

Joint Cutoffs and Negative-Spectrum Decay for Weighted Weil Operators

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Abstract

We obtain a joint finite-rank approximation law for the complete Birman–Schwinger family of a constrained theta-weighted Weil operator. At each fixed absolute error tolerance, the required rank is at most $C \exp(C \sqrt{\log(1/\varepsilon)})$ as the positive resolvent parameter ε tends to zero. The construction pays the complete paired arithmetic tail, high-frequency block and mixed localization errors. It gives a corresponding bound on the entire negative spectral projection and implies that the negative eigenvalue magnitudes decay as $C \exp[-c \log^2(n+1)]$. Thus the full negative part belongs to every Schatten class. We separate two arithmetic cutoff regimes. Effective simultaneous prime-phase returns give signed high-frequency saturation when the retained cutoff grows sufficiently slowly. In the faster regime of the rank construction, theta confinement and compressed-shift estimates instead give a strict high-block norm gap: its normalized upper bound is at most three quarters for every positive limiting arithmetic ratio, with a stronger branch above the theta-overlap threshold. Both arguments retain the original constraints, weighted norms and complete localization errors.

1 Introduction and results

Global conclusion and cutoff regimes The global spectral conclusion uses joint arithmetic, frequency, spatial and approximation cutoffs. Each parameter pays a different remainder. The slow-regime saturation and the fast-regime high-block gap therefore belong in a separate comparison: saturation of one slow block does not establish an optimal global rank law. Approximation rank, carrier height and a frequency-limit constant have different roles.

The spectral effect of an arithmetic truncation depends on the resolvent parameter against which its error is measured. A fixed-cutoff approximation theorem therefore needs a joint parameter analysis before it gives information about eigenvalues accumulating at a threshold. We perform that analysis for the constrained theta-weighted realization introduced in [1]. The new recipient is the entire negative part of the original operator, rather than its output on a prescribed null space.

Write $H = A + B$ for the precise minimal-form realization defined in Section 2, with $A \geq 0$ and B bounded selfadjoint. For $\varepsilon > 0$ put

$$K_\varepsilon = (A + \varepsilon)^{-1/2}(-B)(A + \varepsilon)^{-1/2}, \quad n_-(H; -\varepsilon) = \text{rank } 1_{(-\infty, -\varepsilon)}(H).$$

The negative part is $H_- = \max(-H, 0)$ in the spectral calculus. The first theorem gives the main spectral conclusion. The second identifies a strict distinction between the fast cutoff family used in its proof and the slower saturation family.

Theorem 1.1 (Joint rank and negative-spectrum decay). *For every fixed $0 < \eta \leq 1$ and $0 < \varepsilon \leq 1$, the complete operator K_ε has a selfadjoint finite-rank approximant with norm error less than η . Its rank is at most the explicit integer $N(\varepsilon, \eta)$ in (11). For each fixed η there is C_η such that*

$$N(\varepsilon, \eta) \leq C_\eta \exp\left(C_\eta \sqrt{\log(1/\varepsilon)}\right).$$

There are constants $C, c > 0$ such that

$$n_-(H; -\varepsilon) \leq C \exp\left(C \sqrt{\log(1/\varepsilon)}\right), \quad (1)$$

$$\lambda_n(H_-) \leq C \exp[-c \log^2(n+1)]. \quad (2)$$

Here the negative eigenvalue magnitudes are ordered nonincreasingly, repeated with multiplicity, and padded by zeros if finite. In particular $H_- \in \mathcal{S}_p$ for every $p > 0$.

For the regime comparison write $J_\varepsilon = M_s(A + \varepsilon)^{-1/2}$ and

$$K_{\text{hi}}(M, \varepsilon, L) = J_\varepsilon^*(I - Q_L)V_M(I - Q_L)J_\varepsilon,$$

where V_M and its coefficient d_M are defined in Section 3.

Theorem 1.2 (Fast-regime high-block gap). *Suppose $q \rightarrow \infty$, $\varepsilon = e^{-q}$, $M/q \rightarrow C > 0$, and $\log L/\sqrt{q} \rightarrow K$ with $0 < K < \infty$. Then*

$$\limsup_{q \rightarrow \infty} \frac{a(L) \|K_{\text{hi}}(M, \varepsilon, L)\|}{d_M} \leq \min\left\{\frac{3}{4}, \frac{1}{2} + \frac{1}{4\sqrt{\pi C}}\right\} < 1. \quad (3)$$

The positive spectral edge obeys the same upper bound. For the explicit cutoff family of Theorem 1.1, $C = 1/\log 10$, and the stronger branch is $\frac{1}{2} + \frac{1}{4}\sqrt{\log 10/\pi}$.

The rank bound includes the growing retained arithmetic set and all localization remainders. The parameter η is fixed in the displayed asymptotic; the explicit formula governs its simultaneous variation. The finite-rank construction contains the actual resolvent rows, so the theorem is an approximation-rank statement rather than an estimate for the arithmetic cost of evaluating those rows.

The proof uses the frequency-first enclosure from [1], which we reproduce below in the exact form needed. An elementary Chebyshev estimate removes an unnecessary logarithm from the retained arithmetic coefficient. The form-domain Birman–Schwinger correspondence and finite-rank min–max then control the whole negative spectrum. These spectral tools and Schatten theory are classical; see [3, 4]. Related logarithmic-symbol spectral estimates include [5], whose finite-measure Dirichlet setting differs from the full-line theta-multiplication reference used here. Weighted arithmetic realizations and fixed-depth localization arguments also appear in [6]. The contribution here is the joint cutoff law and its quantitative consequence for H_- in the specified realization.

