

Large negative subspaces for positive Euler deformations

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Luminary Oddities

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Abstract

We construct positive Euler products on the ordinary primes whose associated prime-shift quadratic forms have large negative subspaces despite strong prime-counting control. The prime exponents can be uniformly as close to one as prescribed, agree with one on any prescribed finite prime prefix, and use the same exponent at every power of a prime. Their logarithmic derivatives satisfy a classical subexponential prime number theorem remainder. For one fixed deformation, the negative index on odd polar-free tests of support radius L_j is at least $L_j^{1-o(1)}$ along a growing sequence. The same construction applies to every fixed Hermitian comparison form bounded by a finite Sobolev norm, and survives finite-rank corrections and finitely many additional linear constraints. The proof combines classical positive oscillatory templates with progression-free packet centers. Progression-freeness removes the continuous off-diagonal modulation exactly; a complete Stieltjes matrix estimate and an operator bound for all higher prime powers transfer the negative subspaces to the ordinary-prime model. The resulting index bound quantifies the obstruction to recovering signed form positivity from positive Euler structure, a finite prime prefix and a prime number theorem remainder.

Keywords: positive Euler product; prime-weighted quadratic form; negative index; progression-free set; Sobolev regularization.

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1 The question and the main result

Positivity of Euler coefficients and positivity of an arithmetic quadratic form are different conditions. The first is local and multiplicative: each Euler factor has positive coefficients. The second compares a complete prime-shift operator with an analytic energy, including the interaction between all shifts. Prime number theorem estimates control a cumulative prime measure, whereas a quadratic form can detect coherent oscillations within that measure.

This paper asks how many independent negative directions can remain when the prime weights are positive, uniformly close to their classical values, and constrained by both a finite prime prefix and a prime number theorem remainder. The answer is quantitative. One fixed deformation of the ordinary primes has almost linearly many negative directions, measured in the support radius of the test functions, along a growing sequence of windows. The conclusion holds simultaneously for every fixed comparison energy controlled by a finite Sobolev norm.

1.1 Arithmetic forms and negative index

For $f \in C_c^\infty(\mathbb{R})$, set

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(u) e^{-i\xi u} du, \quad Mf(z) = \int_{\mathbb{R}} f(u) e^{zu} du, \quad (\tau_t f)(u) = f(u - t),$$

and use the inner product $\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(u)} g(u) du$. Our test space is

$$\begin{aligned} \mathcal{V} &= \{f \in C_c^\infty(\mathbb{R}; \mathbb{C}) : f(-u) = -f(u), \quad Mf(1/2) = Mf(-1/2) = 0\}, \\ \mathcal{V}_L &= \{f \in \mathcal{V} : \text{supp } f \subset [-L, L]\}. \end{aligned} \quad (1)$$

The two Mellin conditions will cancel a growing continuous kernel exactly.

Let $b = (b_p)_p$ be a bounded positive sequence indexed by the ordinary primes. Write

$$\Lambda_b(p^k) = b_p \log p \quad (k \geq 1), \quad \Lambda_b(n) = 0 \quad \text{otherwise}, \quad \psi_b(x) = \sum_{n \leq x} \Lambda_b(n). \quad (2)$$

The same b_p occurs at every power of p . Define the Hermitian arithmetic form

$$\mathbf{c}_b[f] = 2 \operatorname{Re} \sum_{n \geq 2} \frac{\Lambda_b(n)}{\sqrt{n}} \langle f, \tau_{\log n} f \rangle. \quad (3)$$

Only finitely many terms contribute to each compactly supported test. For a Hermitian form $\mathbf{a}$ on $\mathcal{V}$, the form to be studied is

$$\mathbf{q}_b^{\mathbf{a}}[f] = \mathbf{a}[f] - \mathbf{c}_b[f]. \quad (4)$$

We use the inhomogeneous Sobolev norm

$$\|f\|_{H^s}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} (1 + \xi^2)^s |\widehat{f}(\xi)|^2 d\xi.$$

Call $\mathbf{a}$ *Sobolev-controlled* if, for some finite $s \geq 0$ and $C_{\mathbf{a}} < \infty$,

$$|\mathbf{a}[f]| \leq C_{\mathbf{a}} \|f\|_{H^s}^2 \quad (f \in \mathcal{V}). \quad (5)$$

The constant is independent of the support of f . This class includes every fixed polynomially bounded Fourier multiplier, and in particular logarithmic multiplier energies.

