Heat-Trace Recovery of a Weyl-Blind Twisting Mode

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Luminary Oddities

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Abstract

A fixed connected polar section can hide every odd logarithmic harmonic of inverse twisting speed from the leading Weyl profile. We show that its original pure cosine perturbation is nevertheless audible in the next heat-trace term. The second coefficient contains the relative heat trace of a connected step strip; replacing the exposed radial face by an independent angular channel gives the wrong model. We derive a uniform momentum-integrated radial expansion and use an exact angular clock, a small real shear and weighted Dirichlet–Neumann bracketing to obtain an actual two-term heat expansion for the Euclidean tube. A positive corner-loop remainder is nonincreasing in heat time. Its Stieltjes representation controls the complex Mellin transform sharply enough to prove noncancellation at the original fundamental frequency. Thus a parameter invisible in the leading Weyl law is determined by the lower-order heat profile, without adding any lower-homogeneity perturbation.

Keywords: twisted tube; heat trace; spectral asymptotics; relative heat invariant; Mellin transform; inverse spectral problem.

1 Geometry, statement and relation to the leading law

Let

$$\omega = \operatorname{int}\left(\overline{(1,3)\times(0,1)} \cup \overline{(3,\sqrt{11})\times(0,2)}\right) \tag{1.1}$$

in polar coordinates (r,q). The aperture $r = 3$, $0 < q < 1$ is interior. A tube end is parametrized by

$$(s,r,q) \longmapsto (s, r\cos(q+\Theta(s)), r\sin(q+\Theta(s))), \qquad s>0, \quad (r,q)\in\omega.$$

The speed $b = \Theta'$ is eventually positive and strictly increasing. Write its inverse in the *absolute axial coordinate* as

$$s = R(y) = yh(\log y), \qquad y \geq y_0, \tag{1.2}$$

where h is smooth, real, L-periodic, and

$$h > 0, \qquad h + h' > 0. \tag{1.3}$$

A fixed bounded core joins the end to the cap at $s = 0$ by sweeping the same cross-section through $0 < s < R(y_0)$ with a smooth twisting angle. Its volume is $R(y_0)\int_\omega r\,\mathrm{d}r\,\mathrm{d}q$, independent of core twisting. This normalized-core assumption is used in every displayed formula with the exact constant $\mathcal{C}(1)$. Changing the cap from Dirichlet to Neumann at this *same location* has only order-T^2 heat effect. Moving the cap changes a constant order-T^3 volume term. The nonzero harmonics below are unaffected by that distinction. Denote the actual Dirichlet tube operator by H_Θ and its heat trace by $\mathcal{H}_h(T) = \operatorname{Tr} e^{-H_\Theta/T^2}$.

Put

$$Z_a(t) = \sum_{n\geq 1} e^{-t(\pi n/a)^2}, \qquad z_k = 1 + 2\pi i k/L, \qquad h(u) = \sum_{k\in\mathbb{Z}} h_k e^{2\pi i k u/L}.$$

We will define a positive relative step heat invariant $D(t)$ in Section 2. For $\Re z > 0$ set

$$M_D(z) = \int_0^\infty v^z D(v^2)\,\mathrm{d}v, \tag{1.4}$$

$$M_1(z) = \frac{\Gamma((z+1)/2)\zeta(z+1)}{2\pi^{z+1}}, \tag{1.5}$$

$$\mathcal{C}(z) = \frac{z}{2\sqrt{\pi}}\left[3M_D(z) - M_1(z) - \frac{3+\sqrt{11}}{4}\,2^{z+1}M_1(z)\right]. \tag{1.6}$$

Theorem 1.1 (Two heat profiles)**.** *Under* (1.2)–(1.3)*, the one-end operator has compact resolvent, finite heat trace, and*

$$\mathcal{H}_h(T) = T^4 A_h(\log T) + T^3 C_h(\log T) + o(T^3), \qquad T \longrightarrow \infty, \tag{1.7}$$

where

$$A_h(u) = \sum_k h_k \frac{z_k}{4\pi}(4 + 2^{z_k+1})M_1(z_k)e^{2\pi i k u/L}, \tag{1.8}$$

$$C_h(u) = \sum_k h_k \mathcal{C}(z_k)e^{2\pi i k u/L}. \tag{1.9}$$

The origin convention is that in (1.2)*; an additional constant volume offset adds a constant to C_h, and no nonzero harmonic. The displayed series converge absolutely.*

