

# Moment certificates for homogeneous stable count laws

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## Abstract

Let  $P$  be a probability law on nonnegative integer count vectors of total size  $n$ , with a homogeneous real-stable probability generating polynomial and positive mean vector  $t$ . We compare  $P$  with  $M = \text{Mult}(n, t/n)$  using the weighted covariance deficit

$$W = \frac{1}{2} \left( m - 1 - \sum_{i=1}^m \frac{\text{Var}(A_i)}{t_i} \right).$$

We prove  $\text{TV}(P, M) \leq \min\{1, (147/68)W\}$ , independently of degree, category count and the mean vector. The argument transfers the centered elementary-symmetric-polynomial inequality of Han and Niles-Weed through Banach's theorem and an orthogonal multinomial density expansion. It retains explicit low orders, handles exceptional discrete root profiles, and combines exact polynomial certificates with positivity of the likelihood ratio. A classical hyperbolic-slice reduction also gives the stronger bound  $\text{TV}(P, M) \leq \min\{1, 2W\}$  for every  $1 \leq n \leq 50$ . Standalone rational-arithmetic code verifies all finite and continuous-domain certificates. Every all-degree linear coefficient must be at least two; whether two suffices in all degrees remains open. The results include nonfactorizable stable generating polynomials and make no algorithmic or statistical-estimation claim.

## 1 Setting and relation to prior work

A nonzero polynomial is *real stable* if it has real coefficients and no zero when every variable has positive imaginary part. Let

$$p(x) = \sum_{|a|=n} P(a)x^a, \quad P(a) \geq 0, \quad p(\mathbf{1}) = 1$$

be homogeneous and real stable. Here  $|a| = \sum_i a_i$  and  $A$  has law  $P$ . Put  $t_i = \mathbb{E}A_i > 0$ ,  $q = t/n$ ,  $L(x) = q \cdot x$ , and  $M = \text{Mult}(n, q)$ . Total variation is  $\text{TV}(P, M) = \frac{1}{2} \sum_a |P(a) - M(a)|$ . Define

$$B = \frac{1}{2} \left( \text{diag}(t) - \frac{tt^\top}{n} - \text{Cov}(A) \right), \quad W = \text{Tr}(\text{diag}(t)^{-1}B). \quad (1)$$

We will prove  $B \succeq 0$ . The identity  $\sum_i t_i = n$  gives the formula for  $W$  in the abstract.

Multinomial approximation for dependent random variables is classical; Loh's Stein-method treatment [7] is a central predecessor. Ehm's binomial approximation [3] already treats the bivariate stable subclass, whose generating polynomials factor into linear forms. Roos [9] develops approximation of convolution products, including independent categorical sums. We do not claim that dependent multinomial approximation, or moment-based approximation in these subclasses, is new.

The closest methodological predecessor is Han and Niles-Weed [5]. Their analysis of permutation mixtures combines centered elementary symmetric polynomials, Banach transfer and orthogonal chi-square expansions. Their Theorem 4.3(2), equation (27), supplies the coefficient inequality used below. Related Banach and coefficient methods occur in Roos [10]; later centered-vector and matrix inequalities appear in [6]. Our argument adapts these tools to the centered hyperbolic expansion of an arbitrary homogeneous stable count law. For categorical observations, a single permutation mixture of independent coordinates has the same count generating polynomial as the underlying product of categorical linear forms. A general exchangeable mixture representation instead yields a mixture of independent reference laws; it does not directly give our one mean-matched multinomial or the deficit in (1).

The contribution here is this full-stable-class transfer, the refined universal covariance certificate with coefficient 147/68, and a reproducible finite-degree verification of coefficient two. Banach's theorem, the centered ESP inequality, hyperbolic-slice extremality, and the orthogonal-expansion architecture are credited inputs rather than new general mechanisms. Fixed degree and category count are separate parameters throughout.

## 2 Centered expansion and the exact density identity

Degree one is exactly multinomial. In the lemmas below we therefore assume  $n \geq 2$ .

**Lemma 1.** *There are homogeneous polynomials  $R_k$  of degree  $k$ ,  $2 \leq k \leq n$ , invariant under  $x \mapsto x + a\mathbf{1}$ , such that*

$$p(x) = L(x)^n + \sum_{k=2}^n R_k(x) L(x)^{n-k}. \quad (2)$$

Moreover  $B \succeq 0$ ,  $B\mathbf{1} = 0$ , and  $R_2(x) = -x^\top Bx$ .

*Proof.* For real  $x$ , set  $y = x - L(x)\mathbf{1}$ . The polynomial  $p(z\mathbf{1} + y)$  is monic of degree  $n$ . Every root is real: an upper-half-plane root contradicts stability, and a lower-half-plane root has an upper conjugate. Its coefficient of  $z^{n-1}$  is  $t \cdot y = 0$ . Write its roots as  $\lambda_1, \dots, \lambda_n$ , so  $\sum_j \lambda_j = 0$ , and let  $R_k(x)$  be its coefficient of  $z^{n-k}$ . This defines homogeneous polynomials, invariant under addition of  $a\mathbf{1}$  because  $y$  is unchanged. Substitution  $z = L(x)$  gives (2).

