

Growing null subtraction and weighted Weil zero detection

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Luminary Oddities

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Abstract

Exact zero detectors for the Weil form can have a small negative Rayleigh quotient in a reciprocal-theta norm even after subtraction of known null vectors. We quantify this effect when both the zero multiplicity and the derivative-null prefix grow with height. For the divided-theta detector pair, the optimal negative depth is at most $e^{-\pi T/4}$ when $(m + N) \log T \leq c_* T$; on the strict subscale its upper logarithmic rate is $-\pi/2$. The proof bounds factorial derivative growth, canceled primitive norms, coefficient extraction and the complete two-column Gram matrix. An explicit finite-remainder norm upper gives a complementary sufficient exclusion inequality. We also give a norm-costed map from compact negative Weil spaces into the original constrained realization. Applied to an existing Gaussian cluster construction, it supplies an amplitude-free weighted depth guarantee of rate $-1/2$ at fixed displacement. The two rates concern different detector families.

1 Introduction and main conclusions

Quantitative use of the converse to Weil's positivity criterion requires a norm as well as a negative energy. In the weighted realization studied here the physical test is $f = sh$, whereas the receiving norm is $\|h\|_2 = \|f/s\|_2$. The weight is fixed by the Riemann theta density, rather than chosen for a particular test.

The exact divided-theta pair and its fixed-prefix height barrier were proved in [4, zero-witness appendix]. The constants there depend on fixed multiplicity and prefix length. We determine a range in which the obstruction survives a growing budget, paying both columns and every common coefficient combination.

Let $\sigma = \delta + iT$ be a centered zero of $\xi(1/2 + z)$ of exact order $m \geq 1$, with $\delta_* \leq \delta < 1/2$, $\delta_* > 0$ and $T \geq 2$. Put $\tau = -\bar{\sigma}$. Using the canceled primitive, the original space and the weight defined below, set

$$P_0 = D^2 - 1/4, \quad q_k = P_0 D^k \psi, \quad \phi_k = q_k/s, \quad Z_N = \text{span}\{\phi_0, \dots, \phi_{N-1}\}, \quad Z_0 = \{0\}.$$

Write Π_N for original Hilbert-space orthogonal projection onto Z_N , and define

$$h_\sigma = R_\sigma^m q_0/s, \quad A_\sigma = (\sigma^2 - 1/4) \frac{\xi^{(m)}(1/2 + \sigma)}{m!}, \quad r_\sigma = (I - \Pi_N)h_\sigma. \quad (1)$$

Use the analogous definitions at τ . The optimal negative depth is

$$\mathcal{D}_N(\sigma) = \max \left\{ 0, \sup_{c \in \mathbb{C}^2 \setminus \{0\}} \frac{-q_H[c_1 r_\sigma + c_2 r_\tau]}{\|c_1 r_\sigma + c_2 r_\tau\|_2^2} \right\}. \quad (2)$$

The quotient pair is independent, as shown in Section 3.

Theorem 1.1 (Growing null budget). *There is an effective theta constant $c_* > 0$ such that for each fixed $\delta_* > 0$ there is an effective $T_0(\delta_*)$ with*

$$T \geq T_0(\delta_*), \quad (m + N) \log T \leq c_* T \quad \implies \quad \mathcal{D}_N(\sigma) \leq e^{-\pi T/4}.$$

Along any sequence of these zeros and integers $N \geq 0$, allowing m to vary, if $(m + N) \log T = o(T)$, then

$$\limsup_{T \rightarrow \infty} T^{-1} \log \mathcal{D}_N(\sigma) \leq -\pi/2, \quad \log 0 = -\infty. \quad (3)$$

Corollary 1.2 (Prefix cost without fixed multiplicity). *Fix $\delta_* > 0$ and $p > 0$. At all sufficiently high zeros in this range, $\mathcal{D}_N(\sigma) \geq T^{-p}$ implies $N > c_* T / (2 \log T)$.*

There are effective theta constants $A, A_1, B \geq 1$ such that, for $N \geq 1, J \geq m + N - 1, K = J - m$ and $R = |\sigma|$, put

$$U_{m,N,J}(R) = A \sum_{k=N}^K \binom{m+k-1}{k} B^k k! R^{-m-k} + A_1 2^m B^J J! R^{-J}, \quad (4)$$

where an empty sum is zero.

