Phase transitions and effective curvature bounds in Fourier density models

Scott Schubr

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Abstract

We study the intrinsic Wasserstein metric on affine Fourier-density models on the unit circle. For three distinct frequencies, with one arbitrary fixed phase per frequency, every sectional curvature at the uniform density is strictly negative unless the frequencies have ratio $1:2:3$. In that exceptional family we determine the exact attained curvature range and a phase-dependent sign criterion: the same frequency magnitudes can have mixed curvature or strictly negative curvature solely through relative phase. For the aligned four-frequency family $\{m, 2m, n, n+m\}$, $n > 5m$, we determine the exact attained maximum sectional curvature at every frequency ratio. Two sharp algebraic thresholds separate mixed signs from strict negativity and then mark saturation of the maximum at $-18\pi^2 m^2$. Finally, exact rational Fourier integration with a bounded infinite reciprocal-series remainder certifies nonuniform closed parameter cubes: the $\{m, 2m, 3m\}$ model has both signs throughout $[-.01, .01]^3$, whereas every sectional curvature of the $\{m, 2m, 4m\}$ model is less than $-.98\pi^2 m^2$ there. The phase and four-frequency classifications are analytic; the explicit neighborhood bounds are computer-assisted and accompanied by reproducible exact-arithmetic certificates.

1 Introduction and relation to prior work

A density family can inherit nontrivial intrinsic curvature from its transport metric even when its densities are described by a small number of Fourier coefficients. General pullback metrics, transport natural gradients and the distinction between a model manifold and its ambient Wasserstein space are established subjects [2, 3, 1]. Here we ask how frequency resonance, tangent projection and relative phase determine the sectional curvature of affine Fourier-density models.

For a finite set S of distinct positive integers, fix real phases φ_j and consider

$$f_\theta(x) = 1 + \sum_{j \in S} \theta_j \cos(2\pi j x + \varphi_j), \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad \sum_j |\theta_j| < 1. \tag{1}$$

The parameters are amplitudes. Each phase is held fixed when the metric and curvature are computed. This is not the model with both sine and cosine amplitudes at each frequency, nor a manifold whose phase variables are additional coordinates.

The predecessor [9] classified aligned three-frequency models at their uniform member: the harmonic ratio $1:2:3$ is the sole exception to strict negative sectional curvature. It also gave higher-dimensional spacing conditions and four-dimensional counterexamples to interpreting arbitrary curvature-operator eigenvectors as planes. This paper extends that work in three directions. First, we classify every three-frequency model with arbitrary fixed phases, including an exact sign transition within the exceptional harmonic spectrum. Second, we obtain the exact maximal sectional curvature throughout an infinite resonant four-frequency family, with an attaining plane and two sharp ratio thresholds. Third, we replace the predecessor's existential continuity neighborhoods by explicit closed cubes with quantitative bounds in two benchmark models and all

their integer dilations. The determinant estimates needed from the aligned classification are proved again, making the phase proof self-contained.

Methodological boundaries. The primitive-over-density metric and submanifold viewpoint are prior [2, 3]. The Gauss equation, Fourier orthogonality, interval enclosure and Gershgorin tests are classical. Four-form modification in dimension four is the Finsler–Thorpe mechanism; exact sectional-curvature decision methods are already available in general algebraic form [4, 6, 5]. Our certificates and exact maximum are specializations to a frequency family, rather than a new general decision algorithm. Circle topology requires care: ambient circle Wasserstein geometry is not globally a flat fixed-cut quantile space [7]. We retain the periodic flux correction and justify precisely why it does not affect the metric two-jet at the uniform density in the fixed-phase models.

These results supply exact geometric benchmarks and show how a Fourier parametrization can change intrinsic curvature. They may inform the selection and testing of finite transport models, including circular-distribution models [8]. They do not by themselves imply faster optimization, better statistical accuracy, geodesic convexity of a parameter cube, or finite-bandwidth Sinkhorn curvature bounds.

2 Periodic metric and a local Fourier Gauss formula

Put $k = 2\pi$, $F_j(x) = \cos(kjx + \varphi_j)$ and $B_j(x) = \sin(kjx + \varphi_j)/(kj)$, so $B_j' = F_j$. For periodic potentials satisfying $-(f_\theta \psi_j')' = F_j$, define

$$G_{ij}(\theta) = \int_{\mathbb{T}} f_\theta \psi_i' \psi_j' \, \mathrm{d}x.$$

All integrals use normalized Lebesgue measure on the unit circle. We use the convention that the round sphere has positive sectional curvature.

Lemma 2.1 (Periodic flux and the uniform two-jet)**.** *Let $A_j = \int B_j/f_\theta$ and $J = \int 1/f_\theta$. Then*

$$G_{ij} = \int_{\mathbb{T}} \frac{B_i B_j}{f_\theta} \, \mathrm{d}x - \frac{A_i A_j}{J}. \tag{2}$$

At $\theta = 0$, $A_j = 0$ and every first parameter derivative of A_j vanishes. Hence the second term in (2) is $O(|\theta|^4)$, and G has the same two-jet at zero as $\widetilde{G}_{ij} = \int B_i B_j/f_\theta$.