Write  $H = \nabla^2 p(\mathbf{1})$ . The coefficient of  $z^{n-2}$  is  $\frac{1}{2}y^\top Hy$ . Euler's identity gives  $H\mathbf{1} = (n-1)t$  and  $\mathbf{1}^\top H\mathbf{1} = n(n-1)$ , so

$$R_2(x) = \frac{1}{2}x^\top \left( H - \frac{n-1}{n} tt^\top \right) x.$$

Since  $H = \text{Cov}(A) + tt^\top - \text{diag}(t)$ , this is  $-x^\top Bx$ . On the other hand the root identity gives

$$R_2(x) = -\frac{1}{2} \sum_j \lambda_j^2, \quad \sum_j \lambda_j^2 = 2x^\top Bx. \quad (3)$$

Thus  $B \succeq 0$ ; the displayed formula or translation invariance gives  $B\mathbf{1} = 0$ . □

If  $W = 0$ , positive  $t$  and  $B \succeq 0$  imply  $B = 0$ . Equation (3) then makes every centered root zero for every real  $x$ , so all  $R_k$  vanish and  $P = M$ . This proves rigidity without dividing by  $W$ . The cases  $n = 1$  or  $m = 1$  are also exact multinomial laws. Zero-mean categories can be deleted because their nonnegative counts vanish almost surely.

For a homogeneous polynomial  $R_k$ , let  $T_k$  denote its symmetric  $k$ -linear form, with  $R_k(x) = T_k(x, \dots, x)$ . Our ordered-index convention means that a monomial coefficient at  $\alpha$  is  $(k!/\alpha!)T_{k,\alpha}$ .

**Lemma 2.** Let  $\nu_k$  be the signed coefficient measure of  $R_k L^{n-k}$ . There exist mutually orthogonal mean-zero densities  $U_k = d\nu_k/dM$  satisfying

$$\frac{P}{M} - 1 = \sum_{k=2}^n U_k, \quad \chi^2(P||M) = \sum_{k=2}^n \mathbb{E}_M U_k^2, \quad (4)$$

and

$$\mathbb{E}_M U_k^2 = \binom{n}{k}^{-1} \sum_{i_1, \dots, i_k} \frac{T_k(e_{i_1}, \dots, e_{i_k})^2}{q_{i_1} \cdots q_{i_k}}. \quad (5)$$

*Proof.* Translation invariance of  $R_k$  gives  $T_k(\mathbf{1}, v_2, \dots, v_k) = 0$ : first differentiate the diagonal polynomial and then polarize its remaining arguments. Let  $X_1, \dots, X_n$  be independent categorical variables of law  $q$  and set

$$H_k(i_1, \dots, i_k) = \frac{T_k(e_{i_1}, \dots, e_{i_k})}{q_{i_1} \cdots q_{i_k}}, \quad U_k = \binom{n}{k}^{-1} \sum_{|S|=k} H_k(X_S).$$

Symmetry makes  $U_k$  a function of the count vector. Summing any one argument of  $H_k$  against  $q$  gives zero by the contraction identity. Hence the kernel is completely conditionally degenerate. For each fixed subset  $S$ ,

$$\mathbb{E}[H_k(X_S) x^{\text{counts}(X)}] = R_k(x) L(x)^{n-k}.$$

Coefficient comparison after symmetrization proves  $\nu_k(a) = M(a)U_k(a)$ . All  $M(a)$  on  $|a| = n$  are positive. In a product of two kernels associated with different subsets, an index in the symmetric difference occurs in only one kernel; integrate that independent categorical observation to obtain zero. This applies both within an order and between distinct orders. Thus only identical subsets contribute to the square norm, giving (5), while (2) gives (4). All operations are finite sums.  $\square$

### 3 Hilbert tensor transfer and a universal certificate

We use Banach's classical real Hilbert-space theorem [1, 2]: the norm of a symmetric multilinear form on arbitrary unit-vector tuples equals its norm on diagonal tuples. It is also the input in [5, Lemma 4.2].

