m_0 M_s^2 - \sum_{n\geq 2} c_n M_s(S_{\log n} + S_{-\log n})M_s\right]|_{\mathcal{K}}, \\ H &= A + B. \end{aligned} \tag{6}$$

The series converges in operator norm. Section2 proves the applicable bounds and identifies $H = T - I$ with the chosen minimal positive-jump closure.

Theorem 1.1 (Entire-kernel quantitative compression)**.** *For $0 < \tau \leq 1$ set*

$$\begin{aligned} M &= 20 + \lceil \log_2(1/\tau)\rceil, \qquad D = \min\{2M+4,\, 6 + 4\lceil\sqrt{M}\rceil\lceil\log_2 M\rceil\}, \\ R &= \max\{2, \lceil\log_2(384D/\tau)\rceil\}, \qquad L = 2^{64D}, \qquad n = 6RL. \end{aligned}$$

If $z \in \ker H$ satisfies the n bounded observations

$$\int_{-L}^{L} \xi^j \widehat{sz}(\xi)\,d\xi = 0, \qquad 0 \leq j < n, \tag{7}$$

then $\|Az\| \leq \tau\|z\|$ and $\|A^{1/2}z\| \leq \tau\|z\|$. The row-zero subspace has codimension at most n in any closed $Y \subseteq \ker H$, and

$$\log n \leq C\sqrt{1 + \log(1/\tau)}\,\log(3 + \log(1/\tau)) \tag{8}$$

for an absolute constant C. In particular, for such Y, the bounded maps $A\Pi_Y$ and $A^{1/2}\Pi_Y$ are compact and their singular values satisfy, for sufficiently large N,

$$s_N \leq C_1 \exp\left[-c_1\left(\frac{\log N}{\log\log(N+3)}\right)^2\right]. \tag{9}$$

Both maps belong to $\mathcal{S}_p$ for every $p > 0$.

The rows involve the known two-row compression only. The finite-rank approximant in the conclusion uses an orthogonal projection *inside* Y; constructing that projection is a different problem. The tolerance parameters are deliberately coarse and extremely large.

Predecessors and scope

The weighted ground-state realization, derivative null family, qualitative phase-space compactness and full-kernel spectral viewpoint are inherited from the weighted Weil analysis of van Hemert[1]. They are restated with the proofs needed for our estimates. Fourier tightness, finite-rank kernel approximation, compact-product arguments and approximation

numbers are classical tools[5, 6]. Our specific increment is a complete output estimate over the *entire* kernel with an explicit observation budget. The sublinear arithmetic bound $D_M = O(\sqrt{M}\log M)$ is what makes the inverse-tolerance rank subpolynomial and produces membership in every Schatten class.

Related Weil-Hilbert constructions and logarithmic compactness appear in Suzuki[7, 8]; their norms and quotient identifications differ. Connes–Consani[9] use prolate space and frequency localization and radical-range constructions. The finite-prime semilocal operator in[10] has a different Schatten threshold and is not the map $A\Pi_{\ker H}$ considered here. Classical Fourier–Laplace division and multiplicity questions are developed in[11]; AppendixB pays their application in the present physical weighted minimal domain.

Recent fixed zero-free half-plane results[13, 14] and the improved bound of Liu[15] concern a different arithmetic recipient. Liu states the boundary $B = (18\xi + 7)/(18\xi + 9) \approx 0.874957019420098946$, where $\xi \in (0.38858, 0.38859)$ solves $1188\xi^3 - 2160\xi^2 + 171\xi + 190 = 0$, and proves optimality within a specified exponent comparison system. These works report accompanying formalizations. Our main theorem uses the stated operator and arithmetic hypotheses. The required sign is $H \geq 0$. The separate polynomial-synthesis work[12] has a different topology and recipient; it does not identify our entire operator kernel.

After proving the main result, we derive finite-deflation norm control and a signed residual-Gram receiver. The explicit lower bound in AppendixA is an analytic application on the entire chosen domain, not a finite matrix computation or a sequence of certificates approaching $H \geq 0$.

2 The exact realization and complete source bounds

2.1 Weil normalization and the minimal core

Write $\mathcal{M}f(z) = \int_{\mathbb{R}} f(u)e^{zu}\,du$ and $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. The classical theta identity gives

$$\mathcal{M}\psi(z) = \xi(1/2 + z), \qquad \mathcal{M}\psi(1/2) = \mathcal{M}\psi(-1/2) = 1/2. \tag{10}$$

All fixed derivatives of ψ decrease faster than every exponential at both ends. We use the standard complete Weil explicit formula[2, 3, 7] with the following precise convention. For smooth f, g all of whose derivatives decrease faster than every exponential, put

$$Q(f,g) = \int_{\mathbb{R}} m(\xi)\overline{\widehat{f}(\xi)}\widehat{g}(\xi)\frac{d\xi}{2\pi} - \sum_{n\geq 2} c_n\big(\langle f, S_{\log n}g\rangle + \langle f, S_{-\log n}g\rangle\big), \tag{11}$$

$$m(\xi) = m_0 + a(\xi). \tag{12}$$

Its zero-side expression is

$$\begin{aligned} Q(f,g) = &\sum_{\rho} \mathcal{M}g(\rho - 1/2)\overline{\mathcal{M}f(1/2 - \overline{\rho})} \\ &- \mathcal{M}g(1/2)\overline{\mathcal{M}f(-1/2)} - \mathcal{M}g(-1/2)\overline{\mathcal{M}f(1/2)}. \end{aligned} \tag{13}$$

Every nontrivial zero is counted with multiplicity. The terms on the second line are polar terms; they vanish on the two-moment subspace. These are the standard explicit-formula identities. Integration by parts makes the transforms rapidly decreasing on each fixed vertical strip. The classical zero count $O(T\log(T+2))$ then gives absolute convergence of the zero pairing; exponential physical tails give absolute convergence of the prime pairing. The same estimates prove continuity under convergence of every exponential derivative seminorm.