Theorem 1.2 (The original blind cosine is audible)**.** *Let $\tau = \pi/\log 2$ and*

$$h_\varepsilon(u) = 1 + \varepsilon\cos(\tau u), \qquad |\varepsilon| < (1+\tau^2)^{-1/2}.$$

Then $\mathcal{C}(1+i\tau) \neq 0$ and

$$\mathcal{H}_{h_\varepsilon}(T) = \frac{T^4}{6\pi} + T^3\left[\mathcal{C}(1) + \varepsilon\Re\{\mathcal{C}(1+i\tau)e^{i\tau\log T}\}\right] + o(T^3). \tag{1.10}$$

Consequently the complete spectrum determines ε in this known one-parameter family. Distinct parameters are not isospectral. For two identical ends the two displayed profiles double.

The first theorem is proved in Sections 3–4; Section 5 proves the noncancellation in the second. At odd k, $2^{1+z_k} = -4$ when $L = 2\log 2$, so (1.8) vanishes on every odd harmonic. In particular its constant is $1/(6\pi)$ for the cosine family. This is the thermal version of the leading Weyl blindness established in [6], following the geometric Weyl analysis in [7]. For the symmetric two-end example the preceding counting law is $N(\Lambda) = \Lambda^2/(6\pi) + o(\Lambda^2)$. The mixed-homogeneity theorem in [6] detects an added lower-order modulation; here no such perturbation is present.

Exploding twisting and spectral confinement have earlier foundations in [2, 1]; the latter provides spectral moment bounds, not the second coefficient here. Heat decay in locally twisted tubes [3] concerns a different, large-time regime. Polygon heat coefficients and localization used below are classical [4]. Mellin/Fourier methods for logarithmic periodicity are likewise established; see [5]. The contribution here is the actual connected boundary term, its global uniform transfer and its nonvanishing at the frequency hidden by the leading width moment.

2 The connected step invariant

Let S be the interior of the union of the closures of $(-\infty, 0) \times (0, 1)$ and $(0, \infty) \times (0, 2)$. Let A be its Dirichlet Laplacian. Inserting a Dirichlet cut on $\{0\} \times (0, 1)$ gives the orthogonal sum B of the two halfstrip operators. Add caps at $X = \pm a$ and write A_a, B_a.

Lemma 2.1. *The finite positive limit*

$$D(t) = \lim_{a\to\infty} \left(\operatorname{Tr} e^{-tA_a} - \operatorname{Tr} e^{-tB_a} \right)$$

exists for every $t > 0$. Its finite-cap approximants increase with a, and

$$\tfrac{1}{2} Z_1(t) \le D(t) \le \tfrac{1}{2} Z_2(t). \tag{2.1}$$

It has the short-time expansion

$$D(t) = \frac{1}{4\sqrt{\pi t}} - \frac{2}{9} + O(t^{-1} e^{-c/t}) \tag{2.2}$$

and decays exponentially at infinity.

Proof. The difference of the connected and cut diagonal kernels is precisely the free Brownian bridge weight for paths that survive in S and hit the aperture. The events increase with the distance of the caps, so monotone convergence applies, without subtracting infinite traces. Restrict these loops to the full narrow strip for the lower bound; enlarge to the full wide strip and allow any crossing of $X = 0$ for the upper bound. The two coordinates are independent. A bridge of generator ∂_X^2, from X back to X in time t, hits zero with probability $e^{-X^2/t}$. Thus

$$\int_{\mathbb{R}} (4\pi t)^{-1/2} e^{-X^2/t} \, \mathrm{d}X = \tfrac{1}{2},$$

which proves (2.1) and finiteness. In particular $D(t) = O(t^{-1/2})$ at zero and $D(t) = O(e^{-\pi^2 t/4})$ at infinity.