Section 6 proves saturation uniformly in a slow-growing arithmetic regime, paying every-interval prime-phase coverage and the original packet norm. Section 7 then proves Theorem 1.2: a smooth highpass transfers the theta-induced spatial confinement without a false sharp-cutoff commutator estimate, and medium and long translations have smaller correlations on that confined core. Thus the fast rank regime genuinely fails to saturate the fixed-cutoff high-band coefficient. This comparison concerns that retained block.

The finite observation-norm estimate used for the smooth transfer was developed in the author's working note [7]; we reproduce its required upper proof in Lemma 2.2. The compact-window autocorrelation half bound has a direct predecessor in [8, Lemma 1] and [10, Lemmas 1 and 1']. The square-zero numerical-radius estimate is a special case of the classical theorem of Haagerup and de la Harpe [9]; its elementary proof and the projection bounds are included. The prime number theorem used to evaluate the arithmetic mass is the classical unconditional theorem [2, §§25.16, 27.2]. The new fast-regime conclusion is the transfer of these ingredients through the complete weighted receiver and its joint scales.

2 Exact realization and inherited budgets

Inner products are conjugate-linear first, and

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-iu\xi} f(u) du, \quad \|f\|^2 = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \frac{d\xi}{2\pi}.$$

Use the full theta kernel

$$\psi(u) = 2 \sum_{n \geq 1} (2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}) e^{-\pi n^2 e^{2u}}. \quad (4)$$

The locally uniformly differentiable series is even and strictly positive. For normalization, with $\vartheta(x) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x}$ and $b(u) = e^{u/2}(\vartheta(e^{2u}) - 1)$, Poisson summation gives $b(u) - b(-u) = e^{-u/2} - e^{u/2}$ and $\psi = \frac{1}{2}(\partial_u^2 - 1/4)b$. Positivity follows from the individual terms on $u \geq 0$, then reflection.

Put $c(u) = \cosh(u/2)$, $w = \psi c$, $s = \sqrt{\psi/c}$, and

$$\mathcal{K} = \{h \in L^2(\mathbb{R}) : \langle \sqrt{w}, h \rangle = \langle \sqrt{w} \tanh(u/2), h \rangle = 0\}.$$

The two rows are smooth, in L^2 , and linearly independent. The weighted unitary coordinate is $h = \sqrt{w}g$, with $\psi g = sh$.

Writing $\psi_0 = \Gamma'/\Gamma$, define

$$m_0 = \psi_0(1/4) - \log \pi < 0, \quad a(\xi) = \Re \psi_0(1/4 + i\xi/2) - \psi_0(1/4).$$

The digamma series and asymptotic [2, §§5.7, 5.11] give

$$a(\xi) = \sum_{j \geq 0} \frac{t^2}{(j + 1/4)((j + 1/4)^2 + t^2)}, \quad t = |\xi|/2, \quad a(\xi) = \log |\xi| + O(1). \quad (5)$$

It is even, positive away from zero, and increasing in $|\xi|$.

On $\mathcal{C} = C_c^\infty(\mathbb{R}) \cap \mathcal{K}$ define

$$\mathfrak{a}[h] = \int a(\xi) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi}.$$

The core is dense: compact smooth approximation followed by correction using two fixed compact smooth vectors with invertible row matrix proves this. The maximal graph map $h \mapsto a(D)^{1/2}(sh)$ is closed, so its restriction is closable. Let $\mathcal{V}$ be the closure of $\mathcal{C}$ in $\|h\|^2 + \mathfrak{a}[h]$, and let A be its associated nonnegative selfadjoint operator. This chosen core closure specifies the realization.

Let $S_b f(u) = f(u + b)$ and $c_n = \Lambda(n)/\sqrt{n}$. The complete bounded perturbation is

$$B = P_{\mathcal{K}} \left[m_0 M_s^2 - \sum_{n \geq 2} c_n M_s (S_{\log n} + S_{-\log n}) M_s \right] \Big|_{\mathcal{K}}. \quad (6)$$

The paired series converges in norm by the estimate below. Thus $H = A + B$ is selfadjoint on $\text{Dom } A$, with form domain $\mathcal{V}$.

Lemma 2.1 (Source bounds). *The following complete-kernel bounds hold:*

$$\|s\|_{\infty} \leq 1, \quad \sup_{|u| > R} s(u) \leq \sigma_R := \min\{1, 13e^{2R-(3/2)e^{2R}}\}, \quad (7)$$

$$\|B - B_M\| \leq \delta_M := 640(M+1)^2/20^{M+1}, \quad (8)$$

$$|m_0| < 6, \quad a(L) \geq \frac{1}{2} \log(2L) \quad (L \geq 1/2), \quad a(\xi) \leq 5 + \log(1 + |\xi|). \quad (9)$$

Here B_M retains the same diagonal term and all $n \leq M$.

Proof. The elementary full-series proof of the unit bound in [1, Lemma 2.1] gives the stronger explicit value $\psi(u) < 11161963/11500000 < 1$. We use this published estimate with exactly the normalization (4). Its companion envelope is

$$\psi(u) \leq 8\pi^2 e^{9u/2 - \pi e^{2u}}, \quad s(u)^2 \leq 16\pi^2 e^{4u - \pi e^{2u}} \quad (u \geq 0).$$

For completeness, the first follows by bounding the theta sum after its first exponential using

$$\sum_{n \geq 1} n^4 q^{n-1} < 2, \quad q = e^{-3\pi} < 1/100.$$

Division by $c(u) \geq e^{u/2}/2$ gives the second. Its square root, reflection and $\pi > 3$, $4\pi < 13$ give (7).