For a Hermitian form q on a complex vector space E , its *negative index* is

$$\operatorname{ind.}(q; E) = \sup\{\dim W : W \subset E, \dim W < \infty, q[f] < 0 \text{ for every } 0 \neq f \in W\}.$$

This is an algebraic form definition, so it applies directly to the spaces in (1).

Theorem 1.1 (Large negative subspaces). *Fix $0 < \epsilon < 1$ and $P_0 \geq 2$. There are ordinary-prime weights b_p , support radii $L_j \rightarrow \infty$, and subspaces $W_j \subset \mathcal{V}_{L_j}$, of dimensions N_j , such that:*

1.

$$1 - \epsilon \leq b_p \leq 1 + \epsilon, \quad b_p = 1 \quad (p \leq P_0), \quad (6)$$

and, for one $c_b > 0$ and one global implied constant,

$$\psi_b(x) = x + O\left(xe^{-c_b \sqrt{\log x}}\right) \quad (x \geq 2). \quad (7)$$

2. *The Euler product*

$$Z_b(s) = \prod_p (1 - p^{-s})^{-b_p} = \sum_{n \geq 1} \frac{a_b(n)}{n^s}, \quad \operatorname{Re} s > 1, \quad (8)$$

converges absolutely, has $a_b(n) > 0$, and satisfies $-Z'_b/Z_b = \sum_{n \geq 2} \Lambda_b(n)n^{-s}$. Moreover $a_b(n) = 1$ for $n \leq P_0$.

3.

$$\frac{\log N_j}{\log L_j} \longrightarrow 1. \quad (9)$$

There are constants $c_\epsilon, c_* > 0$ such that, for every fixed Sobolev-controlled Hermitian form $\mathbf{a}$, all sufficiently large j satisfy

$$\mathbf{q}_b^{\mathbf{a}}[f] \leq -c_\epsilon e^{c_* L_j} \|f\|_2^2 \quad (f \in W_j). \quad (10)$$

The weights, windows and subspaces are chosen independently of $\mathbf{a}$; the threshold in j may depend on the bound in (5).

Consequently $\text{ind.}(\mathbf{q}_b^{\mathbf{a}}; \mathcal{V}_{L_j}) \geq N_j = L_j^{1-o(1)}$ along the declared support sequence.

The universal choice of weights is important. Given any fixed Sobolev-controlled form $\mathbf{a}_0$, adding a second such form R amounts to applying the theorem to $\mathbf{a}_0 + R$; the deformation b and the spaces W_j stay the same. Section 6 also gives the precise dimension surviving finitely many linear constraints or a finite-rank correction.

1.2 Why a whole negative subspace is a stronger conclusion

Independent negative test vectors need not give a large negative index. For example, the form

$$q(z) = -|z_1 + \cdots + z_N|^2$$

is negative at every coordinate vector, but has negative index one. To establish (9), the entire mixed matrix must therefore be controlled, rather than only its diagonal entries.

Our tests are pairs of compact packets centered at $\pm t/2$. The self-correlation of the t -th pair detects a modulation band near the positive shift t . For two distinct packet centers t_i, t_k , the mixed correlations instead occur near

$$\frac{t_i + t_k}{2}, \quad \frac{|t_i - t_k|}{2}.$$

Thus a three-term progression of centers creates an unwanted mixed interaction: the midpoint of two active centers is another active center. We remove this interaction with a progression-free set.

For example, in a block of centers $T + KB$, the set $B = \{0, 1, 3\}$ has pairwise midpoint labels $1/2, 3/2, 2$, all outside B . For $B = \{0, 1, 2\}$, the pair $0, 2$ has midpoint 1, which is an active band. A sufficiently large spacing K separates the mixed-correlation intervals from the bands themselves. Dense classical progression-free sets then produce large negative spaces, while a complete Stieltjes estimate transfers their mixed matrix to ordinary primes.

1.3 Relation to earlier work

Positive oscillatory prime templates have a substantial history. Diamond, Montgomery and Vorhauer [5] constructed generalized prime systems with large oscillation; Broucke [3] develops the template method for prescribed zero-free contours. Broucke and Vindas [4] give a strong discrete approximation theorem by choosing generalized prime locations. The present construction keeps the ordinary prime locations fixed and assigns one positive exponent to every power of each prime. Classical PNT estimates provide the required transfer to those locations. The additional quantitative step is the complete mixed packet estimate, which turns a scalar oscillatory construction into a large negative subspace.