For completeness the cap error is bounded by

$$0 \le D(t) - D_a(t) \le Z_2(t) \left[\frac{a}{\sqrt{\pi t}} e^{-a^2/(4t)} + \tfrac{1}{2} \operatorname{erfc}\left(\frac{a}{2\sqrt{t}} \right) \right]. \tag{2.3}$$

For $|X| \le a/2$, use the union bound for hitting either cap; its integral is the first term. Outside this interval use the aperture-hit Gaussian, giving the second term. The vertical domain may be enlarged to width two in both estimates.

Now fix a, say $a = 2$, and apply the bounded polygon heat expansion [4]. Areas and outer corners cancel. Removing two copies of the aperture removes boundary length two, producing $1/(4\sqrt{\pi t})$. The cut geometry has four right corners; the joined geometry has one right and one reflex corner. With $c(\alpha) = (\pi^2 - \alpha^2)/(24\pi\alpha)$, their difference is

$$c(3\pi/2) - 3c(\pi/2) = -5/144 - 3/16 = -2/9.$$

The polygon remainder and (2.3) are exponentially small, which proves (2.2). □

The anisotropic junction $-\partial_X^2 - v^2 \partial_q^2$ is unitarily equivalent, under $X' = vX$, to $v^2 A$. Its relative response at time one is exactly $D(v^2)$. Thus (1.4) is absolutely convergent and holomorphic on $\Re z > 0$. Termwise Gaussian integration similarly gives

$$\int_0^\infty v^z Z_a(v^2) \, \mathrm{d}v = a^{z+1} M_1(z). \tag{2.4}$$

Remark 2.2 (Angular transmission). The junction does not separate into independent radial faces. For example, if J is zero extension from $(0,1)$ to $(0,2)$, its negative-energy aperture form is

$$M_\kappa = \sqrt{A_1+\kappa^2} + J^*\sqrt{A_2+\kappa^2}J, \qquad A_a = v^2(-\partial_q^2)_{\mathrm{D},(0,a)}.$$

Indeed the minimizing halfstrip extensions are the decaying square-root semigroups; integrating their energy gives this form on the aperture trace space. Reflection about $q=1$ further gives $J^*\sqrt{A_2+\kappa^2}J = (Q_{DD}+Q_{DN})/2$, where the two roots act on $(0,1)$. The aperture is therefore $(3Q_{DD}+Q_{DN})/2$; the two roots need not commute. The relative invariant retains this coupling. An isolated-width replacement of the exposed half-face would not.

3 A uniform integrated radial expansion

For $\eta>0$, $v>0$ and $\xi\in\mathbb{R}$ let

$$K_\eta(v,\xi) = -\eta^2\partial_r^2 - v^2\partial_q^2 + \frac{\xi^2}{v^2r^2}$$

be the Dirichlet operator on the connected section (1.1), with measure $\mathrm{d}r\,\mathrm{d}q$. Put

$$J_\eta(v) = \frac{1}{2\pi}\int_{\mathbb{R}} \operatorname{Tr} e^{-K_\eta(v,\xi)}\,\mathrm{d}\xi, \quad G(v) = 3D(v^2) - Z_1(v^2) - \frac{3+\sqrt{11}}{4}Z_2(v^2).$$

Proposition 3.1. *Uniformly for $v>0$ and sufficiently small $\eta>0$,*

$$J_\eta(v) = \frac{v}{4\pi\eta}\left[Z_1(v^2)\int_1^3 r\,\mathrm{d}r + Z_2(v^2)\int_3^{\sqrt{11}} r\,\mathrm{d}r\right] + \frac{v}{2\sqrt{\pi}}G(v) + E_\eta(v), \tag{3.1}$$

$$|E_\eta(v)| \le C\eta v Z_2(v^2), \qquad \int_0^\infty |E_\eta(v)|\,\mathrm{d}v \le C\eta/3. \tag{3.2}$$

