Optimal unbiased Gaussian quantum estimation and the cost of unknown squeezing

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Luminary Oddities

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Abstract

We determine minimum-variance unbiased estimators of scalar observables from N copies of an unknown Gaussian quantum state. Real orthogonal mixing of the copies defines an expectation-complete invariant model. Its completeness makes the averaged observable optimal at every centered Gaussian state, including squeezed, multimode and pure states, among all globally unbiased integrable quantum procedures. The theorem covers bounded copy observables and number-preserving polynomials, with the unbounded case justified by absolute first-moment forms and exponential photon-number domains. Polynomial energies admit an exact finite Wick formula. For the single-mode quartic energy, we derive the optimal unbiased observable and its variance, then prove a sharp resource consequence allowing arbitrary bias: with occupation at least a, both collective quantum measurements and standard heterodyne have minimax relative risk of order $\min\{1, (Na^2)^{-1}\}$ uniformly for $0 < a \leq 1/4$. Thus admitting unknown squeezing changes the low-brightness quantum copy cost from inverse-linear to inverse-quadratic occupation, matching the heterodyne order. An explicit heterodyne pair statistic attains the upper order, while a squeezed-thermal testing experiment gives the quantum lower bound. Joint location-scatter completeness extends pointwise unbiased optimality to centered targets with unknown displacement using the residual copies.

Keywords. Gaussian quantum states; unbiased estimation; statistical completeness; squeezing; minimax risk.
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1 Introduction

High-order optical correlations can be easy to specify and difficult to estimate. Their statistical difficulty depends on the family of states that the estimator must accommodate. A thermal model fixes the anomalous correlations to zero; a general Gaussian model allows those correlations to vary. We study how that change affects both exact unbiased estimation and the number of copies needed for relative accuracy.

The scalar example already exhibits the issue. Write $t = \langle b^\dagger b \rangle$ and $c = \langle bb \rangle$ for a centered single-mode Gaussian state. The normally ordered quartic energy is

$$F = \langle (b^\dagger)^2 b^2 \rangle = 2t^2 + |c|^2. \tag{1}$$

For a thermal state, $c = 0$. An estimator optimized for that restriction uses the total photon count over the copies. When c is unknown, its contribution to (1) must also be estimated. We give the exact optimal unbiased observable for the larger model and quantify the resulting resource change.

There are two main conclusions. First, the full centered Gaussian family has a concrete expectation-completeness property after real orthogonal mixing of the copies. Consequently an averaged observable is the minimum-variance estimator of its expectation at every state among all procedures unbiased throughout that family. This includes bounded observables on the full copy space and number-preserving polynomial observables. The proof treats arbitrary integrable real-decision POVMs and mixed and pure Gaussian comparison states.

Second, for (1) with $t \geq a > 0$, collective quantum procedures and standard heterodyne have the same low-brightness minimax order, $\min\{1, (Na^2)^{-1}\}$. Both bounds allow bias. This differs from the thermal experiment, where the degree-two quantum and heterodyne resource scales are respectively a^{-1} and a^{-2} [5]. The change is explained by Gaussian states with anomalous covariance of order a: its squared contribution is comparable to the thermal energy a^2, and resolving that contribution costs order a^{-2} copies.

Relation to prior methods and results. Quantum symmetry averaging and its variance-reduction interpretation are established by the quantum Rao–Blackwell theorem [7, Section 5.3]. General minimum-variance sufficiency principles based on complete operator algebras appear in [3, Section 5]. Orthogonal copy mixing, its metaplectic commutation and the centroid stabilizer for displacement are described by Dai and Zhang [1, Sections III and V]. These provide the symmetry underlying our construction.

The additional statistical step here is completeness of the orthogonal invariant model. Coherent lower symbols reduce it to complete real Gaussian scatter laws, including singular scatter. Absolute first-moment forms then identify the averaged first moment of every globally unbiased POVM and justify the second-moment comparison in Fock space. This establishes an exact finite-copy optimum throughout the full Gaussian covariance family. The earlier thermal result [5] uses complex copy mixing and a complex scatter model. Real mixing preserves the anomalous covariance as well, while the functional-analytic argument below also handles the cross-photon-number entries of a general competitor.

Local Gaussian information bounds are developed in [4]; efficient moment-based Gaussian characterization is studied in [6]. Those results give relevant local information and measurement benchmarks. Our quartic conclusion instead bounds a globally uniform relative-risk problem from above and below, including arbitrary biased decisions and collective measurements. The upper uses an explicit pair statistic, and the lower uses finite-separation quantum testing. Tsuda's polynomial estimation results [8] concern displacement with known thermal occupation. Here the estimated functionals depend on the unknown covariance, and the displacement extension concerns their centered values. Lahiry's scalar displaced-thermal reduction [2, Section 6] supplies related nuisance-invariance context; the full-covariance pointwise conclusion below follows from joint location-scatter completeness.

Organization. Section 2 states the exact estimation theorem and the quartic resource law. Section 3 proves completeness and optimality, with the domain estimates stated separately. Section 4 gives the polynomial construction and exact quartic variance. Section 5 proves the quantum and heterodyne minimax bounds. Section 6 treats unknown displacement, and Section 7 gives an exact quartic model-mismatch identity.