Negative index is an established object in the study of explicit-formula quadratic forms. Bombieri [2, Section 8] determines the inertia of finite zero-side approximations using the symmetries of the zero set. Here the index bound is obtained directly from the arithmetic shifts, with an explicit support scale and a fixed finite prime prefix. The proof uses the classical progression-free construction of Behrend [1]; we include a short sphere-layer proof of the needed bound. The stronger quantitative constructions of Elkin and Green–Wolf [6, 7] are compatible with the argument, although their improved constants are not needed for (9).

The proof proceeds as follows. Section 2 establishes the continuous cancellation and the exact packets. Section 3 constructs progression-free blocks and proves their support exclusions. Section 4 verifies the global PNT remainder and the Euler coefficients. Section 5 controls the full actual-prime matrix and completes the proof of Theorem 1.1.

2 Continuous cancellation and exact packets

For a real locally bounded v on $[0, \infty)$, write

$$\mathfrak{c}_v[f] = 2 \operatorname{Re} \int_0^\infty e^{t/2} v(t) \langle f, \tau_t f \rangle dt. \quad (11)$$

This is the continuous prime-measure comparison obtained from $d\psi(x) = dx$ after the change of variable $t = \log x$. The support of f makes the integral finite.

Lemma 2.1 (Exact main-measure cancellation). *Let K be convolution by $e^{-|u|/2}$. For $f \in \mathcal{V}$,*

$$\mathfrak{c}_1[f] = -\langle f, Kf \rangle, \quad 0 \leq K \leq 4I. \quad (12)$$

Proof. The two-sided kernel in (11) with $v = 1$ is $e^{|u-w|/2}$. Use the identity

$$e^{|x|/2} = 2 \cosh(x/2) - e^{-|x|/2}.$$

The first term integrates to $\overline{Mf(1/2)}Mf(-1/2) + \overline{Mf(-1/2)}Mf(1/2) = 0$. The remaining term is $-\langle f, Kf \rangle$. The Fourier multiplier of K is $(\xi^2 + 1/4)^{-1}$, which lies in $[0, 4]$. $\square$

Fix $r > 0$, a real even nonnegative nonzero $\phi \in C_c^\infty(-r, r)$, and a real cutoff $\chi \in C_c^\infty(-3r, 3r)$ such that

$$0 \leq \chi \leq 1, \quad \chi = 1 \text{ on } [-2r, 2r]. \quad (13)$$

Write $\Phi = M\phi$. For $\omega \geq 1$, define

$$g_\omega(u) = -\omega^{-2} \left(\frac{d^2}{du^2} - \frac{1}{4} \right) [\phi(u) \cos(\omega u)], \quad G_\omega = M g_\omega. \quad (14)$$

Lemma 2.2 (Packet identities and bounds). *The function g_ω is real, even and supported in $[-r, r]$. It satisfies*

$$G_\omega(1/2) = G_\omega(-1/2) = 0, \quad (15)$$

$$G_\omega(1/2 + i\omega) = \frac{1 - i/\omega}{2} \{ \Phi(1/2) + \Phi(1/2 + 2i\omega) \} \longrightarrow \frac{\Phi(1/2)}{2} > 0, \quad (16)$$

$$\|g_\omega\|_2^2 \longrightarrow \frac{\|\phi\|_2^2}{2}, \quad \|g_\omega\|_{H^s} \leq C_s(1 + \omega)^s \quad (s \geq 0). \quad (17)$$

The constants in (17) may depend on s and ϕ .

Proof. Integration by parts in the Mellin transform multiplies a function by $z^2 - 1/4$ when the differential operator in (14) is applied. This proves (15). Since

$$M[\phi \cos(\omega u)](z) = \{\Phi(z + i\omega) + \Phi(z - i\omega)\}/2,$$

substitution of $z = 1/2 + i\omega$ proves the equality in (16). The remaining oscillatory integral tends to zero by the Riemann–Lebesgue lemma applied to $e^{u/2}\phi(u)$. The limit is positive because ϕ is nonnegative and nonzero.

