Positive Operator Cocycles in Shrinking Corridors

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Abstract

We characterize trace-norm convergence of positive operator sequences whose cocycle subadditivity is required only near the diagonal. For a positive trace-preserving map on trace class, every such sequence has convergent normalized slopes exactly when the map is strongly mean ergodic and the scalar corridor's square-width integral diverges. The scalar width threshold is inherited; the operator transfer combines recurrent-support absorption, compact positive orbits and a regularized root-fidelity energy with a linear trace-slack bound. The criterion covers faithful stationary states and absorbing channels with nonfaithful stationary states. A summable dyadic trace-error condition gives robustness for positive errors with varying directions. We also prove a finite-dimensional selfadjoint criterion, recover reducing-corner convergence, and obtain the unitary classification by pure-point spectrum and square width. Explicit examples describe infinite-dimensional escape and the role of trace preservation. The inequalities are imposed in both orders on a measurable nonincreasing relative-width corridor with eventual adjacent integer access. Strong mean ergodicity means pointwise trace-norm convergence of the Cesaro averages on every trace-class input. The proof identifies a recurrent support carrying a faithful stationary density and controls transient mass and coherences before synchronizing the normalized slopes.

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1 Restricted subadditivity and the convergence question

Let $\mathcal{H} \neq \{0\}$ be separable and let $\mathcal{S}_1(\mathcal{H})$ denote its trace class. Let Φ be a bounded positive trace-preserving linear map on $\mathcal{S}_1(\mathcal{H})$. We study sequences satisfying

$$A_{n+m} \leq A_n + \Phi^n(A_m) \tag{1}$$

for sufficiently large ordered pairs with

$$|m - n| \leq \min(m, n) w(\log \min(m, n)). \tag{2}$$

Here $w : [T, \infty) \to [0, 1]$ is measurable and nonincreasing, and $nw(\log n) \geq 2$ eventually. The order is Loewner order; both orders of every admissible pair are required. Write $X_n = A_n/n$ and

$$\int_T^\infty w(t)^2 \, dt = \infty. \tag{3}$$

Our main theorem identifies precisely the maps for which every positive corridor sequence has convergent slopes: strong mean ergodicity of Φ, together with (3), is necessary and sufficient. A property tested by additive averages therefore controls the larger class of ordered restricted-subadditive inputs.

Scalar input and operator transfer. The scalar corridor theorem [9] supplies the width threshold. Eventual real subadditivity on (2) and (3) imply $a_n/n \to L \in \mathbb{R} \cup \{-\infty\}$. If the square integral is finite, it supplies a nonnegative linearly bounded counterexample with nonconvergent slopes. The integer condition admits the balanced and adjacent splits used below.

Our operator argument first extracts a stationary density faithful on the recurrent support. Mean ergodicity forces absorption into that support; balanced legal expansions transfer compactness and absorption to arbitrary corridor slopes. A regularized root-fidelity energy prices positive order slack linearly by trace. Its strict convexity on the limiting support forces asymptotic invariance, and invariant observables determine the limit. This includes transient absorption when a faithful stationary density on the entire space is unavailable.

Classical predecessors and the restricted-pair increment. Derriennic and Krengel [2, Theorems 4.6 and 5.2] are direct antecedents. Their Theorem 4.6 transfers convergence of positive additive averages to all-pair positive subadditive processes in uniformly monotone Banach lattices. Its first-input domination and lattice weak-compactness argument are unavailable for a shrinking corridor. Their Theorem 5.2 gives commutative L^1 convergence for a positive contraction with a strictly positive invariant element; its fixed-small-block concatenations likewise require pairs excluded by (2). Trace-class Loewner order is not a Banach-lattice order. Jajte [4] develops noncommutative subadditive ergodic theory with all-pair inequalities for trace-preserving automorphisms. The present contribution is the restricted-pair noncommutative transfer: balanced legal leaves, absorption and trace-priced fidelity recover convergence in the original trace norm, with strong mean ergodicity as the exact map condition.

The commutative Markov lift in [10] gives near-diagonal L^p convergence under its uniform-integrability criterion. Its balanced-leaf and strict-convexity synchronization motivate the present extension; trace-class order and nonfaithful stationary support require the confinement and fidelity arguments developed here.

Füredi and Ruzsa [3] study scalar restricted subadditivity and defects. Kulikov and Shao [6] prove a Banach-valued norm version of Fekete's lemma and analyze comparable-pair difficulties. Untwisted finite matrix order also scalarizes through quadratic forms and polarization [7]. The twisted restricted inequality (1) is handled here by confinement and trace-paid order slack.

Operator tools and consequences. Positive trace-preserving fidelity monotonicity is classical [12]. Absorption into recurrent domains, normal ergodic projections and stationary-support decompositions are established themes in quantum Markov dynamics [1, Section 4]. Our proof supplies the elementary merely-positive-map support argument needed for the corridor transfer. Related T-admissible processes [8] have a different prescribed input structure.

The main criterion yields faithful-state and reducing-corner cases and absorbing channels beyond both sufficient hypotheses. We also give a finite-dimensional selfadjoint convergence criterion and an exact unitary spectral/width classification. Explicit examples exhibit the roles of trace preservation and infinite-dimensional escape.