Proof. Integrate ξ in the killed Brownian bridge formula first. If ρ_u is the radial path, the exact Gaussian integration is

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{-(\xi/v)^2\int_0^1\rho_u^{-2}\,\mathrm{d}u}\,\mathrm{d}\xi = \frac{v}{2\sqrt{\pi}}F[\rho], \qquad F[\rho] = \left(\int_0^1 \rho_u^{-2}\,\mathrm{d}u\right)^{-1/2}.$$

On surviving paths $1\le\rho\le\sqrt{11}$, so F is bounded and Lipschitz in the path supremum norm. Extend r^{-2} to a positive bounded C^2 function on the real line, agreeing near this interval. For an unrestricted centered bridge $\rho_u = r+\eta\beta_u$, Taylor expansion gives, uniformly in r in the interval,

$$F[r+\eta\beta] = r + \eta\int_0^1 \beta_u\,\mathrm{d}u + O(\eta^2\sup|\beta|^2). \tag{3.3}$$

The first variation has mean zero. Multiplying by the radial free diagonal $(4\pi\eta^2)^{-1/2}$, the unrestricted radial bulk is $(2\sqrt{\pi}\eta)^{-1}\int r\,\mathrm{d}r + O(\eta)$.

First cut at $r=3$. On an arm $(l,u)\times(0,a)$ the angular trace factors as $Z_a(v^2)$. Kill radial bridges hitting l or u. At the lower endpoint replace F by l. The reflected Gaussian gives

$$l\int_l^u (4\pi\eta^2)^{-1/2}e^{-(r-l)^2/\eta^2}\,\mathrm{d}r = l/4 + O(e^{-c/\eta^2}).$$

The replacement error is $O(\eta)$. In fact, setting $x=(r-l)/\eta$, its bound is a constant times

$$\eta\int_0^\infty \mathbb{E}[(x+\sup|\beta|)\mathbf{1}_{\{\min\beta\le -x\}}]\,\mathrm{d}x < \infty.$$

Gaussian bridge supremum tails make this bound uniform. The upper endpoint contributes $u/4$ with the same error; bridges hitting both endpoints are exponentially small. Each cut arm therefore has weighted radial trace

$$\frac{1}{2\sqrt{\pi}\eta}\int_l^u r\,\mathrm{d}r - \frac{l+u}{4} + O(\eta).$$

Reconnect the aperture. The difference counts surviving paths hitting $r = 3$, now with weight F. Freeze F to 3. Setting $x = (r-3)/\eta$, the weight error is bounded by $C\eta(|x| + \sup|\beta|)$ on the hit event. Enlarge the angular domain to width two. For a fixed bridge the set of x for which it hits $-x$ has length at most $2\sup|\beta|$. Its integrated weighted error is therefore $O(\eta Z_2(v^2))$. The unweighted reconnection, in $X = (r-3)/\eta$, differs from $D(v^2)$ by $O(\eta^{-1}e^{-c/\eta^2}Z_2(v^2))$ by (2.3) at radial time one. This is uniform in v: after the anisotropic dilation, cap distance divided by square root of time remains a constant times η^{-1}. Thus reconnection contributes $v[3D(v^2) + O(\eta Z_2(v^2))]/(2\sqrt{\pi})$. The cut endpoint factors are $(1+3)Z_1/4$ and $(3+\sqrt{11})Z_2/4$. This proves (3.1) and its error. Finally (2.4) gives $\int_0^\infty vZ_2(v^2)\,\mathrm{d}v = 1/3$. □

4 From frozen fibres to the actual tube

We give the global comparison in detail because an error larger than T^3 would not identify the coefficient. Write $D_j = -i\partial_j$. The actual form in measure $r\,\mathrm{d}r\,\mathrm{d}q\,\mathrm{d}s$ is

$$\int \left(|\partial_r u|^2 + r^{-2}|\partial_q u|^2 + |\partial_s u - b(s)\partial_q u|^2\right) r\,\mathrm{d}r\,\mathrm{d}q\,\mathrm{d}s. \tag{4.1}$$

The derivative margin in (1.3) implies $0 < c \le b' \le C$, $b'' = O(b^{-1})$, with analogous finite-order derivative bounds as needed below.

