Prescribed-Gradient Coefficient Stability for Lorentzian Polynomials

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Luminary Oddities

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Abstract

For normalized Lorentzian polynomials of degree n in n variables, we prove that the power of a linear form with prescribed gradient is the unique minimizer of the squarefree coefficient whenever that gradient is sufficiently close to the all-ones vector. The coefficient excess controls the full coefficient ℓ^1 distance, uniformly in each fixed degree. The proof gives an explicit separation bound for the cone determined by the transverse quadratic derivatives at the balanced power. This bound also yields an explicit local stability estimate when both gradient and polynomial are close to that power. Variable splitting extends the result to positive integer compositions, and volume polynomials give a mixed-volume coefficient stability bound.

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1 The prescribed-gradient minimum

The coefficients of a normalized homogeneous polynomial form a probability law on weak integer compositions. Specifying its gradient at $\mathbf{1} = (1, \dots, 1)$ specifies the mean of this law. A natural extremal question asks whether the matching multinomial law minimizes the probability of the central composition. For Lorentzian polynomials, Gurvits and Leake proved the balanced case of this minimum and its uniqueness [1, Theorem 1.4]. Their Question 1.8 asks for a near-balanced prescribed-gradient extension in the real-stable class. The near-balanced prescribed-gradient minimum and a linear coefficient-distance reserve for real-stable polynomials, with effective global constants, are established in [2]. Here we extend both conclusions to the larger Lorentzian class.

We use the closed class of homogeneous Lorentzian polynomials with nonnegative coefficients, as defined in [3]. In degree at least two it is characterized by M-convex support and the condition that every quadratic coordinate derivative has a Hessian with at most one positive eigenvalue. Singular derivatives are included. Throughout the first four sections $n \geq 2$, the degree and number of variables both equal n, and $p(\mathbf{1}) = 1$. Set

$$t = \nabla p(\mathbf{1}), \quad L_t(x) = \frac{t^T x}{n}, \quad L = L_{\mathbf{1}}, \quad v_n = \frac{n!}{n^n}, \quad \mathcal{A}(p) = [x_1 \cdots x_n]p.$$

Euler's identity gives $\sum_i t_i = n$. For a signed polynomial r, write $\|r\|_{\mathrm{coef},1} = \sum_\alpha |[x^\alpha]r|$.

Theorem 1.1. *For every $n \geq 2$ there are $\varepsilon_n, c_n > 0$ such that every normalized Lorentzian p with $\|t - \mathbf{1}\|_1 \leq \varepsilon_n$ satisfies*

$$\mathcal{A}(p) - v_n \prod_i t_i \;\geq\; c_n \, \|p - L_t^n\|_{\mathrm{coef},1} \, . \tag{1}$$

Thus L_t^n is the unique squarefree-coefficient minimizer over the entire normalized Lorentzian class with that prescribed gradient.

The main issue is linear growth of the excess near the minimizer: uniqueness alone does not imply such growth. We prove an explicit linear bound on a convex cone containing all limiting deviations from L^n. The proof uses permutation averaging only on this cone. A transverse Hessian trace controls the full unsymmetrized polynomial before averaging, so cancellation cannot hide a nonzero first-order deviation. Section 4 gives explicit constants on a joint coefficient and gradient neighborhood. The global radius in Theorem 1.1 is obtained by compactness at fixed n. The nonstable cubic in Section 6 shows why a first-order argument must retain cubic as well as quadratic deviations.

Recent bounded-ratio results [4, 5, 6] concern multiplicative coefficient inequalities and compatibility of quadratic slices. The additive estimate here is centered at a prescribed-gradient power. Its proof requires only a local derivative condition at that power, rather than a global generation assertion for bounded ratios.

2 An explicit separation of the derivative cone

Put $E = \mathbf{1}^\perp$ and $P = I - \mathbf{1}\mathbf{1}^T/n$. Let K_n be the closed convex cone of homogeneous degree-n signed polynomials r such that

$$r(\mathbf{1}) = 0, \qquad \nabla r(\mathbf{1}) = 0, \qquad S_\beta := P\nabla^2\partial^\beta r P \preceq 0 \quad (|\beta| = n-2). \tag{2}$$

The Hessians in this definition are constant matrices.