Expansion of the differential operator gives

$$g_\omega = \phi \cos(\omega u) + 2\omega^{-1}\phi' \sin(\omega u) - \omega^{-2}(\phi'' - \phi/4) \cos(\omega u).$$

The last two terms tend to zero in L^2 ; the squared norm of the first tends to $\|\phi\|_2^2/2$. For the Sobolev bound at every real $s \geq 0$, use

$$\hat{g}_\omega(\xi) = \frac{\xi^2 + 1/4}{2\omega^2} \{\hat{\phi}(\xi - \omega) + \hat{\phi}(\xi + \omega)\}.$$

After setting $\xi = \zeta \pm \omega$, the squared weighted factor is bounded by $C_s(1 + \omega^2)^s(1 + \zeta^2)^{s+2}$. The Schwartz decay of $\hat{\phi}$ proves (17). $\square$

Let $c_0 > 0$ be a constant in the classical PNT bound

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = x + E(x), \quad |E(x)| \leq Cxe^{-c_0\sqrt{\log x}} \quad (x \geq 2). \quad (18)$$

This standard de la Vallée Poussin form is recalled, for example, in [3, Introduction]. Choose once and for all

$$0 < d < c_0/4, \quad \omega_t = e^{d\sqrt{t}}. \quad (19)$$

For a sufficiently large center t , put

$$f_t = \tau_{t/2}g_{\omega_t} - \tau_{-t/2}g_{\omega_t}. \quad (20)$$

These are real odd elements of $\mathcal{V}$, supported in $[-t/2 - r, t/2 + r]$. Their L^2 norms are bounded above and below by fixed positive constants for large t . Choose a phase α_t so that

$$\operatorname{Re}\{e^{i\alpha_t}G_{\omega_t}(1/2 + i\omega_t)^2\} = -|G_{\omega_t}(1/2 + i\omega_t)|^2. \quad (21)$$

Equation (16) makes this choice possible with a nonzero value for every sufficiently large t .

Lemma 2.3 (Diagonal modulation). *Suppose a band centered at t is the only modulation band meeting the positive-shift correlation of f_t away from zero, and no band meets its correlation near zero. If that band is $\epsilon\chi(u - t)\cos(\omega_t(u - t) + \alpha_t)$, then*

$$\mathbf{c}_v[f_t] - \mathbf{c}_1[f_t] = 2\epsilon e^{t/2}|G_{\omega_t}(1/2 + i\omega_t)|^2. \quad (22)$$

Proof. For real even $g = g_{\omega_t}$, write $c_g(u) = \langle g, \tau_u g \rangle$. Near the positive shift t , the autocorrelation of $\tau_{t/2}g - \tau_{-t/2}g$ is $-c_g(u - t)$. Its other positive-shift component is near zero. The cutoff equals one on the entire cross-correlation interval. For $z = 1/2 + i\omega_t$,

$$\int_{\mathbb{R}} e^{zu} c_g(u) du = Mg(z)Mg(-z) = G_{\omega_t}(z)^2,$$

where evenness was used in the last equality. The contribution to $\mathbf{c}_v - \mathbf{c}_1$ is therefore $-2\epsilon e^{t/2} \operatorname{Re}(e^{i\alpha_t}G_{\omega_t}(z)^2)$. Apply (21). $\square$

3 Progression-free blocks of centers

Lemma 3.1 (Classical dense progression-free sets). *There is an absolute constant C such that every sufficiently large integer M has a subset $B \subset \{0, \dots, M-1\}$ with*

$$|B| \geq Me^{-C\sqrt{\log M}}, \quad a + c = 2b, \quad a, b, c \in B \implies a = b = c. \quad (23)$$

Proof. This is the classical sphere construction of Behrend [1]. We include a sufficient estimate. Set $h = \lfloor \sqrt{\log M} \rfloor$ and $q = \lfloor M^{1/h}/2 \rfloor$. For large M ,

$$q \geq 2, \quad (2q)^h \leq M, \quad q \geq M^{1/h}/4.$$

Partition $\{0, \dots, q-1\}^h$ by the integer $\sum_{\nu} x_{\nu}^2$. Some sphere layer S has size at least $q^h/[h(q-1)^2+1]$. Encode a point by

$$x \mapsto \sum_{\nu=0}^{h-1} x_{\nu}(2q)^{\nu}.$$

The encoding is injective and lies below M . In an equation $a + c = 2b$, both sides have digits below $2q$; there are no carries, so the equation holds coordinatewise. On a sphere, strict convexity of the squared norm forces the three coordinate vectors to coincide. Finally,

