Full-phase permanent-derivative compensation under real-part spectral pinching

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Luminary Oddities

25 October 2026

Abstract

We prove positivity of a compensated homogeneous permanent derivative for centered complex positive semidefinite matrices of order six whose real parts satisfy a fixed spectral pinch. With trace $r > 0$ and $P = I - J/6$, the closed inequalities $(7r/40)P \preceq \operatorname{Re} K \preceq (9r/40)P$ imply $F_4(K) > 487r^4$. Every positive-semidefinite-allowed imaginary part is included, without a small-phase assumption. The matrices may be singular or arbitrarily ill-conditioned. The proof combines a universal quartic phase bound, certified by an exact rational sum of squares, with the complete quadratic phase coefficient, its operator-norm variations and monotonicity of actual omission coefficients. On the isotropic real base the full phase fiber has the stronger bound $24455r^4$. An inherited negative mixed coefficient also excludes a class of global independent-factor tensor certificates, without refuting the repeated-state inequality.

1 A full-phase domain beyond a small imaginary tube

All matrices have order six. Write $J = \mathbf{1}\mathbf{1}^*$, $P = I - J/6$ and $W = 6I - J$. Centered means $K\mathbf{1} = 0$. Positive semidefinite means Hermitian positive semidefinite, a bar means entrywise conjugation and $\circ$ denotes entrywise multiplication. With entrywise absolute square, define

$$f_K(v) = \operatorname{per}(K + vJ), \quad \mathcal{P}(K) = f_K(1) + f_K'(1), \quad D(K) = \mathcal{P}(K)^2 - \operatorname{per}|K + \sqrt{2}J|^2,$$

and

$$E_j(K) = [x^j]\, \partial_v D(xK + vW)|_{v=0}\,, \qquad F_4(K) = E_4(K) + \frac{\operatorname{tr} K}{2} E_3(K). \tag{1}$$

The direction vW is additive after xK; inserting another x would change the coefficient. The functional F_4 is homogeneous of degree four over nonnegative real scalars. Put

$$\alpha_j(K) = [x^j] \operatorname{per}(J + xK), \qquad \delta_j(K) = \partial_v \alpha_j(K + vW)|_{v=0} \quad (j = 4, 5).$$

The actual omission quantities δ_4, δ_5 have degrees three and four respectively.

The real-section predecessor [1] proved a global real bound and positivity in the complex tube $\|\operatorname{Im} K\|_{\mathrm{F}} \le \operatorname{tr} K/1000$. The earlier theorem [2] covers the exact conjugate-disjoint locus $K\overline{K} = 0$, which forces rank at most two and real-part rank at most four. Neither result covers the domain below: its real parts have centered rank five, while their imaginary parts can approach the full boundary allowed by positive semidefiniteness.

Theorem 1.1 (Full-phase spectral domain)**.** *Let $K = R + iS \succeq 0$ be centered, where R is real symmetric and S is real antisymmetric. Let $r = \operatorname{tr} K > 0$. If*

$$\frac{7r}{40}P \preceq R \preceq \frac{9r}{40}P, \tag{2}$$

then

$$F_4(K) > \frac{124803}{256}r^4 > 487r^4.$$

Both spectral endpoints are included, and every imaginary part compatible with $K \succeq 0$ is allowed.

The condition makes R positive on the centered five-space, but does not bound the smallest positive eigenvalue of K away from zero. Explicit singular and arbitrarily ill-conditioned examples are given in Section 7. At $r = 0$, positive semidefiniteness gives $K = 0$ and $F_4(K) = 0$, rather than strict positivity.

The new ingredients are a universal quartic phase payment, an exact quadratic coefficient and quantitative control of its base variation. All use the same feasible R, S; no independent choice of adverse moments or tensor factors is made. The quartic bound is a computer-assisted theorem: its rational Gram certificate and exact verification are part of the proof, not numerical evidence for a sign.

The result concerns the auxiliary derivative (1). Complete permanent comparisons, including the border identity in [7, Proposition3.2(iii)] and the comparison-matrix-positive class in [8], concern different statements and do not imply coefficientwise positivity here. The theorem establishes the compensated degree-four bound on the stated closed real-part spectral domain.