Lemma 2.1. *For $r \in K_n$, put $s(r) = -\operatorname{tr}(P\nabla^2 r(\mathbf{1})P)$. Then*

$$\|r\|_{\mathrm{coef},1} \le \frac{n}{2} s(r). \tag{3}$$

Proof. Homogeneity gives

$$P\nabla^2 r(x)P = \sum_{|\beta|=n-2} S_\beta \frac{x^\beta}{\beta!}, \qquad s(r) = \sum_{|\beta|=n-2} \frac{-\operatorname{tr} S_\beta}{\beta!}.$$

For a negative semidefinite matrix S, $\sum_{i,j}|S_{ij}| \le n(-\operatorname{tr} S)$, by Cauchy–Schwarz applied to its diagonal entries. Define $T_u x = L(x)\mathbf{1} + uPx$ for $0 \le u \le 1$. Each coordinate of $T_u x$ is a nonnegative linear form whose coefficients sum to one. Its monomials therefore have coefficient norm one, so the triangle inequality makes substitution by T_u contract the coefficient norm of any signed polynomial. Taylor's formula along this segment, and the two vanishing conditions in (2), give the polynomial identity

$$r(x) = \int_0^1 (1-u)x^T P\nabla^2 r(T_u x)Px\, du.$$

The sum of the coefficient norms of the matrix entries in the integrand is at most $ns(r)$. Multiplication by $x_i x_j$ preserves each entry's coefficient norm. Integration proves (3). □

Proposition 2.2. *For every $r \in K_n$,*

$$\mathcal{A}(r) \ge a_n \|r\|_{\mathrm{coef},1}, \qquad a_n = \frac{n!}{n^{n+2}(n-1)}. \tag{4}$$

Proof. First let r be invariant under coordinate permutations and write

$$r(x) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} b_\alpha x^\alpha.$$

Testing (2) on $e_i - e_j$ yields

$$b_{\beta+2e_i} + b_{\beta+2e_j} \le 2b_{\beta+e_i+e_j}. \tag{5}$$

For fixed remaining coordinates this is a concave sequence in the exponent of x_i, symmetric about its midpoint. A transfer from a coordinate at least two larger to a smaller coordinate therefore increases, or preserves, b_α. Repeated transfers reach $\alpha = \mathbf{1}$, since n units occupy n coordinates. Thus $b_\alpha \le b_{\mathbf{1}}$ for every α. Their weighted sum is $r(\mathbf{1}) = 0$, so $b_{\mathbf{1}} \ge 0$. The sum of the positive coefficients is at most $n^n b_{\mathbf{1}}$, and their total sum is zero. Consequently

$$\|r\|_{\mathrm{coef},1} \le 2n^n b_{\mathbf{1}} = \frac{2n^n}{n!}\mathcal{A}(r). \tag{6}$$

For arbitrary $r \in K_n$, let $\bar{r}$ be its permutation average. Then $\bar{r} \in K_n$, $\mathcal{A}(\bar{r}) = \mathcal{A}(r)$ and $s(\bar{r}) = s(r)$. Euler's identity gives $\mathbf{1}^T \nabla^2 \bar{r}(\mathbf{1})\mathbf{1} = n(n-1)\bar{r}(\mathbf{1}) = 0$, whence

$$s(r) = -\operatorname{tr} \nabla^2 \bar{r}(\mathbf{1}) \le n(n-1)\,\|\bar{r}\|_{\mathrm{coef},1}\,.$$

Apply Lemma 2.1 to r and (6) to $\bar{r}$. Their combination is $\|r\|_{\mathrm{coef},1} \le n^{n+2}(n-1)\mathcal{A}(r)/n!$, as asserted. In particular, symmetrization cannot annihilate a nonzero element of K_n. □

3 From derivative cones to prescribed gradients

Lemma 3.1. *Suppose p_j are normalized Lorentzian polynomials, $t_j = \nabla p_j(\mathbf{1}) \to \mathbf{1}$, and $d_j = \left\|p_j - L^n_{t_j}\right\|_{\mathrm{coef},1} > 0$ tends to zero. Every coefficient limit of $r_j = (p_j - L^n_{t_j})/d_j$ belongs to K_n and has norm one.*

Proof. The value and gradient of r_j at $\mathbf{1}$ vanish exactly. Fix $|\beta| = n-2$. With $R_j = \nabla^2 \partial^\beta r_j$,

$$H_j := \nabla^2 \partial^\beta p_j = v_n t_j^\beta t_j t_j^T + d_j R_j.$$

For $v \in E$, project to $v_j = v - (t_j^T v)t_j/\|t_j\|_2^2 \in t_j^\perp$. The quadratic form of H_j is positive on $w_j = t_j$ for large j. Its restriction to the span of w_j, v_j has nonpositive determinant, since H_j has at most one positive eigenvalue. Hence

$$d_j v_j^T R_j v_j \le \frac{d_j^2 (w_j^T R_j v_j)^2}{w_j^T H_j w_j}.$$

The denominator stays bounded below and R_j stays bounded. Divide by d_j and take the limit to obtain $v^T R v \le 0$ for every $v \in E$, where $R = \lim_j R_j$. This is the transverse condition; coefficient-norm one passes to the limit. □