**Lemma 3.** If  $|R_k(x)| \leq c(x^\top Bx)^{k/2}$  for all real  $x$ , then

$$\mathbb{E}_M U_k^2 \leq \frac{c^2 (nW)^k}{\binom{n}{k}}. \quad (6)$$

*Proof.* For  $v \in \ker B$ , the polynomial  $R_k(x + sv)$  is bounded independently of real  $s$ , hence is constant. Differentiation and polarization give  $T_k(v, v_2, \dots, v_k) = 0$ . Thus  $T_k$  descends to the real Hilbert quotient by  $\ker B$ , with squared norm  $x^\top Bx$ . Banach's theorem there implies

$$|T_k(v_1, \dots, v_k)| \leq c \prod_j \sqrt{v_j^\top B v_j}.$$

Apply this to coordinate vectors in (5) and sum the products to obtain

$$\mathbb{E}_M U_k^2 \leq \binom{n}{k}^{-1} c^2 \left( \sum_i \frac{B_{ii}}{q_i} \right)^k = \binom{n}{k}^{-1} c^2 (nW)^k.$$

No inverse of  $B$  is used.  $\square$

Han and Niles-Weed [5, Theorem 4.3(2)] prove that a real centered vector  $z$  with  $\sum_j z_j^2 = n$  satisfies  $|e_k(z)|^2 \leq 10 \binom{n}{k}$ . Scaling the centered roots in (3) therefore yields, with  $u = x^\top Bx$ ,

$$|R_k(x)|^2 \leq 10 \binom{n}{k} (2u/n)^k.$$

The zero case  $u = 0$  follows directly from the roots. Lemmas 2 and 3 now give

$$\mathbb{E}_M U_k^2 \leq 10(2W)^k, \quad \chi^2(P\|M) \leq \frac{40W^2}{1-2W} \quad (2W < 1). \quad (7)$$

The next elementary bound retains positivity of the actual density. It is the upper-distance branch of the classical sharp relation between chi-square divergence and total variation; see Guntuboyina, Saha and Schiebinger [4, Corollary 5.3], which credits Gilardoni's Proposition 2.1. We give the short proof needed here and make no claim of a new general divergence inequality.

**Lemma 4.** *For probability measures  $P \ll M$ , write  $v = \chi^2(P\|M)$  and  $\delta = \text{TV}(P, M)$ . If  $\delta > 1/2$ , then  $v \geq \delta/(1-\delta)$ .*

*Proof.* Let  $Y = dP/dM - 1$  and  $a = M(Y < 0)$ . Then  $\mathbb{E}Y_+ = \mathbb{E}Y_- = \delta$  and  $Y_- \leq 1$ , so  $a \geq \delta$ . Cauchy-Schwarz on the two parts of the partition gives

$$v \geq \frac{\delta^2}{a} + \frac{\delta^2}{1-a} = \frac{\delta^2}{a(1-a)}.$$

For  $a \geq \delta > 1/2$ ,  $a(1-a) \leq \delta(1-\delta)$ . The degenerate partition cases force  $\delta = 0$  and do not occur here.  $\square$

The uniform coefficient estimate (7) is sufficient for an absolute linear certificate. To obtain the sharper coefficient 147/68, we retain individual low-order terms before using its geometric tail. We first establish the finite-degree result and then give the full large-degree argument.

## 4 Exact finite-degree certificates

The coefficient extremum reduction used next is classical: see Riener and Schabert [8, Theorem 2.8 and Corollary 2.10], as well as the real-vector reduction in [5]. We include the needed special proof.

**Lemma 5.** *For fixed  $n \geq 3$  and  $3 \leq k \leq n$ , the maximum of  $|e_k(\lambda)|$  on  $\sum_i \lambda_i = 0$ ,  $\sum_i \lambda_i^2 = 2$  is attained at a vector with two distinct values.*

*Proof.* The slice is nonempty and compact. Maximize either sign of  $e_k$ , and among maximizers select one with the largest number  $d$  of distinct roots of  $f(z) = \prod_i (z - \lambda_i)$ . If  $3 \leq d < n$ , write  $f = hg$ , where  $g$  contains each distinct linear factor once and  $h$  retains multiplicity minus one. For either sufficiently small sign of  $\varepsilon$ ,  $g + \varepsilon$  has  $d$  simple real roots near the old roots. Thus  $f + \varepsilon h = h(g + \varepsilon)$  is real-rooted and preserves the first two coefficients, since  $\deg h = n - d \leq n - 3$ . If the target coefficient of  $h$  is nonzero, one sign improves the objective. Otherwise the objective is constant. Every repeated old root remains through  $h$  and also supplies a new nearby root through  $g + \varepsilon$ ; no new root equals an old one. Taking disjoint neighborhoods of distinct old roots shows the distinct-root count strictly increases, a contradiction. If  $d = n$ , perturbing by  $\varepsilon z^{n-k}$  preserves the first two coefficients and real-rootedness for both small signs, but changes the objective. Hence  $d \leq 2$ . The positive square sum rules out  $d = 1$ . Apply this argument to both signs to obtain the absolute-value assertion.  $\square$

For  $r + s = n$ , a normalized two-valued vector has values  $a$  repeated  $r$  times and  $b = -ra/s$  repeated  $s$  times, with  $a^2 = 2s/(nr)$ . Thus define the exact rational constants

$$c_{n,k}^2 = \max_{1 \leq r < n} \left( \frac{2s}{nr} \right)^k \left[ \sum_{j=\max(0,k-s)}^{\min(k,r)} \binom{r}{j} \binom{s}{k-j} (-r/s)^{k-j} \right]^2, \quad (8)$$

$$A_{n,k} = \frac{n^k c_{n,k}^2}{\binom{n}{k}}, \quad V_n(w) = \sum_{k=2}^n A_{n,k} w^k.$$

For  $k = 2$ ,  $e_2 = -1$  on the normalized slice, so the same formula applies. Homogeneity, Lemma 5 and the tensor transfer prove

$$\chi^2(P \| M) \leq V_n(W). \quad (9)$$

In particular these constants come from proved extrema and finite arithmetic, not an unproved fit to sampled roots.

