

Observation norms under double-exponential confinement

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Luminary Oddities

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Abstract

We determine the small-regularization norm of a weighted observation of a positive Fourier-energy operator under double-exponential spatial confinement. The leading term is the logarithm of the logarithm of the inverse regularization, with an explicit constant given by the quadratic part of the symbol and a constrained Dirichlet eigenvalue on an interval. Finitely many fixed nonzero polynomial-exponential constraint chains leave this constant unchanged; zero-exponent moment chains survive in the limiting problem. In particular a mean-zero constraint halves the leading observation norm. For the original two-row theta-weighted reference, the result gives the exact constant together with finite explicit bounds. Applications include a full factored-residual covariance enclosure and a complete-arithmetic sandwich bound. The proofs retain the specified compact-core closure and all mixed residual terms.

1 The observation problem

A small positive regularization makes an inverse energy operator bounded, but the norm relevant to a weighted observation can be much smaller than the inverse-energy norm itself. We study this distinction when the observation weight decays doubly exponentially. The active interval then has length comparable to $\log \log(1/\varepsilon)$. A rescaled interval problem determines the exact constant, including the effect of finite constraints.

The concentration estimate used below is classical time–frequency projection machinery [1]. Large barriers and expanding-domain nonlocal limits have classical Dirichlet-limit arguments [2, 3]. Our recipient is a weighted observation of the specified closed form, with finite polynomial-exponential rows. The constraint classification and its exact return to that form are essential: rows with nonzero exponent can be enforced without changing the limiting profile, whereas zero-exponent moments constrain that profile.

The theta application uses the reference of [4]. That paper studies fixed-regularization frequency localization; [5] treats jointly varying cutoffs and the complete negative spectrum. Here the regularization itself tends to zero in the observation norm. After proving the general law, we give finite theta bounds and a complete residual-covariance application.

Asymptotic law and finite certificates

Sharp limiting constant	Theorem 2.1 uses the constrained interval eigenvalue. Fixed nonzero exponential chains disappear from the limiting profile; zero-exponent moment chains remain.
Finite theta bounds	The bounds in Section 3 are explicit inequalities usable at a chosen positive regularization.
Residual covariance	Theorem 4.1 uses a full coefficient-vector covariance, including mixed entries; the signed certificate also pays its separate high-band error.

2 Finite constraints and the sharp law

Our Fourier transform is $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u) e^{-iu\xi} du$, with Plancherel measure $d\xi/(2\pi)$. Inner products are conjugate-linear first. Let $s \in C^\infty(\mathbb{R})$ be strictly positive. Suppose there are constants $\beta, c, C, A, a > 0$, $d \geq 0$ and U_0 such that

$$ce^{-Ce^{\beta|u|}} \leq s(u)^2 \leq Ae^{d|u| - ae^{\beta|u|}}, \quad |u| \geq U_0. \quad (1)$$

In particular s is bounded and every polynomial-exponential multiple of s lies in $L^2(\mathbb{R})$. Let v be continuous, nonnegative, even and nondecreasing in $|\xi|$, with

$$\frac{v(\xi)}{\xi^2} \longrightarrow \kappa \in (0, \infty) \quad (\xi \rightarrow 0), \quad v(\xi) \leq C_v \xi^2 \quad (\xi \in \mathbb{R}). \quad (2)$$

Fix a nonnegative integer m , distinct nonzero complex numbers $\lambda_1, \dots, \lambda_J$ and positive integers $r_1, \dots, r_J$; the collection can be empty. Define the closed subspace $\mathcal{K} \subset L^2(\mathbb{R})$ by

$$\int_{\mathbb{R}} u^k s(u) h(u) du = 0 \quad (0 \leq k < m), \quad (3)$$

$$\int_{\mathbb{R}} u^k e^{\lambda_j u} s(u) h(u) du = 0 \quad (1 \leq j \leq J, 0 \leq k < r_j). \quad (4)$$

The first chain is absent when $m = 0$. These are bounded complex-linear functionals; writing them as integrals fixes their convention even for complex exponents.