Theorem 1.3 (Finite receiving inequality). *For every allowed J ,*

$$\|r_\sigma\|_2, \|r_\tau\|_2 \leq U_{m,N,J}(R), \quad \mathcal{D}_N(\sigma) \geq \frac{m|A_\sigma|^2}{2U_{m,N,J}(R)^2}.$$

If $H \geq -\nu I$, the strict inequality

$$m|A_\sigma|^2 > 2\nu U_{m,N,J}(R)^2 \quad (5)$$

excludes that zero. A minimum over any specified finite collection of allowed J is also a valid norm upper.

The simplest choice $J = m + N - 1$ leaves only the remainder in (4). The receiving inequality retains the individual zero derivative in A_σ . Section 7 gives a whole-cluster alternative not requiring such a bound.

1.1 Previous work and the quantitative question

The explicit formula and Weil's criterion are classical [1, 2, 3]. Van Hemert [5] proves exact all-zero interpolation, pole-free detectors and exhausted negative-index results, including arbitrary multiplicities.

Our direct baseline is [4], which fixes the realization and proves the exact pair and its fixed- m, N obstruction. The new conclusion is simultaneous uniformity in the growing parameters.

The Gaussian cluster construction, its compact support and its physical L^2 margin belong to [6]. We compose it with the existing theta radical repair, paying the original inverse-theta norm. The resulting weighted guarantee is different from its physical guarantee and concerns a different detector family. Cauchy estimates, Young's inequality, finite inverse algebra, Legendre coefficient bounds and vertical-strip Stirling estimates are standard tools; their full parameter accounting in this quotient is the issue here.

2 The original constrained form

Inner products are conjugate-linear first. We use

$$\mathcal{M}f(z) = \int_{\mathbb{R}} f(u)e^{zu} du, \quad \widehat{f}(t) = \int_{\mathbb{R}} f(u)e^{-itu} du,$$

with Plancherel measure $dt/(2\pi)$. Set

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \Xi(t) = \xi(1/2 + it).$$

The positive even theta density is

$$\psi(u) = \sum_{n \geq 1} (4\pi^2 n^4 e^{9u/2} - 6\pi n^2 e^{5u/2}) e^{-\pi n^2 e^{2u}}. \quad (6)$$

Theta inversion gives evenness and $\mathcal{M}\psi(z) = \xi(1/2 + z)$. Put $c(u) = \cosh(u/2)$, $s^2 = \psi/c$ and $w = \psi c$. The receiving space is

$$\mathcal{K} = \{h \in L^2(\mathbb{R}) : \langle \sqrt{w}, h \rangle = \langle \sqrt{w} \tanh(u/2), h \rangle = 0\}. \quad (7)$$

Equivalently $f = sh$ has $\mathcal{M}f(\pm 1/2) = 0$, and

$$\|h\|_2^2 = \int_{\mathbb{R}} |f(u)|^2 \frac{\cosh(u/2)}{\psi(u)} du. \quad (8)$$

Let $\Psi_0 = \Gamma'/\Gamma$ and

$$a(t) = \operatorname{Re} \Psi_0(1/4 + it/2) - \Psi_0(1/4), \quad m_0 = \Psi_0(1/4) - \log \pi.$$

The positive series

$$a(t) = \sum_{j \geq 0} \frac{(t/2)^2}{(j + 1/4)((j + 1/4)^2 + (t/2)^2)} \quad (9)$$

gives $0 \leq a(t) \leq 17t^2$, and $1 + a(t)$ is comparable to $\log(2 + |t|)$. Define the minimal domain $\mathcal{V}$ as the closure of $\mathcal{C} = C_c^\infty(\mathbb{R}) \cap \mathcal{K}$ in

$$\|h\|_2^2 + q_A[h], \quad q_A[h] = \int_{\mathbb{R}} a(t) |\widehat{sh}(t)|^2 \frac{dt}{2\pi}.$$

The graph is closable: multiplication by s is bounded and the multiplier $\sqrt{a}$ is closed. Two fixed compact bumps with independent row values give a right inverse of the row map, so compact constrained tests are dense in $\mathcal{K}$.