2 Experiment and principal results

Let $\mathcal{F}(\mathbb{C}^d)$ be the bosonic Fock space of d modes, with annihilation operators $b_1, \dots, b_d$ satisfying $[b_i, b_j^\dagger] = \delta_{ij}\mathbb{I}$. We use $b_i = (q_i + ip_i)/\sqrt{2}$ and $[q_i, p_j] = i\delta_{ij}\mathbb{I}$. For $R = (q_1, p_1, \dots, q_d, p_d)$, the centered covariance is $V_{ij} = \frac{1}{2}\operatorname{Tr}\rho(R_iR_j + R_jR_i)$, and $\Omega = \bigoplus_{i=1}^d \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. A centered Gaussian state is specified by a finite V satisfying $V + i\Omega/2 \succeq 0$. Denote the family of all such states by $\mathcal{G}_d$. Its boundary includes pure Gaussian states.

The observation is $\rho^{\otimes N}$ for an unknown $\rho \in \mathcal{G}_d$. An estimator consists of a POVM and a measurable real decision; we absorb the decision into a POVM $M(dy)$ on $\mathbb{R}$. It is globally integrable when $\int |y| \operatorname{Tr}(\rho^{\otimes N} M(dy)) < \infty$ for every $\rho \in \mathcal{G}_d$. Global unbiasedness for a scalar target $\theta(\rho)$ means that its expectation equals $\theta(\rho)$ throughout this family. Variance comparisons below are pointwise in ρ, within this globally unbiased class.

For $O \in O(N)$, let U_O be passive real mixing of the copy index:

$$U_O^\dagger b_{i,r} U_O = \sum_{s=1}^N O_{rs} b_{i,s}.$$

Every centered Gaussian product is invariant under these unitaries. On real quadratures their action is $O \otimes I_{2d}$, which preserves $I_N \otimes V$. For a bounded observable B, define its weak operator average

$$\mathcal{T}_N(B) = \int_{O(N)} U_O^\dagger B U_O \, dO, \tag{2}$$

where dO is normalized Haar measure.

We also use a polynomial class on the dN modes. A *balanced Hermitian polynomial* is a finite Hermitian linear combination of normal-ordered monomials with equal total numbers of creators and annihilators. It preserves total photon number L. Its restrictions B_m to the finite m-photon sectors satisfy

$$\|B_m\| \le C(m+1)^p \tag{3}$$

for some $C < \infty$ and nonnegative integer p. We give B its self-adjoint direct-sum realization. Formula (2) then defines a finite matrix on each sector and hence a self-adjoint direct sum as well. Finite-particle vectors form a core. Gaussian states have finite polynomial photon moments.

Theorem 2.1 (Exact Gaussian unbiased optimum)**.** *Fix $d, N \ge 1$. Let B be bounded and self-adjoint, or a balanced Hermitian polynomial as above, and put*

$$\theta_B(\rho) = \operatorname{Tr}(\rho^{\otimes N} B), \qquad T_B = \mathcal{T}_N(B).$$

The spectral measurement of T_B is globally unbiased and has finite variance at every $\rho \in \mathcal{G}_d$. For every globally integrable POVM M unbiased for θ_B,

$$\operatorname{Var}_{\rho^{\otimes N}}(M) \ge \operatorname{Var}_{\rho^{\otimes N}}(T_B) \qquad (\rho \in \mathcal{G}_d). \tag{4}$$

The competitor may have infinite variance. In particular (4) *is an exact minimum unbiased variance for each specified scalar target and covariance.*

For a nonzero homogeneous holomorphic polynomial f of degree n, take $B = f(b_{\cdot,1})^\dagger f(b_{\cdot,1})$. Its target is the single-copy energy $F_f(\rho) = \operatorname{Tr}\rho f(b)^\dagger f(b)$. Its optimal observable is positive and has the concrete form

$$T_f^O = \mathbb{E}_{v \sim \mathrm{Unif}(S^{N-1})} f\left(\sum_r v_r b_{\cdot,r}\right)^\dagger f\left(\sum_r v_r b_{\cdot,r}\right). \tag{5}$$

The expectation in (5) is an average of observables. Its spectral measurement is the procedure in Theorem 2.1. Relative variance is obtained by dividing (4) by $F_f(\rho)^2$ at states with $F_f(\rho) > 0$.

2.1 The quartic energy and its minimax cost

For one mode put

$$t = \operatorname{Tr} \rho b^\dagger b, \quad c = \operatorname{Tr} \rho b^2, \quad u = |c|^2, \qquad 0 \le u \le t(t+1). \tag{6}$$

The centered Gaussian energy is $F = 2t^2 + u$. On the copies define $K = \sum_r b_r^\dagger b_r$ and $Q = \sum_r b_r^2$.

Theorem 2.2 (Exact quartic optimum)**.** *The minimum-variance globally unbiased observable for F is*

$$T_2^O = \frac{Q^\dagger Q + 2K(K-1)}{N(N+2)}. \tag{7}$$

Its absolute variance is

$$\operatorname{Var}(T_2^O) = \frac{A_1(t,u)}{N} + \frac{3A_2(t,u)}{N(N+2)}, \tag{8}$$

where

$$A_1(t,u) = 16t^3(t+1) + (52t^2 + 20t + 2)u + 4u^2, \tag{9}$$

$$A_2(t,u) = 4t^2(t+1)^2 + 16t(t+1)u + 4u^2. \tag{10}$$

These formulas hold for every centered single-mode Gaussian state, including $u = t(t+1)$.

