

# Holomorphic certification on real subspaces

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October 27, 2026

## Abstract

We study a family of nonnegative Hermitian quartics on complexifications of real linear subspaces of dimension at least two. A certificate expresses the quartic, after multiplication by a power of the squared Euclidean norm, as a sum of squared absolute values of holomorphic polynomials. We characterize certificate existence by a simple geometric condition: the real subspace contains no vector whose coordinates are all plus or minus one. When such a vector exists, every nonzero intrinsic polynomial multiplier is excluded. This classification yields coNP-hardness even for integer quartics with polynomially bounded coefficients and exactly one negative coefficient eigenvalue. For fixed ambient and proper subspace dimensions and a fixed positive parameter, the least certificate degree grows as the inverse square of the distance to the failure set as that distance tends to zero. We identify that distance exactly in the operator norm on subspace projections. A parallel law holds for bounded matrices imposing soft constraints. The lower bounds use logarithmic curvature; classical quantitative stabilization supplies matching upper bounds. A weighted two-dimensional family gives the exact leading constant and an example with short rational input but exponentially growing certificate degree.

**Keywords:** Hermitian forms, holomorphic squared norms, certificate complexity, stabilization degree, positive polynomials.

**Mathematics Subject Classification (2020):** 32A99, 68Q17, 15B48.

## 1 The family and the main results

For a fixed multiplier degree, a holomorphic squared-norm certificate is a positive-semidefinite coefficient matrix. The union over all degrees raises a different question: does this certificate hierarchy eventually succeed? We study that question for quartics that are already pointwise nonnegative. Even when the coefficient matrix has only one negative direction, eventual success remains a coNP-hard decision.

The geometric mechanism also explains how the hierarchy becomes expensive. Real linear constraints reduce positivity to avoidance of the sign cube. The least certificate degree grows uniformly as the inverse square of the subspace distance to the nonstabilizing locus. Soft penalties for the constraints obey a corresponding law in their signed residual. A two-dimensional weighted restriction gives a complementary sharp result: its full coefficient matrices determine the exact leading constant. Thus the decision theorem and the quantitative results describe the same distinction between positivity of values and eventual positivity of coefficient matrices. The results address three questions: whether certification eventually succeeds, how difficult that decision is, and how

large the required degree becomes near failure. We now define the family and state the precise answers.

Let  $V \subset \mathbb{R}^n$  be a real linear subspace of dimension  $d \geq 2$ , and write  $V_{\mathbb{C}} = V + iV$ . Equip it with the inherited Hermitian metric. On  $V_{\mathbb{C}}$  define

$$\begin{aligned} N(v) &= \sum_i |v_i|^2, & p_2(v) &= \sum_i v_i^2, \\ A(v) &= n \sum_i |v_i|^4 - |p_2(v)|^2 = \sum_{i < j} |v_i^2 - v_j^2|^2, \\ Q(v) &= N(v)^2 - |p_2(v)|^2, & q_\rho &= A + \rho Q, \quad \rho > 0. \end{aligned} \tag{1}$$

Both  $A$  and  $Q$  are nonnegative. A *holomorphic squared norm* means a finite sum  $\sum_\nu |h_\nu|^2$  with holomorphic polynomial  $h_\nu$ . For a quartic define its norm stabilization degree by

$$\mu_V(\rho) = \min\{m \geq 0 : N^m q_\rho \text{ is a holomorphic squared norm on } V_{\mathbb{C}}\},$$

with value  $+\infty$  when the set is empty.

**Theorem 1.1** (One negative direction and bounded-coefficient hardness). *At  $\rho = 2n^2$ , the Hermitian coefficient matrix of  $q_\rho$  in any independent linear coordinates on  $V_{\mathbb{C}}$  has exactly one negative eigenvalue and all other eigenvalues positive.*

*Deciding whether some finite norm-power certificate exists is coNP-hard even for explicitly expanded, pointwise nonnegative, integer bihomogeneous quartics with exactly one negative coefficient eigenvalue. In the reduction, each coefficient has absolute value at most  $486n^4$ , where  $n$  is the size of the ambient subspace construction. For the structured input given by signed triples defining that construction, the decision language is coNP-complete.*

A fixed-dimensional consequence also concerns succinct input: at  $\rho = 1$ , the family  $b_L = 2 + 2^{-L}$  has rational coefficients of bit length  $O(L)$  but requires stabilization degree  $\Theta(4^L)$  (Corollary 6.1).

The geometric criterion used in the reduction is the following.

**Theorem 1.2** (Sign-cube classification). *The following are equivalent:*

1.  $V \cap \{-1, 1\}^n = \emptyset$ .
2.  $q_\rho(v) > 0$  for every nonzero  $v \in V_{\mathbb{C}}$ .
3.  $\mu_V(\rho) < \infty$ .
4.  $q_\rho$  is a quotient of holomorphic squared norms.
5. There is a Hermitian polynomial  $B$ , nonzero in the intrinsic coordinate ring of  $V_{\mathbb{C}}$ , such that  $Bq_\rho$  is a holomorphic squared norm.

*These equivalences hold for every  $\rho > 0$ .*

The intrinsic qualification excludes ambient polynomials that vanish identically on the subspace. The last assertion permits multipliers of arbitrary degree, sign and vanishing order.

The classification has a uniform metric refinement. Let  $P_V$  be orthogonal projection onto a real  $d$ -plane  $V$ , and define

$$\eta(V) = \min_{s \in \{-1, 1\}^n} \frac{\|(I - P_V)s\|}{\sqrt{n}}, \quad \mathcal{S}_{n,d} = \{W \in \text{Gr}_d(\mathbb{R}^n) : W \cap \{-1, 1\}^n \neq \emptyset\}.$$

Equip the real Grassmannian with  $d_{\text{op}}(V, W) = \|P_V - P_W\|_{\text{op}}$ .