Proof. Integrating the potential equation gives $f_\theta \psi_j' = -B_j + c_j$. Periodicity gives $\int \psi_j' = 0$ and $c_j = A_j/J$, yielding (2). At zero the sine integral vanishes and

$$\partial_r A_j(0) = -\int_{\mathbb{T}} B_j F_r \, \mathrm{d}x = 0.$$

For $r \neq j$ this follows from distinct-frequency orthogonality, and for $r = j$ from the matching-phase sine–cosine integral. Smoothness while $f_\theta > 0$ gives $A_j = O(|\theta|^2)$. Curvature depends only on the metric two-jet. □

Let $H_S = \mathrm{span}\{\sin(kjx + \varphi_j) : j \in S\} \subset L^2(\mathbb{T})$ and let P_S be its orthogonal projection. In velocity coordinates H_S is the tangent space at zero, with orthonormal basis $e_j = \sqrt{2}\sin(kjx + \varphi_j)$.

Proposition 2.2 (Uniform-density Gauss formula)**.** *For $u, v \in H_S$, set $\mathcal{A}(u, v) = (I - P_S)(uv)'$. For independent u, v, the intrinsic sectional curvature of (1) at zero is*

$$K(u, v) = \frac{\langle \mathcal{A}(u, u), \mathcal{A}(v, v)\rangle - \|\mathcal{A}(u, v)\|_2^2}{\|u\|_2^2 \|v\|_2^2 - \langle u, v\rangle^2}. \tag{3}$$

Writing $a_{ij} = \mathcal{A}(e_i, e_j)/k$, define

$$M_{ij,\ell r} = \langle a_{i\ell}, a_{jr}\rangle - \langle a_{ir}, a_{j\ell}\rangle, \qquad i < j, \quad \ell < r. \tag{4}$$

Then $K(u, v) = k^2 w^T M w/\|w\|^2$, where $w = u \wedge v$ in the orthonormal wedge basis.

Proof. Define $C_\theta(x) = x + \sum_j \theta_j B_j(x)$ on $\mathbb{R}$ and let $X_\theta = C_\theta^{-1}$. Both increasing lifts commute with translation by one. The derivatives X_i are periodic and

$$X_i = -B_i(X_\theta)/f_\theta(X_\theta).$$

The products $X_i X_j$ have period one, so their integrals are independent of the chosen unit interval. Changing variables $t = C_\theta(x)$ gives their $L^2(\,\mathrm{d}t)$ Gram matrix $\widetilde{G}$, even when $C_\theta(0) \neq 0$. Thus the periodic displacements $X_\theta(t) - t$ give a local immersion into an affine Hilbert space with metric $\widetilde{G}$. At zero,

$$X_i = -B_i, \qquad X_{ij} = (B_i B_j)'.$$

Passing to velocity coordinates and taking normal components yields $\mathcal{A}$. The Hilbert-space Gauss equation and Lemma 2.1 give (3). Expanding its numerator gives (4). This is a two-jet comparison, not a global isometry for circle transport space. □

An output $\sin(k\ell x + \alpha)$ at $\ell \in S$ loses its component parallel to $\sin(k\ell x + \varphi_\ell)$; its retained normal component is

$$\sin(\alpha - \varphi_\ell)\cos(k\ell x + \varphi_\ell). \tag{5}$$

An output outside S survives unchanged. Thus arbitrary phases require the tangent projection to be recomputed.

3 Complete three-frequency classification with fixed phases

Theorem 3.1 (Classification and exact exceptional range)**.** *Let $S = \{a, b, c\}$ with distinct positive integers $a < b < c$, and fix arbitrary phases in (1). At the uniform density every sectional curvature is strictly negative unless $S = \{m, 2m, 3m\}$ for a positive integer m.*

In that case translate the circle to write the phases as $(0, p, q)$, with $p = \varphi_{2m} - 2\varphi_m$ and $q = \varphi_{3m} - 3\varphi_m$ modulo 2π. Put

$$D_{12} = \tfrac{1}{2}\big(9\sin^2(p-q) + \sin^2 p\big), \qquad D_{13} = 8 + 2\sin^2(q-p),$$
$$D_{23} = \tfrac{1}{2}\big(25 + \sin^2(q-p)\big), \qquad T = \tfrac{1}{2}\sin p\sin(q-p) - 8\cos(q-2p).$$

The exact attained curvature range is

$$4\pi^2 m^2[\lambda_-, \lambda_+], \qquad \lambda_\pm = -\frac{D_{12} + D_{23}}{2} \pm \frac{\sqrt{(D_{12} - D_{23})^2 + 4T^2}}{2}. \tag{6}$$

There is a positive plane exactly when $T^2 > D_{12}D_{23}$, every plane is strictly negative exactly when $T^2 < D_{12}D_{23}$, and equality gives nonpositive curvature with an attained zero plane. A strictly negative plane exists in every case.

