

Prescribed-gradient minimizers and coefficient stability for homogeneous real-stable polynomials

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Abstract

For each fixed degree n , we prove that among homogeneous real-stable polynomials in n variables with nonnegative coefficients, value one at the all-ones vector, and a prescribed gradient sufficiently close to the all-ones vector, the power of the corresponding linear form uniquely minimizes the squarefree coefficient. This answers Question 1.8 of Gurvits–Leake, arXiv:2609.21083v2. We strengthen the minimum statement to a uniform lower bound for the coefficient deficit by the coefficient ℓ^1 distance to that power. The distance order is optimal. We also classify strict local minimality at every positive gradient and give a cubic example separating local and global minimization. The proof uses centered real-root dispersion, an exact coefficient identity and the classical balanced uniqueness theorem. The neighborhood and constants are dimension-dependent and qualitative. We also give a total-variation interpretation for stable count distributions and an illustrative nondeterminantal quartic family.

1 Introduction and main result

A nonzero polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ is *real stable* if $p(z_1, \dots, z_n) \neq 0$ whenever $\operatorname{Im} z_i > 0$ for every i . Throughout, the polynomials under consideration have nonnegative coefficients and are homogeneous of degree n in n variables. Write $\mathbf{1} = (1, \dots, 1)^T$ and

$$\mathcal{A}(p) = [x_1 \cdots x_n]p, \quad \|p\|_{\text{coef},1} = \sum_{|\alpha|=n} |[x^\alpha]p|.$$

For $t \in \mathbb{R}_{>0}^n$ with $\sum_i t_i = n$, set

$$L_t(x) = \frac{t^T x}{n}, \quad F(t) = \mathcal{A}(L_t^n) = \frac{n!}{n^n} \prod_{i=1}^n t_i.$$

The balanced coefficient inequality and its equality classification are due to Gurvits [1]; we use the following formulation, also recalled in Theorem 1.3 of Gurvits–Leake [2].

Theorem 1.1 (Balanced uniqueness). *If p is homogeneous of degree n , real stable, has nonnegative coefficients, $p(\mathbf{1}) = 1$, and $\nabla p(\mathbf{1}) = \mathbf{1}$, then $\mathcal{A}(p) \geq n!/n^n$, with equality if and only if $p = L_{\mathbf{1}}$.*

Theorem 1.1 is a prior result, not a contribution of this paper. The upper-half-plane convention above and the right-half-plane convention in [1] are equivalent for homogeneous polynomials by multiplication of every variable by i . To relate the formulation directly to the coefficient–capacity inequality and its equality case in [1, Theorem 5.7], write $p(x) = \sum_{\alpha} c_{\alpha} x^{\alpha}$. Here $\sum_{\alpha} c_{\alpha} = 1$ and $\sum_{\alpha} c_{\alpha} \alpha = \mathbf{1}$. Weighted AM–GM gives $p(x) \geq \prod_i x_i$ for $x_i > 0$, so $\text{Cap}(p) = \inf_{x>0} p(x)/\prod_i x_i = 1$. The equality classification in that theorem forces a power of a positive linear form; normalization and the balanced gradient fix the form to $L_{\mathbf{1}}$.

Theorem 1.2. *For each fixed integer $n \geq 1$, there exist constants $\varepsilon_n, c_n > 0$ such that the following holds. Suppose p is homogeneous of degree n in n variables, real stable, has nonnegative real coefficients, $p(\mathbf{1}) = 1$, and $t = \nabla p(\mathbf{1})$ satisfies $\|t - \mathbf{1}\|_1 < \varepsilon_n$. Then*

$$\mathcal{A}(p) - F(t) \geq c_n \|p - L_t^n\|_{\text{coef},1}. \quad (1)$$

In particular, L_t^n is the unique minimizer of $\mathcal{A}$ among polynomials with this normalization and prescribed gradient.

Euler's identity gives $\sum_i t_i = n$; the neighborhood will be chosen so that every $t_i > 0$. The theorem includes all feasible polynomials with such a gradient, not only polynomials in a chosen coefficient neighborhood of L_t^n . The constants may depend on n . No useful explicit global value of ε_n or c_n is computed here.