2 A mean-ergodic characterization

Say that Φ is strongly mean ergodic if $C_N A = N^{-1} \sum_{j<N} \Phi^j A$ converges in trace norm for every $A \in \mathcal{S}_1$. Write $\mathrm{UC}(\Phi, w)$ for trace-norm convergence of every positive corridor slope A_n/n.

Theorem 2.1. *For w as in Section 1, $\mathrm{UC}(\Phi, w)$ holds if and only if Φ is strongly mean ergodic and (3) holds. Every limit F is positive, satisfies $\Phi F = F$, and has trace $\lim_n \operatorname{Tr} X_n$.*

2.1 Stationary support and confinement

Positivity and trace preservation give $\|\Phi H\|_1 \le \operatorname{Tr} H_+ + \operatorname{Tr} H_- = \|H\|_1$ for selfadjoint H. For $B \ge 0$ and an orthogonal projection R we use

$$\|B - RBR\|_1 \le 2\sqrt{\operatorname{Tr} B \operatorname{Tr}((I-R)B)}. \tag{4}$$

Indeed $B - RBR = (I-R)B + RB(I-R)$; factor each term through $B^{1/2}$ and apply the Hilbert–Schmidt product inequality.

Lemma 2.2. *A faithful stationary density implies relative compactness of every positive orbit in trace norm.*

Proof. For finite spectral projections R_r of the density σ, $R_r A R_r \to A$ in trace norm and $R_r A R_r \le c_r \sigma$. The whole orbit of each compression lies in $[0, c_r\sigma]$. This interval is compact: its tails are bounded by $c_r \operatorname{Tr}((I - R_s)\sigma)$, and (4) reduces compactness to bounded finite-dimensional corners. Contraction propagates the compression approximation uniformly along the orbit. □

Lemma 2.3. *Strong mean ergodicity is equivalent to relative compactness of every positive orbit. In that case the mean projection $P = \lim_N C_N$ has range supported on $R = \operatorname{supp}\sigma$ for a stationary density σ. With $Q = I - R$,*

$$\Phi^* Q \le Q, \qquad \operatorname{Tr}(Q\Phi^j A) \longrightarrow 0 \quad (A \ge 0). \tag{5}$$

Proof. If the means converge, their limit P is positive and trace-preserving with $P\Phi = \Phi P = P$. Choose a faithful density ρ and set $\sigma = P\rho$. Its support is R. The finite ρ-spectral compressions A_r of $A \ge 0$ satisfy $A_r \le c_r \rho$ and converge to A. Hence $PA_r \le c_r\sigma$ and PA is R-supported.

Using instead spectral corners of σ proves that Φ preserves R-supported inputs. The positive unital dual satisfies $0 \le \Phi^* Q \le I$ with zero R-corner. Positivity forces zero cross corners, so $\Phi^* Q \le Q$. Thus $q_j = \operatorname{Tr}(Q\Phi^j A)$ decreases, and its Cesaro means tend to $\operatorname{Tr}(QPA) = 0$. Therefore $q_j \to 0$.

On $R\mathcal{H}$ the restricted map has faithful stationary density σ. Lemma 2.2 makes the orbit of $R\Phi^N AR$ compact. Equation (4) approximates $\Phi^N A$ by that input as $N \to \infty$; contraction propagates the approximation to all later iterates. This proves compactness of the original positive orbit.

Conversely, compact positive orbits give compact closed convex hulls for their Cesaro means. A finite epsilon-net reduces such a hull to an approximating finite-dimensional compact simplex. Every cluster point is fixed, since $\|\Phi C_N A - C_N A\|_1 \le 2 \operatorname{Tr} A/N$. Each $C_N A - A$ belongs to $\operatorname{Ran}(I - \Phi)$, and

$$\operatorname{Fix}\Phi \cap \overline{\operatorname{Ran}(I-\Phi)} = \{0\} :$$

the bounded Cesaro means vanish on the closed range and equal the identity on fixed vectors. All cluster points agree. Jordan decomposition and complex linearity extend convergence to every trace-class input. □

Lemma 2.4. *Under strong mean ergodicity and* (3)*, (X_n) and its full orbit family $\{\Phi^j X_n : j \ge 0, n \ge 1\}$ are relatively compact. Moreover*

$$\operatorname{Tr}(QX_n) \to 0, \qquad \sup_{j \ge 0} \operatorname{Tr}(Q\Phi^j X_n) \to 0. \tag{6}$$

Proof. The scalar theorem gives $\operatorname{Tr} X_n \to L \geq 0$ and bounded traces. Balanced legal splitting into consecutive leaves gives

$$X_N \leq Y_N = \sum_i (k_i/N)\Phi^{b_i} X_{k_i}, \qquad K \leq k_i \leq 2K, \quad b_i = \sum_{h<i} k_h. \tag{7}$$

For fixed sufficiently large K every child gap is at most one. The finite union of leaf orbits has compact closure and compact closed convex hull. The positive slack satisfies

$$\|Y_N - X_N\|_1 \leq \max_{K\leq k\leq 2K} \operatorname{Tr} X_k - \operatorname{Tr} X_N.$$

Choosing K and then N large proves compactness of the slopes. A finite positive net for the slopes, contraction and compactness of each net point's orbit give compactness of the full orbit family.