3.1 The harmonic matrix and phase transition

Put $s_\ell(\alpha) = \sin(km\ell x + \alpha)$ and $c_\ell(\alpha) = \cos(km\ell x + \alpha)$. For the ordered frequencies $(m, 2m, 3m)$ the six normal forms are

$$
\begin{aligned}
&a_{11}/m = -2\sin p\, c_2(p), && a_{22}/m = 4s_4(2p), && a_{33}/m = 6s_6(2q),\\
&a_{12}/m = 3\sin(p-q)c_3(q) - \sin p\, c_1(0),\\
&a_{13}/m = 4s_4(q) - 2\sin(q-p)c_2(p),\\
&a_{23}/m = 5s_5(p+q) - \sin(q-p)c_1(0).
\end{aligned}
$$

Differentiate the basis-sine products and apply (5) to obtain these formulas. Orthogonality, with squared sine and cosine norms $1/2$, then gives in wedge order $(12, 13, 23)$

$$
M/m^2 = \begin{pmatrix} -D_{12} & 0 & T \\ 0 & -D_{13} & 0 \\ T & 0 & -D_{23} \end{pmatrix}. \tag{7}
$$

For example, the off-diagonal is $\big(\langle a_{12}, a_{23}\rangle - \langle a_{13}, a_{22}\rangle\big)/m^2$; all other off-diagonal products have disjoint support. The coupled eigenvalues are $\lambda_\pm$. Since

$$
D_{12} \le 5, \qquad 8 \le D_{13} \le 10, \qquad D_{23} \ge 25/2,
$$

we have $\lambda_- \le -25/2 < -D_{13} < -5 \le \lambda_+$. In dimension three every bivector is decomposable, so the entire matrix Rayleigh interval is the attained sectional-curvature interval. The coupled block has negative trace and determinant $D_{12}D_{23} - T^2$. This proves all the harmonic assertions in Theorem 3.1.

Corollary 3.2 (A sharp one-parameter phase boundary)**.** *For phases $(0, t, t)$ on $\{m, 2m, 3m\}$, positive curvature exists if and only if $\cos^2 t > 25/281$; every sectional curvature is strictly negative if and only if $\cos^2 t < 25/281$. Equality gives nonpositive curvature with a zero plane. The exact range is $4\pi^2 m^2[\lambda_-, \lambda_+]$, where*

$$
\lambda_\pm = -\frac{25 + \sin^2 t}{4} \pm \frac{\sqrt{(25 - \sin^2 t)^2 + 1024\cos^2 t}}{4}.
$$

At $t = 0$ this is $\pi^2 m^2[-25 - \sqrt{1649}, -25 + \sqrt{1649}]$; at $t = \pi/2$ it is $\pi^2 m^2[-50, -2]$.

Proof. The coupled block is $\left(\begin{smallmatrix} -\sin^2 t/2 & -8\cos t \\ -8\cos t & -25/2 \end{smallmatrix}\right)$, with determinant $(25 - 281\cos^2 t)/4$. Apply the preceding calculation. □

3.2 Exhaustion and strict determinant estimates for nonharmonic triples

We finish the proof of Theorem 3.1. For distinct frequencies i, j, pure second forms have distinct doubled frequencies, so their mutual inner product vanishes. Consequently $D_{ij} = -M_{ij,ij} = \|a_{ij}\|^2$. Let D^0_{ij} be its aligned-phase value:

$$
D^0_{ij} = \tfrac{1}{2}\big((i+j)^2 \mathbf{1}_{i+j\notin S} + (i-j)^2 \mathbf{1}_{|i-j|\notin S}\big). \tag{8}
$$

Every output outside S has the same norm at all phases; an output in S contributes a nonnegative residual instead of vanishing. Thus $D_{ij} \ge D^0_{ij}$. Both outputs of a distinct pair can be in a three-element spectrum only when $b - a = a$ and $a + b = c$, giving $\{a, 2a, 3a\}$. Hence all D^0_{ij} are positive in the nonharmonic cases.

A nonzero off-diagonal in (4) requires a common output between a doubled mode and a mixed pair, or between two mixed pairs. Solving the sum/difference equalities for three ordered distinct indices gives

$$
c = 2b - a, \qquad c = 2a + b, \qquad c = a + 2b. \tag{9}
$$

Indeed the possible supports are $2i$, $j+\ell$, $|j-\ell|$, $i+j$, $|i-j|$, $i+\ell$ and $|i-\ell|$; equating supports of the products in (4) gives exactly these three relations after ordering. Projection can delete an output, but cannot create a frequency equality. The third relation cannot overlap the others; the first two overlap exactly at $(a, 3a, 5a)$. Without a relation the matrix is strictly negative diagonal.

If $c = a + 2b$, the only off-diagonal is

$$M_{ab,bc} = t_0 \cos(\varphi_c - \varphi_a - 2\varphi_b), \qquad t_0 = (b-a)(3b+a)/2.$$

The shared outputs $a + b$ and $2b$ are outside S. Here $D^0_{bc} = b^2 + c^2 > 5b^2$, and $D^0_{ab} = a^2 + b^2$ except when $b = 2a$, when $D^0_{ab} = 9a^2/2$. In either event $D^0_{ab} \geq b^2$ and $0 < t_0 < 3b^2/2$, so $D_{ab}D_{bc} > t_0^2 \geq M^2_{ab,bc}$.