2 The exact phase decomposition and inherited reserves

The zero matrix is immediate. Sections 2–6 normalize $\operatorname{tr} R = 1$ unless stated otherwise; general trace follows by homogeneity. Conjugation symmetry and degree give the exact polynomial

$$F_4(R + ixS) = F_4(R) + b(R, S)x^2 + c(S)x^4. \tag{3}$$

Its quadratic coefficient b is not assumed nonnegative. Neither concavity nor monotonicity along a feasible phase pencil is required.

The real-section theorem [1, Theorem1.1] supplies the following inherited reserve, stated here at general trace:

$$\begin{aligned} F_4(R) \geq\ & (20160 - 464\sqrt{2})\delta_5(R) + 14800r\delta_4(R) \\ & + (39744 - 10368\sqrt{2})r^2 \operatorname{tr}(R^2). \end{aligned} \tag{4}$$

Both omission quantities are nonnegative. To recall their meaning, choose centered Gram forms g_i, let $p_j = \sum_i g_i^j$, and use the Fischer pairing $\|z^\gamma\|^2 = \gamma!$ [4]. With

$$\xi_i = g_i^4 - \tfrac{1}{2}g_i^2 p_2 - \tfrac{1}{3}g_i p_3, \qquad \psi_i = g_i^3 - \tfrac{1}{2}g_i p_2,$$

their exact representations are

$$\delta_5 = 6\sum_i \|\xi_i\|^2 - \left\|\sum_i \xi_i\right\|^2,$$

$$\delta_4 = 2\left[6\sum_i \|\psi_i\|^2 - \left\|\sum_i \psi_i\right\|^2\right].$$

The factor two is the residual-slot factor $(6-4)!$. The predecessor proves (4) by cancelling the paired quartic with $58\delta_5$ and paying the ordinary cubic term with $14800\delta_4$. We retain that theorem rather than reclassifying its real-section result as new.

3 A universal quartic phase certificate

For real centered antisymmetric S, put

$$B = S \circ S, \quad b_S = B\mathbf{1}, \quad q = \|S\|_{\mathrm{F}}^2, \quad u = \|b_S\|^2, \quad v = \|B\|_{\mathrm{F}}^2, \quad w = \mathrm{tr}(S^4), \quad X = \mathrm{tr}(S^2 B).$$

Here B is entrywise nonnegative; it is not presumed positive semidefinite. All powers not marked entrywise are ordinary products.

Proposition 3.1. *The actual pure quartic in* (3) *is*

$$\begin{aligned} c(S) = {} & (604800 - 960\sqrt{2})u + (-483840 + 192\sqrt{2})v \\ & + (-46080 + 192\sqrt{2})q^2 - 80640w - 241920X. \end{aligned} \tag{5}$$

For every centered real antisymmetric S,

$$c(S) \geq -30000q^2. \tag{6}$$

Proof. The pure quartic is $F_4(iS) = E_4(iS)$ because $\mathrm{tr}(iS) = 0$. This is an algebraic evaluation at an indefinite Hermitian matrix; no positivity of an omission variance at iS is assumed. Substitution into the unrestricted coefficient identities of [1, AppendicesA–C] gives

$$\delta_5(iS) = 30u - 24v - 2q^2 - 4w - 12X, \qquad \frac{T_4(iS)}{8\sqrt{2}} = 120u - 24v - 24q^2,$$

where T_4 denotes the paired quartic derivative. In the ordinary quartic, the trace term vanishes and $5760\eta(6d_2 - \eta) = -5760q^2$, with $\eta = \mathrm{tr}((iS)\overline{iS}) = -q$. Thus $c = 20160\delta_5(iS) - 5760q^2 - T_4(iS)$, which proves (5). Appendix A gives a direct original coefficient rule for independently reconstructing these polynomials.