To allow bias, for $a > 0$ define $\mathcal{G}_1(a) = \{\rho \in \mathcal{G}_1 : t \ge a\}$ and

$$R_Q(N,a) = \inf_M \sup_{\rho \in \mathcal{G}_1(a)} \mathbb{E}_\rho \left(\frac{Y}{2t^2 + |c|^2} - 1 \right)^2. \tag{11}$$

The infimum includes every collective POVM and real decision. Define $R_H(N,a)$ by restricting measurements to standard heterodyne of every copy and permitting arbitrary processing of all records. Thus $R_Q \le R_H$.

Theorem 2.3 (Cost of unknown squeezing)**.** *For every $0 < a \le 1/4$ and integer $N \ge 1$,*

$$\frac{3}{819200} \min\left\{1, \frac{1}{Na^2}\right\} \le R_Q(N,a) \le R_H(N,a) \le \min\left\{1, \frac{106(1+a^{-1})^2}{N}\right\}. \tag{12}$$

Consequently both risks are of order $\min\{1, (Na^2)^{-1}\}$ with universal constants on this range. Along any sequence $0 < a_j \le 1/4$, either measurement class admits uniform relative consistency exactly when $N_j a_j^2 \to \infty$.

This compares the same quartic target, copy budget and full unknown covariance family. At low occupation the thermal model has quantum scale $(Na)^{-1}$ for this degree and heterodyne scale $(Na^2)^{-1}$ [5]. Theorem 2.3 identifies the inverse-quadratic scale for both classes once c is admitted as an unknown parameter. Section 5 gives an explicit heterodyne statistic attaining the stated upper bound.

3 Gaussian completeness and the optimality proof

The group average preserves expectations and reduces second moments. To identify an optimum for every globally unbiased competitor, we must additionally determine its averaged first moment. We first establish the analytic control needed for that determination.

3.1 First-moment forms on Fock space

Write $D = dN$, and let e_α, $\alpha \in \mathbb{N}^D$, be the normalized Fock basis, with $Le_\alpha = |\alpha| e_\alpha$. Let $\mathcal{D}_{\mathrm{fin}}$ denote its finite linear span.

Lemma 3.1 (Absolute moment control)**.** *Let M be globally integrable on $\mathcal{G}_d$, and let*

$$E_M[\psi] = \int |y| \langle \psi, M(dy)\psi \rangle.$$

For every $q \in (0,1)$,

$$C_q := \sum_\alpha q^{|\alpha|} E_M[e_\alpha] < \infty, \qquad E_M[\psi] \le C_q \|q^{-L/2}\psi\|^2 \quad (\psi \in \mathrm{Dom}(q^{-L/2})). \tag{13}$$

The first-moment pairings on $\mathcal{D}_{\mathrm{fin}}$ have matrix entries $b_{\alpha\beta}$ satisfying

$$|b_{\alpha\beta}| \le \sqrt{E_M[e_\alpha] E_M[e_\beta]} \le C_q q^{-(|\alpha|+|\beta|)/2}. \tag{14}$$

Their coherent diagonal is a continuous function on $\mathbb{C}^D$. After multiplication by $e^{\|z\|^2}$ it has an entire complexification in independent z and $\overline{z}$ variables. It equals the actual POVM first moment at the coherent vector $|z\rangle$.

Proof. The isotropic thermal state on D modes is $(1-q)^D q^L$ and belongs to the observation family. Integrability gives the finite sum in (13). Positivity of the measure $|y|M(dy)$ gives scalar Cauchy–Schwarz for its matrix entries. For a finite vector $\psi = \sum \psi_\alpha e_\alpha$,

$$E_M[\psi] \le \left(\sum_\alpha |\psi_\alpha| \sqrt{E_M[e_\alpha]} \right)^2 \le C_q \sum_\alpha |\psi_\alpha|^2 q^{-|\alpha|}.$$

In a Naimark dilation $M(dy) = J^* P(dy) J$, the form E_M is the squared norm of the closed operator $|Y|^{1/2} J$, where $Y = \int y P(dy)$. It is a densely defined closed positive form. Fock cutoffs therefore extend the preceding inequality to $\mathrm{Dom}(q^{-L/2})$, and converge in its form norm on that domain. Applying Cauchy–Schwarz to $|y|M(dy)$ also proves (14) for the signed first moment.

The coherent diagonal is

$$h_M(z) = e^{-\|z\|^2} \sum_{\alpha,\beta} \frac{b_{\alpha\beta} \overline{z}^\alpha z^\beta}{\sqrt{\alpha!\beta!}}. \tag{15}$$

The bound (14) makes this series absolutely and locally uniformly convergent, also when $\overline{z}$ is replaced by an independent complex vector. Indeed each one-variable series $\sum_{m \ge 0} r^m/\sqrt{m!}$ converges for every finite r. Coherent vectors belong to every $\mathrm{Dom}(q^{-L/2})$. Their Fock cutoffs converge in the E_M form norm, so the finite pairings converge to the actual first moment by the mixed form Cauchy–Schwarz inequality. This proves both the asserted analytic properties and the equality in (15). □

The same conclusions apply to bounded observables and balanced polynomial observables, using their spectral POVMs. For a polynomial, (3) follows by estimating each creation or annihilation factor on fixed photon sectors. It also shows that Fock cutoffs converge in its graph norm for every vector in an exponential-number domain.