**Theorem 1.3** (Uniform distance to the nonstabilizing locus). *For  $n \geq 3$ ,  $2 \leq d < n$ , and every real  $d$ -plane  $V$ ,*

$$\eta(V) = \text{dist}_{\text{op}}(V, \mathcal{S}_{n,d}). \quad (2)$$

*Fix these dimensions and  $\rho > 0$ . There are constants  $c, C, \varepsilon > 0$ , depending only on  $n, d, \rho$ , such that*

$$c\eta(V)^{-2} \leq \mu_V(\rho) \leq C\eta(V)^{-2} \quad \text{whenever} \quad 0 < \eta(V) < \varepsilon. \quad (3)$$

Thus the degree scale is uniform over the entire fixed-dimensional subspace family, including simultaneous approaches to several sign vectors. The next theorem treats constraints through a positive penalty. For a real matrix  $B \in \mathbb{R}^{r \times n}$ , where  $n \geq 2$  and  $r \geq 0$ , set

$$F_{B,\rho}(v) = A(v) + \rho Q(v) + N(v)\|Bv\|^2, \quad v \in \mathbb{C}^n, \quad \rho > 0, \quad (4)$$

$$\sigma(B) = \min_{s \in \{-1,1\}^n} \frac{\|Bs\|}{\sqrt{n}}. \quad (5)$$

Let  $\mu_B(\rho)$  be the least  $m \geq 0$  for which  $N^m F_{B,\rho}$  is a holomorphic squared norm on  $\mathbb{C}^n$ , with value  $+\infty$  when the set is empty.

**Theorem 1.4** (Uniform degree law for soft constraints). *The form  $F_{B,\rho}$  is nonnegative for every real  $B$ . It is strictly positive away from zero, and has finite  $\mu_B(\rho)$ , exactly when  $\sigma(B) > 0$ . At  $\sigma(B) = 0$ , no nonzero polynomial multiplier makes  $F_{B,\rho}$  a holomorphic squared norm.*

*Fix  $n \geq 2$ ,  $r \geq 0$ ,  $\rho > 0$  and  $K > 0$ . There are constants  $c, C, \varepsilon > 0$ , depending only on these fixed parameters, such that*

$$c\sigma(B)^{-2} \leq \mu_B(\rho) \leq C\sigma(B)^{-2} \quad \text{if} \quad \|B\|_{\text{op}} \leq K, \quad 0 < \sigma(B) < \varepsilon. \quad (6)$$

The constants are uniform as the matrix rank changes and as several signed constraints approach zero together. The positive-penalty formulation makes the distance to failure explicit. Empty matrices and zero rows follow the same criterion. The positive assumption on  $\rho$  accounts for the anisotropic obstruction; at  $\rho = 0$ ,  $A + N\|Bv\|^2$  is itself a holomorphic squared norm.

A complementary exact leading constant holds on the weighted hyperplane

$$H_b = \{v \in \mathbb{C}^3 : v_1 + v_2 + bv_3 = 0\}, \quad b > 0.$$

Write  $\mu_b(\rho)$  for its stabilization degree.

**Theorem 1.5** (Boundary degree law). *For each fixed  $\rho > 0$ ,*

$$\mu_2(\rho) = +\infty, \quad \lim_{b \rightarrow 2, b \neq 2} (b-2)^2 \mu_b(\rho) = \frac{9}{2}(\sqrt{1+\rho} - 1)^2.$$

*The limit is obtained from matching bounds on the complete coefficient matrices.*

## 1.1 Relation to earlier work

Quillen and Catlin–D’Angelo prove finite stabilization of strictly positive Hermitian forms [13, 2]. D’Angelo’s one-variable examples and Varolin’s jet-pullback and basic-zero criteria describe obstructions at zeros [3, 17]. We apply that classical mechanism directly. D’Angelo [4] compares nonnegativity, plurisubharmonicity, Cauchy–Schwarz positivity and squared-norm representations for bihomogeneous families. Our central object is eventual norm-power certification on varying real

subspaces. In the nonsingular coefficient regime of Theorem 1.1, Varolin’s free-support criterion also identifies quotient existence with absence of nontrivial zeros.

The unweighted odd-coordinate hyperplane was treated in [15]: its least degree is at least  $n - 2$  for every positive perturbation, with a finite first-degree parameter window and a sharp five-coordinate endpoint. The present subspace classification includes even coordinates and arbitrary real linear constraints. Its complexity consequence uses an explicit sparse Boolean encoding while preserving nonnegativity and a single negative coefficient direction.

General quartic and tensor hardness is established in [1, 10, 7]. Those results concern different positivity, convexity, decomposition or norm problems. Here the input is already nonnegative, and the decision asks whether a holomorphic certificate exists at *any* finite degree. Norm-power eigenvalue hierarchies are classical; see [12]. Our construction isolates an exact boundary decision within a restricted coefficient class. General high stabilization degrees and perturbative hierarchy jumps also have substantial precedents [2, 9, 6]; Theorem 1.5 gives a matched order for the specific weighted-subspace boundary and determines its leading constant.

Gaubert and Qu [8, Section 4] prove hardness of strict Kraus-map positivity through rational Boolean encodings and a single-vector-to-bilinear feasibility transfer. Those mechanisms are important predecessors. The direct squared-modulus polynomial of their equations is already a holomorphic squared norm on every instance. Our construction instead preserves nonnegativity while making eventual certification equivalent to Boolean unsatisfiability, with one negative coefficient direction and polynomial coefficient magnitude. The anisotropic term and its local obstruction provide this interface.

Effective reciprocal-margin upper bounds were established by To and Yeung [16]; the quantitative theorem used here is Drouot–Zworski [6, Theorem 2]. Its bound supplies the upper bound in Theorem 1.4. Its lower bound uses a uniform anisotropic slice at a real minimum and the necessary plurisubharmonicity of the logarithm of a squared norm. The soft-constraint result tracks all bounded matrices through their smallest signed residual. Theorem 1.3 uses the same anisotropic calculation after restriction and identifies the residual with exact distance to the nonstabilizing locus; it quantifies the original subspace classification uniformly over all fixed-dimensional subspaces. For Theorem 1.5, complete finite coefficient matrices give the sharper leading constant; the limiting even trial sequence is proved by its explicit recurrence. Quadratic Toeplitz model energies and their squeezed-state descriptions are classical; see [5, Section 3.3]. Here complete finite row bounds and fixed finite trial limits supply the sharp boundary certificate.