**Proposition 6.** *If*

$$S_n = \sum_{k=2}^n A_{n,k} 4^{-(k-2)} \leq 16, \quad D_n = \sum_{k=2}^n A_{n,k} \frac{(k-1)^{k-1}}{2^{k-1} k^k} \leq 2,$$

*then every law of degree  $n$  in our setting satisfies  $\text{TV}(P, M) \leq \min\{1, 2W\}$ .*

*Proof.* For  $0 < W \leq 1/4$ , nonnegative coefficients give  $V_n(W) \leq S_n W^2 \leq 16W^2$ , and  $\delta \leq \sqrt{v}/2$  proves  $\delta \leq 2W$ . For  $1/4 < W < 1/2$ , differentiation gives

$$\max_{0 \leq w \leq 1/2} w^{k-1} (1 - 2w) = \frac{(k-1)^{k-1}}{2^{k-1} k^k}.$$

Consequently  $(1 - 2W)V_n(W)/W \leq D_n \leq 2$ , hence  $v \leq 2W/(1 - 2W)$ . If  $\delta \leq 1/2$  the conclusion is immediate. Otherwise Lemma 4 gives  $\delta \leq v/(1 + v) \leq 2W$ . For  $W \geq 1/2$  use  $\delta \leq 1$ , and for  $W = 0$  use rigidity.  $\square$

**Theorem 7.** *For every degree  $1 \leq n \leq 50$ , every category count, and every positive mean vector in the stable setting,*

$$\text{TV}(P, \text{Mult}(n, t/n)) \leq \min\{1, 2W\}.$$

*Proof.* Degree one is exact. For  $2 \leq n \leq 50$ , the accompanying standalone program `verify_coefficients.py` evaluates every candidate in (8) with integer and rational arithmetic, forms  $S_n, D_n$ , and checks their exact inequalities against 16 and 2. All 49 degrees pass. The largest  $S_n$  is 981/200 at  $n = 6$ , and all  $D_n < 141/100 < 2$ . Proposition 6 proves the result.

For reproducibility, the program uses integer roots  $s$  repeated  $r$  times and  $-r$  repeated  $s$  times. Their squared sum is  $nrs$ ; their degree- $k$  elementary coefficient is

$$b_{r,s,k} = \sum_j \binom{r}{j} \binom{s}{k-j} s^j (-r)^{k-j}.$$

The normalized squared candidate is  $b_{r,s,k}^2 / (nrs/2)^k$ . This is algebraically identical to (8), avoids square roots entirely, and is the full finite computation required for the theorem. The program enumerates all  $r = 1, \dots, n-1$  and all  $k = 2, \dots, n$ , using Python's standard-library arbitrary-precision integers and `Fraction`. It checks the stated maximum of  $S_n$  and the stronger common rational bound  $D_n < 141/100$ . There is no floating-point decision in the certificate. The analytic reduction and finite code are both parts of this proof; no inference to untested degrees is made.  $\square$

For illustration, at  $n = 8$  the coefficients  $A_{8,2}, \dots, A_{8,8}$  are

$$\frac{16}{7}, \frac{256}{49}, \frac{288}{35}, \frac{4096}{189}, \frac{256}{7}, \frac{12288}{125}, 256.$$

Here  $S_8 = 12558191/2646000$  and  $D_8 = 15114877380793361149/15738151816396800000$ . Although  $V_8(1/2)/(1/2)^2 > 16$ , the positive-density criterion succeeds. Thus discarding positivity at the last TV step would leave an avoidable obstruction.

## 5 Sharper low orders in large degree

For the remaining argument suppose  $n \geq 51$ . We normalize centered real root vectors by  $\sum_i z_i^2 = n$ . If

$$\frac{|e_k(z)|^2}{\binom{n}{k}} \leq b_k, \quad (10)$$

then scaling (3) and applying Lemma 3 gives  $\mathbb{E}_M U_k^2 \leq 2^k b_k W^k$ . All root extrema below use Lemma 5 one coefficient at a time; their maximizing profiles need not coincide.