For the sign bound define the rational lower polynomial

$$\begin{aligned} L(S) = {} & (604800 - 9600/7)u + (-483840 + 1344/5)v \\ & + (-46080 + 1344/5)q^2 - 80640w - 241920X. \end{aligned}$$

Since $7/5 < \sqrt{2} < 10/7$ and $u, v, q^2 \geq 0$,

$$c - L = 960(10/7 - \sqrt{2})u + 192(\sqrt{2} - 7/5)(v + q^2) \geq 0.$$

Parametrize every centered skew matrix by the ten free coordinates $x_{ij} = S_{ij}$, $1 \leq i < j \leq 5$, set $S_{ji} = -S_{ij}$, $S_{i6} = -\sum_{j \leq 5} S_{ij}$ and $S_{6i} = -S_{i6}$. This is a bijection onto the full centered skew space. Let $m(x)$ be all 55 quadratic monomials $x_a x_b$, $a \leq b$, with the ten coordinates ordered lexicographically. The supplied rational matrix G satisfies the exact polynomial identity

$$L(S) + 30000q^2 = m(x)^T G m(x), \qquad G \succ 0. \tag{7}$$

The complete 55-by-55 matrix is supplied as `quartic_sos_certificate.json` with the editable source. The accompanying portable checker reconstructs all contractions in L from the ten-coordinate matrix, collects every quartic coefficient, and compares it exactly with $m^T G m$. There are 555 nonzero coefficients; equality of the full dictionaries also checks all zero coefficients.

For positivity, exact rational LDL^T elimination gives 55 strictly positive pivots and reconstructs every entry of G. Equivalently, clearing denominators by 210 and performing exact fraction-free elimination gives 55 positive leading principal determinants. The independent exact checker verifies this second certificate by Sylvester's criterion. These are finite rational identities and sign checks, not floating eigenvalue acceptance. If $G = LDL^T$, (7) is explicitly the sum $\sum_k D_{kk}(\sum_j L_{jk} m_j)^2$. This conventional Gram form of an SOS certificate [5] proves (6) on the entire coordinate space, including all singular supports. □

The coefficient can be genuinely negative. For $i, j = 0, \ldots, 5$, take $S_{ij} = 0$ when $i - j = 0 \pmod 3$, $+1$ when $i - j = 1 \pmod 3$ and -1 otherwise. Then $u = q^2/6$, $v = q^2/24$, $w = q^2/2$, $X = q^2/12$, so

$$c(S) = (-25920 + 40\sqrt{2})q^2 < 0.$$

Thus positivity of the pure quartic itself is not the mechanism.

If $K = R + iS \succeq 0$ has trace r, then

$$q \le \operatorname{tr}(R^2), \qquad \operatorname{tr}(R^2) + q = \operatorname{tr}(K^2) \le r^2, \qquad q^2 \le \tfrac{1}{2} r^2 \operatorname{tr}(R^2).$$

The first inequality is $\operatorname{tr}(K\overline{K}) \ge 0$. Consequently (6) and (4) give the intrinsic payment

$$\begin{aligned} F_4(R) + c(S) \ge\ & (20160 - 464\sqrt{2})\delta_5(R) + 14800 r \delta_4(R) \\ & + (24744 - 10368\sqrt{2}) r^2 \operatorname{tr}(R^2). \end{aligned} \tag{8}$$

This does not yet pay the quadratic coefficient $b(R, S)$.

4 The complete quadratic phase functional

Again normalize $\operatorname{tr} R = 1$. In addition to B, b_S, q write

$$t = \operatorname{diag} R, \quad b_R = \operatorname{diag}(R^2), \quad \beta = \operatorname{tr}(R^2), \quad d_2 = \|t\|^2, \quad D_1 = R \circ S, \quad d = D_1 \mathbf{1}.$$

Define

$$\begin{gathered} H = \sum R_{ij} S_{ij}^2, \quad V = t^T b_S, \quad W_1 = \operatorname{tr}(R(-S^2)), \quad C = \sum t_i R_{ij} S_{ij}^2, \\ H_2 = \sum (R^2)_{ij} S_{ij}^2, \quad J_2 = \sum R_{ij}^2 S_{ij}^2, \quad N_2 = \sum (S^2)_{ij} R_{ij}^2, \quad Z_2 = \sum (RS + SR)_{ij} R_{ij} S_{ij}. \end{gathered}$$

All these quantities use the same R, S.