For fixed K, (5) gives $\operatorname{Tr}(Q\Phi^b X_k) \leq \eta$ for $b \geq J$, uniformly over the finite leaf set. Leaves with $b_i < J$ have total length at most $J + 2K$. If C bounds their traces, (7) gives

$$\operatorname{Tr}(QX_N) \leq C(J + 2K)/N + \eta.$$

Let $N \to \infty$ and then $\eta \to 0$. The uniform orbit assertion follows from $\Phi^* Q \leq Q$. □

2.2 A support-sensitive fidelity energy

In the following lemma σ is the stationary density above, faithful on $R\mathcal{H}$.

Lemma 2.5. *For each $\varepsilon > 0$, the functional*

$$E(X) = -\operatorname{Tr}\sqrt{\sigma^{1/2}(X + \varepsilon\sigma)\sigma^{1/2}}, \qquad X \geq 0,$$

is continuous on trace-bounded positive sets, strictly convex on positive inputs supported on $R = \operatorname{supp}\sigma$, and contractive in the sense $E(\Phi X) \leq E(X)$. Moreover, if $Y \geq X \geq 0$, then

$$0 \leq E(X) - E(Y) \leq c_\varepsilon \operatorname{Tr}(Y - X), \qquad c_\varepsilon = (2\sqrt{\varepsilon})^{-1}. \tag{8}$$

Proof. The quantity whose negative defines E is root fidelity, not squared fidelity [11, 5]. Its monotonicity needs only positivity and trace preservation [12, Sections 3.2–3.3]. Explicitly, for positive trace-class A, B,

$$\operatorname{Fid}(A, B) = \frac{1}{2} \inf_{Z>0} \left(\operatorname{Tr}(AZ) + \operatorname{Tr}(BZ^{-1})\right),$$

where Z ranges over bounded positive invertible operators. Let $\Psi = \Phi^*$, a positive unital map. Its inverse-Jensen inequality gives $\Psi(Z^{-1}) \geq \Psi(Z)^{-1}$, so

$$\begin{aligned}\operatorname{Tr}(\Phi(A)Z) + \operatorname{Tr}(\Phi(B)Z^{-1}) &\geq \operatorname{Tr}(A\Psi(Z)) + \operatorname{Tr}(B\Psi(Z)^{-1}) \\ &\geq 2\operatorname{Fid}(A, B).\end{aligned}$$

Taking the infimum proves monotonicity. The inverse-Jensen inequality for a single Z follows by restricting to its generated commutative algebra and applying the operator-convex inverse inequality. The general variational formula is valid for positive normal functionals and does not require bounded inverses of A or B. Stationarity now gives contraction of E.

Trace-norm continuity follows from the Hilbert–Schmidt square-root inequality and $\operatorname{Fid}(A, B) = \|\sqrt{A}\sqrt{B}\|_1$. Trace concavity of the square root gives convexity. It is strict on distinct sandwich

arguments: the resolvent integral for the square root expresses the concavity gap as an integral of positive inverse-Jensen gaps, with equality only for equal arguments. On $R\mathcal{H}$ the sandwich map is injective because $\sigma^{1/2}$ has dense range there. Convexity on the whole space follows because the sandwich depends only on RXR. All traces in question are finite by $\mathrm{Fid}(X+\varepsilon\sigma,\sigma) \le \sqrt{\operatorname{Tr} X+\varepsilon}$.

For completeness the linear price does not require a bounded inverse of σ. In a finite σ-spectral corner inside R, differentiate the trace square root along $X + t(Y - X)$. Its argument is at least $\varepsilon\sigma^2$. Operator monotonicity of inverse square roots therefore gives

$$\sigma^{1/2}\big[\sigma^{1/2}(X + t(Y - X) + \varepsilon\sigma)\sigma^{1/2}\big]^{-1/2}\sigma^{1/2} \le \varepsilon^{-1/2} I.$$

The derivative is half the trace pairing with $R(Y - X)R$, whose trace is at most $\operatorname{Tr}(Y - X)$. No inverse on $R^{\perp}$ is used. Integration gives (8), uniformly in the corner. Sigma-spectral compression and trace-norm continuity of fidelity pass to the full space. Order monotonicity gives the lower inequality. □

2.3 Proof of the characterization

Proof of Theorem 2.1. Assume strong mean ergodicity and (3). For an admissible pair put $\alpha = n/(n + m)$ and $Y = \alpha X_n + (1 - \alpha)\Phi^n X_m \ge X_{n+m}$. Price, convexity and contraction show that

$$b_n = nE(X_n) + c_\varepsilon \operatorname{Tr} A_n$$

is exactly scalar corridor-subadditive. Its normalized slopes are bounded, so the scalar theorem and trace convergence give $E(X_n) \to \ell \in \mathbb{R}$.