For a shift $b \geq 0$, write $v = |u| + |u + b| \geq b$. The envelopes and arithmetic-geometric mean give

$$\begin{aligned} \|M_s S_b M_s\| &= \sup_u s(u) s(u + b) \\ &\leq 16\pi^2 e^{2b - \pi e^b}. \end{aligned}$$

Indeed the intermediate exponent is at most $2v - \pi e^v$, decreasing for $v \geq 0$. Since $c_n \leq 1$, both directions beyond M cost at most $32\pi^2 \sum_{n > M} n^2 e^{-\pi n}$. For $k = M + 1 \geq 2$, use $e^{-\pi} < 1/20$ and

$$\sum_{n \geq k} n^2 e^{-\pi n} \leq k^2 20^{-k} \sum_{j \geq 0} (j+1)^2 20^{-j} < 2k^2 20^{-k}.$$

This proves (8) with $\pi^2 < 10$. The classical quarter-argument digamma value gives $|m_0| < 6$. Retaining the terms $0 \leq j \leq \lfloor L/2 - 1/4 \rfloor$ in (5) gives the lower bound in (9) by comparison with the reciprocal integral. For the upper bound, the summand decreases in its denominator variable. Its first term is at most 4 and its remaining integral is $\frac{1}{2} \log(1 + 4\xi^2)$. Hence $a(\xi) \leq 4 + \frac{1}{2} \log(1 + 4\xi^2) \leq 5 + \log(1 + |\xi|)$. □

Lemma 2.2 (Finite observation bound). *For the chosen constrained reference and $0 < \varepsilon \leq 1$,*

$$j(\varepsilon) := \|M_s(A + \varepsilon)^{-1/2}\| \leq \min\{\varepsilon^{-1/2}, 4\log(\log(1/\varepsilon) + 4)\}.$$

Proof. We reproduce the upper estimate from [7]. Let $f = sh$ for $h \in \mathcal{V}$. For spatial and frequency cutoffs $R, \ell > 0$, direct integration gives $\|P_R Q_\ell\| \leq \sqrt{2R\ell/\pi} =: \rho$. If $\rho < 1$, the identity

$$f = P_R Q_\ell f + (I - P_R)f + P_R(I - Q_\ell)f$$

and the full tails imply

$$(1 - \rho)\|f\| \leq \sigma_R\|h\| + a(\ell)^{-1/2}\sqrt{\mathfrak{a}[h]}.$$

Cauchy–Schwarz against the two energies therefore gives

$$\|sh\|^2 \leq \frac{\sigma_R^2/\varepsilon + 1/a(\ell)}{(1 - \rho)^2}(\mathfrak{a}[h] + \varepsilon\|h\|^2).$$

The statement extends from the core to its chosen closure, and spectral calculus identifies the best quotient with $j(\varepsilon)^2$. The first term of (5) gives $a(\xi) \geq 16\xi^2/(1 + 4\xi^2)$, hence $a(\xi) \geq (16/5)\xi^2$ for $|\xi| \leq 1$. Put $q = \log(1/\varepsilon)$, $R = \log(q + 4) > 1$, and $\ell = \pi/(8R) < 1$. Then $\rho = 1/2$. With $t = q + 4 \geq 4$, the theta tail gives

$$\sigma_R^2/\varepsilon \leq 169t^4 e^{-3t^2+t-4} < 1.$$

The logarithmic derivative $4/t - 6t + 1$ is negative on $[4, \infty)$, and its endpoint value is $43264e^{-48} < 1$. It follows that

$$j(\varepsilon)^2 \leq 4(1 + 20R^2/\pi^2) < 16R^2.$$

The independent cap $j \leq \varepsilon^{-1/2}$ follows from $\|s\| \leq 1$. □

3 A complete finite-rank enclosure

For $\varepsilon > 0$ put $R_\varepsilon = (A + \varepsilon)^{-1/2}$ and $J = M_s R_\varepsilon$. For unit $h \in \mathcal{K}$,

$$\|Jh\| \leq \varepsilon^{-1/2}, \quad \int a(\xi) |\widehat{Jh}(\xi)|^2 \frac{d\xi}{2\pi} \leq 1.$$

Define

$$V_M = -m_0 I + \sum_{2 \leq n \leq M} c_n (S_{\log n} + S_{-\log n}), \quad d_M = |m_0| + 2 \sum_{2 \leq n \leq M} c_n.$$

Its Fourier multiplier is $v_M(\xi) = -m_0 + 2 \sum c_n \cos(\xi \log n)$, and $\|V_M\| \leq d_M$. Let Q_L denote the band projection and P_R the spatial projection. For $d \geq 0$ define

$$\widehat{E_d f}(\xi) = 1_{[-L, L]}(\xi) \sum_{k=0}^d \frac{(-i\xi)^k}{k!} \int_{-R}^R u^k f(u) du, \quad F_d = E_d J, \quad \mathcal{F} = F_d^* V_M F_d.$$

It is selfadjoint of rank at most $d + 1$.

Lemma 3.1 (Full error). *For $M \geq 1$, $R, L > 0$, and integer $d \geq 0$,*

$$\|K_\varepsilon - \mathcal{F}\| \leq \frac{\delta_M}{\varepsilon} + \frac{d_M}{a(L)} + d_M \tau (2/\sqrt{\varepsilon} + \tau), \quad (10)$$

$$\tau = (\sigma_R + b_d)/\sqrt{\varepsilon}, \quad b_d = \sqrt{2/\pi} \frac{(RL)^{d+3/2}}{(2d+3)(d+1)!}.$$

Proof. The arithmetic tail costs δ_M/ε . The retained sandwich is J^*V_MJ . Since V_M commutes with Q_L , it is the sum of $(Q_LJ)^*V_M(Q_LJ)$ and $((I - Q_L)J)^*V_M((I - Q_L)J)$, with zero cross-frequency blocks. The high block has norm at most $d_M/a(L)$ by the energy bound. The real-phase Taylor remainder, integrated on the space-frequency rectangle, gives $\|Q_LP_R - E_d\| \leq b_d$. Consequently

$$Q_LJ - F_d = Q_L(I - P_R)J + (Q_LP_R - E_d)J$$

has norm at most τ , and $\|F_d\| \leq \varepsilon^{-1/2} + \tau$. Expanding the difference of the two low-band signed products gives the last term of (10). This proves the bound on the entire receiving space. $\square$

4 Joint cutoff selection

We first sharpen the arithmetic coefficient with an elementary prime-power count.