If $c = 2b - a$, with $b \notin \{2a, 3a\}$, only

$$M_{ab,bc} = t_0 \cos(\varphi_a + \varphi_c - 2\varphi_b), \qquad t_0 = (b-a)^2/2 - 2b^2$$

can be nonzero. Both shared outputs $b - a$ and $2b$ lie outside S. Now $D^0_{ab} = a^2 + b^2$, $D^0_{bc} = b^2 + c^2$, and

$$D^0_{ab}D^0_{bc} - t_0^2 = \tfrac{1}{4}(b-a)^2\big(3(b-a)^2 + 8b^2\big) > 0.$$

The remaining diagonal is positive, even when an output contributing to it is projected away. The omitted case $b = 2a$ is the harmonic exception.

If $c = 2a + b$, with $b \notin \{2a, 3a\}$, only

$$M_{ab,ac} = t_0 \cos(\varphi_c - \varphi_b - 2\varphi_a), \qquad t_0 = (b-a)(b+3a)/2$$

can be nonzero. Its shared outputs $2a$ and $a + b$ lie outside S. Here $D^0_{ab} = a^2 + b^2$, $D^0_{ac} = a^2 + c^2$, and

$$0 < t_0 \leq (a+b)^2/2 \leq D^0_{ab} < D^0_{ac}.$$

Thus $M^2_{ab,ac} < D_{ab}D_{ac}$.

At the overlap $(a, 3a, 5a)$ the two off-diagonal magnitudes are bounded by $6a^2$ and $16a^2$. The diagonal magnitudes are at least $10a^2, 26a^2, 34a^2$. Eliminating the leaf coordinates in $-M$ leaves the strict Schur lower bound

$$a^2\left(10 - \frac{36}{26} - \frac{256}{34}\right) = \frac{240}{221}a^2 > 0.$$

The matrix is therefore negative definite for all phases.

The remaining projected case is $(a, 2a, 4a)$. Translate its phases to $(0, p, q)$. Formula (5) gives

$$M_{ab,ac}/a^2 = 2\sin p \sin(q - 2p) + \tfrac{9}{2}\cos(q - p),$$

whose absolute value is at most $13/2$. The first term is from the shared tangent output $2a$, and the second from the shared normal output $3a$. We have $D_{ab} \geq 9a^2/2$, $D_{ac} = 17a^2$, $D_{bc} > 0$, and hence

$$D_{ab}D_{ac} \geq \tfrac{153}{2}a^4 > \tfrac{169}{4}a^4 \geq M^2_{ab,ac}.$$

Every coupled block is strict negative and the cases exhaust (9). This completes the proof of Theorem 3.1.

4 Exact maximal curvature in a resonant four-frequency family

All phases in this section are zero and all curvatures are at $\theta = 0$. Let

$$S = \{m, 2m, n, n+m\}, \quad m, n \in \mathbb{N}, \quad n > 5m, \quad r = n/m.$$

The actual frequencies are integers; rational r is a normalized ratio. Define

$$\begin{aligned} P(r) &= r^4 - 8r^3 - 20r^2 - 48r - 6, \\ T(r) &= r^4 - 20r^3 - 137r^2 - 174r - 57. \end{aligned} \tag{10}$$

Let ρ and σ be their respective unique positive roots.

Theorem 4.1 (Exact attained maximum)**.** *The root σ is irrational and $25 < \sigma < 26$. Put*

$$h = 2r + 1, \quad a = (r-1)^2, \quad d = 2(r^2 + 2r + 5), \quad A = (r+2)^2, \quad c = A + 3h.$$

The exact maximal sectional curvature of the above family is $4\pi^2 m^2 q(r)$, where $q(r) = -9/2$ for $r > \sigma$. For $5 < r < \sigma$, $q(r)$ is the unique real number $q > -9/2$ satisfying

$$\sqrt{(2q+9)(2q+h^2)} + \sqrt{(2q+a)(2q+d)} = c. \tag{11}$$

Every radical is the nonnegative square root. The maximum is attained by the explicit plane constructed below. Thus (11) is an exact algebraic specification, with a unique branch, rather than a numerical approximation.

Corollary 4.2 (Two sharp ratio transitions)**.** *The root ρ is irrational and $10 < \rho < 11$. For $5 < r < \rho$ both signs of sectional curvature occur; for $r > \rho$ every sectional curvature is strictly negative. The maximum is strictly greater than $-18\pi^2 m^2$ when $5 < r < \sigma$, and equals that value when $r > \sigma$.*

In particular $\{1, 2, N, N+1\}$ has mixed signs for $6 \le N \le 10$, has every sectional curvature strictly negative for $N \ge 11$, and has exact maximum $-18\pi^2$ for $N \ge 26$.

No equality at either algebraic root occurs for integer-frequency models, since r is rational. For orientation only, $\rho \approx 10.3781508748$ and $\sigma \approx 25.6166361007$.