On the three pairs $(n, n), (n, n + 1), (n + 1, n)$ the trace slack tends to zero. Also

$$\begin{aligned} E(X_{n+m}) - c_\varepsilon \operatorname{Tr}(Y - X_{n+m}) &\le E(Y) \\ &\le \alpha E(X_n) + (1 - \alpha)E(\Phi^n X_m) \\ &\le \alpha E(X_n) + (1 - \alpha)E(X_m). \end{aligned}$$

Both extremes tend to ℓ, so the actual Jensen gap vanishes. The arguments lie in the compact orbit family of Lemma 2.4; all their cluster points as $n, m \to \infty$ are R-supported by (6). With $\alpha \to 1/2$, strict convexity there yields $\|X_n - \Phi^n X_m\|_1 \to 0$ on each pair. Otherwise a convergent subsequence would supply distinct R-supported arguments with zero gap.

Denote the three distances by r_n, s_n, t_n. Contractivity gives

$$\|X_n - \Phi^{2n} X_n\|_1 \le 2r_n, \qquad \|X_n - \Phi^{2n+1} X_n\|_1 \le s_n + t_n.$$

Comparison of consecutive powers yields $\|\Phi X_n - X_n\|_1 \le 2r_n + s_n + t_n \to 0$. Every cluster point is fixed. For each bounded positive observable B, P^*B is positive and Φ^*-fixed. Thus $\operatorname{Tr}((P^*B)A_n)$ is scalar corridor-subadditive with bounded nonnegative normalized slopes and has a limit. On a fixed cluster point F it equals $\operatorname{Tr}(BF)$. These observables separate selfadjoint trace class, so all cluster points agree. Compactness proves convergence; trace continuity identifies its trace.

Conversely, for each $B \ge 0$, the positive additive sequence $A_n = \sum_{j<n} \Phi^j B$ satisfies the full cocycle equality. Universal convergence implies mean ergodicity on positive inputs, hence on all trace class by linear decomposition. It supplies a stationary density $\sigma = P\rho$. If the square-width integral is finite, the inherited nonnegative scalar corridor counterexample a_n makes $A_n = a_n\sigma$ a nonconvergent positive corridor sequence. This proves both necessities for each fixed map. □

Corollary 2.6. *A positive trace-preserving map with a faithful stationary density has universal positive corridor convergence under* (3)*.*

Proof. Lemmas 2.2 and 2.3 give mean ergodicity; apply Theorem 2.1. □

3 Positive trace-error stability

The criterion also supports order errors with a summable normalized tail. The scalar correction is classical nearly-subadditive theory [3, Section 4]; the statement below retains operator errors with varying directions.

Theorem 3.1. *Assume strong mean ergodicity and* (3). *Let* $\epsilon : [1,\infty) \to [0,\infty)$ *be nonincreasing and suppose*

$$h(t) = \sum_{j\geq 0} \epsilon(2^j t) < \infty$$

for all sufficiently large t. *Suppose positive* A_n *satisfy, on all sufficiently large ordered admissible pairs,*

$$A_{n+m} \leq A_n + \Phi^n A_m + D_{n,m}, \qquad D_{n,m} \geq 0, \quad \operatorname{Tr} D_{n,m} \leq (n+m)\epsilon(n+m).$$

Then A_n/n *converges in trace norm to a positive* Φ*-fixed operator, with trace* $\lim_n \operatorname{Tr}(A_n/n)$.

Proof. The tail $h(t)$ tends to zero. For $n \leq m \leq 2n$ and $N = n+m$,

$$\begin{aligned} nh(n) + mh(m) - Nh(N) &\geq n[h(n) - h(2n)] \\ &= n\epsilon(n) \geq N\epsilon(N)/3. \end{aligned}$$

Thus $a_N \leq a_n + a_m + cN\epsilon(N)$ is repaired to exact scalar corridor-subadditivity by adding $3cnh(n)$. Applied to traces, this gives $t_n = \operatorname{Tr} X_n \to L \geq 0$ and bounded traces.

Balanced expansion down to leaves in $[K, 2K]$ gives $X_N \leq Y_N + Z_{N,K}$, where Y_N is the leaf sum in (7), $Z_{N,K} \geq 0$, and

$$\operatorname{Tr} Z_{N,K} \leq h(K/2), \qquad K \geq 2. \tag{9}$$

To verify the bound, charge each internal-node error of size s to its descendant leaves with weights k_i/N, using that their lengths sum to s. On any leaf path the jth ancestor has size at least $2^j(K-1)+1 \geq 2^j K/2$. Each path therefore has total normalized charge at most $h(K/2)$; the leaf weights sum to one. Trace preservation pays every propagated error, including uneven tree depths.

The positive slack now gives

$$\|X_N - Y_N\|_1 \leq \max_{K\leq k\leq 2K} t_k - t_N + 2h(K/2).$$

The compact-hull argument of Lemma 2.4 proves compactness of the slopes and their full orbit family. With R, Q from Lemma 2.3, its absorption argument gives

$$\operatorname{Tr}(QX_N) \leq C(J+2K)/N + \eta + h(K/2).$$

First let $N \to \infty$, then $\eta \to 0$, and finally $K \to \infty$. Thus (6) persists.