For $S_b f(u) = f(u + b)$ define

$$B = P_{\mathcal{K}} \left[m_0 M_s^2 - \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} M_s (S_{\log n} + S_{-\log n}) M_s \right] \Big|_{\mathcal{K}}. \quad (10)$$

This is bounded and selfadjoint. Indeed theta decay gives $s(u) \leq C \exp(-c_1 e^{2|u|})$. At least one of $|u|, |u + b|$ is at least $|b|/2$, whence $\|M_s S_b M_s\| \leq C' \exp(-c_1 e^{|b|})$. The complete prime-power series therefore converges in norm. The closed semibounded form $q_H = q_A + \langle \cdot, B \cdot \rangle$ on $\mathcal{V}$ defines $H = A + B$.

On compact smooth physical tests the unrestricted form is

$$Q_{\text{full}}[f] = \int_{\mathbb{R}} [\text{Re } \Psi_0(1/4 + it/2) - \log \pi] |\hat{f}(t)|^2 \frac{dt}{2\pi} + 2 \text{Re}(\mathcal{M}f(1/2) \overline{\mathcal{M}f(-1/2)}) - 2 \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \text{Re}\langle f, S_{\log n} f \rangle. \quad (11)$$

Thus $q_H[h] = Q_{\text{full}}[sh]$ on $\mathcal{C}$. The polarized explicit formula on sufficiently decreasing smooth tests is

$$Q_{\text{full}}(f, g) = \sum_z m_z \mathcal{M}g(z) \overline{\mathcal{M}f(-\bar{z})}, \quad (12)$$

over distinct centered nontrivial zeros, with multiplicity m_z . For the compact and theta-derived tests below, weighted integration by parts gives arbitrary inverse powers of height, so the series converges absolutely against the $O(T \log(T+2))$ zero count. The physical form is continuous on

$$X = H^1(\mathbb{R}) \cap L^2(\mathbb{R}, e^{2|u|} du), \quad \|f\|_X^2 = \|f'\|_2^2 + \|e^{|u|} f\|_2^2.$$

For example $|Q_{\text{full}}[f]| \leq 20 \|f\|_X^2$ [6, Lemma2.1]: the digamma series has a quadratic Fourier upper, pole evaluations are bounded by weighted Cauchy–Schwarz, and correlations at shift b are bounded by $e^{-|b|} \|e^{|u|} f\|_2^2$; the full arithmetic sum is paid by $\sum_{n \geq 2} \log n / n^{3/2} < 5$.

3 Exact nulls and the pair

The following recalls [4], including the domain return needed in the growing-parameter proof.

Lemma 3.1. *The independent vectors ϕ_k belong to $\ker H$. The detectors in (1) lie in $\mathcal{V}$ and have transform*

$$\mathcal{M}(sh_\sigma)(z) = \frac{(z^2 - 1/4)\xi(1/2 + z)}{(z - \sigma)^m}.$$

Their form matrix has diagonal zero and off-diagonal modulus $m|A_\sigma|^2$, with $|A_\tau| = |A_\sigma| > 0$. In particular

$$q_H[h_\sigma/A_\sigma - h_\tau/A_\tau] = -2m. \quad (13)$$

Proof. When the first r transform jets of g vanish at σ , put

$$R_\sigma^r g(u) = \int_u^\infty \frac{(t-u)^{r-1}}{(r-1)!} e^{\sigma(t-u)} g(t) dt. \quad (14)$$

Its canceled lower-tail expression is $(-1)^r \int_{-\infty}^u (u-t)^{r-1} e^{\sigma(t-u)} g(t) dt / (r-1)!$. The canceled jets prove equality. Direct differentiation and integration by parts give $(-D - \sigma)R_\sigma g = g$ and $\mathcal{M}(R_\sigma g)(z) = \mathcal{M}g(z)/(z - \sigma)$. Every division through the exact multiplicity is legitimate.

For each fixed derivative order the complete theta series, with $x = \pi e^{2u}$ on $u \geq 0$, is bounded by a fixed polynomial in x times $e^{u/2} e^{-x}$, while $\psi(u) \geq c_0 e^{u/2} x^2 e^{-x}$. In the upper primitive tail, write $t = u + v$ and use $e^{2v} \geq 1 + 2v$. Beyond an effective fixed-order onset, its integrand is bounded by such a polynomial times $e^{u/2} e^{-x} e^{-xv} v^{r-1} / (r-1)!$. Use the lower tail on the negative half-line and bound the remaining compact interval directly. The differential equation gives the corresponding fixed derivative estimates. After division by s , all these theta tails are square integrable. These fixed-order domain estimates are not used as uniform constants later.