Gurvits–Klein–Leake [3, Section 2.1] prove a nearbalanced permanent minimization result and explicitly leave the real-stable extension unproved. Gurvits–Leake [2, Question 1.8] ask for the prescribed-gradient conclusion of Theorem 1.2. Their balanced uniqueness result for the larger Lorentzian class does not supply the real-root estimates used below. Productization [4] gives important capacity bounds and gradient-preserving products matching an evaluation at a fixed positive point. That product may depend on the evaluation point; the result does not directly identify the squarefree-coefficient minimizer considered here.

The earlier mixed-discriminant result [8] treats the matrix subclass and answers Question 1.7 of [2]. Indeed, $p(x) = \det(\sum_i x_i X_i)$, for positive semidefinite Hermitian matrices with $\sum_i X_i = I$, has $p(\mathbf{1}) = 1$ and $\partial_i p(\mathbf{1}) = \text{Tr } X_i$. The present argument does not require a determinant representation. Such a representation cannot be assumed for general stable polynomials [6, 7].

Our main step is to use the variance of centered real roots as a single size parameter controlling all higher centered coefficients. Hyperbolic eigenvalue methods are classical; see, for example, Bauschke–Güler–Lewis–Sendov [5]. Here they yield a positive leading payment for the squarefree coefficient under a perturbed gradient. Compactness and Theorem 1.1 then extend the local estimate over the entire feasible class with nearbalanced gradient.

2 Centered roots and the dispersion matrix

For the rest of the proof let $n \geq 2$, let p satisfy the normalization and stability hypotheses, and put $t = \nabla p(\mathbf{1})$. For real x , define $y = x - L_t(x)\mathbf{1}$. Then $t^T y = 0$.

Lemma 2.1. *There are homogeneous polynomials q_k of degree k , for $2 \leq k \leq n$, invariant under $x \mapsto x + a\mathbf{1}$, such that*

$$p(x) = L_t(x)^n + \sum_{k=2}^n L_t(x)^{n-k} q_k(x). \quad (2)$$

There is a real symmetric positive semidefinite matrix B with $B\mathbf{1} = 0$ such that

$$q_2(x) = -x^T Bx, \quad B = \frac{n-1}{2n} t t^T - \frac{1}{2} \nabla^2 p(\mathbf{1}). \quad (3)$$

Writing $d = \text{Tr } B$, we have

$$|q_k(x)| \leq \binom{n}{k} (2x^T Bx)^{k/2} \leq \binom{n}{k} (2d\|x\|_2^2)^{k/2}. \quad (4)$$

Moreover, $d = 0$ if and only if $p = L_t^n$.

Proof. Expand $p(z\mathbf{1} + y)$ in z . Homogeneity and $p(\mathbf{1}) = 1$ make it monic of degree n . The coefficient of z^{n-1} is $t^T y = 0$, so

$$p(z\mathbf{1} + y) = z^n + \sum_{k=2}^n q_k(x) z^{n-k}.$$

The map $x \mapsto y$ is unchanged by adding $a\mathbf{1}$ to x , since $L_t(\mathbf{1}) = 1$. This proves the stated invariance. Substitute $z = L_t(x)$ to obtain (2).

For every real y , all roots of $z \mapsto p(z\mathbf{1} + y)$ are real. An upper-half-plane root would contradict stability, and a lower-half-plane nonreal root would have an upper-half-plane conjugate. Let these roots be $\lambda_1, \dots, \lambda_n$. Their sum is zero. Hence

$$q_2(x) = \sum_{i < j} \lambda_i \lambda_j = -\frac{1}{2} \sum_i \lambda_i^2 \leq 0.$$

Thus $q_2 = -x^T B x$ for a unique symmetric $B \geq 0$; invariance gives $B\mathbf{1} = 0$. To check its formula, put $H = \nabla^2 p(\mathbf{1})$. Euler's identities give $H\mathbf{1} = (n-1)t$ and $\mathbf{1}^T H\mathbf{1} = n(n-1)$. The coefficient of z^{n-2} is $\frac{1}{2}y^T H y = \frac{1}{2}x^T H x - \frac{n-1}{2n}(t^T x)^2$, which is (3).