Fix a fidelity regularization parameter $\delta > 0$, distinct from ϵ, and use the energy E of Lemma 2.5, with $c_\delta = (2\sqrt{\delta})^{-1}$. For one pair put $Y_0 = (n/N)X_n + (m/N)\Phi^n X_m$ and $e = D_{n,m}/N$. Since $Y_0 + e \geq X_N$, order monotonicity and the price give

$$E(X_N) \leq E(Y_0) + c_\delta[\operatorname{Tr} Y_0 + \operatorname{Tr} e - \operatorname{Tr} X_N].$$

Consequently $b_n = nE(X_n) + c_\delta \operatorname{Tr} A_n$ satisfies

$$b_N \leq b_n + b_m + c_\delta N\epsilon(N).$$

The scalar tail repair makes this exact and proves convergence of $E(X_n)$. On the three adjacent pairs, $\operatorname{Tr} e \to 0$ and $\operatorname{Tr}(Y_0 + e - X_N) \to 0$, hence $\|X_N - Y_0\|_1 \to 0$. Continuity gives the same limiting energy for Y_0. Its Jensen gap tends to zero as in the proof of Theorem 2.1. Compactness, absorption and strict convexity on R imply asymptotic invariance of X_n.

Finally every positive invariant observable P^*B has scalar defect at most $\|B\|N\epsilon(N)$. The same tail repair gives convergence of its normalized values. Fixed cluster points are separated by these observables, proving uniqueness and full convergence. □

Proposition 3.2. *An $o(n+m)$ positive trace-error bound alone permits nonconvergent slopes, even for a stationary rank-one input and the full comparable-pair window.*

Proof. Let σ be a stationary density and set $f(n) = 2 + \sin(\log\log n)$, $A_n = nf(n)\sigma$ for $n \geq 3$. For $n \leq m \leq 2n$, $N = n + m$, the derivative bound $|f'(x)| \leq 1/(x \log x)$ gives

$$|f(N) - f(n)|,\ |f(N) - f(m)| \leq \frac{\log 3}{\log n}.$$

Thus the order inequality holds with

$$D_{n,m} = \frac{N \log 3}{\log(N/3)}\sigma$$

eventually. Its trace is $O(N/\log N) = o(N)$, while $f(n)$ has distinct limiting extrema 1 and 3. The finite prefix can be chosen arbitrarily positive. □

4 Finite-dimensional positive maps

Theorem 4.1. *Let $\Phi : M_D(\mathbb{C}) \to M_D(\mathbb{C})$ be positive and trace-preserving. Let selfadjoint A_n satisfy* (1) *and assume* (3)*. Then X_n converges in operator norm if and only if $\operatorname{Tr} X_n$ has a finite real limit. The limit is fixed by Φ. In particular, convergence is automatic if $A_n \geq 0$ eventually. Neither complete positivity nor unitality is required.*

Proof. For selfadjoint H, positivity and trace preservation give

$$\|\Phi H\|_1 \leq \operatorname{Tr} \Phi H_+ + \operatorname{Tr} \Phi H_- = \|H\|_1.$$

Thus Φ is power-bounded on the real space of selfadjoint matrices. On this finite-dimensional space it admits an equivalent Hilbert norm $\|\cdot\|_*$ in which it is a contraction. Indeed unit-circle Jordan blocks are semisimple; choose a norm making the peripheral real rotation blocks isometric. On the stable spectral subspace put

$$\langle x, y\rangle_s = \sum_{j \geq 0} \langle \Phi^j x, \Phi^j y\rangle_0.$$

The series converges, and its squared norm decreases under Φ. Make the two subspaces orthogonal. The Cesaro means $C_J = J^{-1} \sum_{j<J} \Phi^j$ converge in operator norm to a positive trace-preserving projection P onto $\operatorname{Fix} \Phi$. Also, for a fixed c_*,

$$\|S\|_* \leq c_* \operatorname{Tr} S \qquad (S \geq 0). \tag{10}$$

The ordinary Hilbert–Schmidt norm need not contract for nonunital Φ; the equivalent norm is needed.

Taking traces and applying the scalar input gives $\operatorname{Tr} X_n \to L \in \mathbb{R} \cup \{-\infty\}$. Suppose L is finite. Fix a sufficiently large K. Balanced splitting of N until all leaves lie in $[K, 2K]$ uses child gaps at most one. Iterating (1) gives

$$X_N \le Y_N := \sum_i \frac{k_i}{N} \Phi^{b_i} X_{k_i}, \qquad K \le k_i \le 2K, \quad \sum_i k_i = N, \tag{11}$$

where b_i are the consecutive leaf starts. Trace-norm contraction bounds $\|Y_N\|_1$ uniformly in N. Hence $\lambda_{\max}(X_N)$ is bounded above; the finite lower trace bound then bounds $\lambda_{\min}(X_N)$ below. Thus (X_n) is bounded in every matrix norm.

The sequence PA_n satisfies untwisted order subadditivity since $P\Phi^j = P$. Apply the scalar input to its quadratic forms at $e_i, e_i + e_j, e_i + \mathrm{i}e_j$. Boundedness makes all limits finite, and polarization yields $PX_n \to F$, where $\Phi F = F$ and $\operatorname{Tr} F = L$.