From(10)–(13),

$$Q(\psi, g) = -\int_{\mathbb{R}} c(u)g(u)\,du. \tag{14}$$

The digamma identity

$$a(\xi) = \sum_{j\geq 0} \frac{(\xi/2)^2}{(j+1/4)((j+1/4)^2+(\xi/2)^2)} = \int_{\mathbb{R}} K(t)(1-\cos(\xi t))\,dt, \quad K(t) = \frac{e^{-|t|/2}}{1-e^{-2|t|}} \tag{15}$$

follows from the convergent digamma difference series. It proves positivity, monotonicity in $|\xi|$ and logarithmic divergence. Expanding differences in(11), then using(14), gives for compact smooth v

$$\begin{aligned} D[v] &= \frac{1}{2}\iint_{\mathbb{R}^2} K(u-v')\psi(u)\psi(v')|v(u)-v(v')|^2\,du\,dv' \\ &\quad + \sum_{n\geq 2} c_n \int_{\mathbb{R}} \psi(u)\psi(u+\log n)|v(u)-v(u+\log n)|^2\,du, \end{aligned} \tag{16}$$

$$Q[\psi v] = D[v] - \|v\|^2_{L^2(w)}. \tag{17}$$

One way to check every factor in this calculation is to subtract $Q(\psi, \psi|v|^2)$ from $Q[\psi v]$: each paired shift contributes exactly c_n times its displayed squared difference, and the archimedean term contributes the half double integral. The subtracted term is $-\|v\|^2_{L^2(w)}$. The integrable squared differences justify the singularity at $u = v'$; positive truncations and dominated convergence justify the full channel sum. In particular $D \geq 0$.

Let $\mathcal{C} = C_c^\infty(\mathbb{R}) \cap \mathcal{K}$. Multiplication by $\sqrt{w}$ is a bijection between the compact smooth two-moment core in $L^2(w)$ and $\mathcal{C}$. This core is dense in $\mathcal{K}$: approximate in L^2 by compact smooth functions and repair the two row values with two fixed compact bumps. Such bumps exist because the ratio of the two row profiles is $\tanh(u/2)$, which is nonconstant. Their row matrix can be made invertible by choosing disjoint small supports near two distinct points. The correction coefficients tend to zero.

The maximal graph map $h \mapsto a(D)^{1/2}(sh)$ is closed, since M_s is bounded and the multiplier is closed. Its restriction to $\mathcal{C}$ is therefore closable. Define $\mathcal{V}$ to be that restriction's completion in $\|h\|^2 + \mathfrak{a}[h]$. The boundedness of B proved below and(17) give on $\mathcal{C}$

$$\mathfrak{a}[h] + \|h\|^2 \leq D[h/\sqrt{w}] + (\|B\|+1)\|h\|^2 \leq \mathfrak{a}[h] + (2\|B\|+2)\|h\|^2.$$

Thus the minimal positive-jump closure has exactly this form domain, and its selfadjoint operator is $T = I + A + B \geq 0$. This also proves $\operatorname{Dom} H = \operatorname{Dom} A$ and legitimizes $Az = -Bz$ for every $z \in \ker H$.

2.2 Full theta bounds and the paired arithmetic remainder

We use the known strict log-concavity of the Riemann theta kernel[4, Theorems4.2 and4.5]. Under the amplitude and spatial scaling(1), ψ remains even log-concave; hence so is s, since $\log c$ is convex. In particular

$$\|M_s S_b M_s\| = \sup_u s(u)s(u+b) = s(b/2)^2 \quad (b \geq 0). \tag{18}$$

The upper inequality is midpoint concavity applied to u and $-u-b$; equality occurs at $u = -b/2$. This is the norm of a weighted unitary shift, not an equality for the sum of its two directions.

For $r = \pi e^{2u}$, $u \geq 0$, the complete positive series satisfies

$$\psi(u) \leq 4\pi^2 e^{9u/2} \sum_{k\geq 1} k^4 e^{-rk^2} \leq 8\pi^2 e^{9u/2-\pi e^{2u}}. \tag{19}$$

Indeed $k^2 - 1 \geq 3(k-1)$, and with $q = e^{-3\pi} < 1/100$,

$$\sum_{k\geq 1} k^4 q^{k-1} = \frac{1 + 11q + 11q^2 + q^3}{(1-q)^5} < 2.$$

The last endpoint inequality is rational. Consequently

$$s(u)^2 \leq 16\pi^2 e^{4|u|-\pi e^{2|u|}}, \qquad \|s\|_\infty < \sqrt{8} < 3 =: S. \tag{20}$$

Here $x^2 e^{-\pi x}$ decreases for $x \geq 1$, $\pi^2 < 10$ and $e^\pi > 20$. For an integer $R \geq 2$ this also gives

$$\sigma_R := \sup_{|u|>R} s(u) \leq \frac{13\, 4^R}{2^{(3/2)4^R}} \leq 2^{-R}. \tag{21}$$

Use monotonicity of $xe^{-\pi x/2}$, $e^{2R} \geq 4^R$, $4\pi < 13$, and $e > 2$. The last inequality follows from $13 \leq 16$ and $4 + 3R \leq (3/2)4^R$, first at $R = 2$ and then inductively.

Inserting $u = \frac{1}{2}\log n$ in(18) gives

$$\|M_s S_{\log n} M_s\| \leq 16\pi^2 n^2 e^{-\pi n}. \tag{22}$$

Let B_M retain the exact m_0 term and pairs $2 \leq n \leq M$. As $c_n \leq \log n/\sqrt{n} \leq 1$,

$$\|B - B_M\| < \delta_M := \frac{640(M+1)^2}{20^{M+1}}, \qquad M \geq 1. \tag{23}$$

For completeness, with $k = M + 1$, $q_0 = e^{-\pi} < 1/20$,

$$\sum_{n\geq k} n^2 q_0^n = q_0^k \left[\frac{k^2}{1-q_0} + \frac{2kq_0}{(1-q_0)^2} + \frac{q_0(1+q_0)}{(1-q_0)^3}\right] < 2k^2 20^{-k}.$$

The bracket estimate follows at $q_0 = 1/20$, $1/k \leq 1/2$; multiplication by $32\pi^2 < 320$ proves(23). This proves ordinary operator-norm convergence, boundedness and selfadjointness of the complete B, before compression; compression cannot increase the error. These estimates retain all prime powers. They are not truncations of the nonnegative jump form(16).