On $\mathcal{C} = C_c^\infty(\mathbb{R}) \cap \mathcal{K}$ take

$$\mathfrak{a}[h] = \int_{\mathbb{R}} v(\xi) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi}. \quad (5)$$

Let $\mathcal{V}$ be its closure in $\|h\|^2 + \mathfrak{a}[h]$, and let $A_{\mathcal{K}}$ be the associated nonnegative selfadjoint operator on $\mathcal{K}$. The closure is part of the definition. For completeness, $v(D)^{1/2} M_s$ is closed, so this restriction is closable. Compact constrained tests are dense in $\mathcal{K}$: after removing redundant rows, analytic independence of the polynomial-exponential functions gives a right inverse of the row map using finitely many compact smooth tests. Correcting compact smooth L^2 approximations with that inverse proves density.

Define

$$\Lambda_m = \inf \left\{ \frac{\|g'\|_2^2}{\|g\|_2^2} : 0 \neq g \in H_0^1(-1, 1), \int_{-1}^1 t^k g(t) dt = 0 \ (k < m) \right\}. \quad (6)$$

This number is finite and positive: the constrained space has finite codimension, and the usual Dirichlet Poincaré inequality is positive on all of $H_0^1(-1, 1)$.

Theorem 2.1 (Sharp observation law). *Under (1)–(2), with the fixed constraints (3)–(4),*

$$\lim_{\varepsilon \downarrow 0} \frac{\|M_s(A_{\mathcal{K}} + \varepsilon)^{-1/2}\|}{\log \log(1/\varepsilon)} = \frac{1}{\beta \sqrt{\kappa \Lambda_m}}. \quad (7)$$

In particular $\Lambda_0 = \pi^2/4$ and $\Lambda_1 = \pi^2$. Thus fixed nonzero exponential chains do not alter the leading constant; the additional mean-zero chain does.

2.1 Localization and retention of polynomial moments

Put $U = \log(1/\varepsilon)$, $R = \beta^{-1} \log U$ and

$$\ell(\varepsilon) = \inf_{0 \neq h \in \mathcal{V}} \frac{\mathfrak{a}[h] + \varepsilon \|h\|^2}{\|sh\|^2}.$$

Strict positivity of s makes the denominator nonzero. Spectral calculus identifies $\ell(\varepsilon)$ with the reciprocal square of the norm in (7). We will prove

$$R^2 \ell(\varepsilon) \longrightarrow \kappa \Lambda_m. \quad (8)$$

The trial construction in the next subsection already gives a finite uniform upper bound on the left side. Choose near minimizers with $\|sh\|_2 = 1$ and energy at most $\ell(\varepsilon) + R^{-3}$. Hence their energy is at most C_0/R^2 . Write

$$g_\varepsilon(t) = \sqrt{R}(sh)(Rt), \quad b_R(k) = R^2 v(k/R).$$

Then $\|g_\varepsilon\|_2 = 1$ and $\int b_R |\widehat{g}_\varepsilon|^2 dk / (2\pi) \leq C_0$.

For every fixed $\delta > 0$, the exterior weight bound and $\|h\|^2 \leq C_0/(\varepsilon R^2)$ imply

$$\int_{|t| > 1+\delta} |g_\varepsilon(t)|^2 dt \longrightarrow 0.$$

Indeed the exponential logarithm is at most $U - aU^{1+\delta} + O(\log U)$. For every fixed $K > 0$, monotonicity and (2) give $b_R(k) \geq \kappa K^2/2$ on $|k| \geq K$, once R is sufficiently large. The Fourier L^2 tail is therefore at most $2C_0/(\kappa K^2)$. Spatial tightness and this Fourier bound give L^2 precompactness: truncate both variables, use the Hilbert–Schmidt space–frequency rectangle, then let the cutoffs increase. A subsequence converges strongly to a norm-one g supported in $[-1, 1]$.