Proposition 4.1. *Let $\delta_2 = [x^2]\delta_5(R + ixS)$ and $P_2 = [x^2]T_4(R + ixS)/(8\sqrt{2})$. Their exact formulas are*

$$\begin{aligned} \delta_2 = {} & 144(t^{\circ 2})^T b_S - 60 b_R^T b_S - 72\|d\|^2 - 24 t^T R b_S - 48(St)^T d \\ & - 36 H_2 + 12 N_2 - 24 Z_2 + 288 C + 144 J_2 - 12 H - 6 d_2 q + 4\beta q \\ & + 48 \sum_i t_i (RS^2)_{ii} - 24 \sum_i t_i (SRS)_{ii} - 16 \operatorname{tr}(R^2 S^2) + 8 \operatorname{tr}(RSRS), \end{aligned} \tag{9}$$

$$\begin{aligned} P_2 = {} & -2016(t^{\circ 2})^T b_S + 816 b_R^T b_S + 408 q d_2 - 1728 C \\ & + 672 t^T R b_S - 144\beta q + 240 H + 288 H_2 - 432 J_2, \end{aligned} \tag{10}$$

and

$$\begin{aligned} b(R, S) = {} & 20160\delta_2 + 476352 H + 1097856 V - 137376 q - 181440 W_1 \\ & + 5760 q(2\beta - 6 d_2) - 8\sqrt{2} P_2. \end{aligned} \tag{11}$$

Proof. The unrestricted Fischer correction in [1, AppendixC] retains the quadratic Gram state $p_2 = \sum g_i^2$. For $K = R + ixS$ its basic substitutions are

$$K \circ K = R \circ R - x^2 B + 2ixD_1, \quad (K \circ K)\mathbf{1} = b_R - x^2 b_S + 2ixd,$$
$$K^2 = R^2 - x^2 S^2 + ix(RS + SR), \qquad \operatorname{tr}(K\overline{K}) = \beta - x^2 q.$$

The quadratic part of the null-form contribution $144d_4 + 4S_3 + 12X_K - 96C_3 - 24S_4$, where $d_4 = \sum_i K_{ii}^4$, $S_j = \sum_{a,b} K_{ab}^j$, $X_K = \Re \sum_{i,j} (K^2)_{ij} K_{ij}^2$ and $C_3 = \Re \sum_{i,j} K_{ii} K_{ij}^3$, is

$$-12H - 12H_2 - 12N_2 - 24Z_2 + 288C + 144J_2.$$

The quadratic part of the actual nonzero-p_2 correction is

$$\begin{aligned} &144(t^{\circ 2})^T b_S - 24t^T R b_S - 48(St)^T d - 24H_2 + 24N_2 \\ &\quad - 60 b_R^T b_S - 72\|d\|^2 - 6d_2 q + 4\beta q + 48 \sum_i t_i (RS^2)_{ii} - 24 \sum_i t_i (SRS)_{ii} \\ &\quad - 16 \operatorname{tr}(R^2 S^2) + 8 \operatorname{tr}(RSRS). \end{aligned}$$

Their sum proves (9). In this expansion the quadratic part of $\operatorname{tr}((K\overline{K})^2)$ is

$$4 \operatorname{tr}(R^2 S^2) - 2 \operatorname{tr}(RSRS).$$

The trace pairing of the symmetric commutator $SR - RS$ with skew D_1 vanishes. For (10), keep $A = K + \overline{K} = 2R$ fixed and substitute $|K|^2 = R \circ R + x^2 B$ into the unrestricted mixed paired formula. Finally the ordinary quartic change contributes

$$20160\delta_2 - 69120H + 5760q(2\beta - 6d_2),$$

and half the cubic change contributes

$$545472H + 1097856V - 137376q - 181440W_1.$$

This proves (11) with the actual derivative normalization. □

Equations (9)–(11) define the complete ten-coordinate quadratic form, not just selected phase values. For a centered skew basis E_a, $a = 1, \ldots, 10$, its matrix entries are

$$\mathcal{H}_{aa} = b(R, E_a), \qquad \mathcal{H}_{ab} = \tfrac{1}{2}[b(R, E_a + E_b) - b(R, E_a) - b(R, E_b)].$$