The special value of digamma gives

$$m_0 = -\gamma - \pi/2 - 3\log 2 - \log \pi, \qquad |m_0| < 6.$$

For $M \geq 2$ it follows that

$$d_M := |m_0| + 2 \sum_{2\leq n\leq M} c_n \leq D_M := \min\{2M + 4, 6 + 4\lceil\sqrt{M}\rceil\lceil\log_2 M\rceil\}. \tag{24}$$

The second bound uses $\sum_{2\leq n\leq M} n^{-1/2} \leq 2\sqrt{M}$ and $\log M \leq \log_2 M$. No prime distribution theorem enters it.

Finally, for $L \geq 1$, put $J = \lfloor L/2 - 1/4\rfloor$. Retaining terms $0 \leq j \leq J$ in(15) yields

$$a(L) \geq \frac{1}{2}\sum_{j=0}^{J} \frac{1}{j + 1/4} \geq \frac{1}{2}\log(4J + 5) \geq \frac{1}{2}\log(2L). \tag{25}$$

These full-source estimates are the only numerical analytic inputs for the principal theorem.

3 Finite observations and the whole-kernel bootstrap

3.1 A complete phase-space estimate

Let P_R multiply by $1_{[-R,R]}$ and let Q_L be Fourier projection to $[-L, L]$. For integers $R \geq 1, L \geq 2, n \geq 1$ define

$$(\mathcal{L}_{R,L,n} f)(u) = 1_{[-R,R]}(u) \sum_{j=0}^{n-1} \frac{(iu)^j}{j!} \int_{-L}^{L} \xi^j \widehat{f}(\xi) \frac{d\xi}{2\pi}. \tag{26}$$

This map has rank at most n. Its rows are bounded on L^2 because $\xi^j 1_{[-L,L]}$ belongs to $L^2(\, d\xi/(2\pi))$.

Lemma 3.1. *Put $b_{R,L,n} = \sqrt{2RL/\pi}(RL)^n/n!$. Then*

$$\|P_R Q_L - \mathcal{L}_{R,L,n}\| \leq b_{R,L,n}. \tag{27}$$

If $z \in \mathcal{V}$, $f = sz$, and all rows of $\mathcal{L}_{R,L,n} f$ vanish, then, whenever $b = b_{R,L,n} < 1$,

$$(1 - b)\|f\| \leq \sigma_R \|z\| + \frac{\sqrt{\mathfrak{a}[z]}}{\sqrt{a(L)}}. \tag{28}$$

Proof. Since $u\xi$ is real, the integral Taylor remainder along the real segment gives

$$\left| e^{iu\xi} - \sum_{j=0}^{n-1} \frac{(iu\xi)^j}{j!} \right| \leq \frac{|u\xi|^n}{n!} \leq \frac{(RL)^n}{n!}.$$

Integrate its square over $[-R, R] \times [-L, L]$ with measures du and $d\xi/(2\pi)$. The Hilbert–Schmidt norm bounds the operator norm and gives(27). In particular there is no factor e^{RL}. Decompose f in physical L^2 as

$$f = (I - P_R)f + P_R(I - Q_L)f + P_R Q_L f.$$

The first norm is at most $\sigma_R \|z\|$. Monotonicity of a bounds the second by $\sqrt{\mathfrak{a}[z]/a(L)}$. The last is at most $b\|f\|$ when the rows vanish. This pays both spatial tails and the entire omitted frequency range. □

For clarity the corresponding row vector in $\mathcal{K}$ is $P_{\mathcal{K}} M_s q_j$, where $\widehat{q}_j = \xi^j 1_{[-L,L]}$ up to the harmless phase used in(26). The known projection is explicitly

$$P_{\mathcal{K}} h = h - \sum_{j=0}^{1} r_j \frac{\langle r_j, h \rangle}{\|r_j\|^2}, \quad r_0 = \sqrt{w}, \quad r_1 = \sqrt{w} \tanh(u/2),$$

because the two rows are orthogonal by parity. Both have nonzero finite norm. No projection onto an unknown null space occurs in a row condition.

Proposition 3.2 (General null receiving bound)**.** *Let $z \in \ker H$ satisfy the rows in Lemma3.1, and abbreviate $D = D_M$, $b = b_{R,L,n}$. If*

$$\kappa = \frac{\sqrt{D}}{(1 - b)\sqrt{a(L)}} < 1, \qquad \beta = \frac{\sqrt{D}\sigma_R/(1 - b) + \sqrt{\delta_M}}{1 - \kappa},$$

then

$$\|A^{1/2} z\| \leq \beta \|z\|, \tag{29}$$

$$\|Az\| \leq \left\{ \frac{SD}{1 - b} \left(\sigma_R + \frac{\beta}{\sqrt{a(L)}} \right) + \delta_M \right\} \|z\|. \tag{30}$$

Proof. Every actual null is in $\operatorname{Dom} A$, and $Az = -Bz$. With $F = \|sz\|$, $Z = \|z\|$, $X = \sqrt{\mathfrak{a}[z]}$, finite paired Cauchy–Schwarz and the complete tail give

$$X^2 = -\langle z, Bz\rangle \le DF^2 + \delta_M Z^2, \qquad X \le \sqrt{D}F + \sqrt{\delta_M}Z. \tag{31}$$

Combining with(28) and absorbing κX proves(29). Each finite paired shift has full output at most $2Sc_nF$; the m_0 output is at most $S|m_0|F$. Thus $\|Az\| = \|Bz\| \le SDF + \delta_M Z$. Insert(28) and(29) to obtain(30). Output outside any proposed finite subspace is included. □

