Local comparison counting for linear extensions and a bounded-range balance refinement

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Abstract

For uniform linear extensions of a finite poset with maximum incomparability degree at most D, we give a local stable-partition estimate and track the incomparabilities removed by conditioning. Every consistent specified order of three marked vertices has probability at least F_D^{-1}, where

$$F_D = \binom{D + \lfloor D/2 \rfloor}{\lfloor D/2 \rfloor} \binom{2D - \lfloor D/2 \rfloor}{D}.$$

This denominator has order $\Theta((27/4)^D/D)$. Substitution into the finite-boundary argument of Aires–Chan–Pak–Panova gives a sharper explicit bounded-range balance estimate. A complementary two-chain construction shows that a triple-order estimate cannot universally use even a constant multiple of the sharp pair probability: its rare order has reciprocal probability asymptotic to $16^k/(4\pi)$ at range $D = 2k$. These are quantitative counting refinements and a method limitation.

1 Introduction

The central counting step is the incomparability budget consumed by a first comparison. For a consistent order of three marked vertices, conditioning reduces the budget available to the second comparison. The resulting denominator F_D feeds the inherited bounded-range balance argument. A complementary two-chain construction shows why the sharp pair scale cannot simply be reused for triples.

This gives three connected conclusions: a conditioned triple-order bound, an explicit balance refinement, and a rare-triple obstruction. Their scales remain distinct: the sufficient denominator has order $(27/4)^D/D$, while the constructed rare events at range $D = 2k$ have reciprocal probability asymptotic to $16^k/(4\pi)$. The proofs below identify where the conditioning reduction, the imported structural hypotheses and the exact extension count enter.

1.1 Definitions and prior work

Let $P = (X, <_P)$ be a finite poset, with $e(P)$ linear extensions. A linear extension is an order-preserving bijection $f : X \to [|X|]$; all probabilities below use the uniform distribution on these bijections. Write $x \parallel y$ for incomparability, and put

$$\pi(P) = \max_{x \in X} |\operatorname{Inc}_P(x)|, \qquad \delta(P) = \max_{x \neq y} \min\{\mathbb{P}(f(x) < f(y)), \mathbb{P}(f(y) < f(x))\}.$$

The range condition is $\pi(P) \leq D$, for an integer $D \geq 2$. It is distinct from a width or size bound. Principal sets $x{\downarrow} = \{u : u \leq_P x\}$ and $x{\uparrow} = \{u : x \leq_P u\}$ are closed. Write $\delta(P; x, y) = \min\{\mathbb{P}(f(x) < f(y)), \mathbb{P}(f(y) < f(x))\}$ for the balance of a specified pair.

Fishburn's sharp pair-extremum theorem is an essential antecedent [1, Theorem 1]. With a designated m-antichain in an N-element poset, the largest probability of one orientation of a designated pair is

$$1 - \binom{N-m+2}{\lfloor (N-m+2)/2 \rfloor}^{-1}.$$

His equality classification permits two independent blocks with the designated vertices at opposite extremities, the block sizes including those endpoints differing by at most one; internal block orders are arbitrary, and the other designated antichain vertices are isolated. Thus the global sharp pair extremum and its shuffle examples are classical. Our pair proof below is a direct localization of this classical counting principle.

The specific contribution is the conditioned three-vertex estimate: the first comparison reduces the middle vertex's remaining incomparability budget. We also give a precise rare-triple obstruction at fixed range. These differ from maximizing a specified order on a prescribed antichain at fixed total size, the problem posed in [1, Section 3] and studied further in [2]. In particular, reversing a three-vertex order does not form its complement.

Our balance application refines the small-range estimate in [4, Theorem 1.3], whose gain above $C = (5-\sqrt{5})/10$ is $(D+1)^{-4(D+1)}/4096$. The BFT and finite-boundary mechanism is inherited from [3, 4]. Large-width and large-range qualitative advances already occur in [5]; they are not contributions of this note.