## 1.2 Proof strategy and organization

The local obstruction in Section 2 proves the sign-cube classification. The coefficient inertia in Section 3 and the Boolean encoding in Section 4 then give the restricted hardness theorem. Section 5 obtains the hard- and soft-constraint degree laws from a common positivity margin and logarithmic-curvature estimate. Section 6 uses the complete coefficient matrices to determine the sharper leading constant for the binary family.

## 2 Zeros and the local obstruction

**Lemma 2.1** (An anisotropic leading jet). *Let  $q$  be a Hermitian polynomial on a complex vector space, with zero constant and linear Taylor terms at  $z_0$ . Suppose a nonempty open set of directions  $w$  has leading restriction*

$$q(z_0 + tw) = a|t|^2 - bt^2 - \bar{b}\bar{t}^2 + O(|t|^3), \quad b \neq 0.$$

Then no intrinsic nonzero polynomial  $B$  has  $Bq$  equal to a nonzero holomorphic squared norm.

*Proof.* For a fixed nonzero  $B$ , some direction in the stated open set has  $B(z_0 + tw) \neq 0$ . Otherwise every homogeneous Taylor coefficient of  $B$  at  $z_0$  vanishes on an open set and hence identically. Let  $B_k(t, \bar{t})$  be the lowest nonzero term on this line. The leading product is  $q_2 B_k$ , a nonzero polynomial in  $\mathbb{C}[t, \bar{t}]$ .

Every nonzero squared norm of one-variable holomorphic polynomials has leading term  $c|t|^{2\ell}$ ,  $c > 0$ : take the smallest vanishing order of its components and add the squared leading coefficients. Thus  $q_2$  would divide  $t^\ell \bar{t}^\ell$  in  $\mathbb{C}[t, \bar{t}]$ . Every divisor of a monomial is a monomial up to a constant, by unique factorization. The two nonzero pure-square coefficients of  $q_2$  contradict this.  $\square$

*Proof of Theorem 1.2.* If  $A(v) = 0$  and  $v \neq 0$ , all  $v_i^2$  are equal and nonzero. Hence  $v = c\epsilon$  with  $c \neq 0$  and  $\epsilon \in \{-1, 1\}^n$ . Since  $V$  is real,  $c\epsilon \in V_{\mathbb{C}}$  implies  $\epsilon \in V$ . Conversely every such sign vector satisfies  $A = Q = 0$ . This proves the equivalence of the first two assertions. Strict positivity gives finite stabilization by [13, 2], and a norm-power certificate is a quotient certificate and a polynomial multiplier certificate.

Suppose now  $\epsilon \in V \cap \{-1, 1\}^n$ . For  $w \in V_{\mathbb{C}}$ , put

$$C = n\|w\|^2 - |\epsilon \cdot w|^2, \quad D = n \sum_i w_i^2 - (\epsilon \cdot w)^2.$$

Expansion at the balanced-modulus vector  $\epsilon$  gives

$$\begin{aligned} A(\epsilon + tw) &= 4C|t|^2 + O(|t|^3), \\ Q(\epsilon + tw) &= 2C|t|^2 - Dt^2 - \bar{D}\bar{t}^2 + O(|t|^3). \end{aligned}$$

There is a nonzero real  $w \in V \cap \epsilon^\perp$ , since  $d \geq 2$ . For this direction  $C = D = n\|w\|^2 > 0$ , so  $D \neq 0$  on a nonempty open set of complex directions. Lemma 2.1 excludes every intrinsic nonzero polynomial multiplier and proves the remaining implication.  $\square$

**Corollary 2.2** (Hyperplanes). *For  $a \in \mathbb{R}_{>0}^n$ ,  $n \geq 3$ , the restriction to  $\sum_i a_i v_i = 0$  has finite stabilization exactly when  $\sum_i a_i \epsilon_i \neq 0$  for every sign vector. For the unweighted hyperplane this occurs exactly when  $n$  is odd.*

The weighted condition is the absence of an exact signed partition. For example,  $(1, 1, 2)$  has an obstruction in three coordinates, whereas  $(1, 1, 1, 4)$  admits finite stabilization in four.

### 3 The coefficient inertia

Choose real orthonormal coordinates  $z_1, \dots, z_d$  on  $V$ . Use the Fischer product  $\langle z^\alpha, z^\beta \rangle_F = \alpha! \mathbf{1}_{\alpha=\beta}$ . The coefficient operator of  $N^2$  on holomorphic quadrics is  $2I$ , and that of  $|p_2|^2$  is  $|p_2\rangle\langle p_2|$ , with  $\|p_2\|_F^2 = 2d$ . Thus

$$M_Q = 2I - |p_2\rangle\langle p_2|.$$

Its radial eigenvalue is  $-2(d-1)$ ; every other eigenvalue is 2.

Let  $P_V$  be real orthogonal projection onto  $V$ . The linear form  $l_i(z) = v_i$  has norm  $\|P_V e_i\| \leq 1$ , so  $\|l_i^2\|_F^2 = 2\|P_V e_i\|^4 \leq 2$ . As  $A$  is already a squared norm,

$$0 \preceq M_A, \quad \|M_A\| \leq \text{Tr } M_A = 2n \sum_i \|P_V e_i\|^4 - 2d \leq 2n^2.$$

At  $\rho = 2n^2$ , the  $p_2$  Rayleigh quotient of  $M_A + \rho M_Q$  is at most

$$2n^2 - 2\rho(d-1) < 0.$$

Meanwhile

$$M_A + \rho M_Q = (M_A + 2\rho I) - \rho|p_2\rangle\langle p_2|$$

is a positive definite operator minus a positive rank-one operator. Congruence gives  $I - \rho uu^*$ . The negative Rayleigh direction therefore forces precisely one negative eigenvalue, with every other eigenvalue positive. Independent coordinate changes and monomial normalizations act by invertible congruence and preserve this inertia. This proves the first part of Theorem 1.1.