### 5.1 Discrete exceptional profiles

**Lemma 8.** *If a centered, variance-normalized two-valued vector has either root value with multiplicity one or two, then (10) holds with  $b_k = 1$  at every order.*

*Proof.* Negation and permutation allow us to treat the smaller specified block. For multiplicity one the values are  $\sqrt{n-1}$  once and  $-1/\sqrt{n-1}$  repeated  $n-1$  times. Expansion gives

$$e_k = (-1)^k (1-k) \binom{n}{k} (n-1)^{-k/2}, \quad \frac{e_k^2}{\binom{n}{k}} = \frac{(k-1)^2 \binom{n}{k}}{(n-1)^k}.$$

The ratios at  $k = 0, 1$  are one and zero. At  $k = 2$  the ratio is  $n/[2(n-1)] \leq 1$ . For fixed  $k \geq 3$ , regard the last expression as a function of real  $n \geq k$ . Its logarithmic derivative is

$$-\frac{1}{n(n-1)} + \sum_{j=2}^{k-1} \frac{j-1}{(n-j)(n-1)} > 0,$$

because the  $j = 2$  term exceeds the negative term. It therefore increases to  $(k-1)^2/k! \leq 2/3$ . This last sequence starts at  $2/3$  for  $k = 3$  and decreases thereafter, since its next-term ratio is  $k^2/[(k+1)(k-1)^2] < 1$ .

For multiplicity two, assume  $n \geq 4$ ; the case  $n = 3$  is already multiplicity one after reflection. The values are  $\sqrt{(n-2)/2}$  twice and  $-\sqrt{2/(n-2)}$  repeated  $n-2$  times. Direct expansion gives

$$\frac{e_k^2}{\binom{n}{k}} = \binom{n}{k} \left( \frac{2}{n-2} \right)^k F(n, k)^2, \quad F(n, k) = \frac{(k-1)((k-4)n+4)}{4(n-1)}. \quad (11)$$

The orders zero and one are immediate. At order two the general centered identity gives ratio  $n/[2(n-1)] \leq 1$ . For order three, two-valued extrema have  $e_3 = nc/3$ , with  $c = (s-r)/\sqrt{rs}$ , and  $c^2 \leq (n-2)^2/(n-1)$ . Hence the ratio is at most  $\frac{2}{3}n(n-2)/(n-1)^2 < 1$ . For order four, the centered fourth-moment bound proved below gives ratio at most  $3n(n-2)/[8(n-1)(n-3)] \leq 1$  for  $n \geq 4$ ; the cleared difference  $5n^2 - 26n + 24$  is zero at four and increases thereafter.

For  $5 \leq k \leq 11$ , clearing positive denominators in (11) reduces the assertion to

$$16k!(n-1)^2(n-2)^k - 2^k(n)_k(k-1)^2((k-4)n+4)^2 \geq 0. \quad (12)$$

Put  $n = k + x$ . The companion exact program expands these seven integer polynomials. Every coefficient is strictly positive; their minimum coefficients, in order  $k = 5, \dots, 11$ , are

$$1408, 5120, 39168, 444416, 4986880, 55074816, 628633600.$$

The program constructs both products in (12) by integer linear-factor multiplication and checks every coefficient, so this proves the inequality for all real  $x \geq 0$ , not just a finite list of  $n$ .

For  $n \geq k \geq 12$ , the factor  $F$  is nonnegative and is at most  $nk(k-1)/[4(n-1)] \leq (3/11)k^2 < k^2/3$ . Also  $\binom{n}{k} \leq n^k/k!$  and

$$\left(\frac{n}{n-2}\right)^k \leq \left(\frac{k}{k-2}\right)^k < 9.$$

Indeed  $t \log(t/(t-2))$  decreases for  $t > 2$ , and at twelve  $6^{12} < 9 \cdot 5^{12}$ . Thus the ratio in (11) is at most  $2^k k^4/k! < 1$ . This holds at twelve because  $2^{12}12^4 < 12!$  and decreases thereafter, since its next-term ratio is  $2(k+1)^3/k^4 < 1$ . This completes the all-degree exceptional-profile proof.  $\square$

## 5.2 Orders two through six

**Lemma 9.** *For  $n \geq 51$ , the orthogonal density components satisfy  $\mathbb{E}_M U_k^2 \leq a_k W^k$  at orders two through six, where*

$$a_2 = \frac{51}{25}, \quad a_3 = \frac{16}{3}, \quad a_4 = \frac{2499}{400}, \quad a_5 = \frac{163268}{11421}, \quad a_6 = \frac{52859896832}{1412376245}.$$

*Proof.* Write  $p_j = \sum_i z_i^j$ , so  $p_1 = 0, p_2 = n$ . At order two  $e_2 = -n/2$ , giving  $a_2 = 2n/(n-1) \leq 51/25$ . At order three the computation in Lemma 8 bounds the scalar ratio by  $\frac{2}{3}n(n-2)/(n-1)^2 \leq 2/3$ , giving  $a_3 \leq 16/3$ .