3.2 Explicit choice of all parameters

We now prove the quantitative part of Theorem1.1. Use its parameters and $S = 3$. Since $D \le 2M + 4$,

$$\delta_M D \le \frac{1280(M+2)^3}{20^{M+1}} \le 64\,10^{-M} < \frac{\tau^2}{4096S^2}.$$

Here $(M+2)^3 \le 2^M$ for $M \ge 20$: it holds at20, and the ratio of successive cubic terms is less than2. Also $10^{-M} \le 10^{-20}\tau^{\log_2 10} \le 10^{-20}\tau^2$ and $64 \cdot 4096 \cdot 9 < 10^{20}$. Therefore

$$\sqrt{\delta_M} \le \frac{\tau}{64S\sqrt{D}}, \qquad \sigma_R \le \frac{\tau}{128SD}, \qquad a(L) \ge 16D. \tag{32}$$

The last follows from(25), $L = 2^{64D}$ and $\log 2 > 1/2$.

The integral lower bound on $\log(n!)$ and $e < 3$ give $n! \ge (n/3)^n$. As $n = 6RL$, and RL is a positive integer,

$$b \le RL\left(\frac{3RL}{n}\right)^n = RL\,2^{-6RL} \le 2^{-5RL} < 1/2.$$

We used $\sqrt{2RL/\pi} \le RL$ and $RL \le 2^{RL}$. Hence $\kappa \le 1/2$, and

$$\beta \le 4\sqrt{D}\sigma_R + 2\sqrt{\delta_M} \le \frac{\tau}{16S\sqrt{D}} \le \tau.$$

The full output factor in(30) is consequently at most

$$2SD\sigma_R + \frac{S\sqrt{D}}{2}\beta + \delta_M \le \frac{\tau}{64} + \frac{\tau}{32} + \frac{\tau}{4096} < \tau.$$

This proves both norm claims for every row-zero actual null.

Put $\ell = 1 + \log(1/\tau)$. Then $M = O(\ell)$, $D = O(\sqrt{\ell}\log(\ell+2))$, and $R = O(\ell)$. Thus

$$\log n = \log(6R) + 64D\log 2 = O(\sqrt{\ell}\log(\ell+2)),$$

proving(8). Using only $D = O(M)$ would lose this stronger inverse-tolerance dependence.

3.3 Finite-rank approximation and singular numbers

Let $Y \subseteq \ker H$ be closed. Closedness of H implies that the full kernel lies in $\operatorname{Dom} A$ and $A\Pi_Y = -B\Pi_Y$ is bounded. Moreover

$$\|A^{1/2}\Pi_Y h\|^2 = -\langle \Pi_Y h, B\Pi_Y h\rangle \le \|B\|\|h\|^2.$$

Let Y_τ be the intersection of Y with all row kernels, and let E_τ be the orthogonal projection onto $Y \ominus Y_\tau$. There are n bounded rows, so $\operatorname{rank} E_\tau \le n$. The projection $\Pi_Y - E_\tau$ has range in the row-zero space and norm at most1. The preceding estimates give

$$\|A\Pi_Y - AE_\tau\| \le \tau, \qquad \|A^{1/2}\Pi_Y - A^{1/2}E_\tau\| \le \tau. \tag{33}$$

In particular both maps are compact. Their $(n(\tau)+1)$st approximation numbers are at most τ; for compact Hilbert-space maps these equal singular values[6].

To invert the count, take $\tau = \exp[-c(\log N/\log\log(N+3))^2]$. For sufficiently small fixed $c > 0$ and sufficiently large N,(8) makes $\log n(\tau) < \log N$. Monotonicity of approximation numbers proves(9). Its decay is faster than every fixed inverse power of N, since $\log N/(\log\log N)^2 \to \infty$. Thus $\sum s_N^p < \infty$ for each $p > 0$, completing Theorem1.1. Also $(A^{1/2}\Pi_Y)^*(A^{1/2}\Pi_Y) = \Pi_Y A \Pi_Y$ has eigenvalue tail at most τ^2 at rank $n(\tau)$.

The finite rows and their rank bound are explicit. The vectors spanning $Y \ominus Y_\tau$, and therefore E_τ, are existential in this theorem. Confusing these two assertions would incorrectly turn a whole-kernel estimate into a computed null projection.

4 Finite and complete null deflation

Let $\mathcal{Z}$ be a closed subspace of $\ker H$, let Π project onto it, and put $P = I - \Pi$. In applications $\mathcal{Z}$ can be the full derivative-null closure described in AppendixB, but equality with $\ker H$ is not assumed. Fix increasing finite-rank projections $\Pi_N \to \Pi$ strongly, put $R_N = \Pi - \Pi_N$ and $P_N = I - \Pi_N$.

Proposition 4.1. *Writing $b_* = \|B\|$ and $e_N = \|AR_N\|$, one has $e_N \to 0$ and*

$$A_{\mathrm{def}} = PAP = A + \Pi B + B\Pi - \Pi B \Pi \geq 0, \qquad B_{\mathrm{def}} = PBP, \tag{34}$$

$$A_N = P_N A P_N \geq 0, \qquad B_N = P_N B P_N. \tag{35}$$

All references are selfadjoint on $\operatorname{Dom} A$*; both sums equal H. Moreover*

$$\|A_N - A_{\mathrm{def}}\|,\ \|B_N - B_{\mathrm{def}}\| \leq 3e_N, \tag{36}$$

$$\|(A_N+\varepsilon)^{-1} - (A_{\mathrm{def}}+\varepsilon)^{-1}\| \leq 3e_N/\varepsilon^2. \tag{37}$$