2 Local stable partitioning

Lemma 1. *For $x \parallel y$, define*

$$A = (x\downarrow) \setminus (y\downarrow), \quad B = (y\uparrow) \setminus (x\uparrow), \quad a = |A|, \quad b = |B|.$$

Then

$$\mathbb{P}(f(x) < f(y)) \geq \binom{a+b}{a}^{-1}.$$

In particular, if $\pi(P) \leq D$, every incomparable orientation has probability at least $q_D = \binom{2D}{D}^{-1}$.

Proof. The sets A, B are disjoint. Given an extension L, keep all positions occupied by $A \cup B$ and every outside vertex fixed. Refill those positions with the A-subsequence of L, then its B-subsequence, preserving the internal order in each subsequence. Members of A move weakly left, and members of B move weakly right.

There is no relation from B to A, since $y \leq_P v <_P u \leq_P x$ would contradict $x \parallel y$. Internal orders and relations from A to B remain valid. If $w \notin A \cup B$ is a predecessor of some $u \in A$, then $w \leq_P x$ and $w \notin A$ imply $w \leq_P y$. Thus w precedes every member of B. No earlier occupied B-position into which u moves is before w. Dually, an outside successor of a member of B follows every member of A, and cannot be crossed by moving B right. The other outside relation directions are preserved by the direction of movement. The resulting order is an extension with x before y.

The output together with the original binary A/B membership pattern uniquely recovers L. Every output therefore has at most $\binom{a+b}{a}$ preimages, proving the bound. Finally $A \subseteq \mathrm{Inc}_P(y)$ and $B \subseteq \mathrm{Inc}_P(x)$, so $a, b \leq D$, and the binomial coefficient is increasing in each size. □

Remark 2. For two independent chains of lengths a, b, with x atop the first and y at the bottom of the second, exactly one shuffle puts x before y. This is the classical shuffle extremizer underlying [1]. It proves sharpness of the local pair inequality.

Corollary 3. *A positive event specified solely by comparisons on $m \geq 1$ marked vertices has probability at least q_D^{m-1}.*

Proof. Choose a total order on the marked vertices realized by an extension satisfying the event, and successively impose its $m-1$ adjacent comparisons. Conditional on any consistent earlier comparisons, the distribution is exactly uniform on the augmented poset. A forced comparison costs probability one; otherwise Lemma 1 applies. Adding relations cannot increase incomparability degrees. The resulting total-order event implies the original event. Multiply the conditional bounds. With no marked vertices, a positive such event is the sure event. □

3 The three-vertex budget

Theorem 4. *For three distinct marked vertices a, b, c, if a before b before c is consistent with P and $\pi(P) \leq D$, then*

$$\mathbb{P}(f(a) < f(b) < f(c)) \geq F_D^{-1}, \qquad F_D = \binom{D + \lfloor D/2 \rfloor}{\lfloor D/2 \rfloor}\binom{2D - \lfloor D/2 \rfloor}{D}.$$

Proof. If $a <_P b$, the first comparison is forced and the second has probability at least q_D. The denominator below is at least $\binom{2D}{D}$, so this case is included.

Otherwise let $r = |(a\downarrow) \setminus (b\downarrow)|$, with $1 \leq r \leq D$. Lemma 1 gives $\mathbb{P}(f(a) < f(b)) \geq \binom{D+r}{r}^{-1}$. Every member of this ideal difference was incomparable with b and becomes a predecessor of b after imposing $a < b$. Thus in the conditional poset Q, $|\operatorname{Inc}_Q(b)| \leq D - r$. If $b < c$ is forced in Q, its probability is one. Otherwise the two exclusive principal sets for $b < c$ have sizes at most D and $D - r$. In either case,

$$\mathbb{P}(f(b) < f(c) \mid f(a) < f(b)) \geq \binom{2D - r}{D}^{-1}.$$

Consistency excludes a forced opposite orientation. Consequently the event probability is at least $1/H_D(r)$, where

$$H_D(r) = \binom{D + r}{r}\binom{2D - r}{D}.$$

For consecutive indices,

$$\frac{H_D(r+1)}{H_D(r)} = \frac{(D + r + 1)(D - r)}{(r + 1)(2D - r)}.$$

The numerator minus denominator is $D(D - 1 - 2r)$. Hence the maximum over $1 \leq r \leq D$ occurs at $\lfloor D/2 \rfloor$, with adjacent equal maximizers when D is odd. Its value is F_D. Taking $r = D$ also proves $F_D \geq \binom{2D}{D}$, as used above. □