Lemma 4.1 (Square-root arithmetic bound). *For integer $M \geq 1$, $d_M \leq 18\sqrt{M}$.*

Proof. Let $\Psi(x) = \sum_{n \leq x} \Lambda(n)$. Every prime power $p^j \in (n, 2n]$ contributes one unit to the corresponding summand $\lfloor 2n/p^j \rfloor - 2\lfloor n/p^j \rfloor$ in the valuation of $\binom{2n}{n}$; all other summands are nonnegative. Thus

$$\Psi(2n) - \Psi(n) \leq \log \binom{2n}{n} \leq 2n \log 2.$$

Summing along powers of two gives $\Psi(2^r) < 2^{r+1} \log 2$. If $2^{r-1} < x \leq 2^r$, then $\Psi(x) < 4x \log 2 < 3x$; at $x = 1$ it is zero. Stieltjes partial summation yields

$$\sum_{n \leq M} \frac{\Lambda(n)}{\sqrt{n}} = \frac{\Psi(M)}{\sqrt{M}} + \frac{1}{2} \int_1^M \Psi(t) t^{-3/2} dt \leq 6\sqrt{M} - 3 \leq 6\sqrt{M}.$$

Together with $|m_0| < 6$, this gives $d_M \leq 6 + 12\sqrt{M} \leq 18\sqrt{M}$. $\square$

For $0 < \varepsilon, \eta \leq 1$ define the following positive quantities:

$$\begin{aligned} k &= \max \left\{ 4, \left\lceil \frac{\log(1920/(\varepsilon\eta))}{\log 10} \right\rceil \right\}, & M &= k - 1, & D &= 18\sqrt{M}, \\ L &= \exp(6D/\eta), & X_0 &= \log \frac{312D}{\varepsilon\eta}, & R &= \frac{1}{2} \log(2X_0), \\ z &= RL, & b &= \frac{\varepsilon\eta}{24D}, \\ N(\varepsilon, \eta) &= \max \left\{ 1, \lceil 6z \rceil, \left\lceil \frac{\log(\sqrt{z}/b)}{\log 2} \right\rceil \right\}, & d &= N(\varepsilon, \eta) - 1. \end{aligned} \quad (11)$$

These formulas, including their integer ceilings, specify the joint approximation.

Proposition 4.2. *The cutoffs (11) make the full error (10) strictly less than η . For fixed η , $\log N(\varepsilon, \eta) = O_\eta(\sqrt{\log(1/\varepsilon)})$ as $\varepsilon \downarrow 0$.*

Proof. For integers $k \geq 4$, $k^2 \leq 2^k$ by induction, starting at 4. Thus $\delta_M \leq 640 \cdot 10^{-k} \leq \varepsilon\eta/3$, which pays the arithmetic term. Next, $a(L) \geq \frac{1}{2} \log(2L) > 3D/\eta$ pays the high-frequency term by less than $\eta/3$.

Since $e^{2R} = 2X_0$,

$$\sigma_R \leq 26X_0 e^{-3X_0} \leq 13e^{-X_0} = b,$$

using $2X_0 \leq e^{2X_0}$. Write $n = d + 1 = N(\varepsilon, \eta)$. Stirling's elementary factorial lower bound $n! \geq (n/e)^n$ and $n \geq 6z$, $e < 3$ give

$$b_{n-1} \leq \sqrt{z}(ez/n)^n \leq \sqrt{z}2^{-n} \leq b.$$

The last inequality is precisely the logarithmic ceiling in (11). Hence $\tau \leq \eta\sqrt{\varepsilon}/(12D)$, and the last error term is at most

$$D \frac{\eta\sqrt{\varepsilon}}{12D} \left(\frac{2}{\sqrt{\varepsilon}} + \frac{\eta\sqrt{\varepsilon}}{12D} \right) = \frac{\eta}{6} + \frac{\eta^2\varepsilon}{144D} < \frac{\eta}{3}.$$

This proves the strict total error. All quantities in the formulas are finite, with $M \geq 3$, $X_0 > 1$, $R > 0$ and $z > 1$.

For the rank asymptotic put $u = \log(1/\varepsilon)$. At fixed η , M is comparable to u for large u , $D = O_\eta(\sqrt{u})$, $\log L = O_\eta(\sqrt{u})$ and $R = O_\eta(\log u)$. The third cutoff for n is $O_\eta(u + \log z)$, whereas $z = R \exp(6D/\eta)$ eventually dominates every fixed power of u . Thus $\log n = O_\eta(\sqrt{u})$. Enlarging the multiplicative constant on a remaining compact interval of ε proves the rank assertion in Theorem 1.1. □

5 The entire negative part

Proof of the spectral conclusions. The preceding norm approximations, with arbitrary positive accuracy, imply that K_ε is compact selfadjoint. Fix $\eta = 1/2$ and a rank- N approximant $\mathcal{F}$ with error less than η . If the spectral subspace of K_ε for $(1, \infty)$ had dimension greater than N , it would intersect $\ker \mathcal{F}$ nontrivially. On that intersection its Rayleigh quotient is strictly greater than 1, but $\mathcal{F}u = 0$ and the norm error bound make it at most η . This contradiction bounds that dimension by $N(\varepsilon, 1/2)$.