On bounded frequency intervals $b_R(k) \rightarrow \kappa k^2$ uniformly. Strong Fourier convergence, followed by exhaustion of frequency space, gives

$$\liminf R^2 \mathfrak{a}[h] \geq \kappa \|g'\|_2^2, \quad g \in H_0^1(-1, 1). \quad (9)$$

To retain (3), an L^1 tail is needed in addition to this L^2 compactness. For $k < m$,

$$\left| \int_{|t| > 1+\delta} t^k g_\varepsilon(t) dt \right| \leq R^{-k-1/2} \|h\|_2 \left(\int_{|u| > (1+\delta)R} |u|^{2k} s(u)^2 du \right)^{1/2}.$$

Substitution $x = e^{\beta|u|}$ in (1) bounds the last integral by a polynomial factor in U, R times $\exp[-(a/2)U^{1+\delta}]$. Its product with $\|h\| \leq \sqrt{C_0}/(R\sqrt{\varepsilon})$ tends to zero. Strong convergence on a compact interval now passes each exact zero moment to g . Thus $\int t^k g = 0$ for $k < m$. Equations (6) and (9), with the nonnegative regularization term retained, prove the lower limit in (8).

2.2 Exact constrained trials

Compact smooth functions satisfying the first m moment conditions are dense in the constrained space of (6). To see this, approximate in H_0^1 by compact smooth functions and repair their finitely many moments using a fixed compact smooth right inverse. Choose such a nonzero function ψ . Its m -fold antiderivative from $-\infty$ is compact smooth, because its first m moments vanish. Denote it by ϕ , so $\phi^{(m)} = \psi$; for $m = 0$ set $\phi = \psi$.

Fix $0 < \delta < 1$ and set

$$P(D) = \prod_{j=1}^J (D + \lambda_j)^{r_j}, \quad p_0 = P(0) \neq 0, \quad r = (1 - \delta)R,$$

$$\phi_r(u) = r^{-1/2} \phi(u/r), \quad f_r = r^m P(D) D^m \phi_r, \quad h_r = f_r/s.$$

Integration by parts proves every constraint exactly. The D^m factor kills polynomial moments below m ; the zeros of $P(-\lambda_j)$, with multiplicity, kill the exponential chains. Thus $h_r \in \mathcal{C}$.

In scaled Fourier variables,

$$r^{-1/2} \widehat{f}_r(k/r) = P(ik/r)(ik)^m \widehat{\phi}(k).$$

Schwartz decay and the global quadratic bound give

$$\|f_r\|_2^2 \rightarrow |p_0|^2 \|\psi\|_2^2, \quad r^2 \mathfrak{a}[h_r] \rightarrow \kappa |p_0|^2 \|\psi'\|_2^2.$$

After adjusting constants on a compact set, the lower wall bound yields

$$\frac{\varepsilon \|h_r\|^2}{\|f_r\|^2} \leq c_1^{-1} \exp[-U + C_1 U^{1-\delta}],$$

negligible even after multiplication by R^2 . It follows that

$$\limsup R^2 \ell(\varepsilon) \leq \frac{\kappa}{(1-\delta)^2} \frac{\|\psi'\|^2}{\|\psi\|^2}.$$

Take the constrained smooth infimum, then $\delta \downarrow 0$. This completes (8) and (7).

For $m = 0$ the first Dirichlet cosine gives $\Lambda_0 = \pi^2/4$. For $m = 1$, split g into even and odd parts. The odd restriction to $(0, 1)$ has two zero endpoint traces and obeys the Dirichlet inequality with constant π^2 . The even restriction has zero mean and obeys the mean-zero Neumann inequality with the same constant. The parity pieces are orthogonal in both energy and norm. Hence $\Lambda_1 \geq \pi^2$, with equality at $g(t) = \sin(\pi t)$. This proves the remaining assertion of Theorem 2.1.