Put $f_n = X_n - F$,

$$\delta_K = \sup_{n \ge K} |\operatorname{Tr} X_n - L|, \qquad \epsilon_K = \sup_{n \ge K} \|PX_n - F\|_*.$$

The positive slack in (11) has trace at most $2\delta_K$. Consequently

$$\|C_J f_N\|_* \le \max_{K \le k \le 2K} \|C_J f_k\|_* + 2c_* \delta_K. \tag{12}$$

Choose N_j realizing $V = \limsup_n \|f_n\|_*^2$. For $t = 0, 1$ take $r_j = \lfloor N_j/2 \rfloor + t$ and $s_j = N_j - r_j$. The child gap is at most two, so both orders are admissible. With $\alpha = r_j/N_j$ and $\beta = s_j/N_j$, trace convergence and (10) give

$$f_{N_j} - \alpha f_{r_j} - \beta \Phi^{r_j} f_{s_j} \longrightarrow 0.$$

The bracket has squared norm tending to V, whereas both child norms have squared limsup at most V. The Hilbert identity

$$\alpha\|x\|_*^2 + \beta\|y\|_*^2 - \|\alpha x + \beta y\|_*^2 = \alpha\beta\|x - y\|_*^2$$

forces the parent to approach both terms. The reversed inequality also makes it approach f_{s_j}. Contractivity therefore gives $\Phi^{r_j} f_{N_j} - f_{N_j} \to 0$. The two values of t provide consecutive exponents, so

$$\|\Phi f - f\|_* \le \|f - \Phi^r f\|_* + \|\Phi^{r+1} f - f\|_*$$

implies $\Phi f_{N_j} - f_{N_j} \to 0$. For fixed J, telescoping gives $C_J f_{N_j} - f_{N_j} \to 0$. Apply (12), let $j \to \infty$, then $J \to \infty$ at fixed K, and finally $K \to \infty$. This gives

$$\limsup_j \|f_{N_j}\|_* \le \epsilon_K + 2c_* \delta_K \longrightarrow 0.$$

Thus $V = 0$ and the entire sequence converges. The converse follows from continuity of trace. For positive A_n, the scalar limit cannot be $-\infty$, and the scalar theorem excludes $+\infty$; hence it is finite and $F \ge 0$. □

Corollary 4.2. *The same conclusion holds for a positive map admitting a faithful invariant functional $\ell(A) = \operatorname{Tr}(SA)$ with $S > 0$, with $\ell(X_n)$ replacing the trace criterion.*

Proof. Set $B_n = S^{1/2} A_n S^{1/2}$ and

$$\Psi(B) = S^{1/2} \Phi(S^{-1/2} B S^{-1/2}) S^{1/2}.$$

Then Ψ is positive and trace-preserving, and congruence preserves (1). Apply Theorem 4.1 and invert the congruence. □

5 Trace tightness from reducing corners

Theorem 5.1. *Let $\Phi : \mathcal{S}_1(\mathcal{H}) \to \mathcal{S}_1(\mathcal{H})$ be positive and trace-preserving. Suppose increasing finite-rank projections $P_r \to I$ strongly satisfy*

$$\Phi(P_r A P_r) = P_r \Phi(A) P_r \quad (A \in \mathcal{S}_1(\mathcal{H})). \tag{13}$$

Under (3)*, every positive trace-class sequence satisfying* (1) *has $X_n \to F$ in trace norm, with $F \geq 0$ and $\Phi F = F$.*

Proof. Taking traces and using the scalar theorem gives $\operatorname{Tr} X_n \to L \geq 0$ and a uniform trace bound C. The leaf inequality (11) remains valid in the original trace class. By (13) and trace preservation,

$$\operatorname{Tr}(P_r \Phi A) = \operatorname{Tr} \Phi(P_r A P_r) = \operatorname{Tr}(P_r A).$$

The same holds for every power and for $I - P_r$. Thus for fixed legal K,

$$\operatorname{Tr}((I - P_r) X_N) \leq \max_{K \leq k \leq 2K} \operatorname{Tr}((I - P_r) X_k), \qquad N \geq K. \tag{14}$$

The right side tends to zero as $r \to \infty$. A finite prefix can be included, so the full sequence is uniformly trace-tight.

For positive trace-class X and an orthogonal projection P,

$$\|X - PXP\|_1 \leq 2\sqrt{\operatorname{Tr} X \, \operatorname{Tr}((I - P)X)}. \tag{15}$$

Indeed $X - PXP = (I-P)X + PX(I-P)$, and factoring each term through $X^{1/2}$ gives the bound by the Hilbert–Schmidt product inequality. This controls off-diagonal coherence without assuming X is block diagonal.