The map $h = (A + \varepsilon)^{-1/2}u$ is a bijection from $\mathcal{K}$ onto the chosen form domain $\mathcal{V}$, with its form topology, and

$$\langle h, (H + \varepsilon)h \rangle_{\text{form}} = \|u\|^2 - \langle u, K_\varepsilon u \rangle.$$

Transporting finite-dimensional strictly negative form subspaces in both directions proves equality of the two negative indices. The spectral theorem identifies the form index of $H + \varepsilon$ with $\text{rank } 1_{(-\infty, -\varepsilon)}(H)$. This proves (1) from Proposition 4.2, without replacing a form-domain statement by a finite compression.

Since B is bounded and $A \geq 0$, H is bounded below. Finiteness of every negative-depth projection makes its negative spectrum discrete with finite multiplicities and possible accumulation only at zero. If the count is bounded by $A_0 \exp(B_0 \sqrt{\log(1/\varepsilon)})$, take, for large n ,

$$\varepsilon_n = \exp \left[- \left(\frac{\log(n/(2A_0))}{B_0} \right)^2 \right].$$

The number of eigenvalue magnitudes strictly larger than ε_n is at most $n/2$, so the n th magnitude is at most ε_n . Adjusting constants for finitely many remaining indices proves (2), including finite or empty spectra by zero padding. Every positive power of this bound is summable: for fixed $p > 0$, eventually $cp \log^2(n+1) > 2 \log(n+1)$. Hence $\sum_n \lambda_n(H_-)^p < \infty$, proving the stated Schatten conclusion. $\square$

6 Effective phase returns and growing-cutoff sharpness

For the retained operator define

$$K_{\text{hi}}(M, \varepsilon, L) = J^*(I - Q_L)V_M(I - Q_L)J, \quad b_M(L) = \max \left\{ 0, \sup_{|\xi| > L} \frac{v_M(\xi)}{a(\xi)} \right\}.$$

The energy estimate gives $\|K_{\text{hi}}\| \leq d_M/a(L)$ and $b_M(L) \leq d_M/a(L)$.

Theorem 6.1 (Uniform growing-cutoff saturation). *Let $X = \log \log \log L$ for sufficiently large L , and let $\omega(L) \geq \log 2$ be any specified envelope satisfying $\omega(L) = o(X)$. Uniformly over integers $2 \leq M \leq e^{\omega(L)}$ and $0 < \varepsilon \leq 1$,*

$$\frac{a(L) \sup \sigma(K_{\text{hi}}(M, \varepsilon, L))}{d_M} \rightarrow 1, \quad \frac{a(L) \|K_{\text{hi}}(M, \varepsilon, L)\|}{d_M} \rightarrow 1.$$

The scalar caps also satisfy $a(L)b_M(L)/d_M \rightarrow 1$ uniformly in that range.

6.1 Explicit every-interval prime-phase coverage

Lemma 6.2. *Let $p_1, \dots, p_m$ be the primes at most $M \geq 2$, and put $P = \prod p_j$. For $0 < \rho < 1$ define*

$$h = \lceil 256(m+1)^2 \rho^{-4} \rceil, \quad T = 8P^h(2h+1)^m.$$

Every real interval of length T contains N such that $\max_{p \leq M} |e^{iN \log p} - 1| < \rho$.

Proof. The nonnegative trigonometric polynomial

$$F(t) = \prod_{p \leq M} \left(\frac{1 + \cos(t \log p)}{2} \right)^h$$

has nonnegative Fourier coefficients summing to 1. Its constant coefficient is

$A_h = [(\frac{2^h}{h})/4^h]^m \geq (2h+1)^{-m}$. Every other frequency is $\sum k_p \log p$, with $|k_p| \leq h$. Unique factorization makes it nonzero. Write it as $\log(A/B)$, where A, B are distinct positive integers at most P^h . The inequality $\log(A/B) \geq (A-B)/A$ for $A > B$ gives absolute frequency at least P^{-h} . Thus the average of F on any interval of length T differs from A_h by at most $2P^h/T = 1/[4(2h+1)^m]$.

If one prime phase has distance at least ρ from 1, then $F(t) \leq (1 - \rho^2/4)^h$. This is less than $1/[2(2h+1)^m]$. To check the constants put $B_0 = (m+1)^2 \rho^{-4} \geq 1$. Then $h \leq 257B_0$, $2h+1 \leq 515B_0$, and

$$h\rho^2/4 \geq 64(m+1)^2 \rho^{-2} > \log 2 + m \log(2h+1).$$

For the last comparison use $\log 515 < 7$, $\log(m+1) \leq m+1$ and $\log(1/\rho) \leq 1/\rho$ to bound its right side by $(1 + 13m + 2m^2)\rho^{-2}$. The exponential inequality $1 - x \leq e^{-x}$ now proves the claim. The average lower bound $3/[4(2h+1)^m]$ exceeds the proposed pointwise failure ceiling, so some point of every interval has all phases within ρ . $\square$

6.2 Packets with their original weighted norm

Fix once and for all $f_0 \in C_c^\infty(-1, 1)$ with $\|f_0\| = 1$, and write $D_0 = \|f_0'\|$. For $R \geq 1$ let $f_R(u) = R^{-1/2}f_0(u/R)$. Then

$$\int_{|\xi|>r} |\hat{f}_R(\xi)|^2 \frac{d\xi}{2\pi} \leq D_0^2/(R^2 r^2). \quad (12)$$

On $|u| \leq R$ the first positive theta term gives

$$s(u)^2 \geq e^{-4e^{2R}}, \quad \|f_R/s\| \leq e^{2e^{2R}}. \quad (13)$$

Indeed $4x^2 - 6x \geq 2x^2$ for $x \geq \pi > 3$ implies $\psi(u) \geq 2\pi^2 e^{9u/2 - \pi e^{2u}}$ for $u \geq 0$. Divide by $\cosh(u/2) \leq e^{u/2}$, use $\pi < 4$, and reflect.