## 4 Sparse Boolean encoding

Consider a not-all-equal three-literal instance with  $k$  variables and  $m$  clauses. A signed truth assignment is  $x_i \in \{-1, 1\}$ . For each clause define

$$l_c(x) = \sigma_{c,1}x_{i_1} + \sigma_{c,2}x_{i_2} + \sigma_{c,3}x_{i_3}, \quad \sigma_{c,j} \in \{-1, 1\}.$$

Set  $n = k + m$ , and use the graph subspace

$$V = \{(x_1, \dots, x_k, l_1(x), \dots, l_m(x)) : x \in \mathbb{R}^k\}.$$

Its displayed coordinates are independent, and each embedding row has coefficient  $\ell^1$  norm at most 3. Two unused variables may be added to ensure  $k \geq 2$ . The sum of three signs belongs to  $\{-1, 1\}$  exactly when those signs are not all equal. Hence

$$V \cap \{-1, 1\}^n \neq \emptyset \iff \text{the not-all-equal instance is satisfiable.} \quad (7)$$

For completeness, ordinary three-satisfiability reduces to these instances by a two-clause gadget. Introduce a global reference bit  $f$  and one auxiliary bit  $a_c$  for each clause  $u \vee v \vee w$ . Replace it by

$$\text{NAE}(f, u, a_c), \quad \text{NAE}(v, w, \neg a_c).$$

Simultaneous complementation preserves every NAE predicate, so assume  $f$  is false. If  $u$  is false, the first predicate forces  $a_c$  true, and the second is equivalent to  $v \vee w$ . If  $u$  is true, the first allows either  $a_c$ , and the second can always be satisfied. This proves the reduction from classical three-satisfiability [11, 14].

In the graph coordinates take  $\rho = 2n^2$ . Theorem 1.2 and (7) identify eventual certifiability with unsatisfiability. The construction expands in polynomial time. To check its numerical size, let  $l_i$ ,  $1 \leq i \leq n$ , denote all embedding forms. Then

$$q_\rho = n \sum_i |l_i^2|^2 + \rho \sum_{i,j} |l_i l_j|^2 - (1 + \rho) \left| \sum_i l_i^2 \right|^2.$$

Each  $l_i$  has coefficient  $\ell^1$  norm at most 3. The total coefficient  $\ell^1$  norm is consequently at most

$$81n^2 + 81\rho n^2 + 81(1 + \rho)n^2 = 162n^2 + 324n^4 \leq 486n^4.$$

Every coefficient is an integer, and there are  $O(k^4)$  monomial pairs. The reduction remains polynomial when coefficients are written in unary.

In these graph coordinates the inherited multiplier is  $N(x) = \sum_i |l_i(x)|^2$ , a positive definite quadratic form. For the instances constructed here, eventual existence is independent of the positive definite multiplier norm. On sign-free instances, strict positivity and classical stabilization give a certificate for every such norm after a linear coordinate change. On sign-containing instances, the intrinsic polynomial-multiplier obstruction excludes every norm-power certificate. Thus the same reduction applies when the input convention fixes the standard coordinate norm.

Thus the coefficient-input problem is coNP-hard on the stated nonnegative, one-negative-direction promise class. For the structured signed-triple input, its complement has the Boolean satisfying assignment as a polynomial certificate, giving coNP-completeness. This membership statement concerns the structured language; coefficient-input hardness does not require recognizing the promise class.

#### 4.1 A finite demonstration

The seven triples

$$123, 145, 167, 246, 257, 347, 356$$

form the Fano plane and have no two-coloring. Their graph-subspace quartic is therefore strictly positive and finitely stabilizable. Deleting the last triple gives a sign-feasible subspace and absolute multiplier obstruction. The accompanying exact-arithmetic verifier enumerates all colorings and reconstructs both  $28 \times 28$  coefficient matrices. Their inertias are both  $(27, 1, 0)$ ; their different stabilization behavior is determined by the coherent sign vectors, rather than by the coefficient inertia alone.

### 5 Uniform degree growth for hard and soft constraints

We first prove the analytic estimates for Theorem 1.4, then apply the same slice calculation to prove Theorem 1.3. Throughout this section  $B$  is real. The penalty  $N\|Bv\|^2$  is a holomorphic squared norm, so  $F_{B,\rho} \geq 0$ .

#### 5.1 When a sign vector lies in the constraint kernel

If  $A(v) = 0$  and  $v \neq 0$ , the zero-set argument above gives  $v = cs$  for a sign vector  $s$ . At this vector

$$Q(cs) = 0, \quad N(cs)\|Bcs\|^2 = n|c|^4\|Bs\|^2.$$

Consequently  $\sigma(B) > 0$  gives strict positivity, and classical stabilization gives a finite norm-power certificate.

Suppose instead that  $Bs = 0$ . Choose a nonzero real  $w \perp s$ . On the complex line  $s + zw$ , where  $z = x + iy$ , the quadratic Taylor part is

$$\alpha|z|^2 + 4\rho n\|w\|^2 y^2, \quad \alpha = 4n\|w\|^2 + n\|Bw\|^2 > 0. \quad (8)$$

The  $x^2$  and  $y^2$  coefficients differ, and hence the pure  $z^2$  coefficient is nonzero. This remains true on an open set of complex directions around  $w$ . Lemma 2.1 excludes every nonzero polynomial multiplier. This proves the qualitative assertions.