Each $|\lambda_i| \leq (2x^T B x)^{1/2}$. The elementary-symmetric root formulas give (4); positivity of B implies $x^T B x \leq d\|x\|_2^2$. If $d = 0$, then $B = 0$, all the roots vanish for every real x , and all q_k vanish. Equation (2) gives $p = L_t^n$. The converse follows directly. $\square$

3 A uniform local coefficient estimate

Lemma 3.1. *Suppose $t_i > 0$, put $v_i = 1/t_i$ and $D = \text{diag}(v)$. Then*

$$\mathcal{A}(L_t^{n-2} q_2) = \frac{(n-2)! \prod_i t_i}{n^{n-2}} \left(\text{Tr}(D^2 B) - v^T B v \right). \quad (5)$$

If $\|t - \mathbf{1}\|_\infty \leq 1/(8\sqrt{n})$, then

$$\mathcal{A}(L_t^{n-2} q_2) \geq a_n d, \quad a_n = \frac{(n-2)!(7/8)^n}{2n^{n-2}} > 0. \quad (6)$$

Proof. In $-x^T B x$, a diagonal term cannot contribute to a squarefree monomial. The ordered pair $i \neq j$ contributes

$$-B_{ij} \frac{(n-2)!}{n^{n-2}} \prod_{\ell \neq i, j} t_\ell.$$

Summing these terms proves (5). Since $B\mathbf{1} = 0$,

$$v^T B v = (v - \mathbf{1})^T B (v - \mathbf{1}) \leq d\|v - \mathbf{1}\|_2^2, \quad \text{Tr}(D^2 B) \geq (\min_i v_i^2) d.$$

Under the stated neighborhood condition, $7/8 \leq t_i \leq 9/8$, so $\min_i v_i^2 \geq 64/81$. Also

$$|v_i - 1| \leq \frac{1/(8\sqrt{n})}{1 - 1/(8\sqrt{n})} \leq \frac{1}{7\sqrt{n}}, \quad \|v - \mathbf{1}\|_2^2 \leq \frac{1}{49}.$$

The difference $64/81 - 1/49$ exceeds $1/2$, and $\prod_i t_i \geq (7/8)^n$. This proves (6). $\square$

Lemma 3.2. *There is $\eta_n \in (0, 1]$, depending only on n , such that*

$$\mathcal{A}(p) - F(t) \geq \frac{a_n}{2} d$$

whenever $\|t - \mathbf{1}\|_\infty \leq 1/(8\sqrt{n})$ and $d \leq \eta_n$.

Proof. For every homogeneous polynomial R of degree n ,

$$\mathcal{A}(R) = \sum_{S \subseteq [n]} (-1)^{n-|S|} R(\mathbf{1}_S). \quad (7)$$

Indeed, a monomial survives the alternating sum exactly when all variables occur; degree n then makes it squarefree. Here $\mathbf{1}_S$ is the indicator vector of S .

Nonnegativity of t and $\sum_i t_i = n$ imply $0 \leq L_t(\mathbf{1}_S) \leq 1$, while $\mathbf{1}_S^T B \mathbf{1}_S \leq nd$. Thus, for $k \geq 3$, Lemma 2.1 and (7) give

$$|\mathcal{A}(L_t^{n-k} q_k)| \leq 2^n \binom{n}{k} (2nd)^{k/2}.$$

For $0 \leq d \leq 1$, the sum is at most $C_n d^{3/2}$, where

$$C_n = 2^n \sum_{k=3}^n \binom{n}{k} (2n)^{k/2}.$$

For $n = 2$, the sum is empty and $C_2 = 0$. Choose $\eta_2 = 1$, and for $n \geq 3$ take $\eta_n = \min\{1, (a_n/(2C_n))^2\}$. Combining (2) and Lemma 3.1 proves the assertion. $\square$

4 Compactness and the global deficit

Let $\mathcal{K}_n$ be the set of homogeneous degree- n real-stable polynomials with nonnegative coefficients and $p(\mathbf{1}) = 1$.