Let $\eta = R_\sigma^m \psi$. Commutation gives $sh_\sigma = P_0\eta = R_\sigma^m q_0$ and the stated transform. For smooth compact cutoffs χ_R , use $f_R = P_0(\chi_R\eta)$. Both pole moments vanish exactly; the theta tails imply $f_R/s \rightarrow h_\sigma$ in L^2 and $(f_R - sh_\sigma)' \rightarrow 0$ in ordinary L^2 . Since $a(t) \leq 17t^2$, this is convergence in the specified minimal form norm. The complete formula passes to the limit by weighted derivative decay. Taking $\eta = D^k\psi$ proves the same domain return for ϕ_k .

Its transform is $(z^2 - 1/4)(-z)^k \xi(1/2 + z)$, so (12) gives zero pairing with every compact constrained test, then with all $\mathcal{V}$. Form representation puts ϕ_k in $\ker H$. Independence follows by Fourier transformation on a small interval where $\hat{\psi} \neq 0$: a polynomial cannot vanish on that interval unless it is zero. The detector has its only nonzero zero evaluation at σ , equal to $A_\sigma \neq 0$. The functional equation yields $A_\tau = (-1)^m \overline{A_\sigma}$. Formula (12) now proves the matrix and (13). Subtracting Z_N preserves the entire matrix; its nondegeneracy also proves independence of the quotient pair. $\square$

4 Uniform analytic costs

Constants in this section depend only on the actual theta weight.

Lemma 4.1 (Factorial derivative bound). *There are effective $A, B \geq 1$ such that*

$$\|\phi_k\|_2 \leq AB^k k! \quad (k \geq 0). \quad (15)$$

One may take $B = 40$ after enlarging A .

Proof. Take $\rho = 1/10$ and $\kappa_0 = e^{-2\rho} \cos(2\rho) > 1/2$. For $u \geq 0$, $x = \pi e^{2u}$ and complex $|v| \leq \rho$,

$$|\psi(u+v)| \leq C e^{u/2} x^2 e^{-\kappa_0 x}, \quad \psi(u) \geq c_0 e^{u/2} x^2 e^{-x}.$$

For the first inequality, factor the first Gaussian and sum $\sum n^4 e^{-\kappa_0 \pi(n^2-1)}$. It follows that

$$|\psi(u+v)|^2/s(u)^2 \leq C' e^{u/2} \cosh(u/2) x^2 e^{-(2\kappa_0-1)x},$$

an integrable uniform majorant. Reflect for $u < 0$. Hence $M_\rho = \sup_{|v| \leq \rho} \|\psi(\cdot + v)/s\|_2$ is finite and effective. Hilbert-valued Cauchy and Minkowski give $\|D^j \psi/s\|_2 \leq M_\rho j! \rho^{-j}$. Apply P_0 and $(k+1)(k+2) \leq 2 \cdot 4^k$; for example $A \geq M_\rho(2\rho^{-2} + 1/4)$ and $B = 4/\rho$ suffice. The denominator remains the positive real weight, not a possibly vanishing complex theta value. $\square$

Lemma 4.2 (Canceled primitive bound). *There is an effective $C_s \geq 1$ such that, when the first r moments cancel and $|\operatorname{Re} \sigma| \leq 1/2$,*

$$\|R_\sigma^r g/s\|_2 \leq C_s 2^r \|g/s\|_2. \quad (16)$$

This applies to all the canceled primitives of q_k below.

Proof. The actual first-atom asymptotic gives $(\log s)'(u) = 2 - \pi e^{2u} + o(1)$ as $u \rightarrow \infty$. Thus the effective constant

$$C_s = \exp \left(\int_0^\infty \max\{1 + (\log s)'(u), 0\} du \right)$$

is finite. Its integrand vanishes beyond an effective tail; the positive denominator has a lower bound on the remaining compact interval. Consequently $s(t)/s(u) \leq C_s e^{-(t-u)}$ for $0 \leq u \leq t$. After division by s , the positive-half primitive kernel is dominated by $C_s v^{r-1} e^{-v/2}/(r-1)!$, whose L^1 norm is $C_s 2^r$. Apply Young's inequality. On the negative half-line reflect the canceled lower-tail formula. The orthogonal direct sum has the same constant; cancellation identifies both formulas with the actual primitive. $\square$