At order four Newton's identity gives  $e_4 = (n^2 - 2p_4)/8$ . Centering implies  $z_i^2 \leq n-1$ , by Cauchy-Schwarz on the other coordinates. Therefore  $n \leq p_4 \leq n(n-1)$  and  $|e_4| \leq n(n-2)/8$ . After multiplication by  $2^4$  and division by  $\binom{n}{4}$ , the bound is  $6n(n-2)/[(n-1)(n-3)]$ . This decreases for  $n \geq 51$  and its value at 51 is  $2499/400$ .

For higher orders use the two-valued reduction with  $a = \sqrt{s/r}$ ,  $b = -\sqrt{r/s}$ ,  $r + s = n$ , and  $c = a + b$ . Its generating function  $F(T) = (1 + aT)^r(1 + bT)^s$  obeys

$$(1 + cT - T^2)F'(T) = -nTF(T), \quad (k+1)e_{k+1} = -cke_k - (n-k+1)e_{k-1}. \quad (13)$$

Put  $x = c^2$  and  $a_0 = 5n - 12$ . Recurrence (13) gives  $e_5 = nc(6x - a_0)/30$ , with  $0 \leq x \leq (n-2)^2/(n-1)$ . The maxima of  $x(6x - a_0)^2$  on this interval occur at its right endpoint or at  $a_0/18$ , if that point belongs to the interval: its derivative is  $(6x - a_0)(18x - a_0)$ , while zero and  $a_0/6$  give zero values. At the right endpoint the scalar ratio is

$$\frac{2}{15} \frac{n(n-2)(n-3)(n-4)}{(n-1)^4} < \frac{2}{15}.$$

At  $a_0/18$  it is

$$\frac{4}{1215} \frac{n(5n-12)^3}{(n-1)(n-2)(n-3)(n-4)} \leq \frac{100}{243} \frac{n(n-2)^2}{(n-1)(n-3)(n-4)}.$$

The right side decreases with  $n$ , since it is a constant times the product  $n/(n-1)$ ,  $(n-2)/(n-3)$  and  $(n-2)/(n-4)$ . Multiplying by  $2^5$ , its value at 51 is 163268/11421. The other endpoint contributes less than 64/15. The critical-point bound exceeds 64/15, so this proves the stated exact  $a_5$ .

At order six set  $y = c^2/n$ ,  $z_0 = 1/n$ . The recurrence gives

$$\frac{e_6}{n^3} = f_6(y, z_0) = -\frac{y^2}{6} + \frac{13 - 36z_0}{72}y - \frac{(1 - 4z_0)(1 - 2z_0)}{48}.$$

For profiles with  $r, s \geq 3$ , negation allows  $r \leq s$  and gives

$$0 \leq y = \frac{(s-r)^2}{nrs} \leq \frac{1}{r} \leq \frac{1}{3}, \quad 0 < z_0 \leq \frac{1}{51}.$$

The derivative of  $f_6$  in  $y$  is at least  $5/72 - z_0/2 > 0$ . Furthermore  $f_6(0, z_0) \geq -1/48$  and  $f_6(1/3, z_0) = 1/48 - z_0/24 - z_0^2/6 \leq 1/48$ . Hence  $|f_6| \leq 1/48$ . The corresponding energy coefficient is at most

$$\frac{2^6 6!}{48^2 \prod_{j=0}^5 (1 - j/51)} = \frac{115008417}{4237520}.$$

For multiplicity one the exact formula in Lemma 8 gives  $a_6 \leq 2^6(6-1)^2/6! = 20/9$ . For multiplicity two, (11) simplifies to

$$a_6(n) = \frac{320}{9}F_6(n), \quad F_6(n) = \frac{n(n-3)(n-4)(n-5)(n+2)^2}{(n-1)(n-2)^5}.$$

This function decreases for  $n \geq 51$ . Indeed

$$(\log F_6)' = -\frac{1}{n(n-1)} + \frac{1}{n-2} \left( \frac{1}{n-3} + \frac{2}{n-4} + \frac{3}{n-5} - \frac{8}{n+2} \right).$$

The bracket is at most  $6/(n-5) - 8/(n+2) \leq 0$  for  $n \geq 26$ . At  $n = 51$  the coefficient is 52859896832/1412376245, which exceeds both other profile bounds; the code checks these exact comparisons. Lemma 5 and the tensor transfer prove the result for every centered root vector.  $\square$

### 5.3 Fixed orders through twelve on an infinite multiplicity domain

We next prove a scalar coefficient-one estimate uniformly in both integer root multiplicities. It supplies the energy bound needed at orders seven through twelve. The finite polynomial computations below, including all exceptional rows, are generated by the accompanying standalone code.