3 The original theta reference

Let

$$\theta(u) = 2 \sum_{n \geq 1} (2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}) e^{-\pi n^2 e^{2u}}, \quad (10)$$

$$c(u) = \cosh(u/2), \quad w(u) = \theta(u)c(u), \quad s(u) = \sqrt{\theta(u)/c(u)}.$$

The Poisson theta identity makes θ even. Every atom is positive on $u \geq 0$, so reflection gives strict positivity. We use the two original rows

$$\mathcal{K} = \{h : \langle \sqrt{w}, h \rangle = \langle \sqrt{w} \tanh(u/2), h \rangle = 0\}.$$

Since $\sqrt{w}/s = c$, these are precisely the two nonzero exponential rows $e^{u/2}$ and $e^{-u/2}$ on $f = sh$. Take

$$v(\xi) = \Re \Psi(1/4 + i\xi/2) - \Psi(1/4) = \sum_{j \geq 0} \frac{(\xi/2)^2}{(j+1/4)((j+1/4)^2 + (\xi/2)^2)}, \quad (11)$$

where $\Psi = \Gamma'/\Gamma$; see [6, §5.7]. The positive series shows monotonicity and

$$\kappa = \frac{1}{4} \sum_{j \geq 0} (j+1/4)^{-3}, \quad \frac{16\xi^2}{1+4\xi^2} \leq v(\xi) \leq 17\xi^2. \quad (12)$$

For the upper bound the $j = 0$ coefficient is 16, and the remainder is at most $\frac{1}{4} \sum_{j \geq 1} j^{-3} \leq 3/8$.

The complete theta estimates, in the normalization (10), are

$$\|s\|_\infty \leq 1, \quad \sigma_R := \sup_{|u|>R} s(u) \leq 13e^{2R-(3/2)e^{2R}}, \quad s(u)^2 \geq 18e^{-\pi e^{2|u|}}. \quad (13)$$

The unit bound is the explicit full-series estimate of [4, Lemma 2.1]. For the tail, the full series is bounded by $8\pi^2 e^{9u/2 - \pi e^{2u}}$ on $u \geq 0$: after factoring the first exponential, bound the remaining n^4 series by 2, using $e^{-3\pi} < 1/100$. Divide by $c(u) \geq e^{u/2}/2$ and take square roots. The last inequality follows from the first positive atom, $\pi > 3$, and $c(u) \leq e^{u/2}$, then reflection.

Write A for the specified compact-core closure of (5) on this $\mathcal{K}$.

Corollary 3.1 (Sharp theta law). *For $j_\varepsilon = \|M_s(A + \varepsilon)^{-1/2}\|$,*

$$\frac{j_\varepsilon}{\log \log(1/\varepsilon)} \longrightarrow \frac{1}{\pi\sqrt{\kappa}}.$$

Proof. Apply Theorem 2.1 with $\beta = 2$, $m = 0$ and exponents $\pm 1/2$. The exact annihilator is $D^2 - 1/4$. $\square$

3.1 Finite observation bounds

Let P_R be spatial restriction to $[-R, R]$ and Q_L the Fourier band projection. Its integral kernel gives $\|P_R Q_L\| \leq \sqrt{2RL/\pi} = q$. If $q < 1$, the exact identity $f = P_R Q_L f + (I - P_R)f + P_R(I - Q_L)f$ yields

$$j_\varepsilon^2 \leq \frac{\sigma_R^2/\varepsilon + 1/v(L)}{(1 - \sqrt{2RL/\pi})^2}. \quad (14)$$

Indeed apply the triangle inequality, the two complete tail estimates and Cauchy–Schwarz with energy $\mathfrak{a}[h] + \varepsilon\|h\|^2$. The bound extends from the core by closure.