The compressed sequence satisfies the finite-dimensional theorem for the restriction of Φ to $P_r\mathcal{H}$. Each fixed compression therefore converges. Equations (14)–(15) show that (X_n) is trace-norm Cauchy. Its limit is positive and has trace L. Writing $F_r = P_r F P_r$, fixedness of every compressed limit gives

$$P_r(\Phi F - F)P_r = 0.$$

Finite corners recover every trace-class operator in trace norm, so $\Phi F = F$. □

Condition (13) is stronger than invariance of a chosen subspace. It is an identity for the whole map and its source coupling. A numerical finite-section approximation alone does not establish it. The theorem permits coherences in A_n across the reducing sectors and requires no prior uniform-integrability hypothesis.

6 Exact unitary classification

Let $\mathcal{H} \neq \{0\}$ be separable and let U be unitary. Fix a nonincreasing measurable $w : [T, \infty) \to [0, 1]$ with $nw(\log n) \geq 2$ eventually, without yet assuming its square integral diverges. Say that $\mathrm{UC}(U, w)$ holds if every positive trace-class sequence satisfying (1) for $\Phi = \mathrm{Ad}_U$ converges after division by n in trace norm.

Theorem 6.1. *The property $\mathrm{UC}(U, w)$ holds if and only if U has a complete orthonormal eigenbasis and $\int_T^\infty w(t)^2 \, dt = \infty$. In that case the limit commutes with U and has trace $\lim_n \operatorname{Tr} X_n$.*

Proof. If U has a complete eigenbasis, the projections onto its first r basis vectors commute with U and satisfy (13). Theorem 5.1 proves sufficiency.

If U lacks a complete eigenbasis, choose a unit vector x in its nonzero continuous spectral subspace. Its probability spectral measure μ on the unit circle is nonatomic. Set

$$A_n = \sum_{k=0}^{n-1} U^k |x\rangle\langle x| U^{-k}, \qquad X_n = A_n/n.$$

These operators are positive, $\operatorname{Tr} X_n = 1$, and the full cocycle equality holds for every pair. Expansion by the spectral theorem gives

$$\begin{aligned} \|X_n\|_2^2 &= \frac{1}{n^2} \sum_{k,l<n} |\langle U^k x, U^l x\rangle|^2 \\ &= \iint \left| \frac{1}{n} \sum_{k<n} (z\overline{v})^k \right|^2 d\mu(z) d\mu(v) \longrightarrow 0. \end{aligned}$$

The integrand is bounded by one and tends to zero off the diagonal; the diagonal has product measure zero because μ has no atoms. Thus $\|X_n\|_{\mathrm{op}} \to 0$. A trace-norm convergent subsequence would have limit zero but trace one, a contradiction. Therefore no corridor width can repair continuous spectral escape.

Finally suppose U is pure point and $\int w^2 < \infty$. Choose a normalized eigenvector and its rank-one projection B. Then $U^n B U^{-n} = B$. The positive scalar counterexample from the scalar input yields $A_n = a_n B$ satisfying both ordered corridor inequalities with nonconvergent normalized slopes. This proves necessity for every fixed pure-point U, not merely for the identity map. □

Corollary 6.2. *A unitary U is pure point if and only if for every unit vector x the rank-one additive means $n^{-1} \sum_{k<n} U^k |x\rangle\langle x| U^{-k}$ converge in trace norm.*

Proof. Use Theorem 5.1 with $w = 1$ for the forward implication and the continuous spectral construction above for the converse. □

The ordinary rank-one Cesaro characterization and its spectral tools are classical. Theorem 6.1 identifies their role in the universal restricted-pair problem. Pure point here means spectral type, not discreteness of the spectral set or compactness of a resolvent. For a selfadjoint H_0 with a complete eigenbasis, the conclusion applies to $U = e^{-itH_0}$ at each fixed t, even when H_0 has essential spectrum. The finite projections are basis-index projections; they need not be finite-rank energy cuts.

7 Absorbing channels with nonfaithful stationary states

On $\ell^2(\{0, 1, \ldots\})$ choose $p_j \in (0, 1]$, $j \geq 1$, and define Kraus operators

$$K_0 e_0 = e_0, \qquad K_0 e_j = \sqrt{1 - p_j}\, e_j, \qquad K_j = \sqrt{p_j}\, |0\rangle\langle j|.$$

Their adjoint products sum strongly to I, defining a completely positive trace-preserving map. For every $A \geq 0$,

$$\operatorname{Tr}\big((I - |0\rangle\langle 0|)\Phi^n A\big) = \sum_{j \geq 1} (1 - p_j)^n \langle e_j, A e_j\rangle \longrightarrow 0.$$

Dominated convergence and (4) imply $\Phi^n A \to \mathrm{Tr}(A)|0\rangle\langle 0|$ in trace norm. In particular the map is strongly mean ergodic and its unique stationary density is the nonfaithful ground state. Theorems 2.1 and 3.1 therefore give, respectively, exact and trace-error corridor convergence to $L|0\rangle\langle 0|$.

This example also separates the criterion from a reducing finite-corner hypothesis. If a projection P satisfied (13), then $P|0\rangle\langle 0|P$ would be fixed, so either $Pe_0 = 0$ or $e_0 \in \mathrm{Ran}\, P$. A strongly exhaustive increasing family must eventually take the latter case. For any such proper finite P, choose a unit vector $x \in \ker P$. Its excited mass gives

$$\langle e_0, \Phi(|x\rangle\langle x|)e_0\rangle = \sum_{j\geq 1} p_j |\langle e_j, x\rangle|^2 > 0,$$

contradicting (13), whose left side vanishes on this input. Hence no reducing finite-corner exhaustion exists.