Lemma 4.3 (Coefficient extraction). *There are effective $C \geq 1$, $\ell \geq 6$ such that for $0 \leq k \leq K$ a complex-linear functional $L_{k,K}$ on $\mathcal{K}$ satisfies*

$$L_{k,K}(\phi_j) = \mathbf{1}_{j=k} \quad (0 \leq j \leq K), \quad \|L_{k,K}\| \leq C\ell^K. \quad (17)$$

Proof. Let $M_0 = \int \psi > 0$, $M_2 = \int u^2 \psi < \infty$, and choose $0 < b \leq 1$ with $b^2 M_2 \leq M_0$. Then $\hat{\psi}(t) \geq M_0/2$ on $[-b, b]$. For $h = \sum_{j=0}^K a_j \phi_j$, $p(z) = \sum a_j z^j$,

$$\widehat{sh}(t) = -(t^2 + 1/4)\hat{\psi}(t)p(it), \quad \|h\|_2^2 \geq c_0 \int_{-b}^b |p(it)|^2 dt, \quad c_0 = \frac{M_0^2}{128\pi\|s\|_\infty^2}.$$

The Legendre polynomial P_j has coefficient ℓ^1 norm at most 3^j by its recurrence. Its normalized $L^2(-1, 1)$ version has factor $\sqrt{(2j+1)/2}$. Orthonormal expansion and Cauchy–Schwarz bound any coefficient of $p(ibx)$ by $(K+1)3^K\|p(ib\cdot)\|_2$. Undo scaling and absorb $K+1 \leq 2^K$ for $K \geq 1$. Thus $|a_k| \leq C(6/b)^K\|h\|_2$; enlarge C for $K=0$. Orthogonal extension of the finite-span coefficient functional proves the result with $\ell = 6/b$. This explicitly pays conditioning rather than assuming a high-order Gram floor. $\square$

5 Finite expansion and the complete pair

For $J \geq m$, $K = J - m$, the exact finite integration-by-parts identity is

$$\begin{aligned} h_\sigma &= \sum_{k=0}^K c_{m,k} \phi_k \sigma^{-m-k} + E_{\sigma,J}, \\ c_{m,k} &= (-1)^{m+k} \binom{m+k-1}{k}, \\ E_{\sigma,J} &= (-1)^J \sigma^{-J} \sum_{a=0}^{m-1} \binom{J}{a} R_\sigma^{m-a} q_{J-a}/s. \end{aligned} \quad (18)$$

Every remaining primitive has the necessary cancellations. The endpoint signs can also be checked by the polynomial identity

$$1 = (1+x)^m \sum_{k=0}^K (-1)^k \binom{m+k-1}{k} x^k + (-1)^J \sum_{a=0}^{m-1} (-1)^a \binom{J}{a} x^{J-a} (1+x)^a. \quad (19)$$

Multiplication by $(\sigma + D)^m$ and use of the actual canceled inverse gives (18); equivalently integrate the kernel in (14) by parts J times and retain every endpoint jet. No infinite derivative series is used.

Lemma 5.1 (Uniform complete remainder). *For $A_1 = \max\{1, AC_s e\}$ and $R = |\sigma|$,*

$$\|E_{\sigma,J}\|_2 \leq E_J(R) := A_1 2^m B^J J! R^{-J}. \quad (20)$$

The same upper holds at τ .

Proof. Lemmas 4.1 and 4.2, with $\binom{J}{a}(J-a)! = J!/a!$, bound the norm by

$$AC_s 2^m B^J J! R^{-J} \sum_{a=0}^{m-1} \frac{(2B)^{-a}}{a!} \leq AC_s e 2^m B^J J! R^{-J}.$$

$\square$

Proof of Theorem 1.3. The terms below N in (18) lie in Z_N . Contractivity of $I - \Pi_N$, (15) and (20) give (4) for both columns. The normalized negative pair (13), after subtraction, has squared norm at most $4U^2/|A_\sigma|^2$ and energy $-2m$. This proves the lower depth and the strict contradiction in (5). Each finite choice bounds the same two norms, so their finite minimum is valid. $\square$

Proposition 5.2 (Whole two-column Gram). *Take $K \geq N + 1$, $J = m + K$, $h = m + N$. If*

$$2C\ell^K E_J(R) \leq \frac{\delta R^{-h-2}}{2\sqrt{2}}, \quad (21)$$

then

$$\text{Gram}(r_\sigma, r_\tau) \succeq \frac{\delta^2 \ell^{-2K} R^{-2h-4}}{16C^2} I_2. \quad (22)$$