For the zero-extended sandwiches

$$K_{N,\varepsilon} = (A_N+\varepsilon)^{-1/2}(-B_N)(A_N+\varepsilon)^{-1/2}, \quad K_{\mathrm{def},\varepsilon} = (A_{\mathrm{def}}+\varepsilon)^{-1/2}(-B_{\mathrm{def}})(A_{\mathrm{def}}+\varepsilon)^{-1/2},$$

one has

$$\|K_{N,\varepsilon} - K_{\mathrm{def},\varepsilon}\| \leq \frac{3e_N}{\varepsilon} + \frac{3b_* e_N}{\varepsilon^2}. \tag{38}$$

Proof. $A\Pi = -B\Pi$ is bounded, and its adjoint gives the bounded extension $\Pi A = -\Pi B$. Thus Π, P preserve $\operatorname{Dom} A$. The bounded map $A^{1/2}\Pi$ similarly shows preservation of the form domain. Expanding PAP gives(34), a bounded selfadjoint perturbation of A with form $\mathfrak{a}[Ph] \geq 0$. The same proof applies to Π_N. Theorem1.1 makes $A\Pi$ compact, so its composition with the strongly vanishing R_N converges in norm to zero. This gives $e_N \to 0$.

Expand $P_N = P + R_N$. The three differences are PAR_N, $R_N AP$ and $R_N A R_N$, each of norm at most e_N. The analogous expansion for B has the same bound because $BR_N = -AR_N$ and the adjoint identity holds. All differences extend boundedly; no unbounded product is left undefined. The resolvent identity proves(37).

For nonnegative selfadjoint C, D with bounded difference Δ, the Stieltjes formula and resolvent identity yield

$$\|(C+\varepsilon)^{-1/2} - (D+\varepsilon)^{-1/2}\| \leq \frac{\|\Delta\|}{\pi}\int_0^\infty \frac{t^{-1/2}}{(t+\varepsilon)^2}\,dt = \frac{\|\Delta\|}{2\varepsilon^{3/2}}.$$

Split the sandwich difference into the two square-root changes and the one perturbation change, use(36), $\|B_N\|, \|B_{\mathrm{def}}\| \leq b_*$ and inverse-square-root norms at most $\varepsilon^{-1/2}$. This gives(38). Both sandwiches annihilate their removed null spaces because their middle factors do, so these are exactly their zero extensions. □

For fixed ε this supplies full operator-norm convergence, including every cross output. For example, on ℓ^2 the compact diagonal model $A = \mathrm{diag}(t_j)$, $B = -A$, $H = 0$ permits $t_j \downarrow 0$ arbitrarily slowly; then a coordinate-prefix tail has $e_N = t_{N+1}$. This is a warning about abstract compactness, not a counterexample to the actual-source estimate of Theorem1.1. Its explicit observations need not coincide with any chosen derivative prefix.

5 A global signed finite-matrix receiver

This section states a sufficient finite certificate, with its whole-space error budget. Let $\mathcal{S}$ be either $\mathcal{K}$ or $P\mathcal{K}$ from(34). On the latter space use the restrictions $A_{\mathcal{S}}, B_{\mathcal{S}}$ of the deflated reference and perturbation. Their form is the original $\mathfrak{a}$ restricted to $P\mathcal{K}$. Set

$$R_\varepsilon = (A_{\mathcal{S}} + \varepsilon)^{-1/2}, \quad J = M_s R_\varepsilon : \mathcal{S} \longrightarrow L^2(\mathbb{R}), \quad K_\varepsilon = R_\varepsilon(-B_{\mathcal{S}})R_\varepsilon.$$

Positive-form congruence gives the exact equivalence

$$H|_{\mathcal{S}} \geq -\varepsilon I \quad \Longleftrightarrow \quad K_\varepsilon \leq I. \tag{39}$$

There is no omitted polar term: the original two rows are already imposed.

Theorem 5.1 (Complete residual-Gram receiver)**.** *Choose $\sigma \geq \|s\|_\infty$, $\sigma_{R_0} \geq \sup_{|u|>R_0} s(u)$, $R_0 > 0$, $L \geq 1$ and an integer $d \geq 0$. Put*

$$j_\varepsilon = \sigma/\sqrt{\varepsilon}, \qquad b_d = \sqrt{2/\pi}\frac{(R_0 L)^{d+3/2}}{(2d+3)(d+1)!},$$

$$t_d = \frac{\sigma_{R_0}}{\sqrt{\varepsilon}} + \frac{1}{\sqrt{a(L)}} + b_d j_\varepsilon, \tag{40}$$

$$\Gamma_d = \frac{\delta_M}{\varepsilon} + d_M t_d(2j_\varepsilon + t_d). \tag{41}$$

Define $p_k = 1_{[-R_0,R_0]}u^k/k!$ and $\widehat{q}_k = 1_{[-L,L]}(-i\xi)^k$, $0 \leq k \leq d$. Let

$$Ye_k = p_k, \qquad Qe_k = \mathrm{Proj}_{\mathcal{S}}(sq_k),$$
$$V_M = -m_0 I + \sum_{2\leq n\leq M} c_n(S_{\log n} + S_{-\log n}),$$
$$C = Y^* V_M Y, \qquad G = Q^*(A_{\mathcal{S}} + \varepsilon)^{-1}Q.$$

For any trial map $X : \mathbb{C}^{d+1} \to \mathrm{Dom}\, A_{\mathcal{S}}$ put

$$E = Q - (A_{\mathcal{S}} + \varepsilon)X,$$
$$D_X = Q^*X + X^*Q - X^*(A_{\mathcal{S}} + \varepsilon)X, \qquad G_U = D_X + E^*E/\varepsilon. \tag{42}$$

If rigorously enclosed finite Hermitian matrices and a scalar satisfy

$$\overline{C} \geq C, \qquad \overline{G} \geq G_U, \quad \overline{G} > 0, \qquad \overline{C} \leq \beta\overline{G}^{-1}, \quad \beta \geq 0, \qquad \beta + \Gamma_d \leq 1, \tag{43}$$

then $H|_{\mathcal{S}} \geq -\varepsilon I$. All inequalities are in Loewner order. The covariance E^*E refers to full residual columns in $\mathcal{S}$.