3.2 The complete real scatter family

Lemma 3.2 (Real scatter completeness)**.** *Let X have N independent $\mathcal{N}(0, W)$ columns in $\mathbb{R}^m$. The laws of XX^{T}, with W varying over any nonempty open subset of positive definite real symmetric matrices, are complete for functions integrable throughout that subset.*

Proof. Fix W_0 in the subset. Relative to its law, the likelihood ratio is

$$\left(\frac{\det W_0}{\det W}\right)^{N/2} \exp\left\{-\frac{1}{2}\operatorname{Tr}[(W^{-1} - W_0^{-1})XX^{\mathsf{T}}]\right\}.$$

Suppose an integrable function $g(XX^{\mathsf{T}})$ has zero expectation throughout the subset. The finite signed measure with density g relative to the scatter law at W_0 has a Laplace transform that is finite and zero on a neighborhood of zero in the real symmetric matrices. Uniqueness of the Laplace transform gives the zero measure. The likelihood ratios have common support; this reasoning applies also when $N < m$ and the scatter matrices have rank at most N. □

For real matrices X, Y of size $m \times N$, the identity $XX^{\mathsf{T}} = YY^{\mathsf{T}}$ is equivalent to $Y = XO$ for some $O \in O(N)$. This follows by extending the isometry between their row spans to an orthogonal map on $\mathbb{R}^N$. A Borel choice of Gram factors on each rank stratum shows that an invariant Borel function is a Borel function of XX^{T}.

Proposition 3.3 (Expectation completeness after orthogonal averaging)**.** *Let M be a globally integrable POVM. Suppose its first-moment matrix is invariant under $O(N)$ and has the same expectation as an invariant bounded or balanced-polynomial observable T at every centered Gaussian product state. Then its matrix entries on $\mathcal{D}_{\mathrm{fin}}$ equal those of T.*

Proof. Write a coherent amplitude matrix as $z \in \mathbb{C}^{d\times N}$ and let $X \in \mathbb{R}^{2d\times N}$ contain its real and imaginary parts. By Lemma 3.1, the difference of the coherent first moments is continuous and $O(N)$-invariant, hence has the form $g(XX^{\mathsf{T}})$.

Consider the Gaussian states obtained by mixing coherent vectors over an arbitrary real centered amplitude law $\mathcal{N}(0, W)$, $W \succ 0$, in $\mathbb{R}^{2d}$. These are centered Gaussian states: Gaussian coherent displacement adds a positive real covariance to the vacuum covariance. Their products correspond to independent Gaussian columns of X. Integrability of the POVM absolute moment and Tonelli's theorem justify the coherent integral; positive and negative decision parts give Fubini for the first moment. Thus

$$\mathbb{E}_W g(XX^{\mathsf{T}}) = 0, \qquad \mathbb{E}_W |g(XX^{\mathsf{T}})| < \infty \qquad (W \succ 0).$$

Lemma 3.2 gives zero almost surely. The Gaussian law has a strictly positive density on the full amplitude space, so continuity makes the coherent difference zero everywhere. The entire complexification in Lemma 3.1, or equivalently uniqueness of its real-analytic power series in $z, \overline{z}$, identifies every Fock matrix entry. This proves the proposition. □

The proposition establishes expectation completeness of the invariant model. Its proof uses a classically represented Gaussian subfamily to identify matrix entries; the variance comparison will apply to every Gaussian state in $\mathcal{G}_d$.

3.3 Comparison at squeezed and pure states

Lemma 3.4 (Gaussian number domains)**.** *Every finite-covariance Gaussian density operator σ on finitely many modes satisfies $\operatorname{Tr}\sigma e^{\epsilon L} < \infty$ for some $\epsilon > 0$. Every eigenvector with a positive eigenvalue belongs to $\operatorname{Dom}(e^{\epsilon L/2})$.*

Proof. The Husimi density $Q_\sigma(z) = \pi^{-D}\langle z|\sigma|z\rangle$ is a nondegenerate real Gaussian density, including for pure Gaussian σ. It has a finite exponential moment $\int Q_\sigma(z)e^{s\|z\|^2}dz$ for sufficiently small $s > 0$. Integrating the coherent series in each complex coordinate gives

$$\int Q_\sigma(z)e^{s\|z\|^2}dz = (1-s)^{-D}\operatorname{Tr}[\sigma(1-s)^{-L}] \qquad (0 < s < 1).$$

The identity can also be obtained by first integrating the positive rank-one coherent operators and then taking their expectation. Choose $\epsilon = -\log(1-s)$. If $\sigma = \sum_j \lambda_j|\psi_j\rangle\langle\psi_j|$, positivity gives $\sum_j \lambda_j\|e^{\epsilon L/2}\psi_j\|^2 < \infty$, proving the second assertion whenever $\lambda_j > 0$. □

Proof of Theorem 2.1. Average the actual competitor POVM:

$$\overline{M}(dy) = \int_{O(N)} U_O^\dagger M(dy)U_O\, dO.$$

This is a normalized POVM with the same outcome law as M at each centered Gaussian product. Its absolute moment satisfies Lemma 3.1, with the same constants, since U_O commutes with L. Its first-moment matrix is invariant. Gaussian invariance also gives $\operatorname{Tr}\rho^{\otimes N}T_B = \theta_B(\rho)$. Proposition 3.3 therefore identifies the first-moment entries of $\overline{M}$ with those of T_B on finite Fock vectors.