$$\frac{q^h}{h(q-1)^2+1} \geq \frac{M}{2h4^h M^{2/h}} \geq Me^{-C\sqrt{\log M}},$$

after enlarging C . □

Set $\eta = 1/4$. We construct finite blocks $\mathcal{T}_j$ recursively. Let U_{j-1} be the largest earlier center plus $3r$, with $U_0 = 0$. Choose an integer

$$K_j > 4U_{j-1} + 20r + 2. \quad (24)$$

Choose an integer T_j so large that

$$T_j > 8U_{j-1} + 100r, \quad K_j \leq T_j^{1/j} \quad (j \geq 2), \quad M_j = \left\lfloor \frac{T_j}{4K_j} \right\rfloor \quad (25)$$

is in the range of Lemma 3.1. The first T_j is also chosen beyond $\log P_0 + 3r$ and the packet threshold in (16). For B_j supplied by that lemma, define

$$\mathcal{T}_j = \{T_j + K_j b : b \in B_j\}, \quad N_j = |B_j|. \quad (26)$$

Every center in the block lies in $[T_j, 5T_j/4]$. All centers in the entire construction have a fixed positive minimum spacing.

Lemma 3.2 (Support separation). *Let $t_i \neq t_k$ belong to the same block. The intervals of radius $2r$ about $(t_i + t_k)/2$ and $|t_i - t_k|/2$ are disjoint from every modulation interval $[t - 3r, t + 3r]$, for every center t in every block. For the diagonal packet f_{t_i} , the only band meeting its positive-shift correlation away from zero is its own band. No band meets the correlation near zero.*

Proof. The positive-shift mixed correlation has exactly the two possible centers stated in the lemma, since the packets are supported near $\pm t_i/2$ and $\pm t_k/2$. For a current center $t_\ell = T_j + K_j b_\ell$,

$$\left| \frac{t_i + t_k}{2} - t_\ell \right| = \frac{K_j}{2} |b_i + b_k - 2b_\ell| \geq K_j/2 > 5r$$

by progression-freeness. The sum midpoint is above all previous bands and below all future bands by (25). The difference midpoint is at least $K_j/2 > U_{j-1} + 5r$, and at most $T_j/8 < T_j - 5r$, after the initial choice of T_j . Hence it lies between the previous and current bands, with the required margins. Future bands are still farther away. For a diagonal, the positive-shift correlation is confined to radius $2r$ about zero and its own center. Distinct centers have spacing greater than $5r$, and the first center is greater than $5r$. $\square$

Define the single function and prime weights

$$v(u) = 1 + \epsilon \sum_{t \in \bigcup_j \mathcal{T}_j} \chi(u - t) \cos(\omega_t(u - t) + \alpha_t), \quad b_p = v(\log p). \quad (27)$$

The sum is locally finite, and the bands are disjoint. Consequently $1 - \epsilon \leq v \leq 1 + \epsilon$, and (6) holds.

4 The global prime estimate and all Euler powers

Lemma 4.1 (A global weighted PNT estimate). *For the function v in (27),*

$$\sum_{n \leq x} \Lambda(n) v(\log n) = x + O\left(x e^{-(d/2)\sqrt{\log x}}\right) \quad (x \geq 2).$$

The implied constant is independent of the final band or endpoint.

Proof. For a center t , integrate one oscillatory band against dx . In logarithmic coordinates, a complete or partial band has the form

$$\int e^u \chi(u - t) \cos(\omega_t(u - t) + \alpha_t) du.$$

One integration by parts, including the partial-band endpoint, bounds this by $Ce^t/\omega_t = Ce^{t-d\sqrt{t}}$. Integration against dE , with E as in (18), costs

$$C(1 + \omega_t) e^{t - c_0 \sqrt{t}}.$$

Indeed the boundary term has this bound, and the total variation of the band factor is $O(1 + \omega_t)$. Shifting t by at most $3r$ changes the exponent by a bounded amount. As $d < c_0/2$, both contributions are $O(e^{t-d\sqrt{t}})$.