The parameters p_j may tend to zero, allowing slow absorption of individual excited levels. Independently, every decreasing $r_n \to 0$ gives the exact stationary cocycle $A_n = n(L + r_n)|0\rangle\langle 0|$. These examples exhibit convergence with arbitrarily slow normalized slopes under the qualitative criterion.

8 Explicit boundaries

8.1 Unitality does not replace a faithful invariant trace

On $M_3(\mathbb{C})$ define $\Phi(A) = a_{00} I_3$. This map is completely positive, unital and idempotent, with Kraus operators $|i\rangle\langle 0|$, $i = 0, 1, 2$, but it is not trace-preserving. Let

$$V_n = \cos(\log n)\sigma_z + \sin(\log n)\sigma_x, \quad X_n = 2 \oplus (I_2 + V_n/5), \quad A_n = nX_n.$$

The Pauli matrices obey $V_n^2 = I_2$. Thus $X_n > 0$, its eigenvalues are $2, 6/5, 4/5$, and $\mathrm{Tr}\, X_n = 4$ for every n. For $n/2 \leq m \leq 2n$, put $N = n + m$ and $\alpha = n/N \in [1/3, 2/3]$. Since $\Phi^n X_m = 2I_3$,

$$\frac{A_n + \Phi^n A_m}{N} = \alpha X_n + (1 - \alpha)2I_3.$$

Its lower block has minimum eigenvalue $2 - 6\alpha/5 \geq 6/5$, whereas the lower block of X_N is at most $6I_2/5$. The top blocks agree. Hence the exact order inequality holds for every comparable pair, including both orders, and thus for every allowed shrinking corridor.

Nevertheless integers nearest $e^{2\pi k}$ and $e^{\pi+2\pi k}$ produce distinct limits of V_n. The sequence does not converge. It is genuinely noncommuting: $[V_n, V_m] = \sin(\log m - \log n)[\sigma_z, \sigma_x]$. Also $\Phi^*(S) = \mathrm{Tr}(S)|0\rangle\langle 0|$ admits no faithful positive fixed density. This counterexample distinguishes the invariant faithful-functional condition of Corollary 4.2 from unitality and an ordinary constant trace slope.

8.2 A channel with metastable escape

There is a normal completely positive trace-preserving map on $\mathcal{S}_1(\ell^2(\mathbb{N}))$ for which positive full-additive cocycles can fail even operator-norm convergence. Set

$$H_j = 2^{2^{j+3}}, \quad p_j = H_j^{-1}, \quad q_j = 1 - p_j,$$

and use Kraus operators $K_j = \sqrt{q_j}|j\rangle\langle j|$ and $L_j = \sqrt{p_j}|j + 1\rangle\langle j|$. Their adjoint squares sum strongly to I. The map kills off-diagonal entries and evolves diagonal states by a pure-birth chain

with geometric holding mean H_j at state j. The positive Kraus series converges in trace norm by monotone trace summation.

Take $B = |1\rangle\langle 1|$ and $A_n = \sum_{k<n} \Phi^k B$. The full additive identity holds and $\operatorname{Tr} X_n = 1$. If T_j is the arrival time at state j, then $\mathbb{E}T_j = \sum_{i<j} H_i = o(H_j)$. Therefore $\mathbb{P}(T_j \le H_j/4) \to 1$. Independently, the chance of holding at j for at least H_j further steps is $q_j^{H_j} \ge 1/4$. At $N_j = H_j$ this gives $(X_{N_j})_{jj} \ge 1/16$ for all sufficiently large j, with rounding absorbed by the slack in the constant. Hence $\limsup \|X_n\|_{\mathrm{op}} \ge 1/16$.

For each fixed j, total expected occupation is H_j, so $(X_n)_{jj} \le H_j/n \to 0$. All matrices are diagonal, and any operator-norm limit would be zero, contradicting the lower bound. Trace-norm convergence also fails. The channel has no nonzero normal stationary state: its fixed diagonal recursion forces every entry to vanish. This classical metastable mechanism, expressed as a channel, shows why the infinite-dimensional structural hypotheses cannot simply be discarded.

8.3 A nonunital channel within the finite theorem

The qubit amplitude-damping channel with Kraus operators $\operatorname{diag}(1, 1/2)$ and $(\sqrt{3}/2)|0\rangle\langle 1|$ is trace-preserving but nonunital. For $B = |+\rangle\langle +|$, its additive cocycle has

$$A_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 11/8 & 3/4 \\ 3/4 & 5/8 \end{pmatrix}, \quad [A_1, A_2] = \begin{pmatrix} 0 & -3/8 \\ 3/8 & 0 \end{pmatrix}.$$

Nevertheless $A_n/n \to |0\rangle\langle 0|$. This additive example exhibits noncommuting inputs and nonunital evolution within Theorem 4.1.

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