## 5.2 A uniform positivity margin

Let

$$\delta(B) = \min_{v \in \mathbb{R}^n, \|v\|=1} (A(v) + \|Bv\|^2).$$

Assume  $\|B\|_{\text{op}} \leq K$ . For a real unit vector  $v$ , choose  $s_i = \text{sign}(v_i)$ , with either choice at zero, and put  $u = s/\sqrt{n}$ . Then

$$\begin{aligned} A(v) &= n \sum_i (v_i^2 - 1/n)^2 \\ &\geq \sum_i (|v_i| - 1/\sqrt{n})^2 = \|v - u\|^2. \end{aligned}$$

It follows that  $\sigma(B) \leq \|Bu\| \leq \|Bv\| + K\sqrt{A(v)}$ . Inserting a minimizing sign vector for the converse bound gives

$$\frac{\sigma(B)^2}{2 \max\{1, K^2\}} \leq \delta(B) \leq \sigma(B)^2. \quad (9)$$

When  $\delta(B) \leq 1$ , it is also the exact minimum of  $F_{B,\rho}$  on the complex unit sphere. Indeed rotate a vector by a scalar phase so that  $v = x + iy$  has  $x \cdot y = 0$ , and put  $a = \|x\|^2$ ,  $b = \|y\|^2$ . Homogeneity gives

$$A(x) + a\|Bx\|^2 \geq \delta(B)a^2, \quad A(y) + b\|By\|^2 \geq \delta(B)b^2.$$

Because  $B$  is real, direct expansion yields

$$\begin{aligned} A(v) + N(v)\|Bv\|^2 &\geq \delta(B)(a^2 + b^2) + 2ab + 2n \sum_i x_i^2 y_i^2 \\ &\quad + b\|Bx\|^2 + a\|By\|^2 \\ &\geq \delta(B)(a + b)^2. \end{aligned}$$

Adding  $\rho Q \geq 0$  preserves the bound. A real minimizing unit vector attains equality in the resulting sphere minimum.

For a quartic  $F = \sum c_{\alpha\beta} z^\alpha \bar{z}^\beta$ , write

$$\Lambda(F)^2 = \sum_{|\alpha|=|\beta|=2} \frac{\alpha! \beta!}{4} |c_{\alpha\beta}|^2.$$

The coefficients of  $F_{B,\rho}$ , and hence  $\Lambda(F_{B,\rho})$ , are uniformly bounded for the fixed parameters and  $\|B\|_{\text{op}} \leq K$ . Drouot–Zworski [6, Theorem 2] bounds a sufficient multiplier exponent by a universal constant times

$$\frac{\Lambda(F_{B,\rho})}{\delta(B)} (n+2)^3 \log^3 n.$$

Together with (9), this proves the upper bound in (6) for small positive  $\sigma(B)$ .

## 5.3 The matching logarithmic-curvature obstruction

The upper bound uses the minimum value of the quartic. For a matching lower bound, we find a complex line along which the logarithm has large negative curvature. A norm-power certificate

must compensate for that curvature, forcing a large multiplier degree. Take a real minimizing unit vector  $v$ , a real unit  $w \perp v$ , and abbreviate  $\delta = \delta(B)$ . Define

$$S_{22} = \sum_i v_i^2 w_i^2, \quad S_{13} = \sum_i v_i w_i^3, \quad S_{04} = \sum_i w_i^4,$$

$$b_0 = \|Bv\|^2, \quad b_1 = \langle Bv, Bw \rangle, \quad b_2 = \|Bw\|^2.$$

Stationarity of the real sphere minimum kills the linear term of the restriction to  $v + zw$ . Direct expansion gives the exact polynomial

$$\begin{aligned} q(z) &:= F_{B,\rho}(v + zw) = \delta + \alpha x^2 + \beta y^2 + \gamma x(x^2 + y^2) + \eta(x^2 + y^2)^2, \\ \alpha &= 6nS_{22} - 2 + b_0 + b_2, \\ \beta &= 2nS_{22} + 2 + b_0 + b_2 + 4\rho, \\ \gamma &= 4nS_{13} + 2b_1, \quad \eta = nS_{04} - 1 + b_2. \end{aligned} \tag{10}$$

The small constant term controls every coefficient required below:

$$|nS_{22} - 1| \leq \sqrt{n\delta}, \quad b_0 \leq \delta, \quad b_2 \leq K^2, \quad |b_1| \leq K\sqrt{\delta}.$$

Thus, uniformly as  $\delta \downarrow 0$ ,

$$\alpha = 4 + b_2 + O_n(\sqrt{\delta}), \quad \beta = 4 + b_2 + 4\rho + O_n(\sqrt{\delta}),$$

while  $\gamma$  and  $\eta$  remain bounded. In particular,  $\alpha \geq 2$  and  $\beta - \alpha \geq 2\rho$  once  $\delta$  is sufficiently small, with a uniform onset.

Set

$$t^2 = \frac{2(\alpha + \beta)}{\beta(\beta - \alpha)} \quad \text{and evaluate at} \quad x = 0, \quad y = t\sqrt{\delta}.$$

The numbers  $t$  stay bounded above and away from zero. Writing  $\Delta_{\mathbb{R}^2} = \partial_x^2 + \partial_y^2$ , differentiation of (10) gives, with uniform remainders,

$$\begin{aligned} q\Delta_{\mathbb{R}^2}q - |\nabla_{\mathbb{R}^2}q|^2 &= \delta\{2(\alpha + \beta) + 2\beta(\alpha - \beta)t^2\} + O(\delta^2) \\ &= -2\delta(\alpha + \beta) + O(\delta^2), \\ q &= \delta\{1 + \beta t^2 + O(\delta)\}. \end{aligned}$$

Consequently some uniform  $c_0 > 0$  satisfies

$$\partial_z \partial_{\bar{z}} \log q \leq -\frac{c_0}{\delta}. \tag{11}$$

The logarithm of a positive holomorphic squared norm is plurisubharmonic. Along this line  $N(v + zw) = 1 + |z|^2$ , and  $\partial_z \partial_{\bar{z}} \log N = (1 + |z|^2)^{-2} \leq 1$ . A certificate for  $N^m F_{B,\rho}$  would therefore imply, at the specified point,

$$0 \leq m \partial_z \partial_{\bar{z}} \log N + \partial_z \partial_{\bar{z}} \log q \leq m - \frac{c_0}{\delta}.$$

Hence  $m \geq c_0/\delta \geq c_0/\sigma(B)^2$ , by (9). This proves the lower bound and completes Theorem 1.4. When  $r = 0$  or  $B = 0$ , its positive-residual regime is empty and the qualitative sign-null alternative applies.