For even $D = 2j$, $F_D = \binom{3j}{j}^2$; Stirling's formula, and the neighboring odd case, give

$$F_D = \Theta((27/4)^D/D).$$

This improves the denominator rate 16^D from applying the sharp range-only pair estimate twice without retaining the consumed budget.

4 The balance consequence

We record precisely the inherited analytic input. Set $K = 5 + 3\sqrt{5}$, and $h(x) = \mathbb{E}f(x)$. A BFT triple satisfies $h(x) \le h(y) \le h(z) \le h(x) + 2$ and is not a chain. Under $\delta(P) < 0.2764$, its surviving case is $x <_P z$, $y \parallel x, z$. The ACP defect R satisfies

$$R \le K(\delta(P) - C), \quad \mathbb{P}(f(y) < f(w) < f(x)) \le R, \quad \mathbb{P}(f(z) < f(w) < f(y)) \le R. \tag{I}$$

These use [4, Section 2.1 and Lemmas 2.3–2.4]. If no such positive three-vertex event exists, their structural proof gives $x \lessdot z$, y incomparable only with x, z, and a partition $L, \{x, y, z\}, U$, where $L = \{w : w <_P y\}$, $U = \{w : y <_P w\}$, with $L < z,\ x < U,\ L < U$.

For a connected nontrivial incomparability graph, order the vertices v_i by h, and let $M = \max_i |h(v_i) - i|$. Their Lemma 4.1 selects a consecutive BFT triple attaining M at an endpoint. Once its structural conclusions hold, the proof of their Lemma 4.2 gives

$$\max\{\delta(P; x, y), \delta(P; y, z)\} \ge C + M^2/441. \tag{II}$$

The original smaller range-dependent hypothesis is used there to obtain structure. After structure, put $\varepsilon = \max\{\delta(P; x, y), \delta(P; y, z)\} - C$. Its displayed estimates require only $\delta < 0.2764$ and $0 \le \varepsilon < 10^{-5}$: $M \le 20\sqrt{\varepsilon} + 11\varepsilon \le 21\sqrt{\varepsilon}$. Thus (II) applies with our re-established structure and these cutoffs; we do not invoke a changed-hypothesis statement as an unaudited black box.

Lemma 5. *If $\pi(P) \le D$ and its incomparability graph is connected and nontrivial, then $M \ge 1/(D + 1)$.*

Proof. The least-height vertex v_1 is minimal. A unique minimal vertex of a finite poset lies below every vertex and would be isolated in the incomparability graph. Choose a different minimal vertex u. Deleting u from an extension leaves at most $D + 1$ possible reinsertion slots, because every vertex preceding u must be incomparable with it. Prepending u is always valid. Therefore

$$\mathbb{P}(u \text{ is first}) = \frac{e(P - u)}{e(P)} \ge \frac{1}{D + 1}, \quad M \ge h(v_1) - 1 \ge \mathbb{P}(f(u) < f(v_1)) \ge \frac{1}{D + 1}.$$

□

Theorem 6. *For every finite non-chain P with $\pi(P) \le D$, $D \ge 2$,*

$$\delta(P) > C + \frac{1}{KF_D}. \tag{1}$$

Proof. For $D \le 6$, the known six-thin theorem [6] gives $\delta(P) \ge 1/3$, which is stronger: $C < 7/25,\ K > 11,\ F_D \ge 9$ imply $C + 1/(KF_D) < 7/25 + 1/99 < 1/3$.