**Lemma 10.** *For every real vector  $z$  with  $\sum_i z_i = 0$ ,  $\sum_i z_i^2 = n$ , and  $5 \leq k \leq \min(12, n)$ ,*

$$|e_k(z)|^2 \leq \binom{n}{k}.$$

*Consequently, for every law under consideration,*

$$\mathbb{E}_M U_k^2 \leq 2^k W^k, \quad 7 \leq k \leq \min(12, n). \quad (14)$$

*Proof.* By Lemma 5, it suffices to treat two-valued profiles. Reflect and permute so their multiplicities satisfy  $s \geq r$ ,  $n = r + s$ . Let  $E_k(r, s)$  be the elementary coefficient of the integer root vector consisting of  $r$  copies of  $s$  and  $s$  copies of  $-r$ . Its squared norm is  $nrs$ , so the desired inequality is precisely

$$D_k(r, s) = (n)_k (rs)^k - k! E_k(r, s)^2 \geq 0. \quad (15)$$

For each  $k = 5, \dots, 12$ , put  $a = \lceil k/2 \rceil$ ,  $r = a + u$ ,  $s = a + u + x$ . Newton's identities reconstruct the exact polynomial  $E_k$  from

$$p_j = rs^j + s(-r)^j, \quad E_0 = 1, \quad E_h = \frac{1}{h} \sum_{j=1}^h (-1)^{j-1} E_{h-j} p_j.$$

The program expands (15) under this substitution and checks that every coefficient in  $u, x$  is nonnegative. The numbers of nonzero coefficients, in increasing order of  $k$ , are

$$121, 168, 225, 288, 361, 440, 529, 624.$$

Thus (15) holds for the whole real quadrant  $u, x \geq 0$ , covering every integer  $s \geq r \geq \lceil k/2 \rceil$ , with no upper cutoff on either multiplicity or on  $n$ .

The construction also has an independent exact polynomial-identity check. Since  $p_1 = 0$ , each Newton expansion term in  $E_k$  has at most  $\lfloor k/2 \rfloor$  power-sum factors. Its total degree is at most  $k + \lfloor k/2 \rfloor$  and its  $x$ -degree at most  $k$ . Therefore  $D_k$  has  $u$ -degree at most  $3k$  and  $x$ -degree at most  $2k$ . On the complete grid  $0 \leq u \leq 3k$ ,  $0 \leq x \leq 2k$ , the program independently compares the polynomial with the direct formula

$$E_k(r, s) = \sum_j \binom{r}{j} \binom{s}{k-j} s^j (-r)^{k-j}.$$

These 4068 exact comparisons determine the eight polynomials by interpolation. Coefficient non-negativity, rather than those finite evaluations alone, proves the infinite-domain inequality.

It remains to cover  $r < \lceil k/2 \rceil$ . Multiplicities one and two are already covered by Lemma 8. The remaining pairs are exactly

|     |   |   |      |      |         |          |
|-----|---|---|------|------|---------|----------|
| $k$ | 7 | 8 | 9    | 10   | 11      | 12       |
| $r$ | 3 | 3 | 3, 4 | 3, 4 | 3, 4, 5 | 3, 4, 5. |

We give the cleared polynomial construction for these twelve cases. Write

$$E_k = (-r)^k \binom{n}{k} \frac{N_{r,k}(n)}{r^r (n)_r},$$

$$N_{r,k}(n) = \sum_{j=0}^{\min(r,k)} (-1)^j \binom{r}{j} n^j r^{r-j} (k)_j (n-j)_{r-j}.$$

To derive this formula, express the generating function as  $(1-rT)^n [1+nT/(1-rT)]^r$  and compare coefficients; then use  $\binom{n-j}{k-j} / \binom{n}{k} = (k)_j / (n)_j$ . Dividing by  $\sqrt{rs}$  normalizes the roots to squared norm  $n$ . Clearing positive denominators in the scalar inequality gives

$$k! r^{2r} (n)_r^2 (n-r)^k - r^k (n)_k N_{r,k}(n)^2 \geq 0. \quad (16)$$

For every listed pair the code substitutes  $n = \max(k, 2r) + x$ , expands both products, and checks that all coefficients are nonnegative. Since  $x \geq 0$  is exactly the admissible range, this covers all remaining degrees. It independently checks the displayed coefficient factorization against the direct binomial formula on a degree-sized grid. The two-root reduction and Lemma 3 finish the proof.  $\square$

For clarity, the finite computations here prove only the stated fixed orders. No induction in symbolic  $k$  or extrapolation to arbitrary order is asserted.

We also specify the final probability certificate mechanism. On  $[a, b]$  substitute  $w = a + (b - a)t$ . If a polynomial of degree  $d$  becomes  $\sum_j v_j t^j$ , its Bernstein coefficients are

$$\beta_i = \sum_{j=0}^i v_j \frac{\binom{i}{j}}{\binom{d}{j}}, \quad p(a + (b - a)t) = \sum_{i=0}^d \beta_i \binom{d}{i} t^i (1 - t)^{d-i}. \quad (17)$$

All basis functions are nonnegative on  $[0, 1]$ . Nonnegative coefficients therefore certify nonnegativity on the whole interval. Exact interval bisection with both children retained preserves coverage; no midpoint test is an acceptance gate.