Lemma 4.1. *The set $\mathcal{K}_n$ is compact in coefficient space. The functions $t(p)$, $B(p)$, $d(p)$, and $\mathcal{A}(p)$ are continuous on it, and*

$$0 \leq d(p) \leq M_n := \frac{n(n-1)}{2}. \quad (8)$$

Proof. The coefficients lie in a fixed finite-dimensional probability simplex. It remains to show stability is closed within that simplex. Let $p_j \rightarrow p$ coefficientwise, with $p_j \in \mathcal{K}_n$. Fix real a and strictly positive real b . The polynomial $z \mapsto p_j(a + zb)$ has only real roots by the stability and conjugation argument. Its leading coefficient satisfies

$$p_j(b) \geq (\min_i b_i)^n > 0.$$

Therefore its coefficient limit still has degree n , and closure of the real-rooted univariate polynomials shows that $z \mapsto p(a + zb)$ has only real roots. If $p(a + ib) = 0$, this polynomial would have the nonreal root i . As a, b were arbitrary, p is stable. The other defining conditions are closed, proving compactness.

Continuity follows from coefficient differentiation and (3). Nonnegative coefficients give $H_{ii} \geq 0$, where $H = \nabla^2 p(\mathbf{1})$. Also $t_i \geq 0$ and $\sum_i t_i = n$, so

$$d = \frac{n-1}{2n} \|t\|_2^2 - \frac{1}{2} \text{Tr } H \leq \frac{n-1}{2n} n^2 = M_n.$$

$\square$

Proposition 4.2. *There exist $\varepsilon_n, b_n > 0$ such that*

$$\mathcal{A}(p) - F(t) \geq b_n d(p) \quad (9)$$

for every $p \in \mathcal{K}_n$ with $\|t - \mathbf{1}\|_1 < \varepsilon_n$.

Proof. Take ε_n no larger than $1/(8\sqrt{n})$. Lemma 3.2 handles $d \leq \eta_n$. Consider the compact set

$$E = \{p \in \mathcal{K}_n : t(p) = \mathbf{1}, d(p) \geq \eta_n\}.$$

If $E \neq \emptyset$, Theorem 1.1 and Lemma 2.1 give

$$m_n := \min_{p \in E} (\mathcal{A}(p) - F(\mathbf{1})) > 0.$$

For a sufficiently small gradient neighborhood, every $p \in \mathcal{K}_n$ with $d(p) \geq \eta_n$ has $\mathcal{A}(p) - F(t) \geq m_n/2$. Otherwise a sequence with $t \rightarrow \mathbf{1}$ would have a convergent subsequence with limit in E and gap at most $m_n/2$, a contradiction. By (8), take $b_n = \min\{a_n/2, m_n/(2M_n)\}$.

If $E = \emptyset$, compactness instead implies that in a sufficiently small gradient neighborhood there is no $p \in \mathcal{K}_n$ with $d(p) \geq \eta_n$: a contrary sequence would converge to a member of E . Then $b_n = a_n/2$ suffices. $\square$

Lemma 4.3. *There is $K_n > 0$, depending only on n , such that*

$$\|p - L_t^n\|_{\text{coef},1} \leq K_n d(p) \quad (p \in \mathcal{K}_n).$$

Proof. For real $\|x\|_2 \leq 1$, nonnegativity and $\sum_i t_i = n$ give $|L_t(x)| \leq \|t\|_2/n \leq 1$. Using the exact quadratic term in (2) yields

$$|p(x) - L_t(x)^n| \leq d + \sum_{k=3}^n \binom{n}{k} (2d)^{k/2} \leq K'_n d,$$

because $0 \leq d \leq M_n$. For example, one can take $K'_n = 1 + \sum_{k=3}^n \binom{n}{k} 2^{k/2} M_n^{k/2-1}$.