### 5.4 Distance to the nonstabilizing subspaces

*Proof of Theorem 1.3.* Write  $u = s/\sqrt{n}$  for a normalized sign vector. If  $u \in W$ , then

$$\|(I - P_V)u\| = \|(P_W - P_V)u\| \leq \|P_W - P_V\|_{\text{op}}.$$

Taking the minimum over signs and the infimum over  $W \in \mathcal{S}_{n,d}$  gives one direction of (2). For the other, choose a minimizing sign  $u$ , put  $p = P_V u$ , and set

$$W = (V \cap p^\perp) \oplus \text{span}_{\mathbb{R}}\{u\}.$$

Here  $p \neq 0$ : averaging  $\|P_V s/\sqrt{n}\|^2$  over all signs gives  $d/n > 0$ , so a minimizing-distance sign has nonzero projection. For  $w \in V \cap p^\perp$ ,  $\langle w, u \rangle = \langle w, p \rangle = 0$ , so  $W$  is a real  $d$ -plane containing  $u$ . The two planes have this common  $(d-1)$ -plane, while the angle between their remaining directions  $p/\|p\|$  and  $u$  has sine  $\eta(V)$ . Hence  $\|P_V - P_W\|_{\text{op}} = \eta(V)$ , proving the equality.

Let

$$\delta_V = \min_{v \in V, \|v\|=1} A(v),$$

where the minimum is over real vectors. For such  $v$ , the normalized coordinate sign  $u$  gives  $A(v) \geq \|v - u\|^2 \geq \eta(V)^2$ . Conversely, for a minimizing sign  $u$ , normalize its projection:  $v = P_V u / \|P_V u\|$ . Since  $\|v - u\|^2 = 2(1 - \sqrt{1 - \eta(V)^2}) \leq 2\eta(V)^2$  and  $|v_i^2 - u_i^2| \leq 2|v_i - u_i|$ ,

$$\eta(V)^2 \leq \delta_V \leq 8n\eta(V)^2. \quad (12)$$

The phase-rotation argument in Section 5.2 applies inside  $V_{\mathbb{C}}$ : the orthogonal real and imaginary parts remain in  $V$ . When  $\delta_V \leq 1$ , it proves that the complex-sphere minimum of  $q_\rho$  is exactly  $\delta_V$ . In intrinsic real orthonormal coordinates the quartic coefficient norm is bounded uniformly over  $V$ . The cited quantitative stabilization theorem in dimension  $d$  therefore gives  $\mu_V(\rho) \leq C/\delta_V \leq C/\eta(V)^2$ .

For the lower bound choose a real minimizing unit  $v \in V$  and a real unit  $w \in V \cap v^\perp$ , possible since  $d \geq 2$ . Stationarity within  $V$  removes the linear term on  $v + zw$ . Formula (10) now holds with  $B = 0$  and  $\delta = \delta_V$ . Its coefficients satisfy

$$\alpha = 4 + O_n(\sqrt{\delta_V}), \quad \beta = 4 + 4\rho + O_n(\sqrt{\delta_V}),$$

and the cubic and quartic coefficients are uniformly bounded. The calculation leading to (11) consequently gives  $\partial_z \partial_{\bar{z}} \log q_\rho(v + zw) \leq -c_0/\delta_V$  at the specified point, uniformly over  $V$ . Since  $N(v + zw) = 1 + |z|^2$ , logarithmic plurisubharmonicity again forces

$$\mu_V(\rho) \geq c_0/\delta_V \geq \frac{c_0}{8n\eta(V)^2}.$$

This proves (3). □

### 5.5 An explicit soft-penalty example

Let  $n = 3$ ,  $\rho = 1$ , and  $B_\epsilon = (\epsilon, 0, 0)$ , where  $0 < \epsilon \leq 1$ . Then  $\sigma(B_\epsilon) = \epsilon/\sqrt{3}$ . On  $v = (1, 1, 1) + z(0, 1, -1)$ , with  $e = \epsilon^2$ ,

$$\begin{aligned} N &= 3 + 2|z|^2, \\ F_{B_\epsilon, 1} &= 3e + (24 + 2e)x^2 + (48 + 2e)y^2 + 2(x^2 + y^2)^2. \end{aligned}$$

At  $z = i\epsilon$ ,

$$\partial_z \partial_{\bar{z}} \log F_{B_\epsilon, 1} = \frac{-468 + 78e + 4e^2}{e(51 + 4e)^2}, \quad \partial_z \partial_{\bar{z}} \log N = \frac{6}{(3 + 2e)^2}.$$

The resulting exact necessary bound is

$$\mu_{B_\epsilon}(1) \geq \frac{(468 - 78e - 4e^2)(3 + 2e)^2}{6e(51 + 4e)^2} \sim \frac{78}{289\epsilon^2}. \quad (13)$$

For  $\epsilon = 1/10$  and  $1/100$ , it gives the integer lower bounds 28 and 2700, respectively. This example displays a computable necessary degree inside the uniform law; the next section determines a leading constant by complete coefficient matrices for a weighted restriction.