Proof of Theorem 1.1. The normalized nonnegative-coefficient Lorentzian class is compact: its coefficients lie in a finite simplex, and the class is closed. If no positive radius and reserve existed, we could choose p_j with $\|t_j - \mathbf{1}\|_1 \le 1/j$, $d_j > 0$, and

$$\frac{\mathcal{A}(p_j) - \mathcal{A}(L^n_{t_j})}{d_j} < \frac{1}{j}.$$

After passage to a subsequence, $p_j \to p_\infty$ with balanced gradient. If $p_\infty \neq L^n$, the balanced uniqueness theorem of Gurvits–Leake gives a strictly positive limiting numerator. This contradicts the displayed inequality, since $d_j \leq 2$. Thus $p_\infty = L^n$ and $d_j \to 0$. Pass to a coefficient limit $r_j \to r$. Lemma 3.1 and Proposition 2.2 give $\mathcal{A}(r) \geq a_n > 0$, again a contradiction. The radius may be decreased below $1/2$ to keep all t_i positive. Finally L_t^n is itself feasible, and $\mathcal{A}(L_t^n) = v_n \prod_i t_i$. □

4 An explicit joint neighborhood

The separation bound also gives a finite-neighborhood estimate without a limiting argument. The following deliberately conservative constants are rational and depend only on n:

$$h = \frac{(n-2)!}{n^{n-2}}, \qquad b = v_n 2^{-(n-2)}, \qquad B = n\,n!,$$
$$J = \frac{2B^2}{bn}, \qquad V = 2(n-1), \qquad D = a_n V + n v_n,$$
$$d_* = \min\left\{\frac{b}{2n!}, \frac{h a_n}{2JD}\right\}, \qquad \delta_* = \min\left\{\frac{1}{2}, \frac{h a_n}{4BD}\right\}.$$

Proposition 4.1. *If p is normalized Lorentzian, $\|t - \mathbf{1}\|_1 \leq \delta_*$ and $\|p - L_t^n\|_{\mathrm{coef},1} \leq d_*$, then*

$$\mathcal{A}(p) - v_n \prod_i t_i \geq \frac{a_n}{2} \|p - L_t^n\|_{\mathrm{coef},1}.$$

Proof. Write $d = \|p - L_t^n\|_{\mathrm{coef},1}$. The assertion is immediate for $d = 0$. Otherwise let $r = (p - L_t^n)/d$ and $\delta = \|t - \mathbf{1}\|_1$. For $|\beta| = n-2$, put $R = \nabla^2 \partial^\beta r$. Every entry of R is a coefficient of r multiplied by a factorial at most $n!$, so $\|R\|_{\mathrm{op}} \leq B$. In $H = v_n t^\beta t t^T + dR$, the scalar $v_n t^\beta$ is at least b, and $\|t\|_2^2 \geq n$. The unit vector $w = t/\|t\|_2$ therefore satisfies $w^T H w \geq bn/2$. The determinant argument in Lemma 3.1 gives $v^T R v \leq Jd\|v\|_2^2$ for $v \perp t$. For a unit vector $v \in E$ its projection v_t onto $t^\perp$ has $\|v - v_t\|_2 \leq \delta/\sqrt{n}$ and $\|v_t\|_2 \leq 1$. Thus

$$PRP \preceq \eta P, \qquad \eta = Jd + 2B\delta.$$

This estimate retains both the coefficient perturbation and rotation of the positive direction.