On the affine trace-one section these entries have degree at most two in R. For homogeneous unnormalized formulas, restore $-12rH$ in (9), $+240rH$ in (10), multiply the H, V, W_1 terms of (11) by r and the standalone q term by r^2. The resulting coefficient has exact bidegree $(2, 2)$ in R, S.

5 The full isotropic fiber and nonflat variations

Set $R_0 = P/5$. Substitution into Proposition 4.1, using $S\mathbf{1} = 0$, gives

$$\delta_2(R_0, S) = \tfrac{13}{25} q, \quad P_2(R_0, S) = \tfrac{48}{5} q, \quad b(R_0, S) = \kappa q, \qquad \kappa = \tfrac{2304 - 384\sqrt{2}}{5} > 0. \tag{12}$$

There is also a structural explanation. A row-permutation-invariant quadratic form on skew edges is a combination of squared Frobenius norm and squared row sums. A disjoint-edge cross coefficient vanishes by exchanging only one edge's endpoints; all shared-edge coefficients agree up to orientation. Centering eliminates the row-sum term. Thus the form is scalar on the complete centered skew space, and (12) identifies that scalar.

Corollary 5.1 (Complete isotropic phase fiber)**.** *For $K = R_0 + iS \succeq 0$ centered,*

$$F_4(K) \geq \frac{3081216 - 17152\sqrt{2}}{125} > 24455.$$

Proof. The original coefficient at R_0 is $a = 25344 - (15616/125)\sqrt{2}$. Since $-R_0 \preceq iS \preceq R_0$, $\|S\|_{\mathrm{op}} \leq 1/5$. Centered skew S has rank at most four, so $0 \leq q \leq 4/25$. The concave polynomial $a + \kappa q - 30000q^2$ is minimized at an endpoint of this interval; its smaller endpoint is the displayed value. Use $\sqrt{2} < 99/70$ for the strict numerical bound. □

Now write $R = R_0 + E$, with E centered real symmetric and $\operatorname{tr} E = 0$, and set $\varepsilon = \|E\|_{\mathrm{op}}$. The exact quadratic-base polarization is

$$b(R, S) = \kappa q + L(E, S) + M(E, S), \tag{13}$$

where L is linear and M quadratic in E. Put $H_E = \sum E_{ij} S_{ij}^2$, $V_E = \operatorname{diag}(E)^T b_S$ and $W_E = \operatorname{tr}(E(-S^2))$. Exact first variation gives

$$L = (524736 - 768\sqrt{2})H_E + (1097856 + (6144/5)\sqrt{2})V_E - 149184W_E. \tag{14}$$

In particular, the first variations of δ_2 and P_2 are $12H_E/5+8W_E/5$ and $96H_E-768V_E/5$ respectively. The diagonal mixed term cancels completely in the first expression. These equalities follow by differentiating the explicit contractions, not by fitting feasible phase samples.

For a precise formula for M, evaluate (9) and (10) at E in place of R, and remove their trace-one linear terms: set $\delta_{\mathrm{hom}} = \delta_2(E, S) + 12H_E$ and $P_{\mathrm{hom}} = P_2(E, S) - 240H_E$. Then

$$M = 20160\delta_{\mathrm{hom}} + 5760q(2\operatorname{tr} E^2 - 6\|\operatorname{diag} E\|^2) - 8\sqrt{2}P_{\mathrm{hom}}. \tag{15}$$

Lemma 5.2 (Uniform phase-tensor variation)**.** *For all such E, S,*

$$|L(E, S)| \leq 1772500\varepsilon q, \qquad |M(E, S)| \leq 23000000\varepsilon^2 q.$$

The coefficient in the first bound has a strict upper bound as shown below.