4.1 Three complementary curvature blocks

Let $Q = M/m^2$. Physical curvature is $4\pi^2 m^2$ times its Rayleigh quotient on decomposable bivectors. In wedge order $(12, 13, 14, 23, 24, 34)$, its only entries are the following complementary blocks:

$$\begin{aligned} Q|_{12,34} &= -\tfrac{1}{2}\begin{pmatrix} 9 & 3h \\ 3h & h^2 \end{pmatrix}, \\ Q|_{13,24} &= -\tfrac{1}{2}\begin{pmatrix} a & A \\ A & d \end{pmatrix}, \\ Q|_{14,23} &= -\tfrac{1}{2}\begin{pmatrix} A & a \\ a & 2r^2+8 \end{pmatrix}. \end{aligned} \tag{12}$$

To verify these blocks explicitly, write $s_\ell = \sin(km\ell x)$. The labels $m\ell$ are integer frequencies. The normal forms a_{ij}/m are

$$\begin{aligned} &a_{11}/m = 0, && a_{12}/m = 3s_3, && a_{13}/m = -(r-1)s_{r-1}, \\ &a_{14}/m = (r+2)s_{r+2}, && a_{22}/m = 4s_4, \\ &a_{23}/m = (r+2)s_{r+2} - (r-2)s_{r-2}, \\ &a_{24}/m = (r+3)s_{r+3} - (r-1)s_{r-1}, \\ &a_{33}/m = 2rs_{2r}, && a_{34}/m = (2r+1)s_{2r+1}, && a_{44}/m = (2r+2)s_{2r+2}. \end{aligned}$$

Substitution into (4), with squared sine norm $1/2$, gives (12). The restriction $r > 5$ avoids additional resonances. The incidental equality $r - 2 = 4$ at $r = 6$ occurs only in products with a repeated wedge index and adds no curvature entry.

In four dimensions a nonzero bivector is decomposable if and only if its Plücker expression vanishes:

$$p(w) = w_{12}w_{34} - w_{13}w_{24} + w_{14}w_{23} = 0. \tag{13}$$

Let H be the symmetric matrix whose only nonzero upper-triangular entries are $H_{12,34} = 1/2$, $H_{13,24} = -1/2$, $H_{14,23} = 1/2$. Thus $w^T Hw = p(w)$, and any multiple of H leaves sectional numerators unchanged. This is the classical four-form modification [4, 6, 5]. We prove the required bounds and attainment directly, without appealing to a general optimization equivalence.

4.2 The mixed-sign threshold

The first block of Q is negative semidefinite of rank one. Its kernel vector is

$$z = (h/3, 0, 0, 0, 0, -1), \qquad p(z) = -h/3 \neq 0.$$

Four times the determinant of the third block is

$$2(r+2)^2(r^2+4) - (r-1)^4 = r^4 + 12r^3 + 10r^2 + 36r + 31 > 0.$$

Four times the determinant of the second block is $P(r)$. Descartes' rule gives exactly one positive root; $P(10) = -486$ and $P(11) = 1039$ isolate it. A rational root of this monic integer polynomial would be an integer, impossible in $(10, 11)$.

For $P(r) > 0$, Q is negative semidefinite with kernel exactly $\operatorname{span}\{z\}$. Since the kernel has no nonzero decomposable vector, every actual plane has strictly negative curvature. Compactness of the unit decomposable-bivector set gives a strictly negative maximum for each fixed spectrum, despite the singularity of Q.

For $P(r) < 0$, put $B = r^2 + 2r + 5$ and $t = (r+2)^2/(2B)$. Set $y_{13} = 1$, $y_{24} = -t$, with other entries zero. Then

$$y^T Qy = -P(r)/(4B) > 0, \qquad p(y) = t.$$

With $s = \sqrt{3t/h}$, the vector $w = y + sz$ has $p(w) = t - s^2 h/3 = 0$. Since $Qz = 0$, its numerator stays positive. The 14 coordinate plane has negative curvature, giving both signs. This proves the first part of Corollary 4.2.

For an explicit example, in $S = \{1, 2, 10, 11\}$ take

$$u = (1, 0, 0, 2/7), \quad v = (0, 2, 1, 0), \quad w = u \wedge v = (2, 1, 0, 0, -4/7, -2/7).$$

Here $w^T Qw = 95/98$, $\|w\|^2 = 265/49$, and

$$K(u, v) = 38\pi^2/53 > 0. \tag{14}$$

This explicit decomposition supplies the additional information needed to construct a positive plane in dimension four.

There is also a rational strict-negative certificate for every rational $r > \rho$. Set $\lambda = \min\{1, P(r)/(8r^2)\}$. Then $Q + \lambda H$ is negative definite. Its first block is strict for $0 < \lambda < 6h$, and its third remains strict because $\lambda \leq 1 < a$ decreases the magnitude of its off-diagonal. For the second, let $C = ad$. As $r > \rho > 10$,

$$\sqrt{C} < 2r^2, \quad A < 2r^2, \quad \sqrt{C} - A = \frac{P(r)}{\sqrt{C} + A} > \frac{P(r)}{4r^2} \geq 2\lambda.$$

Hence $A + \lambda < \sqrt{C}$ and the second block is strict as well.