Proof. For unit $h \in \mathcal{S}$, spectral calculus gives

$$\|Jh\| \leq j_\varepsilon, \qquad \mathfrak{a}[R_\varepsilon h] \leq 1.$$

Hence the full spatial and frequency tails of J have norms at most $\sigma_{R_0}/\sqrt{\varepsilon}$ and $1/\sqrt{a(L)}$. Taylor expansion through degree d, now integrating $|u\xi|^{2d+2}$ exactly, gives

$$\|P_{R_0}Q_L - \mathcal{L}_{R_0,L,d+1}\| \le \left(\frac{2(R_0L)^{2d+3}}{\pi(2d+3)^2((d+1)!)^2}\right)^{1/2} = b_d.$$

Thus $J_d = \mathcal{L}_{R_0,L,d+1}J$ obeys $\|J - J_d\| \le t_d$ and $\|J_d\| \le j_\varepsilon + t_d$. The factor j_ε on the Taylor error is necessary because it is an error before composition with J.

The finite-prime sandwich is exactly J^*V_MJ. By(23), its difference from K_ε is at most δ_M/ε. Expanding the two product differences, including mixed outputs, proves

$$\|K_\varepsilon - J_d^*V_MJ_d\| \le \Gamma_d. \tag{44}$$

Our phases give the coefficient $\langle q_k, f\rangle$ in the Taylor map, so $J_d = YQ^*R_\varepsilon$. Consequently

$$J_d^*V_MJ_d = WCW^*, \qquad W = R_\varepsilon Q, \qquad W^*W = G.$$

C is signed: its entries are finite polynomial integrals over overlaps of intervals translated by $\log n$. The coefficient retains its sign.

Expanding(42) proves the exact identity

$$G - D_X = E^*(A_{\mathcal{S}} + \varepsilon)^{-1}E, \qquad 0 \le G - D_X \le E^*E/\varepsilon.$$

Thus $G \le G_U \le \overline{G}$. With $W_0 = W\overline{G}^{-1/2}$ one has $W_0^*W_0 \le I$, and

$$WCW^* \le W\overline{C}W^* = W_0[\overline{G}^{1/2}\overline{CG}^{1/2}]W_0^* \le \beta W_0W_0^* \le \beta I.$$

The final step uses $\beta \ge 0$. Combining with(44) proves $K_\varepsilon \le I$, hence(39). □

One may take $\sigma = 3$ and $\sigma_{R_0} = \min\{3, 13e^{2R_0-(3/2)e^{2R_0}}\}$ from(20). At fixed ε any prescribed norm tolerance in(41) is possible: choose M, then R_0, then L, and finally d. This proves existence of a finite-rank enclosure scheme, not the signed condition(43). On $P\mathcal{K}$, evaluation of Q requires the full prescribed null projection or a paid replacement. Taking $\mathcal{S} = \mathcal{K}$ avoids that unknown projection at a potentially much larger rank.

Matrix roundoff must be paid in Loewner order, for example by a norm-radius multiple of the identity. Entrywise upper endpoints need not give a Hermitian upper matrix. A residual projected into a trial space does not bound E^*E, and separate favorable diagonal residual estimates omit its mixed covariance.

A An explicit global weighted lower certificate

We prove an application on the original domain, retaining the full arithmetic tail. The constant is deliberately coarse.

Theorem A.1. *For the realization in Section2,*

$$T \ge \frac{1}{100}I, \qquad H \ge -\frac{99}{100}I.$$

The inequalities also hold on every reducing null complement.

Proof. First take g in the original compact smooth two-row core and put $h = \sqrt{w}g$, $f = \psi g = sh$, $N = \|g\|^2_{L^2(w)}$, $D = D[g]$. The $M = 3$ perturbation tail is $\delta_3 = 8/125$. As $|m_0| < 6$ and $c_2, c_3 \le 1$, the full identity gives

$$D \ge \frac{117}{125}N - 10\|f\|_2^2. \tag{45}$$

We need two complete source-tail estimates and a bound at the origin. Even log-concavity of s gives $\|s\|^2_\infty = \psi(0)$. Its first atom is less than

$$\frac{40 - 6(157/50)}{23} = \frac{529}{575},$$

using $\pi^2 < 10$, $\pi > 157/50$ and $e^\pi > 23$. For all remaining atoms, $k^2 - 4 \ge 5(k-2)$, $e^3 > 20$, $e^{15} > 1000$ yield

$$\sum_{k\ge 2} 4\pi^2 k^4 e^{-\pi k^2} < \frac{40}{20^4}\sum_{j\ge 0}\frac{(j+2)^4}{1000^j} < \frac{17}{4000}.$$

The last series is an exact differentiated geometric series. Since $529/575 + 17/4000 < 1$, this proves $\|s\|^2_\infty < 1$ without discarding higher atoms.

On $|u| \ge 1$,(20), $e^2 > 7$, $e < 3$ and $e > 8/3$ give

$$\sup_{|u|\ge 1} s(u)^2 < 12960e^{-21} < 12960(3/8)^{21} < 1/1000. \tag{46}$$

Let $\nu = w\,du$, $I = [-1, 1]$, and $\nu_{\text{out}} = \nu(I^c)$. Since $c(u) \le e^{u/2}$ for $u \ge 0$, the full theta bound and $x = e^{2u}$ give

$$\begin{aligned}\nu_{\text{out}} &\le 16\pi^2 \int_1^\infty e^{5u-\pi e^{2u}}\,du = 8\pi^2\int_{e^2}^\infty x^{3/2}e^{-\pi x}\,dx\\ &< 80\int_{e^2}^\infty x^2 e^{-3x}\,dx = 80e^{-3e^2}(e^4/3 + 2e^2/9 + 2/27)\\ &< 2400e^{-21} < 2400(3/8)^{21} < 1/1000.\end{aligned}$$

Here the bracket is less than30 because $e^2 < 9$, $e^4 < 81$. From(10), $\nu(\mathbb{R}) = 1/2$, so $\nu_I > 499/1000$.