The function $F(t) = t - d\sqrt{t}$ has $F'(t) \geq 1/2$ eventually. If successive centers have spacing at least $H > 0$, completed-band bounds sum backwards with ratio at most $e^{-H/2}$. A partial final band obeys the same bound. Between bands use monotonicity of F . Thus there is one estimate for every x , even though the number of centers in a block grows. The unweighted PNT error is smaller. Weakening d to $d/2$ absorbs fixed initial and endpoint constants. $\square$

Lemma 4.2 (The complete higher-power correction). *Replacing $\Lambda(p^k)v(k \log p)$ by $b_p \log p$ at all $k \geq 2$ has cumulative absolute cost*

$$O(\sqrt{x} \log^2 x) \quad (n \leq x). \quad (28)$$

On tests whose shifts satisfy $n \leq X$, the difference of the complete paired arithmetic forms has absolute value at most

$$C(1 + \log^2 X) \|f\|_2^2. \quad (29)$$

Proof. Each changed coefficient is at most $2\epsilon \log p$ in absolute value. At most $\log_2 x$ exponents contribute, each with bases at most $\sqrt{x}$, which gives (28). For the form estimate, use $|\langle f, \tau_t f \rangle| \leq \|f\|_2^2$. The square terms are bounded by

$$\sum_{p \leq \sqrt{X}} \frac{\log p}{p} \leq \sum_{2 \leq n \leq \sqrt{X}} \frac{\log n}{n} = O(1 + \log^2 X).$$

For all powers $k \geq 3$ together, use the convergent majorant

$$\sum_{n \geq 2} \frac{(\log n) n^{-3/2}}{1 - n^{-1/2}}.$$

This bounds the entire operator difference, including all its mixed entries on any subspace. $\square$

Proposition 4.3. *The weights (27) satisfy parts (1) and (2) of Theorem 1.1.*

Proof. The prefix and positivity statements follow from disjoint bands. Lemma 4.1 and (28) give (7), since $\sqrt{x} \log^2 x = o(xe^{-(d/2)\sqrt{\log x}})$. One may take a smaller fixed $c_b > 0$ to include all initial values with one constant.

For $\operatorname{Re} s > 1$, the series

$$\sum_p b_p \sum_{k \geq 1} \frac{p^{-ks}}{k}$$

is absolutely convergent. Its exponential defines (8) and permits termwise logarithmic differentiation. Each factor has the binomial expansion

$$(1 - z)^{-b_p} = \sum_{k \geq 0} \frac{(b_p)_k}{k!} z^k,$$

whose coefficients are positive because $b_p > 0$. Multiplication over prime factors gives $a_b(n) > 0$. The absolute Dirichlet series at a real $\sigma > 1$ is bounded by $\zeta(\sigma)^{1+\epsilon}$. Every prime factor of $n \leq P_0$ has exponent $b_p = 1$; all local coefficients at these primes are one. Hence $a_b(n) = 1$ for $n \leq P_0$. $\square$

5 The complete mixed matrix

Fix a block and abbreviate $T = T_j$, $N = N_j$. List its centers as $t_1, \dots, t_N$, put

$$\omega_{\max} = e^{d\sqrt{5T/4}}, \quad w_i = e^{t_i/4}, \quad F_z = \sum_{i=1}^N z_i \frac{f_{t_i}}{w_i}, \quad z \in \mathbb{C}^N.$$

The packets have disjoint supports within the block. Thus

$$\|F_z\|_2^2 \leq C e^{-T/2} \|z\|_{\ell^2}^2. \quad (30)$$

For every fixed finite s , the triangle inequality and Cauchy–Schwarz give

$$\|F_z\|_{H^s}^2 \leq C_s N e^{2sd\sqrt{5T/4}-T/2} \|z\|_{\ell^2}^2 = o(1) \|z\|_{\ell^2}^2, \quad (31)$$

because $N \leq M_j \leq T$. For $s = 0$, the sharper estimate (30) is available.

5.1 The continuous matrix

Let

$$\kappa_0 > 0, \quad 2|G_{\omega_t}(1/2 + i\omega_t)|^2 \geq \kappa_0$$

for all centers, increasing the first center if necessary. Lemma 2.3 gives a normalized modulation entry at least $\kappa_0\epsilon$ on the diagonal of $\mathbf{c}_v - \mathbf{c}_1$. For distinct centers, Lemma 3.2 shows that $v = 1$ on the entire mixed-correlation support. The corresponding off-diagonal modulation entry is therefore exactly zero. Lemma 2.1 consequently gives

$$-\mathbf{c}_v[F_z] \leq -\kappa_0\epsilon \|z\|_{\ell^2}^2 + 4\|F_z\|_2^2. \quad (32)$$