Proof. Apply $L_{N,K}, L_{N+1,K}$ to both columns. They annihilate Z_N and every other term through K . With $a = c_{m,N}$, $b_1 = c_{m,N+1}$, the leading matrix is

$$M = \begin{pmatrix} a\sigma^{-h} & a\tau^{-h} \\ b_1\sigma^{-h-1} & b_1\tau^{-h-1} \end{pmatrix}.$$

Its determinant has modulus $2\delta|ab_1|R^{-2h-2}$ and its operator norm is at most $\sqrt{2}(|a| + |b_1|)R^{-h}$. Since both integer coefficients have modulus at least one, $|ab_1|/(|a| + |b_1|) \geq 1/2$. Therefore $s_{\min}(M) \geq \delta R^{-h-2}/\sqrt{2}$. The error matrix has norm at most $2C\ell^K E_J(R)$. Under (21) the actual least singular value is at least half that lower bound. The extracting map has norm at most $\sqrt{2}C\ell^K$. Divide by that norm and square, obtaining (22) for every common coefficient vector. $\square$

6 Height and the growing schedules

Lemma 6.1 (Uniform multiplicity amplitude). *An absolute constant C_a satisfies*

$$|A_\sigma| \leq C_a 2^m T^5 e^{-\pi T/4}. \quad (23)$$

Proof. For $\text{Re } z > 0$ the continued Euler integral is

$$\zeta(z) = \frac{z}{z-1} - z \int_1^\infty \{x\} x^{-z-1} dx.$$

Differentiate $m \geq 1$ times under the absolutely convergent integral. Since $\int_1^\infty (\log x)^j x^{-\beta-1} dx = j!/\beta^{j+1}$, for $1/2 \leq \beta \leq 1$ and $T \geq 2$,

$$\frac{|\zeta^{(m)}(\beta + iT)|}{m!} \leq T^{-m-1} + |\beta + iT|\beta^{-m-1} + \beta^{-m} \leq CT2^m.$$

At an exact order- m zero only the top zeta derivative survives on differentiating the completion. Uniform vertical-strip Stirling [7, Section 5.11] gives $|\Gamma(\beta/2 + iT/2)| \leq CT^{\beta/2-1/2} e^{-\pi T/4} \leq Ce^{-\pi T/4}$. The completion costs T^2 , the Euler integral T , and the additional P_0 factor another T^2 . This proves (23), with no hidden multiplicity constant. $\square$

The pair's exact form matrix has operator norm $m|A_\sigma|^2$. Proposition 5.2 and Lemma 6.1 therefore give, whenever (21) holds,

$$\mathcal{D}_N(\sigma) \leq C' m 4^m \delta_*^{-2} \ell^{2K} R^{2m+2N+4} T^{10} e^{-\pi T/2}. \quad (24)$$

Its constants are independent of m, N, T .

Proof of Theorem 1.1. Set

$$\kappa = \min \left\{ \frac{1}{64\ell B}, \frac{1}{64(1 + \log \ell)} \right\}, \quad c_* = \kappa/4.$$

For the first conclusion choose $J = \lfloor \kappa T \rfloor$, $K = J - m$. The budget with $h = m + N$ implies $K \geq N + 1$ at a uniform sufficiently large onset. Using $J! \leq J^J$ and $R \geq T$, the logarithm of the left side of (21) is at most

$$O(1) + m \log 2 + J \log(\ell B J / T) \leq -\kappa(\log 64)T + o(T).$$

The right side has logarithm at least $-c_*T + o(T)$, uniformly for $\delta \geq \delta_*$. Thus the required inequality holds eventually. In (24) the positive linear exponent is at most

$$2\kappa \log \ell + 2c_* \leq \frac{1}{32} + \frac{1}{128} = \frac{5}{128}.$$

The remaining $m4^m$ and polynomial factors have logarithm $o(T)$ uniformly under the budget. The margin from this bound to $\pi/4$ proves the first conclusion. The majorants are explicit scalar functions of T and fixed constants; their dominance has an effective common onset.