## 6 The universal coefficient 147/68

**Theorem 11.** *For every homogeneous stable count law in our setting,*

$$\text{TV}(P, \text{Mult}(n, t/n)) \leq \min\{1, (147/68)W\}.$$

*Proof.* If  $n \leq 50$ , apply Theorem 7. Suppose  $n \geq 51$ , and let  $a_2, \dots, a_6$  be as in Lemma 9, with  $a_k = 2^k$  for  $7 \leq k \leq 12$ . Define

$$V(w) = \sum_{k=2}^{12} a_k w^k + \frac{81920w^{13}}{1 - 2w}, \quad N(w) = (1 - 2w)V(w).$$

Orthogonality, Lemma 9, (14), and the credited Han–Niles–Weed estimate  $\mathbb{E}U_k^2 \leq 10(2W)^k$  for  $k \geq 13$  give  $v = \chi^2(P\|M) \leq V(W)$  for  $0 \leq W < 1/2$ . The finite tail is bounded by the infinite geometric tail, since all its terms are nonnegative; no unproved scalar coefficient-one estimate is used at higher orders.

Put  $C = 147/68$ . On  $0 < W \leq 1/(2C) = 34/147$ ,  $V(W)/W^2$  increases, and its exact value at  $34/147$  is less than  $4C^2$ . The accompanying code checks this rational comparison directly. Hence  $\delta \leq \sqrt{v}/2 \leq CW$  throughout this interval.

On  $34/147 < W < 68/147$ , suppose  $\delta > CW$ . Lemma 4 implies  $v > CW/(1 - CW)$ . The opposite weak inequality follows from  $v \leq V(W)$  and

$$Q(W) = C(1 - 2W) - (1 - CW)\frac{N(W)}{W} \geq 0. \quad (18)$$

Here  $N(W)$  is divisible by  $W^2$ , so  $Q$  is a polynomial; both original rational denominators are positive. The code constructs  $Q$  from the displayed constants. The exact Bernstein transformation (17) certifies (18) on  $[34/147, 68/147]$ . Six covering subintervals suffice. It checks their endpoints and exact adjacency, and every coefficient on every accepted interval is nonnegative. Therefore (18) is a proved continuous-domain inequality, contradicting the assumed strict lower bound on  $v$ . Finally,  $W \geq 68/147$  is covered by  $\delta \leq 1 \leq CW$ , and  $W = 0$  by rigidity.  $\square$

The program `verify_refined.py` checks every finite computation used here. These are the degrees two through fifty, the seven polynomial identities for multiplicity two, the eight fixed-order quadrant polynomials, their twelve remaining exceptional rows, and the final global gap and endpoint comparison. The program is self-contained and uses only the Python standard library. The option `--output`, followed by a filename, saves every exact rational certificate. Explicit failure checks remain active under Python optimization. The analytic reductions, the factorial tail for multiplicity two, and the finite certificate calculations are all dependencies of the theorem. A finite root sample or extrapolation cannot replace them.

## 7 Limitations and the coefficient-two question

Every universal coefficient is at least two. For even  $n$ , take the stable monomial  $p(x_1, x_2) = x_1^{n/2} x_2^{n/2}$ . Then  $W = 1/2$  and

$$\text{TV}(P, \text{Mult}(n, 1/2)) = 1 - 2^{-n} \binom{n}{n/2} \rightarrow 1.$$

This elementary binomial example gives the lower barrier, not a claim of a new classical asymptotic. Consequently the optimal all-degree coefficient satisfies

$$2 \leq C_{\text{opt}} \leq \frac{147}{68}.$$

Whether  $C_{\text{opt}} = 2$  remains open here. Finite certification through fifty is not an all-degree proof, and the number fifty is a verified computational cutoff rather than a structural boundary. In particular no monotonicity of the finite criteria beyond that cutoff is asserted. An all-order constant-one centered ESP bound would strengthen one route, but is not assumed or proved in this paper.

The mean vector and covariance are treated as known inputs. We prove no sample-complexity result for estimating  $W$ , no practical testing guarantee, no sampler construction and no computational speedup. Stability is a substantive hypothesis; matching a few moments without it does not certify the full law. The finite result and universal theorem allow singular covariance deficit, repeated roots and zero coefficients. Signed centered-expansion terms are not themselves probability laws, so the positivity lemma is applied only to the full density  $P/M$ .

## Computational and AI assistance

The certificates use only the accompanying standard-library Python sources, which terminate with an explicit failure if a required inequality fails. The finite-degree calculation is in `verify_coefficients.py`; the complete refined verification, including that calculation, is independently self-contained in `verify_refined.py`. The latter optionally writes all exact rational certificate data to a reader-selected output file. No random sampling or external numerical package is used. Generative AI systems assisted with mathematical exploration, literature discovery, proof drafting, exact-code development and independent adversarial proof checks. These checks are internal AI assistance and are not external human refereeing. Mathematical claims are presented with their proofs and reproducible computation so that they can be assessed independently; the author retains responsibility for the manuscript. Classical inputs and the close Han–Niles–Weed methodology are explicitly attributed.

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