Proposition 3.2. *For $0 < \varepsilon \leq 1$,*

$$j_\varepsilon \leq \min\{\varepsilon^{-1/2}, 4\log(\log(1/\varepsilon) + 4)\}.$$

Moreover $j_\varepsilon \leq 7/4$ for every $\varepsilon \geq 10^{-6}$, and $j_\varepsilon \leq 2$ for every $\varepsilon \geq 10^{-7}$.

Proof. For $U = \log(1/\varepsilon)$ set $R = \log(U + 4)$ and $L = \pi/(8R)$, so $q = 1/2$ and $L < 1$. Equations (12)–(13) give $v(L) \geq 16L^2/5$ and $\sigma_R^2/\varepsilon < 1$. For the latter, with $t = U + 4 \geq 4$, use $169t^4 e^{-3t^2+t-4} < 1$; its logarithmic derivative is negative, and the endpoint is $43264e^{-48} < 1$. Thus (14) gives $j_\varepsilon^2 \leq 4(1 + 20R^2/\pi^2) < 16R^2$. The other bound follows from $\|s\|_\infty \leq 1$.

For the finite caps use exact $R = 11/10$, $L = 2/5$, the lower $v(L) \geq 16L^2/(1 + 4L^2)$ and the full σ_R envelope. At $\varepsilon = 10^{-6}$ the right side of (14) is less than $2.999229 < 49/16$; at 10^{-7} it is less than $3.974030 < 4$. The supplied directed-arithmetic program checks these strict enclosures from the displayed formulas. The right side decreases with ε , proving both entire ranges. $\square$

4 Full residual covariance and signed certificates

The following application keeps the whole common-state covariance, including singular image Grams. Let $Q : \mathbb{C}^d \rightarrow \mathcal{K}$ be a finite column map and $X : \mathbb{C}^d \rightarrow \text{Dom } A$. Put

$$E = Q - (A + \varepsilon)X, \quad D = Q^*X + X^*Q - X^*(A + \varepsilon)X, \quad G = Q^*(A + \varepsilon)^{-1}Q.$$

The exact residual identity is $G - D = E^*(A + \varepsilon)^{-1}E$.

Theorem 4.1 (Factored covariance). *If $E = P_{\mathcal{K}}M_s e$ for $e : \mathbb{C}^d \rightarrow L^2(\mathbb{R})$, then*

$$0 \leq G - D \leq j_\varepsilon^2 e^* e. \quad (15)$$

There is also the Fourier upper bound

$$G - D \leq \int_{\mathbb{R}} \frac{\widehat{e}(\xi)^* \widehat{e}(\xi)}{v(\xi) + \varepsilon} \frac{d\xi}{2\pi}. \quad (16)$$

The same upper conclusions hold on a legitimate restricted form space with its own projection and reference identity.

Proof. With $J = M_s(A + \varepsilon)^{-1/2}$, one has $(A + \varepsilon)^{-1/2}E = J^*e$. Testing an arbitrary coefficient vector proves (15). For (16), variational duality and $\|s\|_\infty \leq 1$ bound the denominator $\mathfrak{a}[h] + \varepsilon\|h\|^2$ below by $\int (v + \varepsilon)|\widehat{sh}|^2 d\xi / (2\pi)$. Enlarge the allowed sh only for this upper bound, then apply Fourier Cauchy–Schwarz. No equality of constrained and unconstrained domains is used. $\square$

For the columns $Q_k = P_{\mathcal{K}}M_s q_k$, $q_k = 1_{[-R, R]} u^k$, a compact smooth constrained trial has the explicit complete factor

$$e_k = q_k - v(D)(sX_k) - \varepsilon X_k / s.$$

Its terms lie in L^2 : X_k/s is compact smooth, and the quadratic symbol upper suffices for $v(D)(sX_k) \in L^2$. The form representation gives $AX_k = P_{\mathcal{K}}M_s v(D)(sX_k)$ and hence operator-domain membership. Under another restriction this identity and membership must be paid separately.