On the finite-dimensional space of real homogeneous degree- n polynomials, coefficient ℓ^1 norm is bounded by a constant times the supremum norm on the Euclidean unit ball. For completeness, on the compact coefficient- ℓ^1 unit sphere the latter norm is continuous and strictly positive: a nonzero polynomial cannot vanish on the entire ball. It therefore has a positive minimum. This proves the required norm comparison, and hence the lemma. $\square$

Proof of Theorem 1.2. For $n \geq 2$, combine Proposition 4.2 and Lemma 4.3, taking $c_n = b_n/K_n$. The power L_t^n is stable, has nonnegative coefficients, value one at $\mathbf{1}$, and gradient t . Equality in the minimum forces $\|p - L_t^n\|_{\text{coef},1} = 0$. For $n = 1$, normalization forces $p = x_1$, so the theorem holds with any positive constants. $\square$

5 Optimal distance order and a probability formulation

Proposition 5.1. *For every $n \geq 2$, no positive uniform lower bound by $\|p - L_t^n\|_{\text{coef},1}^\alpha$ with $\alpha < 1$ can replace the right side of (1), even when $t = \mathbf{1}$.*

Proof. Put $L = L_1$, $a = x_1 - x_2$, and, for $0 < s < 1/n$, let

$$p_s = L^{n-2}(L + sa)(L - sa) = L^n - s^2 L^{n-2} a^2.$$

Every linear factor has nonnegative coefficients and positive coefficient sum, so the product is stable. Moreover, $p_s(\mathbf{1}) = 1$ and $\nabla p_s(\mathbf{1}) = \mathbf{1}$. Only the term $-2x_1x_2$ in a^2 can contribute to the squarefree coefficient, giving

$$\mathcal{A}(p_s) - \mathcal{A}(L^n) = \frac{2(n-2)!}{n^{n-2}} s^2, \quad \|p_s - L^n\|_{\text{coef},1} = s^2 \|L^{n-2} a^2\|_{\text{coef},1} > 0.$$

The ratio of the gap to this distance raised to $\alpha < 1$ tends to zero. This proves the assertion. $\square$

This establishes optimal order, not the best constant in Theorem 1.2.

Corollary 5.2. *Let N take values in $\{\alpha \in \mathbb{Z}_{\geq 0}^n : \sum_i \alpha_i = n\}$, and suppose its probability generating polynomial $p(x) = \mathbb{E}[x^N]$ is real stable. Put $t = \mathbb{E}N$, and let M_t have the multinomial count law for n independent draws with probabilities t_i/n . Under the gradient condition of Theorem 1.2,*

$$\mathbb{P}(N = \mathbf{1}) - \frac{n!}{n^n} \prod_i t_i \geq 2c_n d_{\text{TV}}(\mathcal{L}(N), \mathcal{L}(M_t)).$$

Proof. The coefficients of p are the probabilities of N , $p(\mathbf{1}) = 1$, and $\nabla p(\mathbf{1}) = \mathbb{E}N = t$. The generating polynomial of M_t is L_t^n . Coefficient ℓ^1 distance is twice total variation, where $d_{\text{TV}}(\mu, \nu) = \frac{1}{2} \sum_{\alpha} |\mu(\alpha) - \nu(\alpha)|$. The result is precisely (1). $\square$

The unique minimizing object here is a count law, not an underlying labeled sampling mechanism. Differentiation gives a further interpretation of the dispersion matrix:

$$2B = \text{diag}(t) - \frac{tt^T}{n} - \text{Cov}(N) = \text{Cov}(M_t) - \text{Cov}(N). \quad (10)$$

Indeed, $\nabla^2 p(\mathbf{1}) = \text{Cov}(N) + tt^T - \text{diag}(t)$; substitution into (3) proves (10). This interpretation does not by itself replace the coefficient comparison.

6 Exact local geometry and a global obstruction

The nearbalanced global conclusion does not identify the full gradient region in which the power is a local minimizer. We now classify strict local minimality in the entire stable-polynomial class. The corresponding matrix Hessian criterion already appears in [8]; the point here is that centered-root control suffices without a matrix representation.

Proposition 6.1. *Fix $n \geq 2$ and $t_i > 0$ with $\sum_i t_i = n$, and set*

$$\gamma(t) = \frac{n^2}{\|t\|_2^2} - (n-1).$$

In the coefficient topology on normalized nonnegative homogeneous real-stable polynomials with gradient t , the power L_t^n is a strict local minimizer of $\mathcal{A}$ if and only if $\gamma(t) > 0$. If $\gamma(t) < 0$, it is not a local minimizer. If $\gamma(t) = 0$, a nonconstant stable product curve through L_t^n has the same squarefree coefficient throughout.