4.3 Saturation of the maximal curvature

Consider $L = Q + 3hH + (9/2)I$. Its first block is $\mathrm{diag}(0, (9-h^2)/2)$; its remaining two blocks are

$$-\tfrac{1}{2}\begin{pmatrix} a-9 & c \\ c & d-9 \end{pmatrix}, \qquad -\tfrac{1}{2}\begin{pmatrix} A-9 & r^2-8r-2 \\ r^2-8r-2 & 2r^2-1 \end{pmatrix}.$$

The first of these has fourfold determinant $T(r)$. The second has fourfold determinant

$$U(r) = r^4 + 24r^3 - 71r^2 - 36r + 1.$$

For $r = 5 + t$, $U = t^4 + 44t^3 + 439t^2 + 1554t + 1671 > 0$ when $t > 0$. All displayed diagonal magnitudes are positive for $r > 5$. Thus $T(r) \geq 0$ implies $L \leq 0$, and on decomposable w,

$$w^T Q w \leq -\tfrac{9}{2}\|w\|^2.$$

The plane $w = e_{12}$ attains equality. Descartes' rule and $T(25) = -11907$, $T(26) = 8263$ give its unique positive root σ. Its irrationality follows from the monic rational-root test and $(25, 26) \cap \mathbb{Z} = \varnothing$. This proves the saturated part of Theorem 4.1.

4.4 The exact maximum below saturation

Suppose $5 < r < \sigma$. Let $f_1(\lambda)$ and $f_2(\lambda)$ be the larger eigenvalues of the first two blocks of $Q + \lambda H$. On $[-16, 3h]$, the first is strictly decreasing up to the endpoint; the second is strictly increasing, since $A + \lambda \geq A - 16 > 0$. At $\lambda = -16$ the first block has negative determinant and $f_1(-16) > 0$, whereas the second block is negative definite because

$$\begin{aligned} ad - (A-16)^2 &= r^4 - 8r^3 + 12r^2 + 80r - 134 \\ &= (r-5)^4 + 12(r-5)^3 + 42(r-5)^2 + 100(r-5) + 191 > 0. \end{aligned}$$

Thus $f_2(-16) < 0 < f_1(-16)$. At $\lambda = 3h$, $f_1(3h) = -9/2$. Since $T(r) < 0$, the determinant of the second block plus $(9/2)I$ is negative, giving $f_2(3h) > -9/2$. There is a unique crossing $\lambda_* \in (-16, 3h)$, with common eigenvalue $q > -9/2$.

The determinant equations for those two blocks, and the positivity of $3h - \lambda_*$ and $A + \lambda_*$, give

$$\begin{aligned} \sqrt{(2q+9)(2q+h^2)} &= 3h - \lambda_*, \\ \sqrt{(2q+a)(2q+d)} &= A + \lambda_*. \end{aligned} \tag{15}$$

Adding gives (11). Its left side is strictly increasing for $q > -9/2$, because every factor is positive there. At $q = -9/2$ it is $\sqrt{(a-9)(d-9)} < c$, precisely because $T(r) < 0$, and it tends to infinity with q. Hence its root exists and is unique.

The third block is strictly below this crossing. For $-16 \leq \lambda \leq 3h$,

$$|a - \lambda| \leq a + 16.$$

Indeed the maximum occurs at an endpoint, and $2a + 16 - 3h = 2r^2 - 10r + 15 > 0$. For $q \geq -9/2$, the diagonal entries of twice $qI - Q_3 - \lambda H_3$ are at least $A - 9$ and $2r^2 - 1$. Its determinant numerator is therefore at least

$$\begin{aligned} (A-9)(2r^2-1) - (a+16)^2 &= r^4 + 12r^3 - 49r^2 + 64r - 284 \\ &= (r-5)^4 + 32(r-5)^3 + 281(r-5)^2 + 974(r-5) + 936 > 0. \end{aligned}$$

Both diagonals are positive, so $qI - Q_3 - \lambda H_3$ is positive definite. At the crossing this proves

$$Q + \lambda_* H \preceq qI.$$

It gives the desired upper bound on every decomposable bivector.

To attain it, put

$$\eta = \frac{2q+9}{3h-\lambda_*} > 0, \qquad \zeta = \frac{2q+a}{A+\lambda_*} > 0.$$

The two top block eigenspaces contain $(1, -\eta)$ and $(1, -\zeta)$. Define

$$w = (1, \sqrt{\eta/\zeta}, 0, 0, -\sqrt{\eta\zeta}, -\eta).$$

Its two complementary products cancel, giving $p(w) = 0$. An actual plane in the orthonormal tangent frame is

$$U = (1, 0, 0, \sqrt{\eta\zeta}), \qquad V = (0, 1, \sqrt{\eta/\zeta}, 0), \qquad U \wedge V = w.$$

Each nonzero block component is a q-eigenvector, so $w^T(Q + \lambda_* H)w = q\|w\|^2$. Since $p(w) = 0$, this plane has curvature $4\pi^2 m^2 q$. This proves attainment and completes Theorem 4.1 and Corollary 4.2.

The block square-root feasibility mechanism is classical in four-dimensional curvature theory [5, Section 5.4][6, Section 2.2 and Remark 3.4]. The specialization here identifies the active blocks throughout the entire ratio interval, excludes the third block by explicit positive polynomials, and supplies a decomposable attaining plane. It does not classify arbitrary four-frequency spectra or determine the minimal sectional curvature.