Choose fixed compact smooth repair vectors χ_0, χ_1 with invertible two-row matrix U . The exact identities

$$\sqrt{w}/s = \cosh(u/2), \quad \sqrt{w} \tanh(u/2)/s = \sinh(u/2)$$

make the constraint integrals of $e^{iNu} f_R/s$ those of $e^{iNu} f_R$ against $\cosh$ and $\sinh$. One integration by parts therefore gives a correction $r_{N,R}$ with $v_{N,R} = e^{iNu} f_R/s - r_{N,R} \in \mathcal{C}$ and

$$\|r_{N,R}\| + \|sr_{N,R}\| + \sqrt{\mathfrak{a}[r_{N,R}]} \leq C_0 e^{R/2} \sqrt{R}/N =: \kappa. \quad (14)$$

For an explicit fixed constant, put $g_j = \|\chi_j\| + \|s\chi_j\| + \sqrt{\mathfrak{a}[\chi_j]}$ and take

$$C_0 = 2(1/2 + D_0) \|U^{-1}\|_{\text{op}} \left(\sum g_j^2 \right)^{1/2}.$$

The derivative L^1 norms of each row integrand are at most $\sqrt{2}(1/2 + D_0)e^{R/2}\sqrt{R}$; multiplication by U^{-1} and Cauchy–Schwarz against the fixed vector norms prove (14). The form expression on a repair vector here denotes the same ambient nonnegative Fourier integral; it is defined for compact smooth functions without imposing the two rows.

The symbol upper bound in Lemma 2.1 and Plancherel give the uniform energy estimate

$$\mathfrak{a}[e^{iNu} f_R/s] \leq \log(1 + N) + 6 + D_0^2/R^2.$$

Use $1 + |N + \zeta| \leq (1 + N)(1 + |\zeta|)$ and $\log(1 + |\zeta|) \leq 1 + \zeta^2$ for this comparison. Form Cauchy–Schwarz then gives

$$\begin{aligned} \mathfrak{a}[v_{N,R}] &\leq [\sqrt{\log(1+N) + 6 + D_0^2/R^2} + \kappa]^2, \\ \|v_{N,R}\| &\leq e^{2e^{2R}} + \kappa. \end{aligned} \tag{15}$$

The inverse-theta input norm remains in the second bound.

6.3 A joint Rayleigh bound and the uniform limit

Fix $0 < \theta \leq 1/8$ and let

$$k_M = \lceil \log_2 M \rceil, \quad \rho = \theta/(2k_M), \quad r = \theta/(2 \log M), \quad R = \max\{1, D_0/(r\sqrt{\theta})\}.$$

Choose the return $N \in [L + r + 1, L + r + 1 + T]$ from Lemma 6.2. Each retained nonzero channel is $n = p^k$, $k \leq k_M$. Thus $|e^{iN \log n} - 1| \leq \theta/2$, and the additional phase on a window $|t| \leq r$ is at most $\theta/2$. Consequently $v_M(N + t) \geq (1 - \theta)d_M$ throughout that window. It lies entirely above L , and (12) puts at least $1 - \theta$ of the carrier mass there. On the rest of the high band use $v_M \geq -d_M$. Applying the correction bound yields

$$\langle sv_{N,R}, (I - Q_L)V_M(I - Q_L)sv_{N,R} \rangle \geq d_M(1 - 3\theta - 2\kappa - \kappa^2).$$

No frequency mass outside the central window is assigned an unproved positive sign.

Put

$$\kappa_L = C_0 e^{R/2} \sqrt{R}/L, \quad E_L = \log(1 + L + r + 1 + T) + 6 + D_0^2,$$

$$\mathcal{D}_L = (\sqrt{E_L} + \kappa_L)^2 + \varepsilon(e^{2e^{2R}} + \kappa_L)^2.$$

When $\kappa_L \leq 1/8$, the numerator below is positive. Normalizing $v_{N,R}$ by the actual reference form, as $u = (A + \varepsilon)^{1/2} v_{N,R} / \sqrt{\mathfrak{a}[v_{N,R}] + \varepsilon \|v_{N,R}\|^2}$, and using (15) gives

$$\sup \sigma(K_{\text{hi}}) \geq d_M(1 - 3\theta - 2\kappa_L - \kappa_L^2) / \mathcal{D}_L. \tag{16}$$

Proof of Theorem 6.1. Take $\theta = [(1 + \omega(L))/X]^{1/3}$ for large L , so $\theta \rightarrow 0$ and eventually $\theta \leq 1/8$. The preceding choices give

$$R = O(\sqrt{X(1 + \omega(L))} + 1) = o(X).$$

Since the number of retained primes is at most M , $\log P \leq m \log M$, and the explicit phase budget gives

$$\log T \leq C(M + 1)^3 (\log M + 1)^5 \theta^{-4} \leq e^{O(\omega(L))} (1 + \omega(L))^5 X^{4/3}.$$