Define $r_0(x) = -L(x)^{n-2} x^T P x$. Its value and gradient vanish at $\mathbf{1}$, $\|r_0\|_{\mathrm{coef},1} \leq V$, and $\mathcal{A}(r_0) = n v_n$. For $v, w \in E$, differentiating first in these directions shows

$$v^T \nabla^2 \partial^\beta r_0 w = -2h\, v^T w.$$

Consequently $r + k r_0 \in K_n$ for $k = \eta/(2h)$. By Proposition 2.2 and the reverse triangle inequality,

$$\mathcal{A}(r) \geq a_n(1 - kV) - k n v_n = a_n - kD.$$

The coefficient and gradient restrictions give, respectively, $Jd \leq h a_n/(2D)$ and $2B\delta \leq h a_n/(2D)$. Hence $kD = \eta D/(2h) \leq a_n/2$, and multiplication by d proves the claim. □

5 Compositions and mixed volumes

Corollary 5.1. *Fix positive integers $k_1, \ldots, k_m$ with sum $d \geq 2$. There are $\varepsilon, c > 0$ such that every degree-d Lorentzian polynomial p in m variables with $p(\mathbf{1}) = 1$ and $\|\nabla p(\mathbf{1}) - k\|_1 \leq \varepsilon$ satisfies*

$$[x^k]p - \frac{d!}{\prod_i k_i!} \prod_i (t_i/d)^{k_i} \geq c \left\| p - (t^T x/d)^d \right\|_{\mathrm{coef},1}, \qquad t = \nabla p(\mathbf{1}).$$

One may take $\varepsilon = \varepsilon_d$ and $c = c_d / \prod_i (k_i!/k_i^{k_i})$ from Theorem 1.1.

Proof. Replace x_i by the average of k_i new variables. The resulting q is Lorentzian by nonnegative substitution [3, Theorem 2.10], has degree d in d variables and has mean t_i/k_i on group i. Its mean error is exactly $\|t-k\|_1$. Only the original monomial x^k contributes to the squarefree coefficient, with multiplier $C_k = \prod_i k_i!/k_i^{k_i}$. Moreover, this substitution preserves the coefficient norm of any signed polynomial: different original monomials have disjoint group-multidegree supports, and their expansion weights are nonnegative and sum to one. Apply Theorem 1.1 and divide by C_k. □

For the coefficient law P_p and the matching multinomial law M_t, $\left\|p-(t^Tx/d)^d\right\|_{\mathrm{coef},1} = 2d_{\mathrm{TV}}(P_p,M_t)$. Thus the excess probability of the fixed composition in Corollary 5.1 controls the full law at fixed degree.

For a geometric consequence, let $K_1,\dots,K_n$ be convex bodies in $\mathbb{R}^n$, set $K=K_1+\cdots+K_n$, and assume $W=\mathrm{Vol}(K)>0$. Write $V_\alpha = V(K_1[\alpha_1],\dots,K_n[\alpha_n])$ for the standard mixed volumes.

Corollary 5.2. *Put $t_i = nV(K_i,K[n-1])/W$. If $\|t-\mathbf{1}\|_1\le\varepsilon_n$, then*

$$\frac{n!V(K_1,\dots,K_n)}{W} - v_n\prod_i t_i \ge c_n\sum_{|\alpha|=n}\frac{n!}{\alpha!}\left|\frac{V_\alpha}{W}-\prod_i(t_i/n)^{\alpha_i}\right|.$$

Proof. The normalized volume polynomial $p(x)=\mathrm{Vol}(\sum_i x_iK_i)/W$ is Lorentzian by [3, Theorem 4.1]. Its coefficients are $(n!/\alpha!)V_\alpha/W$, and its gradient is the stated t. Substitute into (1). □

6 A nonstable family and the first-order mechanism

An explicit cubic illustrates why a derivative argument is useful beyond the real-stable setting. In three variables put $r=x_1-x_2$ and, for $0<\delta<1/1000$, define

$$p_\delta = L^3-\delta Lr^2+\frac{\delta}{18}r^3.$$

This is normalized and balanced. To verify Lorentzian membership, put $u=L+r/3$, $v=L-r/3$, $D=9\delta$ and $C=1/6$. The normalized binary coefficients of $8p_\delta$ in u,v are

$$b_0=1-D+CD,\quad b_1=1+D/3-CD,\quad b_2=1+D/3+CD,\quad b_3=1-D-CD.$$

They are positive, and

$$b_1^2-b_0b_2 = 4D(1/3-C)+8D^2/9>0,\quad b_2^2-b_1b_3=4D(1/3+C)+8D^2/9>0.$$

The binary ultra-log-concavity criterion and nonnegative substitution from [3] prove membership. In a direction y with $L(y)=0$ and $r(y)=1$, the line restriction is $p_\delta(z\mathbf{1}+y)=z^3-\delta z+\delta/18$. Its discriminant $\delta^2(4\delta-1/12)$ is negative, so p_δ is not real stable. The squarefree excess is $2\delta/3$. The finite rational checks in `verify_tangent.py` test the separation estimates, the repair polynomial, this family and the splitting identities. The universal conclusions follow from the analytic proofs above.

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