Proof. Let $T = \text{diag}(t)$ and $S = T^{-1}BT^{-1}$. Then $S \geq 0$ and $St = 0$. Define

$$r = \mathbf{1} - \frac{nt}{\|t\|_2^2}, \quad \|r\|_2^2 = 1 - \gamma(t).$$

The exact coefficient identity of Lemma 3.1 becomes

$$A_2 := \mathcal{A}(L_t^{n-2}q_2) = F(t) \frac{n}{n-1} (\text{Tr } S - r^T S r).$$

In particular, the bracket is the negative sum of all ordered off-diagonal entries $B_{ij}/(t_i t_j)$; its diagonal cancels. Since $r^T S r \leq \|r\|_2^2 \text{Tr } S$,

$$A_2 \geq F(t) \frac{n}{n-1} \gamma(t) \text{Tr } S \geq \alpha(t) d, \quad \alpha(t) = \frac{F(t)n\gamma(t)}{(n-1)(\max_i t_i)^2},$$

when $\gamma(t) > 0$. The cube estimate in the proof of Lemma 3.2 required only $t_i \geq 0$ and $\sum_i t_i = n$. It therefore gives, for $d \leq 1$,

$$|\mathcal{A}(p) - F(t) - A_2| \leq C_n d^{3/2}.$$

For $n \geq 3$, put $\eta(t) = \min\{1, (\alpha(t)/(2C_n))^2\}$. For $n = 2$, put $\eta(t) = 1$, since the remainder vanishes. If $d \leq \eta(t)$, then

$$\mathcal{A}(p) - F(t) \geq \frac{1}{2}\alpha(t)d.$$

The rigidity $d = 0 \iff p = L_t^n$ and continuity of d establish strict local minimality. Lemma 4.3 also gives local coefficient-distance coercivity with constant $\alpha(t)/(2K_n)$. This is an explicit local dispersion radius, not an explicit global gradient neighborhood.

For the converse, if $\gamma(t) \leq 0$ then $r \neq 0$. Set $b = Tr$, so $b^T \mathbf{1} = t^T r = 0$. For sufficiently small $s > 0$, all coefficients of $L_t \pm sb^T x$ are positive. Hence

$$p_s = L_t^{n-2}(L_t + sb^T x)(L_t - sb^T x) = L_t^n - s^2 L_t^{n-2}(b^T x)^2$$

is feasible and nonconstant. Here $B = s^2 b b^T$, $S = s^2 r r^T$, and there are no higher centered terms. Consequently,

$$\mathcal{A}(p_s) - F(t) = F(t) \frac{n}{n-1} s^2 \gamma(t) \|r\|_2^2.$$

It is negative when $\gamma(t) < 0$ and zero when $\gamma(t) = 0$. □

This criterion is local. Failure of global rank-one minimization away from balance is already known for permanents; see [2, Section 6] and [3]. The following small example shows that even strict local positivity does not provide a global theorem.

Proposition 6.2. *In degree three there are strictly positive stable product polynomials with $\gamma(t) > 0$ but $\mathcal{A}(p) < F(t)$. Moreover, any universal open gradient neighborhood $\|t - \mathbf{1}\|_1 < \varepsilon$ on which the power is always the unique global minimizer must have $\varepsilon \leq 1$.*

Proof. Let $0 < a < 1/2$, $t = (1 + 2a, 1 - a, 1 - a)$, $L = L_t$ and $b^T x = -2x_1 + x_2 + x_3$. Set

$$p_{a,e} = (L - eb^T x)^2 (L + 2eb^T x).$$

All three factors have coefficient sum one, and their gradients sum to t . Their coefficients are strictly positive if $0 < e < \min\{(1 - a)/3, (1 + 2a)/12\}$. Direct expansion gives

$$p_{a,e} = L^3 - 3e^2 L(b^T x)^2 + 2e^3 (b^T x)^3, \quad \mathcal{A}(p_{a,e}) - F(t) = 6e^2(1 - 2a - 4e).$$

For every $1/4 < a < 1/2$, the interval

$$(1 - 2a)/4 < e < (1 + 2a)/12$$

is nonempty and satisfies the positivity constraints, while the displayed gap is negative. Yet $\|t\|_2^2 = 3 + 6a^2 < 9/2$, so $\gamma(t) > 0$.