5 Effective nonuniform curvature neighborhoods

Strict curvature sign at the uniform density persists on some neighborhood by continuity and compactness of the unit-plane Grassmannian. This gives no explicit radius. We now provide quantitative closed cubes in two aligned models, together with an elementary dilation extension.

Theorem 5.1 (Certified closed cubes and their dilations)**.** *For a positive integer m, take either $S = \{m, 2m, 3m\}$ or $S = \{m, 2m, 4m\}$, with all phases zero in (1). On the entire closed amplitude cube $\theta \in [-1/100, 1/100]^3$ the density is at least $97/100$, and:*

1. *For $S = \{m, 2m, 3m\}$, the fixed density-coordinate plane spanned by $(1, 0, 3)$ and $(0, 1, 0)$ has $K > 4.66\pi^2 m^2$. The first/third density-coordinate plane has $K < -30.76\pi^2 m^2$. Thus both signs occur at every parameter.*

2. *For $S = \{m, 2m, 4m\}$, every sectional curvature satisfies $K < -.98\pi^2 m^2$.*

The base case $m = 1$ is computer-assisted: its proof consists of the analytic enclosure below and finite exact rational inequalities in the companion certificate. The dilation is analytic.

5.1 A finite-spectrum metric-jet enclosure

We state the remainder estimate for any finite aligned spectrum, not only the two examples. It is an elementary negative-binomial bound, rather than a new interval algorithm.

Lemma 5.2 (Reciprocal-series enclosure)**.** *Let S be a finite positive-integer spectrum with zero phases, let $b_j \geq 0$ and $|\theta_j| \leq b_j$, and assume $\delta = \sum_j b_j < 1$. Fix a nonnegative integer L. Set $\overline{G} = k^2 G$ and $\overline{B}_i = \sin(kix)/i$. For a parameter multiindex α, $h = |\alpha|$, one has*

$$\partial^\alpha \overline{G}_{ij} = (-1)^h h! \int_{\mathbb{T}} \frac{\overline{B}_i \overline{B}_j \prod_{\ell \in S} \cos(k\ell x)^{\alpha_\ell}}{f_\theta^{h+1}} \, \mathrm{d}x. \tag{16}$$

Expanding $(1+g)^{-h-1}$ through degree L in $g = f_\theta - 1$ and integrating gives a remainder of absolute value at most

$$E_{ij,h} = \frac{h!}{ij} \binom{h+L+1}{h} \frac{\delta^{L+1}}{(1-\delta)^{h+1}}. \tag{17}$$

The finite integral is computed exactly by its Laurent constant coefficient. Enclosing each parameter monomial over the whole box, then adding $[-E_{ij,h}, E_{ij,h}]$, yields a rigorous full-box derivative interval.

Proof. Parity gives $A_j = 0$ throughout an aligned model, so (2) reduces to its first term. Differentiation under the integral gives (16). These are ordinary derivatives, not factorial-normalized Taylor coefficients. Since $|g| \leq \delta$, the negative-binomial expansion is absolutely convergent. After truncating at L, the absolute integrated remainder is bounded using $|\overline{B}_i \overline{B}_j| \leq 1/(ij)$ and the unit bound on cosine factors. For every $s \geq 0$,

$$\binom{h+L+1+s}{h} \leq \binom{h+L+1}{h} \binom{h+s}{h}.$$

For $h > 0$ compare the h factors $(L+1+s+t)/(L+1+t) \leq (s+t)/t$, $1 \leq t \leq h$; for $h = 0$ the assertion is equality. Summing $\binom{h+s}{h}\delta^s = (1-\delta)^{-h-1}$ proves (17). Finally,

$$\overline{B}_i \overline{B}_j = \frac{\cos(k(i-j)x) - \cos(k(i+j)x)}{2ij},$$

and the multinomial expansion of g^q reduces every finite integral to exact rational Fourier coefficients. Interval evaluation of monomials encloses their full ranges; ignoring their dependencies may enlarge, but cannot invalidate, the enclosure. □

For Theorem 5.1 at $m = 1$, take $b_j = 1/100$, $\delta = 3/100$, $h \leq 2$ and $L = 12$. A nonconstant parameter monomial with all exponents even ranges from zero to $(1/100)^{|\beta|}$; if some exponent is odd it ranges in the symmetric interval with this absolute endpoint. The constant monomial is exactly one. There are six symmetric metric entries and ten multiindices of order at most two, giving 60 derivative intervals for each model. Formula (17) controls every omitted term; no Fourier-tail cutoff, quadrature approximation or parameter sampling is implicit.