The comparison in (15) multiplies the factor covariance e^*e , not the generally different projected covariance E^*E . Valid upper matrices from (15), (16) or their scalar convex combinations can be used; entrywise minima are not Loewner upper bounds.

For clarity, the complete signed receiver of [4] has retained cutoff M , frequency cutoff L , spatial cutoff R and phase error b_d . Write $d_M = |m_0| + 2\sum_{2 \leq n \leq M} \Lambda(n)/\sqrt{n}$, with $m_0 = \Psi(1/4) - \log \pi$, and let δ_M be the complete arithmetic remainder. If $b_M(L)$ is its one-sided high-band cap, the improved observation upper j gives

$$\tau = \sigma_R/\sqrt{\varepsilon} + b_d j, \quad \Gamma = \delta_M/\varepsilon + b_M(L) + d_M \tau(2j + \tau). \quad (17)$$

The spatial and arithmetic denominators retain their displayed values.

Corollary 4.2 (Complete sufficient certificate). *Let B be the complete bounded arithmetic perturbation specified in the next section. Use the signed finite coefficient matrix C and column map Q of that same full frequency-first enclosure. Suppose*

$$\overline{C} \geq C, \quad \overline{R} \geq e^*e, \quad \overline{G} \geq D + j^2 \overline{R}, \quad \overline{G} > 0,$$

and, for $\beta \geq 0$,

$$\overline{C} \leq \beta \overline{G}^{-1}, \quad \beta + \Gamma \leq 1.$$

Then the complete original operator $H = A + B$ satisfies $H \geq -\varepsilon I$ on its specified form domain.

Proof. Theorem 4.1 gives $G \leq \overline{G}$. For $Z = (A + \varepsilon)^{-1/2}Q$, $Z^*Z = G$, so $Z\overline{G}^{-1/2}$ is a contraction. Since $\beta \geq 0$, $ZCZ^* \leq Z\overline{C}Z^* \leq \beta I$, without assuming $C \geq 0$. The full enclosure with error (17) bounds $(A + \varepsilon)^{-1/2}(-B)(A + \varepsilon)^{-1/2}$ above by $(\beta + \Gamma)I$. Form congruence gives the conclusion. $\square$

5 Complete arithmetic norm growth

The same observation law also bounds the full sandwich, rather than only a finite compression. The original paired arithmetic satisfies

$$B = P_{\mathcal{K}} \left[m_0 M_s^2 - \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} M_s (S_{\log n} + S_{-\log n}) M_s \right] \Big|_{\mathcal{K}},$$

where $S_a f(u) = f(u + a)$. Its complete norm tail is $\delta_M = 640(M + 1)^2 / 20^{M+1}$ [4]. Therefore

$$\|(A + \varepsilon)^{-1/2}(-B)(A + \varepsilon)^{-1/2}\| \leq d_M j_\varepsilon^2 + \delta_M / \varepsilon. \quad (18)$$

An elementary complete-prime estimate gives $d_M \leq 6 + 16\sqrt{M}$. Indeed valuations of $\binom{2n}{n}$ give $\Psi_\Lambda(2n) - \Psi_\Lambda(n) \leq 2n \log 2$, where $\Psi_\Lambda(x) = \sum_{n \leq x} \Lambda(n)$. Dyadic summation gives $\Psi_\Lambda(x) \leq 4x \log 2$, and partial summation yields $\sum_{n \leq M} \Lambda(n) / \sqrt{n} \leq 8\sqrt{M} \log 2$. Every prime power is retained.

For $U = \log(1/\varepsilon) \geq 0$ choose $M = \lceil U + 5 \rceil$. Since $\log 20 > 2$,

$$\delta_M / \varepsilon \leq 640(U + 7)^2 e^{-U-12} < 1.$$

Combining (18) and Proposition 3.2 proves

$$\|(A + \varepsilon)^{-1/2}(-B)(A + \varepsilon)^{-1/2}\| \leq 1 + 16(6 + 16\sqrt{U + 6}) \log^2(U + 4). \quad (19)$$

Thus the full arithmetic norm is $O(\sqrt{\log(1/\varepsilon)} [\log \log(1/\varepsilon)]^2)$. This upper uses absolute arithmetic estimates; the signed condition in Corollary 4.2 remains a separate one-sided matrix inequality.