For example, $a = 1/3$, $e = 1/8$ gives categorical factor rows

$$(29/36, 7/72, 7/72), \quad (29/36, 7/72, 7/72), \quad (1/18, 17/36, 17/36).$$

Here $\gamma = 5/11$, $A_2 = 1/32$, the cubic contribution is $-3/64$, and the total gap is $-1/64$. At $a = 1/4$, $e = 1/8$, allowing one zero factor coefficient, the rows are $(3/4, 1/8, 1/8)$ twice and $(0, 1/2, 1/2)$. This nonpower has the same coefficient as L_t^3 and $\|t - \mathbf{1}\|_1 = 1$. Thus every open uniqueness radius greater than one contains a counterexample. Strictly positive failures also approach this gradient distance from above. □

Neither proposition identifies the complete global minimizer region or proves sufficiency of a degree-three radius equal to one.

7 Quartic examples and scope

For $n = 4$, the expansion is

$$p = L_t^4 - L_t^2 x^T B x + L_t q_3 + q_4.$$

At $t = \mathbf{1}$, the leading squarefree contribution is $d/8$. The remaining contributions are $O(d^{3/2})$ and $O(d^2)$; the constants depend only on degree. Thus the argument controls actual stable quartics without choosing a smooth parameterization of their cone.

For an example outside the determinant class, consider the specialized Vámos polynomial from [7, Section 3]:

$$h(x) = x_1^2 x_2^2 + 4(x_1 + x_2 + x_3 + x_4) e_3(x), \quad h(\mathbf{1}) = 65,$$

where $e_3 = \sum_{i < j < k} x_i x_j x_k$. Its real stability (inherited from the stable Vámos matroid polynomial by specialization) and the obstruction to a definite determinantal representation of any positive integer power are prior results; see [6, 7]. Let J be the all-ones matrix and define

$$T_\delta = (1 - \delta)J/4 + \delta I, \quad p_\delta(x) = h(T_\delta x)/65, \quad 0 < \delta \leq 1.$$

Every row of T_δ is nonnegative with sum one, so it maps vectors with positive imaginary parts to such vectors; hence p_δ is stable. The transformation is invertible and fixes $\mathbf{1}$. A definite determinantal representation of a positive power would pull back to one for a power of h , so the obstruction persists. Its gradient is

$$t_\delta = (1 + \delta/65, 1 + \delta/65, 1 - \delta/65, 1 - \delta/65).$$

Direct expansion gives

$$\begin{aligned} \mathcal{A}(p_\delta) &= \frac{3}{32} + \frac{19}{208}\delta^2 + \frac{3}{52}\delta^3 + \frac{7}{2080}\delta^4, \\ \mathcal{A}(p_\delta) - F(t_\delta) &= \frac{3089}{33800}\delta^2 + \frac{3}{52}\delta^3 + \frac{480593}{142805000}\delta^4 > 0. \end{aligned}$$

These are illustrative identities, not premises of the proof.

Stability cannot simply be omitted. Put $L = (x_1 + x_2 + x_3 + x_4)/4$. For $0 < r < 3/32$, the polynomial

$$q_r = \frac{L^4 - r x_1 x_2 x_3 x_4}{1 - r}$$

has nonnegative coefficients, value one at $\mathbf{1}$, and gradient $\mathbf{1}$, but $\mathcal{A}(q_r) = (3/32 - r)/(1 - r) < 3/32$. Under $x = (z + 1, z - 1, z, z)$ it becomes $z^4 + r z^2/(1 - r)$, which has a nonreal upper-half-plane root. All four corresponding coordinates have positive imaginary part; thus q_r is not stable.

The proof does not extend the perturbed-gradient theorem to every Lorentzian polynomial: real-rooted line restrictions are essential to Lemma 2.1. The results do not classify global minimizers at distant gradients, produce a dimension-uniform global neighborhood, or imply an algorithmic improvement for permanent approximation or TSP. Effective radii and dimension dependence remain natural questions.

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