The logarithm of this last bound is $o(X)$, so $\log T = e^{o(X)}$, whereas $\log L = e^{e^X}$. Therefore $T/L \rightarrow 0$ and $\kappa_L \rightarrow 0$ uniformly over the declared range. The normalization term is also paid uniformly for $0 < \varepsilon \leq 1$, because

$$\log \left(\frac{e^{4e^{2R}}}{\log L} \right) = 4e^{2R} - \log \log L = e^{o(X)} - e^X \longrightarrow -\infty.$$

Thus $\mathcal{D}_L/\log L \rightarrow 1$ uniformly. Combine (16) with $a(L)/\log L \rightarrow 1$ and the universal upper $d_M/a(L)$ to prove both operator limits. For the scalar cap, the same returns give $v_M(N) \geq (1 - \theta)d_M$, $N/L \rightarrow 1$ and $a(N)/a(L) \rightarrow 1$. This proves its lower limit; its upper bound is again $d_M/a(L)$. $\square$

7 Fast cutoffs and a strict high-block gap

The proof uses medium as well as disjoint spatial translations. We give the projection facts before transferring them to the original weighted output.

Lemma 7.1 (Projected compact-core correlations). *Let F be supported in $I = [-R, R]$ and let Q be an orthogonal projection on $L^2(\mathbb{R})$. For $b > R$,*

$$|\langle F, QS_b F \rangle| \leq \frac{3}{4} \|F\|^2.$$

For $b > 2R$ the upper coefficient can be replaced by $1/2$. If Q commutes with translations, the same bounds apply to $\langle QF, S_b QF \rangle$.

Proof. The compression $A_b = P_I S_b P_I$ on $L^2(I)$ is a contraction. For $b > R$ it satisfies $A_b^2 = 0$, since $u, u+b, u+2b$ cannot all belong to I . The compact-window half bound is also recorded in [8, Lemma 1] and [10, Lemmas 1 and 1']; it is the order-two case of [9, Theorem 1]. Here is a direct proof. Decompose $L^2(I) = \ker A_b \oplus (\ker A_b)^\perp$. Since $\text{Ran } A_b \subset \ker A_b$, for $F = (x, y)$ one has

$$|\langle F, A_b F \rangle| \leq \|x\| \|y\| \leq \frac{1}{2} \|F\|^2.$$

Now $2Q - I$ is unitary, so

$$|\langle F, QS_b F \rangle| \leq \frac{1}{2} (|\langle F, S_b F \rangle| + \|F\|^2) \leq \frac{3}{4} \|F\|^2.$$

If $b > 2R$, F and $S_b F$ are orthogonal. For normalized orthogonal vectors F, G put $\alpha = \langle F, QF \rangle$, $\beta = \langle G, QG \rangle$. The squared cross term is bounded both by $\alpha\beta$ and $(1 - \alpha)(1 - \beta)$, using projection Cauchy–Schwarz and $\langle F, QG \rangle = -\langle F, (I - Q)G \rangle$. At least one product is at most $1/4$, according as $\alpha + \beta \leq 1$ or $\alpha + \beta \geq 1$. This proves the stronger half bound. Finally, commutation and $Q^2 = Q$ give $\langle QF, S_b QF \rangle = \langle F, QS_b F \rangle$. $\square$

Fix a spatial margin $\delta > 0$. For large q put

$$R_w = \frac{1}{2} \log(q/\pi), \quad R_0 = R_w + \delta/2, \quad R_1 = R_w + \delta.$$

Choose real even $\chi_q \in C_c^\infty$, with $0 \leq \chi_q \leq 1$, equal to 1 on $[-R_0, R_0]$ and supported in $[-R_1, R_1]$. A fixed transition of width $\delta/2$ gives $\text{Lip } \chi_q \leq C_\delta$ independently of q . Fix real even $\eta_0 \in C_c^\infty$ with $0 \leq \eta_0 \leq 1$, equal to 1 on $[-1/2, 1/2]$ and zero outside $[-1, 1]$. Let B_L have Fourier multiplier $1 - \eta_0(\xi/L)$, and set

$$k(x) = \frac{1}{2\pi} \int e^{ix\xi} \eta_0(\xi) d\xi, \quad C_{\eta_0} = \int |x| |k(x)| dx < \infty.$$

The multiplier is a contraction, vanishes below $L/2$, and satisfies $(I - Q_L)B_L = I - Q_L$ exactly. The identity part of B_L commutes with χ_q ; the remaining commutator kernel, up to sign, is

$$Lk(L(u - v))(\chi_q(v) - \chi_q(u)).$$

Its row and column absolute integrals are each at most $C_\delta C_{\eta_0}/L$. The integral Schur test proves

$$\|[B_L, \chi_q]\| \leq C_\delta C_{\eta_0}/L. \quad (17)$$

This estimate concerns the smooth multiplier B_L .

For any unit input $h \in \mathcal{K}$, let $v = (A + \varepsilon)^{-1/2}h$, $f = sv$, $g = B_L f$ and $F = \chi_q g$. The original input satisfies $\|v\| \leq e^{q/2}$ when $\varepsilon = e^{-q}$. The sharper envelope in the proof of Lemma 2.1 gives

$$\sup_{|u| > R} s(u) \leq 4\pi e^{2R - (\pi/2)e^{2R}}.$$

At R_0 this bounds $\|(1 - \chi_q)f\|$ by $4qe^\delta \exp[-q(e^\delta - 1)/2]$. Commuting B_L and χ_q with (17), and using Lemma 2.2 for $\|f\|$, yields the full error

$$\|g - F\| \leq e_q := 4qe^\delta e^{-q(e^\delta - 1)/2} + \frac{4C_\delta C_{\eta_0} \log(q + 4)}{L}. \quad (18)$$

Both terms are uniform in the unit input. The two different norms have distinct roles: $e^{q/2}$ pays the original spatial tail, and the smaller observation norm pays only the commutator. Since g vanishes in the frequency interval $[-L/2, L/2]$, its complete reference energy gives

$$(I - Q_L)g = (I - Q_L)f, \quad \|g\|^2 \leq 1/a(L/2), \quad \|F\| \leq \|g\|. \quad (19)$$

Thus the sharp highpass output is compared to a confined smooth-highpass core without assuming that the sharp projection itself preserves support.