## 6 A sharp fixed-dimensional degree law

We now specialize to a two-dimensional family to determine the leading constant. The first two steps give explicit lower and upper degree bounds from finite coefficient matrices. Weighted row estimates and finite trial vectors then sharpen them to the same limit. Fix  $\rho > 0$ ,  $b > 0$ , and put  $t = 4/b^2$ ,  $c = 2 + t$ . Real orthonormal coordinates on  $H_b$  are

$$v_1 = \frac{Y}{\sqrt{c}} + \frac{X}{\sqrt{2}}, \quad v_2 = \frac{Y}{\sqrt{c}} - \frac{X}{\sqrt{2}}, \quad v_3 = -\frac{2Y}{b\sqrt{c}}.$$

Then  $N = |X|^2 + |Y|^2$ , and

$$q_b = \alpha|X|^4 + \beta|Y|^4 + \chi|XY|^2 + \eta(X^2\bar{Y}^2 + \bar{X}^2Y^2), \quad (14)$$

where

$$\alpha = \frac{1}{2}, \quad \beta = \frac{2(t-1)^2}{(2+t)^2}, \quad \chi = \frac{12}{2+t} + 2\rho, \quad \eta = \frac{3}{2+t} - 1 - \rho.$$

These coefficients follow by expanding  $3 \sum |v_i|^4 - |X^2 + Y^2|^2 + \rho Q$ . They are rational functions of  $b^2$ .

### 6.1 A necessary two-monomial minor

For  $\ell = m + 2$ , use the ordinary monomials  $X^j Y^{\ell-j}$ . The coefficient matrix of  $N^m q_b$  has diagonal

$$D_j = \alpha \binom{m}{j-2} + \chi \binom{m}{j-1} + \beta \binom{m}{j}$$

and off-diagonal entry  $\eta \binom{m}{j}$  between  $j$  and  $j+2$ . Binomial coefficients outside their range are zero. The principal block at indices  $0, 2$  is

$$\begin{pmatrix} \beta & \eta \\ \eta & \alpha + \chi m + \beta m(m-1)/2 \end{pmatrix}.$$

Every certificate therefore requires

$$\beta[\alpha + \chi m + \beta m(m-1)/2] \geq \eta^2. \quad (15)$$

As  $b \rightarrow 2$ ,  $\beta \rightarrow 0$ ,  $\chi \rightarrow 4 + 2\rho$ , and  $\eta \rightarrow -\rho$ . Applying (15) to a finite-liminf subsequence of  $u = \beta \mu_b(\rho)$  gives

$$\liminf_{b \rightarrow 2, b \neq 2} \beta \mu_b(\rho) \geq \sqrt{(4 + 2\rho)^2 + 2\rho^2} - (4 + 2\rho) > 0. \quad (16)$$

If  $u$  is unbounded the inequality is automatic.

## 6.2 An explicit full-matrix certificate

For  $b$  sufficiently close to 2, with  $b \neq 2$ ,  $0 < \beta \leq \alpha$  and  $\chi \geq 2|\eta|$ . Rescale the monomial of index  $j$  by  $\sqrt{\binom{\ell}{j}}$ , an invertible diagonal congruence. Writing  $u = j$ ,  $v = \ell - j$ , the diagonal numerator, after multiplying by  $\ell(\ell - 1)$ , is

$$\alpha u(u - 1) + \chi uv + \beta v(v - 1).$$

The sum of the two possible off-diagonal absolute numerators is at most

$$\begin{aligned} |\eta| \left[ \sqrt{(u+1)(u+2)v(v-1)} + \sqrt{u(u-1)(v+1)(v+2)} \right] &\leq |\eta|[(u+2)v + u(v+2)] \\ &= 2|\eta|uv + 2|\eta|\ell. \end{aligned}$$

Absent endpoint entries are zero. The diagonal minus this row sum is therefore bounded below by

$$\frac{\beta\ell^2/2 - (\alpha + \beta + 2|\eta|)\ell}{\ell(\ell - 1)}.$$

Here we used  $u^2 + v^2 \geq \ell^2/2$ ,  $\beta \leq \alpha$ , and  $\chi - 2|\eta| \geq 0$ . Consequently the complete Hermitian matrix is diagonally dominant with nonnegative diagonal whenever

$$\ell \geq \frac{2(\alpha + \beta + 2|\eta|)}{\beta}.$$

It is then positive semidefinite and is an explicit squared-norm Gram certificate. In particular,

$$\limsup_{b \rightarrow 2, b \neq 2} \beta\mu_b(\rho) \leq 1 + 4\rho. \quad (17)$$

Since  $\beta/(b-2)^2 \rightarrow 2/9$ , (16)–(17) prove the inverse-square order. At  $b = 2$  a sign vector lies in  $H_b$ , so Theorem 1.2 gives infinite degree.

## 6.3 Matching bounds and the exact constant

Write  $e_b = |\eta|$ ,  $\chi_b = \chi$ , and  $\beta_b = \beta$ . Near  $b = 2$ ,  $e_b > 0$ ,  $\chi_b > 2e_b$ , and

$$r_b = \frac{\chi_b - \sqrt{\chi_b^2 - 4e_b^2}}{2e_b} \in (0, 1), \quad \kappa_b = e_b r_b.$$

Thus  $\chi_b = e_b(r_b + r_b^{-1})$ , and

$$\kappa_b \rightarrow \kappa_0 = \frac{4 + 2\rho - \sqrt{(4 + 2\rho)^2 - 4\rho^2}}{2} = (\sqrt{1 + \rho} - 1)^2. \quad (18)$$

We prove  $\beta_b\mu_b(\rho) \rightarrow \kappa_0$ .

Let  $H_{b,\ell}$  be the coefficient matrix after congruence by  $\text{diag}(\binom{\ell}{j}^{-1/2})$  and multiplication by  $\ell(\ell - 1)$ . With  $v = \ell - j$ , its complete entries are

$$\begin{aligned} (H_{b,\ell})_{jj} &= \alpha j(j - 1) + \chi_b jv + \beta_b v(v - 1), \\ (H_{b,\ell})_{j,j+2} &= \eta \sqrt{(j+1)(j+2)v(v-1)}. \end{aligned}$$

It is positive semidefinite exactly when the exponent  $m = \ell - 2$  is a certificate.