Fix a comparison state $\sigma = \rho^{\otimes N}$ for which the competitor second moment is finite. In its positive-eigenvalue decomposition, every ψ_j belongs to an exponential-number domain by Lemma 3.4. Choose q sufficiently near one so that this domain lies in $\mathrm{Dom}(q^{-L/2})$. Its Fock cutoffs converge in the absolute-moment form norm of $\overline{M}$. They also converge in the graph norm of T_B: this is automatic when T_B is bounded and follows from (3) in the polynomial case. Consequently the finite-vector matrix identity extends, for every finite ϕ, to

$$\int y\langle\phi, \overline{M}(dy)\psi_j\rangle = \langle\phi, T_B\psi_j\rangle.$$

In a Naimark dilation the left side is $\langle J\phi, YJ\psi_j\rangle$. Its squared modulus is at most

$$\|\phi\|^2 \int y^2\langle\psi_j, \overline{M}(dy)\psi_j\rangle.$$

Taking the supremum over unit finite vectors ϕ gives the same inequality with left side $\|T_B\psi_j\|^2$. Sum with weights λ_j to obtain

$$\operatorname{Tr}\sigma T_B^2 \le \int y^2 \operatorname{Tr}[\sigma\overline{M}(dy)] = \int y^2 \operatorname{Tr}[\sigma M(dy)].$$

The means agree, so subtracting their square proves (4). An infinite competitor second moment satisfies the inequality directly. The bounded or polynomial construction and Gaussian moments give the finite second moment of T_B. □

4 Polynomial energies and exact quartic risk

The main theorem gives optimality once the averaged observable is constructed. For polynomial energies the construction and its risk are finite calculations.

4.1 A finite all-order formula

Let f have degree n, and put $H_v = f(b(v))^\dagger f(b(v))$, $b_i(v) = \sum_r v_r b_{i,r}$. For real unit directions v, w, every cross contraction in the Gaussian Wick expansion of $\mathbb{E}_\rho H_v H_w$ is multiplied by $s = v^\mathsf{T} w$. There are $2n$ operators in each factor. Pairing within a factor removes an even number of them, so the number of cross pairs is $2k$, $0 \le k \le n$. Hence

$$\mathbb{E}_\rho H_v H_w = F_f(\rho)^2 + \sum_{k=1}^{n} A_k(\rho, f) s^{2k}. \tag{16}$$

This defines A_k explicitly: sum the ordered Gaussian Wick pairings with exactly $2k$ cross-factor pairs, including the polynomial coefficients. Internal and cross contractions use $\langle b_i^\dagger b_j \rangle$, $\langle b_i b_j^\dagger \rangle$ and $\langle b_i b_j \rangle$ with their operator order. The zero-cross term is the product of the two means. The pairing polynomial may be defined using two auxiliary copy directions, so that every real overlap is available. Exchanging those directions conjugates the expectation and is an orthogonal copy symmetry, making its coefficients real. For $N = 1$ the same pairing polynomial is evaluated at $s = \pm 1$.

Write $(x)^{\overline{k}} = \prod_{j=0}^{k-1}(x + j)$ and $(x)^{\overline{0}} = 1$. For independent uniform real unit vectors,

$$\mathbb{E}(v^\mathsf{T} w)^{2k} = \frac{(1/2)^{\overline{k}}}{(N/2)^{\overline{k}}}.$$

Indeed the first coordinate of a uniform real sphere vector has squared value with beta parameters $1/2, (N-1)/2$; for $N = 1$ both sides equal one. Integrating (16) gives the exact optimal unbiased variance

$$\mathrm{Var}_\rho(T_f^O) = \sum_{k=1}^{n} A_k(\rho, f) \frac{(1/2)^{\overline{k}}}{(N/2)^{\overline{k}}}. \tag{17}$$

Finite Gaussian polynomial moments justify the integrations. Formula (17) retains every contraction order and the anomalous covariances.

4.2 The degree-two observable

For $f(z) = z^2$, the real fourth sphere moment is

$$\mathbb{E} v_r v_s v_p v_q = \frac{\delta_{rs}\delta_{pq} + \delta_{rp}\delta_{sq} + \delta_{rq}\delta_{sp}}{N(N+2)}.$$

The first pairing gives $Q^\dagger Q$ in (5). The other two give $K(K-1)$ each, by normal ordering. This proves (7). Theorem 2.1 proves its minimum-variance property.

Its expectation can be checked directly. Gaussian moments and independence give

$$\mathbb{E} Q^\dagger Q = 2Nt^2 + N^2 u,$$
$$\mathbb{E} K(K-1) = N(N+1)t^2 + Nu.$$

Substitution yields $\mathbb{E} T_2^O = 2t^2 + u$.