Let $D \ge 7$, and set $\eta = 1/(KF_D)$. Incomparability components are ordinal-sum factors, so their internal comparison probabilities are unchanged. Work in a nontrivial connected component. A two-vertex component has balance $1/2$, and is immediate. Assume for contradiction $\delta < C + \eta$. The classical BFT theorem ensures $\delta \ge C$. We have $F_7 = 39600$, $K > 11$, and $C < 0.276394$, hence

$$\eta \le \eta_7 < 1/435600 < 0.000006, \qquad C + \eta < 0.2764.$$

The cutoffs in (I)–(II) hold. By (I), $R < 1/F_D$. Theorem 4 excludes every positive bad three-vertex event in (I). Structure therefore holds, so (II) and Lemma 5 give

$$\delta \ge C + \frac{1}{441(D + 1)^2} > C + \frac{1}{KF_D}.$$

For the last comparison at $D = 7$, use $11F_7 = 435600 > 441 \cdot 64 = 28224$. For all larger D, hold a maximizing r fixed in $H_D(r)$. Its first binomial factor increases with D, and Pascal's recurrence gives $\binom{2D+2-r}{D+1} \geq 2\binom{2D-r}{D}$. Thus $F_{D+1} \geq 2F_D$, while $((D+2)/(D+1))^2 < 2$. Therefore $F_D/(D+1)^2$ increases, propagating the strict comparison.

This contradiction proves $\delta \geq C + 1/(KF_D)$. The value δ is rational, whereas

$$C + \frac{1}{KF_D} = \frac{1}{2} - \frac{1}{4F_D} + \frac{3 - 2F_D}{20F_D}\sqrt{5}$$

is irrational for integer $F_D \geq 9$. Equality is impossible. □

At $D = 7$, the added gain is approximately $2.156823147 \cdot 10^{-6}$. It is range dependent and exponentially decreasing. Known stronger bounds should be used in their applicable regimes. Neither (1) nor its counting proof yields an appreciable absolute universal improvement or the conjectured 1/3.

5 Why a triple cannot reuse the pair denominator

Theorem 7. *For every $k \geq 2$, there is a poset P_k on $4k - 1$ vertices, of range $D = 2k$, with a consistent order E_k on three pairwise incomparable marked vertices such that*

$$\mathbb{P}(E_k) = T_k^{-1}, \qquad T_k \sim \frac{16^k}{4\pi}, \qquad \frac{T_k}{\binom{4k}{2k}} \sim \sqrt{\frac{k}{8\pi}}.$$

In particular no fixed $c > 0$ gives a universal lower bound $\mathbb{P}(E) \geq c/\binom{2D}{D}$ for positive triple-order events.

Proof. Put $m = 2k - 1$. Take chains $x_1 < \cdots < x_m$, $y_1 < \cdots < y_m$, and a vertex b, with additional relations $x_{k-1} < b < y_{k+1}$ and all transitive consequences. Vertex b is incomparable with $x_k, \ldots, x_m, y_1, \ldots, y_k$, giving degree $2k$. The same maximum bounds every other vertex: vertices in the terminal x-block and initial y-block have degree $2k$, and the remaining vertices have at most that degree. The event $E_k = \{x_m \text{ before } b \text{ before } y_1\}$ forces the unique extension $x_1, \ldots, x_m, b, y_1, \ldots, y_m$.

If i, j chain vertices precede b, then $k - 1 \leq i \leq 2k - 1$ and $0 \leq j \leq k$. Conversely every prefix and suffix shuffle with these bounds is valid. Thus exactly

$$T_k = \sum_{i=k-1}^{2k-1} \sum_{j=0}^{k} \binom{i+j}{i} \binom{4k-2-i-j}{2k-1-i}. \tag{2}$$

For $k = 2$, this is $71 > \binom{8}{4} = 70$, already disproving the constant-one pair denominator for triples.