A A calibrated scalar high-frequency cap

The finite observation bounds improve the low-band phase and residual factors. The separate high-band cap in (17) is also required. Here is an exact calibration in the same theta normalization. Set

$$v_5(\xi) = -m_0 + 2 \sum_{n=2}^5 \frac{\Lambda(n)}{\sqrt{n}} \cos(\xi \log n), \quad b_5(L) = \max \left\{ 0, \sup_{|\xi| > L} \frac{v_5(\xi)}{v(\xi)} \right\}.$$

In particular $\Lambda(4) = \log 2$. Then

$$b_5(L) > \frac{627}{10000} > \frac{1}{16} \quad (0 < L \leq 10^{66}), \quad b_5(10^{67}) < \frac{31}{500} < \frac{1}{16}. \quad (20)$$

The first inequality uses the single exact rational witness $\xi_* = 10^{66} + 135376/10$. Directed evaluation gives $v_5(\xi_*)/v(\xi_*) > 0.06271633994$. A different enclosure avoids complex digamma: with $t = \xi_*/2$, $F_t(r) = t^2/[r(r^2 + t^2)]$, $c = 128 + 1/4$ and $S = \sum_{j=0}^{127} F_t(j + 1/4)$, monotonicity gives

$$S + \frac{1}{2} \log(1 + t^2/c^2) \leq v(\xi_*) \leq S + \frac{1}{2} \log(1 + t^2/c^2) + F_t(c).$$

Together with $\Psi(1/4) = -\gamma - \pi/2 - 3 \log 2$, this independently gives the ratio lower bound $0.06271476967 > 627/10000$.

For the entire upper half-lines, the same decreasing-sum comparison with one term proves

$$v(L) \geq v_{\text{low}}(L) := \frac{16L^2}{1 + 4L^2} + \frac{1}{2} \log(1 + 4L^2/25).$$

Thus $b_5(L) \leq d_5/v_{\text{low}}(L)$, where $d_5 = |m_0| + 2 \sum_{n=2}^5 \Lambda(n)/\sqrt{n}$. At $L = 10^{67}$, directed evaluation gives $d_5/v_{\text{low}}(L) < 0.061984453 < 31/500$. This covers every omitted frequency, not a grid. Both scalar representations and the finite observation caps are reconstructed by the supplied `verify_bounds.py`.

For comparison, use the particular global Taylor majorant

$$b_d = \sqrt{2/\pi} \frac{(RL)^{d+3/2}}{(2d+3)(d+1)!}$$

in (17), $j = 7/4$, $10^{-6} \leq \varepsilon \leq 1$ and a finite cap $\beta \geq 15/16$. Success of that sufficient inequality requires $L > 10^{66}$ by (20). It also requires $R > 1/10$: the first positive theta atom gives $s(1/10)^2 > 1/4$, so for $R \leq 1/10$ any valid $\sigma_R > 1/2$, and $d_5\tau^2 > 1$ since $d_5 > 5$. Write $Z = RL$, $n = d + 1$. If $n \leq Z/2$, then $n! \leq n^n$ gives

$$b_d \geq \sqrt{2/\pi} \sqrt{Z} \frac{2^n}{2n+1} \geq \frac{2}{3} \sqrt{2/\pi} \sqrt{Z} > \frac{1}{2} \sqrt{Z} > 1,$$

again exceeding the available product-error budget. Hence this specified scalar/Taylor design requires $n > RL/2 > 5 \cdot 10^{64}$. The conclusion concerns its displayed sufficient majorant.

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