Lemma 4.1 (Weighted real normal form)**.** *On a sufficiently remote end, the form* (4.1) *is unitarily equivalent to a form whose local comparison with*

$$k[u] = \int \left(|D_r u|^2 + B(z)^2|D_q u|^2 + \frac{|D_z u|^2}{B(z)^2 r^2}\right) \mathrm{d}r\,\mathrm{d}q\,\mathrm{d}z \tag{4.2}$$

has relative error at most $C/B(z)^2$, plus a bounded absolute potential. The connected radial/angular domain remains fixed.

Proof. Use the exact angular clock

$$x = \Theta^{-1}(\Theta(s) + q), \qquad B = b(x).$$

Then $\partial_s - b(s)\partial_q = -b(s)\partial_q$ in the new coordinates, while the old angular derivative becomes $\partial_q + B^{-1}\partial_x$, and the axial Jacobian is $B/b(s)$. Uniformly for bounded q,

$$b(s)^2 = B^2 - 2b'(x)q + O(B^{-2}), \qquad B/b(s) = 1 + O(B^{-2}).$$

Remove polar measure by multiplication by $r^{1/2}$ and conjugate the positive axial density. The resulting form differs by weighted relative $O(B^{-2})$ and a bounded absolute term from

$$\int \left(|D_r u|^2 + r^{-2}|D_q u + B^{-1}D_x u|^2 + B^2|D_q u|^2\right).$$

For example, logarithmic density derivatives in q are $O(B^{-2})$, in x they are $O(B^{-3})$; Young's inequality absorbs their cross terms in this weighted error. The polar potential $-1/(4r^2)$ is

bounded. These comparisons use forms, so introduce no unaccounted boundary terms on subsequent Neumann caps.

Remove the remaining large mixed direction by a real small shear

$$z = x + qg(x, r), \qquad g = -\frac{1}{B(1 + B^2 r^2)}.$$

Here $g = O(B^{-3})$, $g_x = O(B^{-4})$, $g_r = O(B^{-3})$. For $a = B^2 + r^{-2}$ the leading z, q cross coefficient is $ag + r^{-2}/B = 0$. The residual is $O(B^{-5})$; the z, z coefficient is the Schur complement $(1+B^2r^2)^{-1}$ up to smaller errors. Replacing it by $(B^2r^2)^{-1}$ costs relative $O(B^{-2})$. The radial shear has coefficient $O(B^{-3})D_rD_z$, again relative $O(B^{-2})$ against $|D_r|^2+B^{-2}|D_z|^2$. Also $B(x) - B(z) = O(B^{-3})$ and the shear Jacobian is $1 + O(B^{-4})$, with differentiated bounds of the same harmless type. For sufficiently remote points this is a diffeomorphism. Both changes preserve (r, q) and all physical Dirichlet faces, including the interior aperture. Conjugating the last density proves the lemma. □

Lemma 4.2 (Axial caps and weighted comparison)**.** *Fix a sufficiently large speed cutoff B_0. The actual heat trace on the remote end differs from the continuous frozen diagonal trace*

$$\int_{R(B_0)}^{\infty} \frac{\mathrm{d}z}{2\pi} \int_{\mathbb{R}} \mathrm{d}p \ \mathrm{Tr}_\omega \exp\left[-T^{-2}\left(D_r^2 + B(z)^2 D_q^2 + \frac{p^2}{B(z)^2 r^2}\right)\right] \tag{4.3}$$

by $O(T^3/B_0) + O(T^{5/2}\log(2 + T)) + O(T^2)$, with constants uniform in sufficiently large B_0, except for fixed-core terms.