For the strict subscale put $a_T = (h + 3) \log T$ and choose $J = \lceil \sqrt{T a_T} \rceil$, $K = J - m$. Then $J/T \rightarrow 0$, $J/a_T \rightarrow \infty$, and $K \geq N + 1$ eventually. The complete functional error has logarithm at most

$$O(1) + m \log 2 - J \log[T/(\ell B J)],$$

more negative than any fixed multiple of a_T . It satisfies (21). Every positive exponent in (24) is $o(T)$, proving (3). $\square$

Lemma 6.2 (Local multiplicity). *A nontrivial zero at height $T \geq 20$ has multiplicity at most $140 \log(T + 2)$.*

Proof. We recall the explicit Jensen count in [6, Lemma 4.1, equation (4.2)]. For $Z(z) = \zeta(1/2 + iz)$, anchor at $z_0 = T - 3i/2$, where the Euler product gives $|Z(z_0)| > 1/2$. On its radius-eight disk, Euler–Maclaurin with four Bernoulli terms bounds $|Z|$ by $(T + 2)^{16}$. Jensen then counts at most

$$[16 \log(T + 2) + \log 2] / \log(8/7) \leq 140 \log(T + 2)$$

zeros in the radius-seven disk, with multiplicity. The selected zero lies there; the symmetries cover either displacement. $\square$

Proof of Corollary 1.2. Eventually $140 \log(T + 2) \log T \leq c_* T/2$. If also $N \log T \leq c_* T/2$, the local multiplicity bound puts the total budget inside Theorem 1.1. But $e^{-\pi T/4} < T^{-p}$ for all sufficiently large T . $\square$

7 A paid whole-cluster alternative

The previous upper obstruction concerns a specific pair. The following receiving theorem applies to a different negative space in the same original realization.

Theorem 7.1 (Compact-to-weighted receiving map). *There is an effective theta constant $C_\theta \geq 1$ as follows. Let $L \geq 1$, $r > 0$ and let E be a finite-dimensional complex space of smooth tests supported in $(-L, L)$ with*

$$Q_{\text{full}}[f] \leq -r\|f\|_2^2 \quad (f \in E).$$

There is a complex-linear injective, parity-preserving map $J : E \rightarrow \mathcal{V}$ with

$$q_H[Jf] = Q_{\text{full}}[f], \quad \|Jf\|_2^2 \leq C_\theta e^{\pi e^{2L}} \|f\|_2^2. \quad (25)$$

The same holds for real parity spaces. Its entire image has negative depth at least $rC_\theta^{-1}e^{-\pi e^{2L}}$.

Proof. Use the existing radical row repair [4, Section2]. The normalization is

$$\mathcal{M}\psi(\pm 1/2) = 1/2, \quad \mathcal{M}\psi'(1/2) = -1/4, \quad \mathcal{M}\psi'(-1/2) = 1/4.$$

For $p_\pm = \mathcal{M}f(\pm 1/2)$ put

$$\alpha = p_+ + p_-, \quad \beta = 2(p_- - p_+), \quad F = f - \alpha\psi - \beta\psi', \quad Jf = F/s.$$

Both moments of F vanish. Both corrections have zero evaluation at every nontrivial zero. The complete polarized formula preserves the entire form, including all mixed coefficients. The corrections individually need not be members of the constrained $\mathcal{K}$.

Let $C_0 = \|\psi/s\|_2^2$, $C_1 = \|\psi'/s\|_2^2$; these are finite effective theta integrals. Cauchy–Schwarz gives

$$|p_\pm|^2 \leq 2 \sinh L \|f\|_2^2 \leq e^L \|f\|_2^2, \quad |\alpha|^2 \leq 4e^L \|f\|_2^2, \quad |\beta|^2 \leq 16e^L \|f\|_2^2.$$

Thus, for $W(L) = \max_{|u| \leq L} s(u)^{-2}$,

$$\|Jf\|_2^2 \leq 3[W(L) + (4C_0 + 16C_1)e^L] \|f\|_2^2. \quad (26)$$

The actual first atom, $x = \pi e^{2u}$, gives $\psi(u) \geq (4 - 6/\pi)e^{u/2}x^2e^{-x}$ for $u \geq 0$. Since $\cosh(u/2) \leq e^{u/2}$ and s is even,

$$W(L) \leq C_w e^{\pi e^{2L}}, \quad C_w = [(4 - 6/\pi)\pi^2]^{-1}.$$

One may therefore take $C_\theta = \max\{1, 3(C_w + 4C_0 + 16C_1)\}$.