For the second moment, introduce formal variables x, y, z, w for the two creator and two annihilator groups. With $s = v^\mathsf{T} w$, their ordered Gaussian generating function is

$$\begin{aligned}\mathbb{E}\big[e^{xb(v)^\dagger} e^{yb(v)} e^{zb(w)^\dagger} e^{wb(w)}\big] = \exp\big\{ &\tfrac{1}{2}\overline{c}(x^2 + z^2) + \tfrac{1}{2}c(y^2 + w^2) + t(xy + zw) \\ &+ s[\overline{c}xz + cyw + txw + (t+1)yz]\big\}.\end{aligned} \tag{18}$$

The $t+1$ term comes from commuting the middle annihilator and creator groups. Extracting $x^2y^2z^2w^2$ and multiplying by $(2!)^4$ gives

$$\mathbb{E}H_vH_w = F^2 + A_1(t,u)s^2 + A_2(t,u)s^4,$$

with (9)–(10). For example the three terms are obtained from the zeroth, second and fourth powers of the cross expression in the exponential, divided by their factorials, with the remaining degree supplied by the internal exponential. This finite coefficient extraction supplies the displayed polynomials for complex c as well; phase rotation reduces their dependence to $u = |c|^2$. The sphere moments $\mathbb{E}s^2 = 1/N$ and $\mathbb{E}s^4 = 3/[N(N+2)]$ prove (8) and complete Theorem 2.2.

4.3 The price of the larger unbiasedness model

The thermal optimum for the same quartic target restricted to $c = 0$ is [5]

$$T_2^U = \frac{2K(K-1)}{N(N+1)}.$$

At a general centered Gaussian state its expectation is $2t^2 + 2u/(N+1)$. Thus for $N > 1$ its anomalous contribution differs from that of F. At a thermal state the two global unbiasedness requirements lead to the exact variance difference

$$\mathrm{Var}(T_2^O) - \mathrm{Var}(T_2^U) = \frac{4t^2(t+1)^2(N-1)}{N(N+1)(N+2)}. \tag{19}$$

The identity follows by subtracting the thermal variance

$$\mathrm{Var}(T_2^U) = \frac{16t^3(t+1)}{N} + \frac{8t^2(t+1)^2}{N(N+1)}$$

from (8) at $u = 0$. It quantifies the variance cost of unbiasedness over the enlarged covariance family even when the comparison state is thermal. At $N = 1$ both estimators equal $(b^\dagger)^2b^2$.

5 Minimax resources with unknown squeezing

Theorem 2.3 requires a lower bound for all biased quantum procedures and an upper bound uniform over the allowed covariances. We prove them directly. The exact unbiased result also gives a useful sharper quantum upper bound.

5.1 A uniform quantum upper bound

Put $F = 2t^2 + u$. The inequalities $F \geq 2t^2$, $F \geq u$ and $F^2 \geq 8t^2u$ give, term by term,

$$\frac{A_1}{F^2} \leq \frac{29}{2} + \frac{13}{2t} + \frac{1}{4t^2},$$
$$\frac{A_2}{F^2} \leq 7 + \frac{4}{t} + \frac{1}{t^2}.$$

For the middle term of A_1, use $u/(2t^2+u)^2 \leq 1/(8t^2)$; the other terms follow from the first two inequalities. Since $3/(N+2) \leq 1$, (8) yields

$$\frac{\mathrm{Var}\, T_2^O}{F^2} \leq \frac{C(a)}{N}, \qquad C(a) = \frac{43}{2} + \frac{21}{2a} + \frac{5}{4a^2}. \tag{20}$$

An unbiased estimator with relative variance at most v has risk at most $v/(1+v)$ after division by $1+v$. Hence $T_2^O/(1+C(a)/N)$ has uniform risk at most $C(a)/(N+C(a))$. For $0 < a \leq 1/4$, $C(a)a^2 \leq 167/32$.

5.2 A lower bound against arbitrary quantum procedures

Let ρ_a be the thermal state with occupation a. Define squeezed thermal states

$$\rho_{a,r} = S(r)\rho_a S(r)^\dagger, \qquad S(r) = e^{irG}, \quad G = \frac{b^2 - (b^\dagger)^2}{2i}.$$

Their occupation and anomalous magnitude are

$$t_r = a + (2a+1)\sinh^2 r, \qquad u_r = (2a+1)^2 \sinh^2 r \cosh^2 r. \tag{21}$$

Thus $\rho_{a,r} \in \mathcal{G}_1(a)$. Take

$$r_0 = a, \qquad r_1 = a + h, \qquad h = \tfrac{1}{4}\min\{a, N^{-1/2}\}.$$

Throughout this interval, $a \le r \le 5a/4 \le 5/16$. Write $F_r = 2t_r^2 + u_r$. Differentiating (21) gives $F_r' \ge u_r' \ge 2r$, so

$$F_{r_1} - F_{r_0} \ge 2ah. \tag{22}$$

For $0 \le r \le 5/16$, $\sinh r \le 2r$ and $\cosh r \le 2$. Since $2a + 1 \le 3/2$,

$$t_r \le a + 6r^2 \le \frac{107}{32}a, \qquad u_r \le \frac{225}{4}a^2.$$

Consequently $F_r \le (40249/512)a^2 < 80a^2$. The relative harmonic separation therefore satisfies

$$\delta := \frac{F_{r_1} - F_{r_0}}{F_{r_1} + F_{r_0}} \ge \frac{h}{80a}. \tag{23}$$