Proof. Insert normal z caps *after* the global coordinate changes. Partition the remote axis into intervals of length $\ell = T^{-1/2}$. For the frozen diagonal operator the Dirichlet and Neumann caps separate into sine and cosine momenta $p_j = \pi j/\ell$. The transverse heat trace decreases with $p \geq 0$ by form order. Thus the sum-versus-integral error is bounded by its $p = 0$ term; the D/N difference is exactly that term. Radial Neumann relaxation into the two bands gives

$$\mathrm{Tr}_\omega\, e^{-T^{-2}(D_r^2+b^2D_q^2)} \leq CTZ_2(b^2/T^2) \leq C\frac{T^2}{b}e^{-cb^2/T^2}. \tag{4.4}$$

Relaxing $r^{-2} \geq 1/11$ and summing the axial Gaussian gives the interval trace and its first energy moment bounds

$$C\left(\ell T^3 + \frac{T^2}{b}\right)e^{-cb^2/T^2}. \tag{4.5}$$

For the moment use $xe^{-x} \leq Ce^{-x/2}$; harmless polynomials in b/T may be absorbed by decreasing c.

On the cell with frozen speed b_j, Lemma 4.1 and bounded B' give relative error $e_j \leq C(\ell/b_j + b_j^{-2})$. Min–max and the scalar mean-value bound for the exponential control the trace error by Ce_j times (4.5) at weaker heat time. Crucially, keep this weight cell by cell. Since $\mathrm{d}z$ is comparable to $\mathrm{d}b$, the volume errors sum to

$$CT^3\ell \int_{B_0}^{\infty} b^{-1}e^{-cb^2/T^2}\,\mathrm{d}b \leq CT^{5/2}\log(2 + T),$$
$$CT^3 \int_{B_0}^{\infty} b^{-2}e^{-cb^2/T^2}\,\mathrm{d}b \leq CT^3/B_0.$$

The zero-momentum cap errors sum to $CT^2\ell^{-1}\int_{B_0}^{\infty} b^{-1}e^{-cb^2/T^2}\,\mathrm{d}b = O(T^{5/2}\log(2 + T))$. The second term of (4.5) multiplied by e_j is smaller at fixed B_0. Replacing cell values by the continuous frozen integral uses the same relative energy squeeze. A bounded potential costs $O(T^2)$ since its exponent is $O(T^{-2})$ and the total trace is $O(T^4)$. This proves the claim. The estimates also give finiteness of the remote heat trace for every $T > 0$ and hence compact resolvent. □

Proof of Theorem 1.1. In (4.3) set $b = Tv$, $p = T^2\xi$, $\eta = T^{-1}$. The radial proposition gives its bulk

$$\frac{T^2}{4\pi}\int_{B_0}^{\infty} y\,\mathrm{d}R(y)[4Z_1(y^2/T^2) + Z_2(y^2/T^2)] \tag{4.6}$$

and boundary

$$\frac{T^3}{2\sqrt{\pi}}\int_{B_0/T}^{\infty} vR'(Tv)G(v)\,\mathrm{d}v, \tag{4.7}$$

with integrated error $O(T^2)$, because R' is bounded and (3.2) is integrable.

The transformed cap at speed B_0 differs from the physical axial cap by $O(B_0^{-1})$. Its core-volume discrepancy therefore costs at most CT^3/B_0. For fixed B_0 the core is a bounded Lipschitz domain. Its heat trace has its usual volume term and an $O_{B_0}(T^2)$ remainder. The formal extension of $R(y) = yh(\log y)$ to $y > 0$ has $R(0) = 0$. Its missing measure has $\int_0^{B_0} \mathrm{d}R = R(B_0)$, exactly the fixed-core length in the stated origin convention. Poisson summation for Z_a shows that extending (4.6) to zero adds this same volume term, with only $O_{B_0}(T^2)$ difference. An actual axial offset instead adds the stated constant volume coefficient.