The archimedean term of(16) alone supplies the local variance estimate. $K(t)$ decreases with $|t| > 0$, and on $I \times I$ off its null diagonal,

$$\frac{K(u-v)}{c(u)c(v)} \ge k := \frac{K(2)}{\cosh(1/2)^2} > 1/4.$$

Indeed $K(2) > e^{-1} > 1/3$ and $\cosh(1/2)^2 < 4/3$; the latter follows from $e < 11/4$ and $e^{-1} < 3/8$, which give $\cosh 1 < 25/16$ and $\cosh(1/2)^2 < 41/32$. Writing $N_I = \int_I |g|^2\,d\nu$, the exact complex variance identity gives

$$D \ge k\left[\nu_I N_I - \left|\int_I g\,d\nu\right|^2\right].$$

The first GLOBAL row, not a newly imposed local mean, says $\int_I g\,d\nu = -\int_{I^c} g\,d\nu$. Cauchy–Schwarz therefore gives

$$N_I \le \frac{D}{k\nu_I} + \frac{\nu_{\text{out}}}{\nu_I}N \le \frac{4000}{499}D + \frac{1}{499}N.$$

Using $\|s\|_\infty^2 < 1$ and(46),

$$\|f\|_2^2 \le N_I + \frac{1}{1000}(N - N_I) \le \frac{4000}{499}D + \left(\frac{1}{499} + \frac{1}{1000}\right)N.$$

Insert this UPPER bound into the negative term in(45) to obtain

$$D \ge \frac{117/125 - 10/499 - 1/100}{1 + 40000/499}N = \frac{226037}{20249500}N > \frac{1}{100}N.$$

Convergence in the chosen minimal form norm extends the inequality from the compact core to $\mathcal{V}$, and the representation theorem proves the operator statement. A null subspace reduces H; restriction gives the identical lower bound on its complement. □

All scalar estimates used above have elementary exact controls. For instance, the exponential series through order4 bounds $8/3 < e < 11/4$ by estimating the remaining factorial ratios geometrically; the series through order10 at $157/50$ proves $e^\pi > 23$. The formula for the higher-atom geometric series is

$$\sum_{j\ge 0}(j+2)^4 q^j = \frac{16}{1-q} + \frac{32q}{(1-q)^2} + \frac{24q(1+q)}{(1-q)^3} + \frac{8q(1+4q+q^2)}{(1-q)^4} + \frac{q(1+11q+11q^2+q^3)}{(1-q)^5}.$$

The public rational checker verifies these constants and the final absorption fraction.

The weighted norm is essential: this is not a uniform physical unweighted floor. The RH-sign target would be $T \ge I$, not merely strict positivity of T.

B Multiplicity, divided sources and the derivative closure

This appendix explains why a complete-kernel estimate should not assume that a familiar derivative family is complete. Division and jet separation are classical[11, 12]; the issue here is their physical weighted operator domain. Let $\mathcal{E}$ consist of smooth functions f such that

$$\sup_{u\in\mathbb{R}} e^{a|u|}|f^{(j)}(u)| < \infty \quad \text{for every } a > 0,\ j \ge 0.$$

The formula(13) and its continuity apply to this class. To check the physical-side continuity directly, two integrations by parts bound the gamma integral with logarithmic multiplier; for the shifts, exponential tails give $|\langle f, S_b g\rangle| \le C_A(1+|b|)e^{-A|b|}$. Taking $A > 1/2$ and then a slightly larger exponent makes the prime-power series summable uniformly in bounded seminorm sets. Polar values are continuous by the same tails. On the zero side, integration by parts and the zero count give the corresponding absolute convergence.

Write $\mathcal{P} = \partial_u^2 - 1/4$, $q = \mathcal{P}\psi$ and

$$\mathcal{L}(z) = (z^2 - 1/4)\xi(1/2 + z).$$

The known physical derivative nulls are $g_k = q^{(k)}/\psi$. Under the unitary $U : g \mapsto \sqrt{w}g$, let $\mathcal{Z}$ be the closed span of Ug_k, $k \ge 0$.

Proposition B.1. *The space $\mathcal{Z}$ is an infinite-dimensional closed subspace of $\ker H$. For every finite set F of distinct centered xi zeros $\sigma = \rho - 1/2$ with multiplicities m_σ,*

$$\dim_{\mathbb{C}}(\ker H/\mathcal{Z}) \ge \sum_{\sigma\in F}(m_\sigma - 1). \tag{47}$$

In particular, $\mathcal{Z} = \ker H$ implies that all nontrivial xi zeros are simple.

Proof. We first record a general domain passage. Suppose $\eta \in \mathcal{E}$ and its fixed derivatives satisfy

$$|\eta^{(j)}(u)| \le C_j \psi(u) e^{C_j|u|}. \tag{48}$$

Put $f = \mathcal{P}\eta$, $g = f/\psi$, and assume its Mellin transform $\mathcal{M}f$ vanishes at every centered zero value. The $\mathcal{P}$ factor forces $\mathcal{M}f(\pm 1/2) = 0$, so both Hilbert rows hold. Choose a smooth cutoff χ_R equal to1 on $[-R, R]$, zero outside $[-R-1, R+1]$, with uniformly bounded fixed derivatives, and set

$$f_R = \mathcal{P}(\chi_R \eta), \qquad g_R = f_R/\psi.$$

Both polar moments vanish EXACTLY. Bounds(48) and the double-exponential theta tails give $g_R \to g$ in $L^2(w)$, including the cutoff-derivative terms, and $f_R \to f$ in every seminorm of $\mathcal{E}$. Thus $Q[f_R - f_S] \to 0$. The compact-core identity gives

$$D[g_R - g_S] = \|g_R - g_S\|^2_{L^2(w)} + Q[f_R - f_S] \longrightarrow 0.$$

Hence g belongs to the chosen minimal form domain. Against each compact constrained test v,(13) gives $Q(f, \psi v) = 0$, because the zero set is invariant under $\sigma \mapsto -\overline{\sigma}$ and the two polar moments of $v\psi$ vanish. Pass the mixed core identity to the limit and then use form-core density:

$$D(g, v) = \langle g, v \rangle_{L^2(w)} \quad \text{for every form-domain } v.$$

The operator representation theorem yields $Ug \in \operatorname{Dom} T$ and $T(Ug) = Ug$, so $Ug \in \ker H$. No maximal-domain substitution was made.