For the minimal-domain return, cut off F and correct the vanishing row errors using a fixed compact smooth right inverse. The resulting F_R are smooth compact and exactly pole-null. Theta tails give $F_R/s \rightarrow F/s$ in L^2 , and $(F_R - F)' \rightarrow 0$ in ordinary L^2 . The positive symbol bound in (9) pays convergence in the specified minimal form norm. Bounded B and continuity of the complete physical formula identify the resulting form with $Q_{\text{full}}[f]$. If $Jf = 0$, its form is zero, so strict negativity forces $f = 0$. For even f , $\beta = 0$; for odd f , $\alpha = 0$. This also proves the real parity assertions. $\square$

The Gaussian theorem of [6, Theorems1.3–1.4, Sections7–8] uses exactly the density, variable and full form in (6)–(11). Its whole-cluster contour interpolation retains exact multiplicities without within-cluster separation.

Corollary 7.2 (Weighted Gaussian cluster margin). *Let $H_0 \geq 20$, $0 < \eta_0 < 1/2$ and $K_0 = \log(H_0 + 2)\log(e/\eta_0)$. Suppose $K_0 \leq d^2 H_0/10^7$ for $0 < d \leq 1$, and put*

$$L = \frac{1}{2} \log((1 + d)H_0/(2\pi)), \quad r_H = C_\theta^{-1} e^{-(1+d)H_0/2 - 10000K_0}.$$

If there are k ordinates $T_j + i\eta_j$ of Ξ with $20 \leq T_j \leq H_0$, $\eta_0 \leq \eta_j < 1/2$ and $|T_j - T_\ell| \geq 3$ for $j \neq \ell$, each real parity sector of H contains a k -dimensional space with $q_H[h] \leq -r_H \|h\|_2^2$. Its closed projection on $(-\infty, -r_H]$ has rank at least k , as does its strict projection on $(-\infty, -r_H/2)$.

Proof. The cited theorem supplies the compact spaces at radius L and physical depth $e^{-10000K_0}$. The hypothesis ensures $L \geq 1$. Apply Theorem 7.1 and substitute $\pi e^{2L} = (1+d)H_0/2$. Min-max gives the projection ranks. $\square$

For one zero at height T and fixed displacement floor η_0 , take $H_0 = T$ and $d = \sqrt{10^7 K_0/T} \leq 1$ for sufficiently large T . The guaranteed depth is at least

$$C_\theta^{-1} \exp \left[-T/2 - \frac{1}{2} \sqrt{10^7 T K_0} - 10000 K_0 \right]. \quad (27)$$

Its lower logarithmic rate is $-1/2$, for a different family from (3). This is not an upper bound on the best cluster depth. The source's greedy count also transfers: if $N_{\text{off}}(H_0, \eta_0)$ counts the stated upper-right ordinates with multiplicity, each strict count at $-r_H/2$ is at least $\lceil N_{\text{off}}/[140 \log(H_0 + 2)] \rceil$. Here multiplicity enters via the local cluster count, not as independent value coordinates at a single zero.

8 Consequences and exact algebra checks

The growing-prefix theorem limits a particular divided pair under a growing derivative budget. The finite receiving inequality exposes a complete original-norm amplitude comparison. The compact map supplies an amplitude-free alternative, paying its original norm explicitly. A physical norm or an order-location statement cannot replace that payment.

A fixed allowance $H \geq -\nu I$, $\nu > 0$, cannot by itself contradict only the guarantee (27) at every arbitrarily high hypothetical zero. A sharper norm estimate for the actual cluster selectors could improve it: the worst-support bound in Theorem 7.1 is not an optimality result. The necessary prefix scale is likewise not a sharp transition law. Uniformity as the displacement floor tends to zero is outside Theorem 1.1.

The prefix lower bound in Corollary 1.2 is necessary, not sufficient: it gives no implication that a longer derivative-null prefix restores a polynomial-depth signal. The finite receiving inequality likewise remains conditional on simultaneous amplitude and norm estimates. In the compact-cluster application, compactification does not impose exact zero values at unselected zeros, and the radical row repair leaves all nontrivial-zero evaluations unchanged. The additional zero contributions are controlled by the complete-form cutoff estimates in the cited cluster theorem. Thus that receiver uses a broader class of tests than the exactly isolated divided pair.

The standard-library program `verify.py` checks exact rational polynomial identities for (19), the binomial/factorial remainder normalization, Legendre coefficient and norm algebra, and the determinant/Frobenius inequalities for the complete pair. It also checks the polar repair and parameter algebra of the compact receiver. Its explicit exception gates survive Python's `-O` option. These finite controls supplement the analytic proofs.

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