By (2.1), $vG(v)$ is bounded at zero and exponentially decreasing at infinity. Extending the boundary lower limit to zero costs $O_{B_0}(T^2)$. Insert the absolutely convergent smooth Fourier series of $R'(Tv) = h(\log T + \log v) + h'(\log T + \log v)$. Equation (2.4) yields (1.8), and (1.4)–(1.6) yield (1.9). All integrals with logarithmic derivatives converge absolutely; smooth Fourier coefficients and the uniform absolute integral bounds justify the interchanges. At fixed B_0, the normalized limsup of the remaining error is at most C/B_0 by Lemma 4.2. Let T tend to infinity first, and then B_0 to infinity. This proves (1.7). The opposite twist sign has the same argument with the signed clock; its diagonal form depends on the speed magnitude. Joining two ends across a fixed cap costs $O(T^2)$ and gives the stated addition. □

5 Noncancellation at the original blind frequency

Define

$$E(t) = D(t) - \tfrac{1}{2}Z_1(t). \tag{5.1}$$

Lemma 5.1 (A monotone corner-loop remainder)**.** *E is nonnegative and nonincreasing, $E(0+) = 1/36$, and*

$$0 \le E(t) \le \min\{1/36, Z_2(t)/2\}. \tag{5.2}$$

For $z = 1 + i\tau$, $s = (z+1)/2$ and $M_E(z) = \int_0^\infty v^z E(v^2)\,\mathrm{d}v$,

$$|M_E(z)| \le M_E(1)/|s|. \tag{5.3}$$

Proof. Translate the reflex corner by $Y = q - 1$. The step is $W \cap \{-1 < Y < 1\}$, where $W = \{Y < 0\} \cup \{X > 0\}$ is the $3\pi/2$ wedge. Let the two internal rays be $V = \{X = 0, Y < 0\}$ and $U = \{Y = 0, X > 0\}$. D counts surviving loops hitting V. The subcollection staying below U is precisely the full narrow-strip cut defect $Z_1/2$. Thus E counts loops surviving the step and hitting *both* rays. Touching without crossing an artificial interior line does not contribute an exceptional positive-measure bridge event. Scale a time-t loop and its base point by $\sqrt{t}$. The two-dimensional diagonal factor t^{-1} cancels the area scaling. The wedge and ray events remain fixed, while the strip becomes $-t^{-1/2} < Y < t^{-1/2}$, decreasing with t. Hence E is nonnegative and nonincreasing. Equations (2.2) and $Z_1(t) = 1/(2\sqrt{\pi t}) - 1/2 + O(e^{-c/t})$ give its limit $-2/9 + 1/4 = 1/36$. The theta upper bound follows from (2.1).

Let $\mu = -\mathrm{d}E$ be the positive Stieltjes measure. Since E decays exponentially and is bounded at zero, Fubini gives

$$M_E(z) = \frac{1}{2}\int_0^\infty t^{s-1}E(t)\,\mathrm{d}t = \frac{1}{2s}\int_0^\infty u^s\,\mathrm{d}\mu(u).$$

Its modulus is at most $(2|s|)^{-1}\int u\,\mathrm{d}\mu = M_E(1)/|s|$, proving (5.3). $\square$

Proposition 5.2. *For $\tau = \pi/\log 2$, $\mathcal{C}(1+i\tau) \neq 0$.*

Proof. Put $a = \pi^2/4$, $t_0 = \log 18/a$, and $s = 1+i\tau/2$. Split (5.2) at t_0. Since $n^2 \geq 1+3(n-1)$,

$$\int_{t_0}^{\infty} Z_2(t)\,\mathrm{d}t = \sum_{n\geq 1} \frac{e^{-an^2 t_0}}{an^2} \leq \frac{e^{-at_0}}{a(1-e^{-3at_0})}.$$

Consequently

$$M_E(1) \leq U := \frac{\log 18 + (1-18^{-3})^{-1}}{18\pi^2} < 0.021899692. \tag{5.4}$$

At $z = 1+i\tau$ we have $2^{z+1} = -4$. Combining (1.6) and (5.1), its bracket becomes

$$B(z) := \frac{2\sqrt{\pi}}{z}\mathcal{C}(z) = (7/2+\sqrt{11})M_1(z) + 3M_E(z). \tag{5.5}$$