Define the medium-and-long and long arithmetic masses

$$S_1 = \sum_{\substack{2 \leq n \leq M \\ \log n > R_1}} c_n, \quad S_2 = \sum_{\substack{2 \leq n \leq M \\ \log n > 2R_1}} c_n.$$

Apply Lemma 7.1 with $Q = I - Q_L$. Each short paired channel costs at most $2c_n\|F\|^2$, each medium channel costs $\frac{3}{2}c_n\|F\|^2$, and each long channel costs $c_n\|F\|^2$. The constant term costs $(-m_0)\|F\|^2$. Therefore

$$|\langle F, (I - Q_L)V_M(I - Q_L)F \rangle| \leq (d_M - \frac{1}{2}S_1 - \frac{1}{2}S_2)\|F\|^2.$$

The coefficient is positive, since $S_1, S_2 \leq \sum c_n$. The bounded signed multiplier and (18)–(19) now imply the whole-input norm estimate

$$\|K_{\text{hi}}\| \leq \frac{d_M - \frac{1}{2}S_1 - \frac{1}{2}S_2}{a(L/2)} + d_M e_q \left(\frac{2}{\sqrt{a(L/2)}} + e_q \right). \quad (20)$$

Indeed the difference of the g and F quadratic forms is bounded by the second term, including all mixed terms. The identity in (19) returns this to the original high block, and a uniform absolute quadratic bound on a selfadjoint operator bounds its norm.

Proof of Theorem 1.2. The classical unconditional prime number theorem gives

$$\Psi(x) = \sum_{n \leq x} \Lambda(n) = x + o(x);$$

see [2, 25.16.3 and 27.2.3]. Abel summation gives

$$A_1(x) := \sum_{n \leq x} c_n = \frac{\Psi(x)}{\sqrt{x}} + \frac{1}{2} \int_1^x \Psi(t) t^{-3/2} dt = 2\sqrt{x} + o(\sqrt{x}).$$

For the error, split the integral at any fixed threshold beyond which $|\Psi(t) - t| \leq \tau t$, bound the remainder by a constant times $\tau\sqrt{x}$, then send τ to zero. No uniform quantitative PNT error is required.

Since $M \sim Cq$, $d_M \sim 4\sqrt{Cq}$. Also $e^{R_1} = e^{\delta\sqrt{q/\pi}}$, so $A_1(e^{R_1}) = O_\delta(q^{1/4})$. Consequently $S_1/d_M \rightarrow 1/2$ for every fixed $C > 0$. Put $A_\delta = e^{2\delta}/\pi$. If $C > A_\delta$, subtraction of the two prefix asymptotics gives $S_2/d_M \rightarrow \frac{1}{2}(1 - \sqrt{A_\delta/C})$. If $C < A_\delta$, this mass is eventually zero. At equality it is still $o(d_M)$, since the positive part of the difference of the two prefix asymptotics is $o(\sqrt{q})$. Thus in all cases

$$\frac{S_2}{d_M} \rightarrow \frac{1}{2}(1 - \sqrt{A_\delta/C})_+.$$

Under $\log L \sim K\sqrt{q}$, $a(L)/a(L/2) \rightarrow 1$ and $\sqrt{a(L)}e_q \rightarrow 0$: the first error term decays exponentially in q at fixed δ , and the second is $O_\delta(\log q/L)$. Multiplying (20) by $a(L)/d_M$ therefore gives

$$\limsup \frac{a(L)\|K_{\text{hi}}\|}{d_M} \leq \frac{3}{4} - \frac{1}{4}(1 - \sqrt{A_\delta/C})_+.$$

The target is independent of δ and this holds for every fixed positive δ . Sending $\delta \downarrow 0$ after the q limsup proves (3), including $C = 1/\pi$. The positive spectral edge is at most the norm. For the explicit cutoffs (11) at fixed η , $M/q \rightarrow 1/\log 10$ and $\log L/\sqrt{q} \rightarrow 108/(\eta\sqrt{\log 10})$, which proves the stated specialization. $\square$

8 Parameter regimes and further applications

Theorem 1.1 selects $M \asymp \log(1/\varepsilon)$ and $\log L \asymp \sqrt{\log(1/\varepsilon)}$ at fixed approximation tolerance. Theorem 6.1 proves high-block saturation in the slower range $\log M = o(\log \log \log L)$. Theorem 1.2 now shows a strict failure of saturation throughout the fast rank scale, including the critical theta-overlap ratio. Thus extending the fixed-cutoff sharpness constant to that scale would be incorrect. The high-block upper coefficient there has a genuine reserve; the finite bound (20) states how its medium and long arithmetic masses enter before the asymptotic evaluation.

The optimal approximation rank depends on the low-frequency map and complete arithmetic remainder as well as the retained-block gap. A natural further refinement is a joint bound on the arithmetic sum before separately maximizing its channel correlations. Such a bound must preserve the actual confined smooth-highpass range and its full weighted input norm.

The new negative-spectrum conclusion concerns H_- itself. It uses no identification of an entire kernel or derivative-null closure. Structured finite signed certificates can address individual lower thresholds through a different reference and computational task. The rank theorem and the joint saturation theorem provide analytic information about the full operator family across parameter ranges.

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