To control the distance between the quantum observations, purify ρ_a as $|\Psi_a\rangle = \sum_k \sqrt{p_a(k)}|k,k\rangle$. The overlap of the corresponding squeezed purifications is $\operatorname{Tr}\rho_a S(h)$. Thermal moments give

$$\operatorname{Tr}\rho_a G = 0, \qquad \operatorname{Tr}\rho_a G^2 = a^2 + a + \tfrac{1}{2} \le \tfrac{13}{16}.$$

The spectral inequality $1 - \cos x \le x^2/2$ implies

$$\left|\operatorname{Tr}\rho_a S(h)\right|^2 \ge 1 - h^2 \operatorname{Tr}\rho_a G^2.$$

The right side is positive for the chosen h. Trace-distance contraction under partial trace and the pure-state overlap formula yield

$$\begin{aligned}\frac{1}{2}\left\|\rho_{a,r_0}^{\otimes N} - \rho_{a,r_1}^{\otimes N}\right\|_1 &\le \sqrt{1 - \left|\operatorname{Tr}\rho_a S(h)\right|^{2N}} \\ &\le h\sqrt{N \operatorname{Tr}\rho_a G^2} \le \tfrac{1}{4}.\end{aligned} \tag{24}$$

The second inequality uses $1 - (1-z)^N \le Nz$ for $0 \le z \le 1$.

For any real decision, threshold its output at the harmonic midpoint $2F_{r_0}F_{r_1}/(F_{r_0} + F_{r_1})$. A wrong classification entails relative squared loss at least δ^2. The sum of the testing errors is at least one minus the trace distance in (24). Thus its maximum risk on the two states is at least $3\delta^2/8$. Substituting (23) and

$$\frac{h^2}{a^2} = \frac{1}{16}\min\{1, (Na^2)^{-1}\}$$

proves the lower bound in (12). This argument applies to arbitrary bias, randomization and collective POVMs.

5.3 A matching standard-heterodyne construction

Heterodyne on a centered scalar Gaussian state gives an improper complex Gaussian record Z with

$$\mathbb{E}|Z|^2 = V = t + 1, \qquad \mathbb{E}Z^2 = c.$$

For two independent records define

$$q(Z) = |Z|^2 - 1, \quad p(Z) = Z^2, \quad h(Z, W) = 2q(Z)q(W) + \Re[p(Z)\overline{p(W)}]. \tag{25}$$

This symmetric real kernel has expectation $F = 2t^2 + u$. For $N \geq 2$ use the full pair statistic

$$U_N = \frac{1}{N(N-1)} \sum_{r \neq s} h(Z_r, Z_s). \tag{26}$$

Its first centered Hoeffding projection is

$$h_1(Z) = 2t(q(Z) - t) + \Re[\overline{c}(p(Z) - c)].$$

Gaussian moments give $\operatorname{Var} q = V^2 + u \leq 2V^2$ and $\mathbb{E}|p - c|^2 = 2V^2$. Hence

$$\operatorname{Var} h_1 \leq 16t^2V^2 + 4uV^2 \leq 8V^2F. \tag{27}$$

Also $\mathbb{E}q^2 \leq 3V^2$ and $\mathbb{E}|p|^2 \leq 3V^2$, since $u \leq t(t+1) \leq V^2$. Independence and $(x+y)^2 \leq 2x^2 + 2y^2$ imply

$$\mathbb{E}h(Z, W)^2 \leq 8(\mathbb{E}q^2)^2 + 2(\mathbb{E}|p|^2)^2 \leq 90V^4. \tag{28}$$

Put $h_2(Z, W) = h(Z, W) - F - h_1(Z) - h_1(W)$. Its conditional mean given either coordinate is zero. Orthogonality of these three centered components gives

$$\operatorname{Var} U_N = \frac{4}{N} \operatorname{Var} h_1 + \frac{2}{N(N-1)} \operatorname{Var} h_2, \qquad \operatorname{Var} h_2 \leq \operatorname{Var} h \leq \mathbb{E}h^2.$$

Using (27)–(28), $F \geq 2t^2$ and $\gamma = 1 + a^{-1}$ yields

$$\frac{\operatorname{Var} U_N}{F^2} \leq \frac{16\gamma^2}{N} + \frac{45\gamma^4}{N(N-1)} \leq 16z + 90z^2, \qquad z = \frac{\gamma^2}{N}. \tag{29}$$

When $z \leq 1$ the last expression is at most $106z$. Choose the better of U_N and the zero decision according to these known bounds, using zero for $N = 1$. This proves the upper bound in (12). Finally $(1 + a)^2 \leq 25/16$ for $a \leq 1/4$, so

$$R_H(N, a) \leq \frac{1325}{8} \min\{1, (Na^2)^{-1}\}.$$

The lower and upper bounds give the stated orders and the consistency equivalence, completing Theorem 2.3.