For the asymptotic, introduce *auxiliary unconstrained* uniform chain coordinates in $[0, 1]$, sorted independently within each chain. Their selected order statistics are independent

$$U_k \sim \mathrm{Beta}(k-1, k+1), \qquad V_k \sim \mathrm{Beta}(k+1, k-1).$$

The allowed interval for b's coordinate has length $(V_k - U_k)_+$. The independent variables do not describe the actual induced poset after deleting b, which retains cross-relations. The positive-part window enforces those relations in the integral. Each extension chamber has volume $1/(4k-1)!$, whereas the sorted-chain density is $(m!)^2$. Therefore

$$T_k = \frac{(4k-1)!}{((2k-1)!)^2} \mathbb{E}(V_k - U_k)_+. \tag{3}$$

Represent each beta as a ratio of independent gamma sums of unit exponentials. The independent-sum central limit theorem and denominator law of large numbers give

$$\sqrt{k}(U_k - \tfrac{1}{2}) \;\Rightarrow\; N(0, \tfrac{1}{8}), \quad \sqrt{k}(V_k - \tfrac{1}{2}) \;\Rightarrow\; N(0, \tfrac{1}{8})$$

with independent limits. The shape asymmetry contributes a mean shift of order $1/k$. Thus $Z_k = \sqrt{k}(V_k - U_k) \Rightarrow N(0, 1/4)$. The exact moments are

$$\mathbb{E}(V_k - U_k) = \frac{1}{k}, \qquad \mathbb{E}Z_k^2 = \frac{k^2 + 4k + 1}{2k(2k+1)} \le 1 \quad (k \ge 2).$$

Hence the positive parts are uniformly integrable: $\mathbb{E}[Z_k^+; Z_k^+ > R] \le 1/R$. Truncation and convergence in distribution now yield

$$\sqrt{k}\mathbb{E}(V_k - U_k)_+ \longrightarrow \mathbb{E}N(0, \tfrac{1}{4})^+ = \frac{1}{2\sqrt{2\pi}}.$$

Stirling's formula gives

$$\frac{(4k-1)!}{((2k-1)!)^2} = (4k-1)\binom{4k-2}{2k-1} \sim 16^k \sqrt{\frac{k}{2\pi}}.$$

Substitution into (3) proves the constant $1/(4\pi)$; the central-binomial asymptotic proves the ratio claim. Since $D = 2k$, the event probability divided by q_D tends to zero. □

Equivalently $\mathbb{P}(E_k) \sim 4\pi\, 4^{-D}$ and $\mathbb{P}(E_k)/q_D \sim 4\sqrt{\pi/D}$. This supplies a necessary square-root loss relative to the pair scale, not a matching sufficient bound for arbitrary posets. It neither proves optimality of F_D nor produces a counterexample to the balance conjecture. The BFT expected-rank theorem applies under its stated expected-rank hypotheses.

6 Reproducibility and remaining questions

The accompanying standard-library Python checker uses exact integer extension counts and rational arithmetic. It checks the pair inequalities and sorting map, the three-order bound, the scalar conditions, the seven-vertex counterexample, and independent formulas for the bridge counts. The beta integral also has the exact polynomial form, for $k \ge 2$,

$$\mathbb{E}(V_k - U_k)_+ = B(k-1, k+1)^{-2} \sum_{i=0}^{k} \sum_{j=0}^{k-2} \frac{(-1)^{i+j}\binom{k}{i}\binom{k-2}{j}}{(k-1+i)(k+i)(2k+i+j+1)},$$

where $B(k-1, k+1) = (k-2)!\,k!/(2k-1)!$. This enables an exact check of (3) independent of (2). Finite checks are diagnostics; the proofs above establish the all-size statements and asymptotic limit. The reader instructions specify the executed finite coverage and how to reproduce it.

There remains an exponential gap between the general F_D denominator and the bridge obstruction. A stronger near-extremal stability principle would be needed to turn the local counting gains into a material universal balance advance.

Attribution and assistance. The sharp global pair extremum and shuffle examples are due to Fishburn. The balance mechanism and its constants are due to BFT and ACP. The claimed increment is localized counting with a conditioning budget, its explicit quantitative application, and the displayed bridge limitation. AI tools assisted proof exploration, literature review, verification-code development and manuscript preparation.

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