5.2 Curvature enclosure from the metric two-jet

We give the exact index convention. Write commas for parameter derivatives, let $\overline{G}^{ab}$ be the inverse metric, and set

$$C_{aij} = \tfrac{1}{2}(\overline{G}_{ai,j} + \overline{G}_{aj,i} - \overline{G}_{ij,a}).$$

The covariant wedge curvature matrix is

$$\begin{aligned} \overline{R}_{ij,k\ell} = {} & \tfrac{1}{2}(\overline{G}_{i\ell,jk} + \overline{G}_{jk,i\ell} - \overline{G}_{ik,j\ell} - \overline{G}_{j\ell,ik}) \\ & + \sum_{a,b} \overline{G}^{ab} (C_{ai\ell} C_{bjk} - C_{aik} C_{bj\ell}). \end{aligned} \tag{18}$$

An exact interval cofactor inverse encloses $\overline{G}^{-1}$ after a positive determinant interval is checked. A separate Gershgorin lower bound proves positive definiteness of the metric throughout the cube. Interval evaluation of (18) encloses its symmetric curvature matrix in wedge order $(12, 13, 23)$.

For density-coordinate vectors u, v and $w = u \wedge v$, physical curvature is

$$K(u, v) = 4\pi^2 \frac{w^T \overline{R} w}{(u^T \overline{G} u)(v^T \overline{G} v) - (u^T \overline{G} v)^2}. \tag{19}$$

Indeed $\overline{G} = k^2 G$ and $\overline{R} = k^2 R$, so the sectional quotient for the scaled metric must be multiplied by k^2.

For $S = \{1, 2, 3\}$ the metric Gershgorin lower bound is greater than .05429. The positive-plane directions have wedge $w = (1, 0, -3)$. The exact certificate implies the following coarsened outward intervals:

$$\begin{aligned} w^T \overline{R} w &\in [.14688, .29122], \\ K((1,0,3),(0,1,0))/(4\pi^2) &\in [1.16580, 2.34785], \\ K((1,0,0),(0,0,1))/(4\pi^2) &\in [-8.31304, -7.69193]. \end{aligned}$$

Both plane Gram determinants have strictly positive lower bounds. In particular $4(1.16580) > 4.66$ and $4(-7.69193) < -30.76$.

For $S = \{1, 2, 4\}$ the metric Gershgorin lower bound exceeds .03092. The Gershgorin lower bound μ of $-\overline{R}$ exceeds .06243, so $w^T \overline{R} w \leq -\mu \|w\|^2$ for every bivector. Let L_G be the maximum absolute row-sum bound of $\overline{G}$. The largest eigenvalue of its induced second exterior power is at most L_G^2, so the denominator of (19) is at most $L_G^2 \|w\|^2$. The exact rational output gives

$$-\mu / L_G^2 < -.24593.$$

Hence $K/(4\pi^2) < -.24593$, which implies the stated $-.98\pi^2$ bound. These calculations prove Theorem 5.1 for $m = 1$.

5.3 Integer dilation and reproduction

For an aligned base spectrum S and its dilation mS, identify the amplitude coordinates. Their primitives satisfy $B_{mj}(x) = m^{-1} B_j(mx)$, and their densities satisfy $f_\theta^{mS}(x) = f_\theta^S(mx)$. Periodicity and the change of variable $y = mx$ therefore give

$$G^{mS}(\theta) = m^{-2} G^S(\theta).$$

Multiplying a metric by a positive constant multiplies sectional curvature by its reciprocal. Thus every sectional curvature for mS is m^2 times the corresponding curvature for S at the same amplitude parameters. Density lower bounds and amplitude boxes are unchanged. This proves all dilation assertions in Theorem 5.1.

The public companion program `reproducibility/density_box.py`, with radius `1/100` and degree `12`, generates the certificate `data/density_box.json`. It uses exact rational arithmetic for Fourier coefficients, monomial intervals, the infinite-tail bound, matrix inverse and curvature operations. Decimal endpoints are serialized outward by directed integer division. The accompanying `check_results.py` tests the sign and quantitative gates with exact fractions. All 120 derivative intervals and both resulting metrics and curvature matrices are supplied. The code requires only the Python standard library.

The supplied coordinate metric-two-jet calculations also check the analytic uniform-density matrices. Finite checks supplement the analytic proofs: continuum coverage comes from Lemma 5.2 and interval inclusion. General interval verification and rigorous Fourier continuation have substantial precedents [11, 10]; the contribution here is the complete executed geometric certificate on these full nonuniform domains.

6 Consequences and remaining questions

The phase theorem isolates a structural distinction: a frequency spectrum can be the unique three-mode candidate for nonnegative curvature without forcing mixed signs for every phase choice. Retained cosine normal components and phase-sensitive correlations jointly determine the sign. The four-mode family demonstrates a different mechanism: its curvature operator has a nondecomposable null line, which can complete an operator-positive vector into an actual positive plane below the first threshold. Above that threshold the null line survives but no sectional plane lies in it. The exact maximum identifies a further transition before the lowest-frequency coordinate plane becomes a global maximizer.

The effective cubes address robustness rather than only the uniform-density value. They certify that the mixed-versus-negative distinction is not confined to a single symmetric density. Natural further problems include arbitrary-phase four-frequency families, exact minimal curvature, sharper certified neighborhoods and higher-dimensional frequency criteria. Extending the fixed-phase theorem to phase variables as coordinates requires new treatment of periodic flux and same-frequency sine–cosine interactions. The neighborhood certificates concern the limiting transport metric itself, not finite-bandwidth Sinkhorn approximations.

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