Taking $\eta = \psi^{(k)}$ proves that all Ug_k are null. Their transforms are $(-z)^k \mathcal{L}(z)$, which are linearly independent since $\mathcal{L}$ is not identically zero. Thus $\mathcal{Z}$ is infinite dimensional and, by closedness of H, lies in its kernel.

Now fix a centered zero σ of order $m \ge 2$. For $h \in \mathcal{E}$ with $\mathcal{M}h(\sigma) = 0$, define the canceled primitive

$$\begin{aligned}(\mathcal{R}_\sigma h)(u) &= e^{-\sigma u} \int_u^\infty e^{\sigma t} h(t)\, dt \\ &= -e^{-\sigma u} \int_{-\infty}^u e^{\sigma t} h(t)\, dt.\end{aligned} \tag{49}$$

It satisfies

$$(\partial_u + \sigma)\mathcal{R}_\sigma h = -h, \qquad \mathcal{M}(\mathcal{R}_\sigma h)(z) = \frac{\mathcal{M}h(z)}{z - \sigma}. \tag{50}$$

The positive sign in the quotient follows from $\mathcal{M}\partial_u = -z\mathcal{M}$. Both representations in(49) are needed to control both ends.

Here are the actual weight estimates that pay this division. From the first theta atom and its normally differentiable remaining series,

$$(\log \psi)'(u) = -2\pi e^{2u} + O(1) \quad (u \to +\infty), \qquad |\psi^{(j)}(u)|/\psi(u) \le C_j e^{2j|u|}.$$

Indeed the ratio of every higher atom to the first has an exponentially decreasing factor $e^{-\pi(k^2-1)e^{2u}}$ times a fixed polynomial; its differentiated sum has the same dominance. In particular, for sufficiently large $t \ge u$,

$$\psi(t) \le C\psi(u) \exp[-ce^{2u}(t-u)],$$

and evenness gives the reflected estimate. If $|h(u)| \le C\psi(u)e^{-2r|u|}$, $r \ge 0$, the appropriate tail representation in(49) consequently gives

$$|\mathcal{R}_\sigma h(u)| \le C_{\sigma,r}\psi(u)e^{-2(r+1)|u|}. \tag{51}$$

The exponential tail integral costs $O(e^{-2|u|})$ after increasing the fixed starting point so that $ce^{2|u|}$ dominates $|\operatorname{Re}\sigma| + 2r$. Compact intervals are bounded separately. Differentiating the recurrence(50) gives every required derivative estimate. Thus repeated primitives lie in $\mathcal{E}$ and satisfy(48).

Set $\eta_0 = \psi$, $\eta_\ell = \mathcal{R}_\sigma \eta_{\ell-1}$ for $1 \le \ell < m$, and $f_{\sigma,\ell} = \mathcal{P}\eta_\ell$. At every step the canceled moment is available, because

$$\mathcal{M}\eta_\ell(z) = \frac{\xi(1/2+z)}{(z-\sigma)^\ell}, \qquad \mathcal{M}f_{\sigma,\ell}(z) = \frac{\mathcal{L}(z)}{(z-\sigma)^\ell}.$$

These last transforms vanish at both polar moments and at every centered zero VALUE; at σ the remaining order is $m - \ell \ge 1$. The domain passage above proves $U(f_{\sigma,\ell}/\psi) \in \ker H$.

To separate these vectors from $\mathcal{Z}$, for any $j \ge 0$ consider the jet functional on the physical weighted space

$$\ell_{\sigma,j}(g) = \int_{\mathbb{R}} u^j e^{\sigma u} \psi(u) g(u)\, du, \qquad C_{\sigma,j}^2 = \int_{\mathbb{R}} |u|^{2j} e^{2\operatorname{Re}\sigma u} \frac{\psi(u)}{c(u)}\, du < \infty.$$

Cauchy–Schwarz bounds its norm by $C_{\sigma,j}$. Every vector in $U^{-1}\mathcal{Z}$ has zero jets of orders $j < m_\sigma$, by the derivative transforms and continuity. For $g = f_{\sigma,\ell}/\psi$ and $j = m - \ell$,

$$\ell_{\sigma,j}(g) = \frac{j!(\sigma^2 - 1/4)}{m!} \xi^{(m)}(1/2+\sigma) \neq 0. \tag{52}$$

In particular its distance from $U^{-1}\mathcal{Z}$ is at least the absolute value in(52) divided by $C_{\sigma,j}$.

For a finite collection of zeros, choose the largest ℓ with a nonzero coefficient at one chosen σ in a proposed relation modulo $\mathcal{Z}$. Its first remaining jet $j = m_\sigma - \ell$ is nonzero; smaller ℓ vanish to higher order, and sources divided at other zeros retain all m_σ jets. Thus this coefficient vanishes. Descending in ℓ and then through the other distinct zeros proves independence modulo $\mathcal{Z}$ and(47). □

Multiplicity in the Weil zero pairing is a scalar weight on VALUES, whereas the derivative-generated closure preserves full zero jets. This is the distinction used here. All reflected and conjugate zero locations are included in the complex count(47). Theorem1.1 automatically covers these possible extra null directions. It does not require that their multiplicities or locations be known, nor does it decide whether the derivative family is complete when all zeros are simple.

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AI systems assisted with mathematical exploration, literature comparison, proof checking and manuscript preparation. The author takes responsibility for its statements, proofs, citations and limitations.

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