For the upper bound choose positive weights  $d_0 = d_1 = 1$  and

$$\frac{d_{j+2}}{d_j} = r_b \sqrt{\frac{j+1}{j+2}}.$$

The two weighted off-diagonal row sums are bounded by

$$e_b r_b (j+1)v + \frac{e_b}{r_b} j(v+2) = \chi_b jv + \kappa_b v + \frac{2e_b}{r_b} j.$$

Indeed the exact square-root factors are  $\sqrt{v(v-1)} \leq v$  and  $\sqrt{(v+1)(v+2)} \leq v+2$ . Missing endpoint entries only reduce the sum. The diagonal minus this weighted row sum is therefore at least

$$\begin{aligned} & \alpha j(j-1) + \beta_b v(v-1) - \kappa_b v - 2e_b j/r_b \\ &= \ell[\beta_b(\ell-1) - \kappa_b] + j[-\beta_b(2\ell-1) + \kappa_b - \alpha - 2e_b/r_b] + (\alpha + \beta_b)j^2. \end{aligned} \quad (19)$$

Fix  $\delta > 0$  and take  $\ell_b = \lceil (\kappa_b + \delta)/\beta_b \rceil$ . The linear coefficient in (19) is bounded below by  $-C_\delta$  uniformly near  $b = 2$ . Every row slack is at least

$$\ell_b(\delta - \beta_b) - \frac{C_\delta^2}{4\alpha} > 0$$

eventually. Congruence by  $\text{diag}(d_j)$  now gives a diagonally dominant positive semidefinite matrix. Hence  $\mu_b \leq \ell_b - 2$ , and  $\limsup \beta_b \mu_b \leq \kappa_0 + \delta$ . Letting  $\delta \downarrow 0$  proves the upper limit.

For the lower limit first note that  $\mu_b \rightarrow \infty$ . Otherwise a subsequence has one fixed exponent, and its block (15) has determinant tending to  $-\rho^2$ , a contradiction. Consider a subsequence with  $\beta_b \mu_b \rightarrow u < \infty$ , and put  $\ell_b = \mu_b + 2$ . For every fixed finite set of indices, the entries of  $H_{b,\ell_b}/\ell_b$  tend to those of  $uI + J$ , where

$$J_{jj} = (4 + 2\rho)j, \quad J_{j,j+2} = -\rho\sqrt{(j+1)(j+2)}.$$

This is only a finite-entry limit; no infinite-operator domain assertion is needed.

Set  $r = \lim r_b$ , and define

$$a_{2k} = r^k \frac{\sqrt{(2k)!}}{2^k k!}, \quad a_{2k+1} = 0.$$

The squares satisfy  $a_{2k}^2 \leq r^{2k}$ , so all polynomially weighted square sums converge. The recurrence gives, entry by entry,

$$(Ja)_j = [(4 + 2\rho)j - \rho r(j+1) - \rho j/r]a_j = -\kappa_0 a_j$$

on even indices, including  $j = 0$ ; odd entries vanish. All associated quadratic-form series converge absolutely. Thus the finite truncations  $a^{[K]} = (a_0, \dots, a_{2K})$  obey

$$\frac{\langle a^{[K]}, Ja^{[K]} \rangle}{\|a^{[K]}\|^2} \longrightarrow -\kappa_0.$$

If  $u < \kappa_0$ , choose one finite truncation with Rayleigh quotient below  $-u$ . The finite-entry limit then gives a negative quadratic form for  $H_{b,\ell_b}/\ell_b$ , contradicting its positive semidefiniteness. Therefore  $u \geq \kappa_0$ , proving the lower limit. Finally  $\beta_b/(b-2)^2 \rightarrow 2/9$  and (18) prove Theorem 1.5.

**Corollary 6.1** (Binary input size). *At fixed  $\rho = 1$  and  $b_L = 2 + 2^{-L}$ , the binary quartic (14) has bounded rational coefficients of bit length  $O(L)$ , and its least norm multiplier degree is  $\Theta(4^L)$ .*

*Proof.* The rational functions in (14) have fixed degree and stay bounded near  $b = 2$ . The degree law gives the conclusion.  $\square$

The sharp binary law also has a direct geometric normalization. For the real plane underlying  $H_b$ , the sign-distance formula gives, for  $b$  sufficiently close to 2,

$$\eta(H_b)^2 = \frac{(b-2)^2}{3(2+b^2)}, \quad \lim_{b \rightarrow 2, b \neq 2} \eta(H_b)^2 \mu_b(\rho) = \frac{1}{4}(\sqrt{1+\rho} - 1)^2.$$

Indeed its normal is  $(1, 1, b)$ , and the smallest absolute signed normal sum near  $b = 2$  is  $|b - 2|$ . Substitution into Theorem 1.5 proves the displayed limit. This expresses its exact constant in the same distance used in Theorem 1.3.

## 7 Reproduction and perspective

All results above are analytic. The included standard-library Python programs provide exact finite controls for the Boolean gadget, full integer coefficient matrices, inertia, soft-penalty curvature identities, exact subspace-projector geometry and the binary stabilization matrices. For example, at  $\rho = 1, b = 9/4$ , the exact least exponent is 13, while the displayed diagonal-dominance bound gives 416. These values illustrate the certificate construction; the limiting exponent is proved by the two analytic bounds.

The restricted hardness theorem separates pointwise nonnegativity and one-negative coefficient inertia from eventual holomorphic certifiability. The sign-cube criterion identifies the exact failure set. Its Grassmannian refinement measures certificate cost uniformly by distance to that set; the soft-constraint law gives the parallel signed-residual scale, and the weighted restriction resolves an exact leading constant by full matrices. The geometric, computational and quantitative conclusions thus describe one certificate hierarchy at three complementary levels.

## Data availability

The analytic results are proved in the article. The editable source and verification code are available in the accompanying source package at the Luminary Research Archive.

## Statements and Declarations

**Funding.** This research received no external funding.

**Competing interests.** The author declares no competing financial or nonfinancial interests.

**AI assistance.** ChatGPT (OpenAI) assisted with research, drafting and mathematical checks. The author assumes responsibility for all content.

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