We bound its first term below by Euler products. On the line $\Re(2s) = 2$,

$$|\zeta(2s)| \geq \prod_p (1+p^{-2})^{-1} = \frac{\pi^2}{15}.$$

Improve just the prime-three factor. Since $2\sqrt{2} < 3 < 4$, $\tau\log 3 \in (3\pi/2, 2\pi)$, so its cosine is positive and $|1-3^{-2-i\tau}| \leq \sqrt{82}/9$. Replacing the generic factor $9/10$ by $9/\sqrt{82}$ gives

$$|M_1(1+i\tau)| \geq \frac{|\Gamma(s)|}{3\sqrt{82}}. \tag{5.6}$$

The standard identity $|\Gamma(1+ib)|^2 = \pi b/\sinh(\pi b)$, with $b = \tau/2$, now gives

$$\frac{(7/2+\sqrt{11})|\Gamma(s)|}{3\sqrt{82}} > 0.0269349, \qquad \frac{3U}{|s|} < 0.0265236. \tag{5.7}$$

These strict inequalities can be verified using elementary rational bounds, as detailed below; they do not involve numerical spectra. By (5.3), the second term of (5.5) has modulus smaller than the first. Thus $B(z) \neq 0$ and $\mathcal{C}(z) \neq 0$. Conjugation gives the negative frequency. $\square$

Reproducible finite arithmetic

The accompanying standard-library script `nonzero_certificate.py` checks a stronger-than-needed positive squared margin using exact rational arithmetic. Machin's formula $\pi = 16\arctan(1/5) - 4\arctan(1/239)$ with alternating tails bounds π. The positive atanh series bounds $\log 2$ and $\log 3$ and its geometric tail bounds $\log 18 = \log 2 + 2\log 3$. For $b = \pi/(2\log 2)$ use $|\Gamma(1+ib)|^2 > 2\pi b e^{-\pi b}$, and bound the exponential by its positive Taylor sum and geometric tail. Replacing $\sqrt{11}$ by $331662479/10^8$, whose square is less than 11, the certificate produces the conservative squared bounds

$$\left(\frac{(7/2+\sqrt{11})|\Gamma(s)|}{3\sqrt{82}}\right)^2 > 0.0007254946254, \qquad \left(\frac{3U}{|s|}\right)^2 < 0.0007034989279.$$

It asserts their positive difference as a rational number; decimal printing is not the acceptance criterion. These bounds already imply $|B(1+i\tau)| > 0.0004$ and $|\mathcal{C}(1+i\tau)| > 0.0005$. No finite grid is a proof dependency.

Proof of Theorem 1.2. The derivative margin follows from $h_\varepsilon + h'_\varepsilon = 1 + \varepsilon(\cos \tau u - \tau \sin \tau u) > 0$. In (1.8), the $k = \pm 1$ width factor is zero. Also $M_1(1) = 1/12$, so the constant leading coefficient is $(4+4)/(48\pi) = 1/(6\pi)$. In (1.9) the two coefficients $h_{\pm 1} = \varepsilon/2$ combine to the real oscillation in (1.10), with nonzero multiplier by Proposition 5.2. Two distinct parameters therefore give different heat traces along a sequence of T tending to infinity. The spectrum determines the heat trace and its periodic limiting profile; the nonzero multiplier determines ε. □

6 Scope and further questions

This paper detects the original pure fundamental modulation in the actual lower heat coefficient. The multiplier (1.6) is explicit through an absolutely convergent relative Mellin invariant. For higher odd frequencies, the same formula identifies the noncancellation problem addressed subsequently in [8, Theorem 1.1]. That companion proves a strict, frequency-dependent lower bound at every original odd frequency, giving recovery in each fixed pure family and injectivity on admissible finite odd Fourier profiles for one specified end or identical specified two ends. The later finite-data result [9] gives a deterministic recovery disk under its explicit spectral-prefix, tail-envelope, multiplier and bias-bound input contract; it does not provide numerical values for every geometric constant.

The coefficient depends on the connected junction, not just the radial boundary lengths. The two-term theorem and its fundamental noncancellation supply a focused continuation of [6].

AI assistance. AI tools assisted with exploration, drafting and independent internal checks. Full arguments and the finite arithmetic certificate are supplied for independent scrutiny.

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