6 Centered targets with unknown displacement

Let $\rho_{\mu,V} = D(\mu)\rho_{0,V}D(\mu)^\dagger$ have unknown annihilation mean $\mu \in \mathbb{C}^d$ and unknown Gaussian covariance V, where $D(\mu)^\dagger b_i D(\mu) = b_i + \mu_i$. Fix $N \geq 2$. A real orthogonal transform of the copy index with first row $N^{-1/2}(1,\dots,1)$ gives

$$\rho_{\mu,V}^{\otimes N} \longmapsto \rho_{\sqrt{N}\mu,V} \otimes \rho_{0,V}^{\otimes(N-1)}. \tag{30}$$

The transformed state factorizes because its covariance is block diagonal and it is Gaussian. We call the first block the centroid and the remaining blocks the residual copies.

Theorem 6.1 (Residual-copy pointwise optimum)**.** *Let B be bounded and self-adjoint or a balanced Hermitian polynomial on $N-1$ centered copies. For the centered target*

$$\theta_B(V) = \mathrm{Tr}(\rho_{0,V}^{\otimes(N-1)} B),$$

apply $\mathcal{T}_{N-1}(B)$ to the residual copies in (30)*. Its spectral measurement has minimum variance at every (μ, V) among real-decision POVMs that are integrable and unbiased throughout the full displaced Gaussian family. Its variance is the centered $N-1$-copy variance.*

Proof. Unbiasedness and the variance formula follow from (30). To prove optimality, average the competitor over residual $O(N-1)$ mixing, leaving the centroid fixed. This preserves each transformed Gaussian state. A coherent amplitude matrix $X \in \mathbb{R}^{2d\times N}$ has centroid $X_0 = N^{-1/2}\sum_r X_r$ and residual part $X_\perp$. The invariant coherent symbol is a function of

$$(X_0, X_\perp X_\perp^{\mathsf{T}}),$$

which is equivalent to $(\sum_r X_r, \sum_r X_r X_r^{\mathsf{T}})$.

On the positive coherent-mixture subfamily, the columns have the full real normal location-covariance law $\mathcal{N}(m, W)$. The likelihood ratios depend on this pair of statistics through the natural parameters

$$(W^{-1}m, -\tfrac{1}{2}W^{-1}),$$

which range over an open set. The Laplace-transform argument in Lemma 3.2, now on the product space of vectors and symmetric matrices, proves completeness of these joint statistics, including deficient residual scatter. The invariant first-moment coherent symbol is therefore uniquely determined by the target and equals that of the residual observable. Coherent-series uniqueness gives equality of all finite Fock matrix entries.

The zero-mean isotropic thermal subfamily supplies Lemma 3.1. Every displaced Gaussian state still has a Gaussian Husimi density with a finite exponential-number moment. The domain and Naimark argument in Theorem 2.1 consequently proves the pointwise second-moment comparison here as well. □

For example, the centered quartic energy with unknown displacement is estimated by (7) on the $N-1$ residual modes, with the exact variance (8) after replacing N by $N-1$. The scalar residual transform is familiar from thermal nuisance reduction [2]; joint full-covariance completeness gives its pointwise unbiased optimality here.

Proposition 6.2 (One-copy obstruction)**.** *For $f \neq 0$ homogeneous of positive degree, a single displaced Gaussian copy admits no globally integrable unbiased estimator of the centered energy $F_f(\rho_{0,V})$ throughout all means and covariances.*

Proof. The displaced Gaussian family includes every coherent state. Its centered state is vacuum, whose energy is zero. Global unbiasedness would therefore make the coherent first-moment symbol zero everywhere. Lemma 3.1 and coherent-series uniqueness make every first-moment matrix entry zero. The same domain extension at a positive thermal state gives zero expectation there, contradicting its strictly positive centered energy. □

7 An exact quartic model-mismatch identity

The role of Gaussian moment relations can be tested directly. Let ρ now be any centered single-mode state with finite second photon-number moment, and write

$$t = \operatorname{Tr} \rho b^\dagger b, \qquad c = \operatorname{Tr} \rho b^2, \qquad m_4 = \operatorname{Tr} \rho (b^\dagger)^2 b^2.$$

Independence of the copies gives

$$\mathbb{E}Q^\dagger Q = N m_4 + N(N-1)|c|^2, \quad \mathbb{E}K(K-1) = N m_4 + N(N-1)t^2.$$

Consequently the same observable (7) satisfies the exact bias identity

$$\mathbb{E}T_2^O - m_4 = -\frac{N-1}{N+2}\big(m_4 - 2t^2 - |c|^2\big). \tag{31}$$

Thus its large-copy expectation is the Gaussian moment $2t^2 + |c|^2$. For the Fock state $|2\rangle$, $t = 2$, $c = 0$ and $m_4 = 2$, and (31) gives a bias $6(N-1)/(N+2)$. This explicit comparison isolates the moment relation used by the Gaussian estimator.

Reproducibility. The accompanying standard-library program derives (9)–(10) by all 105 ordered Wick pairings, checks the variance by a separate normal-order calculation at finite copy counts, and integrates the heterodyne pair moments by exact rational Gaussian arithmetic. It also checks the stated variance inequalities, pure-squeezed boundaries and (31) on finite Fock product states. The analytic proofs establish the all-parameter conclusions.

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