5.2 Transfer from the continuous measure to actual primes

We first compare $\mathbf{c}_v$ with the arithmetic form using $\Lambda(n)v(\log n)$, before making the consistent-power replacement. For real basis packets define the polarized correlation

$$h_{ik}(u) = \langle f_{t_i}, \tau_u f_{t_k} \rangle + \langle f_{t_k}, \tau_u f_{t_i} \rangle.$$

The corresponding real symmetric matrix defines the Hermitian form on complex coefficients as well. Uniformly in the block,

$$|h_{ik}| \leq C, \quad |h'_{ik}| \leq C(1 + \omega_{\max}). \quad (33)$$

These follow from Cauchy–Schwarz and the L^2 and H^1 bounds in Lemma 2.2. Its support for $i \neq k$ is confined to the two intervals in Lemma 3.2.

Lemma 5.1 (Normalized mixed PNT error). *For some $\delta > 0$, the operator norm of the full normalized error matrix is bounded by*

$$CNe^{-\delta\sqrt{T}} + CNe^{-T/2+d\sqrt{5T/4}} + CNe^{-T/2}. \quad (34)$$

In particular it tends to zero as $j \rightarrow \infty$.

Proof. For $i \neq k$, $v = 1$ on the mixed support. Near the sum midpoint $S = (t_i + t_k)/2$, use the test function

$$x^{-1/2} h_{ik}(\log x)$$

in Stieltjes integration against $dE(x)$. Its derivative is

$$x^{-3/2} \{h'_{ik}(\log x) - h_{ik}(\log x)/2\}.$$

The endpoints vanish by smooth compact support. On this fixed logarithmic-width interval, (18) and (33) bound the error by

$$C(1 + \omega_{\max})e^{S/2-c_0\sqrt{S}}.$$

Division by $w_i w_k = e^{S/2}$, and $S \geq T$, give at most $Ce^{-\delta\sqrt{T}}$, since $d\sqrt{5/4} < c_0$.

Near the difference midpoint $D = |t_i - t_k|/2$, the cruder bound $|E(x)| \leq Cx$, valid for all $x \geq 1$, is enough. After the same integration by parts and normalization, the error is at most

$$C(1 + \omega_{\max})e^{D/2-(t_i+t_k)/4} \leq Ce^{-T/2+d\sqrt{5T/4}}.$$

One may split the two separated support intervals with fixed smooth cutoffs; their derivatives only change C .

For a diagonal entry, the cross region is near t_i . The differentiated test now contains vh_{ii} . The derivative is $v'h_{ii} + vh'_{ii}$, whose size is $O(1 + \omega_{t_i})$, with no product of derivatives. The normalized error is $O(e^{-(c_0-d)\sqrt{t_i}})$. The self region has fixed logarithmic width near zero. Only finitely many actual prime powers occur there; their contribution and the continuous integral are $O(1)$ by the bound on h_{ii} , so their normalized error is $O(e^{-T/2})$.

These estimates bound every normalized entry, with one constant independent of i, k, j . The norm of a Hermitian matrix is bounded by the maximum absolute row sum. There are N entries in each row, which proves (34). $\square$

5.3 Completion of the negative-subspace estimate

All shifts of a test in the block satisfy $\log n \leq 5T/4 + 2r$. Lemma 4.2 and (30) therefore bound the entire consistent-power replacement by

$$CT^2e^{-T/2}\|z\|_{\ell^2}^2 = o(1)\|z\|_{\ell^2}^2. \quad (35)$$

For any fixed $\mathfrak{a}$ satisfying (5), (31) gives

$$|\mathfrak{a}[F_z]| = o(1)\|z\|_{\ell^2}^2.$$

Combining this with (32), Lemma 5.1 and (35), for all sufficiently large j we obtain the complete estimate

$$\mathfrak{q}_b^{\mathfrak{a}}[F_z] \leq -\frac{\kappa_0\epsilon}{2}\|z\|_{\ell^2}^2. \quad (36)$$

Every term in the actual-prime form is included in this estimate.

Proof of Theorem 1.1. Parts (1) and (2) were proved in Proposition 4.3. Set

$$L_j = \frac{5T_j}{8} + r, \quad W_j = \text{span}_{\mathbb{C}}\{f_t : t \in \mathcal{T}_j\}.$$

The packets are nonzero and disjointly supported within the block, so $\dim W_j = N_j$, and $W_j \subset \mathcal{V}_{L_j}$. Equations (30) and (36) give

$$\mathfrak{q}_b^{\mathfrak{a}}[f] \leq -c_{\epsilon}e^{T_j/2}\|f\|_2^2 \quad (f \in W_j).$$

Since $T_j/2 = (4/5)(L_j - r)$, this implies (10), for example with any fixed $0 < c_* < 4/5$, after changing c_{ϵ} . The block construction, weights and spaces did not use $\mathfrak{a}$; only the eventual threshold uses its Sobolev bound.

Finally, by (25),

$$\frac{\log M_j}{\log T_j} \geq 1 - \frac{1}{j} - o(1).$$

Lemma 3.1 gives $\log N_j / \log M_j \rightarrow 1$, while $\log L_j / \log T_j \rightarrow 1$. As $N_j \leq M_j \leq T_j$, these inequalities prove (9). $\square$

6 Finite constraints, corrections and examples

Corollary 6.1 (Constraint and repair rank). *Let b, L_j, W_j, N_j be as in Theorem 1.1, and let $\mathbf{a}$ be fixed and Sobolev-controlled. For linear maps $\ell_j : \mathcal{V}_{L_j} \rightarrow \mathbb{C}^{r_j}$, all sufficiently large j , with threshold depending on $\mathbf{a}$, satisfy*

$$\text{ind}(\mathbf{q}_b^{\mathbf{a}}; \mathcal{V}_{L_j} \cap \ker \ell_j) \geq N_j - r_j$$

whenever the right side is positive. For these j , the same bound holds after adding any Hermitian correction that factors through ℓ_j . For a fixed finite number of constraints and a fixed finite-rank correction, arbitrarily large negative subspaces and arbitrarily negative L^2 Rayleigh quotients remain.

Proof. Rank-nullity gives $\dim(W_j \cap \ker \ell_j) \geq N_j - r_j$. The estimate (10) holds on this intersection. A correction $H(\ell_j f, \ell_j f)$, for any Hermitian form H on the finite coordinate space, vanishes there. If r_j is fixed, $N_j - r_j \rightarrow \infty$. $\square$

Corollary 6.2 (Fixed differential and logarithmic energies). *The weights in Theorem 1.1 work simultaneously for each fixed Fourier multiplier*

$$\mathbf{a}_a[f] = \frac{1}{2\pi} \int_{\mathbb{R}} a(\xi) |\hat{f}(\xi)|^2 d\xi$$

with real symbol satisfying $|a(\xi)| \leq C(1 + \xi^2)^s$ for a finite s . They also work after adding any fixed multiple of a finite Sobolev norm to such a form.

Proof. The symbol bound gives (5). Sums of two Sobolev-controlled forms satisfy the same condition after increasing the exponent and constant. $\square$

Logarithmic symbols are a useful case: their energy can grow with packet frequency, but that growth is still paid in the whole-space estimate (31). For example, each fixed symbol

$$a(\xi) = \text{Re} \frac{\Gamma'}{\Gamma} \left(\alpha + \frac{i\xi}{2} \right) + c, \quad \alpha > 0, \quad c \in \mathbb{R},$$

has logarithmic growth and belongs to Corollary 6.2. Such multipliers appear in archimedean terms of explicit formulas; see [2, Section 2]. The theorem treats the chosen multiplier and the complete deformed arithmetic form in the same original test norm.

Remark 6.3 (An explicit elementary center family). The sphere construction is needed only for the almost-linear index scale. If one instead uses the integers below 3^h whose base-3 digits lie in $\{0, 1\}$, the same no-carry argument makes them progression-free. Their cardinality is 2^h . Choosing h with $3^h \leq M_j < 3^{h+1}$ gives the fully explicit alternative

$$\frac{\log N_j}{\log L_j} \longrightarrow \frac{\log 2}{\log 3}.$$

All analytic estimates and repair conclusions are unchanged.

6.1 Reproducible finite geometry

The companion program `verify_geometry.py` uses integer and rational arithmetic to check finite sphere-layer sets, their midpoint exclusions, and all mixed-support separations in three explicit blocks. It includes the progression-free example $\{0, 1, 3\}$ and the collision example $\{0, 1, 2\}$ from the introduction. The analytic PNT, Sobolev and all-prime-power estimates are proved in the text; the program supplies finite reproducible examples of their geometric input.

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