Proof. Since $B_{ij} \geq 0$ and $\sum B_{ij} = q$, the direct entry bound $|E_{ij}| \leq \varepsilon$ gives $|H_E| \leq \varepsilon q$. Also $|V_E| \leq \varepsilon q$ and $|W_E| \leq \varepsilon q$, the latter because $-S^2 \succeq 0$. The first two coefficients in (14) are positive, so their absolute coefficient sum, retaining the shared radical, is

$$1771776 + \tfrac{2304}{5}\sqrt{2} < 1772500.$$

This proves the linear bound.

For the second, E acts on a five-dimensional space. Thus $\|E\|_{\mathrm{F}} \leq \sqrt{5}\varepsilon$, $|E_{ij}| \leq \varepsilon$, and $(E^2)_{ii} \leq \varepsilon^2$. Antisymmetry gives $\|S\|_{\mathrm{op}} \leq \sqrt{q/2}$. For $d_E = (E \circ S)\mathbf{1}$, rowwise Cauchy–Schwarz gives $\|d_E\|^2 \leq \varepsilon^2 q$. The complete termwise coefficient budgets for δ_{hom}, divided by $\varepsilon^2 q$, are

contraction	budget	contraction	budget
$(\operatorname{diag} E)^{\circ 2} \cdot b_S$	144	$\operatorname{diag}(E^2) \cdot b_S$	60
$\|d_E\|^2$	72	$(\operatorname{diag} E)^T E b_S$	$24\sqrt{5}$
$(S \operatorname{diag} E)^T d_E$	$48\sqrt{5/2}$	H_2	36
N_2	12	Z_2	48
C	288	J_2	144
$\|\operatorname{diag} E\|^2 q$	30	$\operatorname{tr}(E^2)q$	20
$\sum E_{ii}(ES^2)_{ii}$	48	$\sum E_{ii}(SES)_{ii}$	24
$\operatorname{tr}(E^2S^2)$	16	$\operatorname{tr}(ESES)$	8.

In this table the named contractions are evaluated at E. For N_2, the matrix $E \circ E$ is not declared positive semidefinite: its entries are nonnegative and maximal row sum is at most ε^2, which bounds its operator norm. Pair with $-S^2 \succeq 0$. For Z_2, use $\|ES + SE\|_{\mathrm{F}} \le 2\varepsilon\sqrt{q}$ and $\|E \circ S\|_{\mathrm{F}} \le \varepsilon\sqrt{q}$. Weighted traces use operator/nuclear norm duality; the final trace is bounded by $\|ES\|_{\mathrm{F}}^2$. These arguments justify every row without entrywise positivity of E.

Using $\sqrt{5} < 9/4$ and $\sqrt{5/2} < 8/5$, the sum is less than 1100. The eight analogous budgets for P_{hom} are

$$2016,\ 816,\ 2040,\ 1728,\ 672\sqrt{5},\ 720,\ 288,\ 432,$$

whose sum is less than 10000. Equation (15) now gives

$$|M| \le [20160 \cdot 1100 + 5760 \cdot 40 + 12 \cdot 10000]\varepsilon^2 q = 22526400\varepsilon^2 q < 23000000\varepsilon^2 q,$$

with the zero case understood directly. □

6 Actual omission monotonicity and the closed pinch

Lemma 6.1. *If $A, B \succeq 0$ and $a \ge 0$, then $\delta_j(aA + B) \ge a^{j-1}\delta_j(A)$ for $j = 4, 5$.*

Proof. Every coefficient of $\mathrm{per}(J + xA + yB + vW)$ is nonnegative. Represent its Gram forms in an orthogonal direct sum colored by the four matrices. In the Fischer product norm, the multihomogeneous components are orthogonal, so each coefficient is a squared norm. Now

$$\delta_j(A) = [x^{j-1}v]\,\mathrm{per}(J + xA + vW).$$

Substitute $aA + B$ and retain its nonnegative all-A term. This is a mixed permanent coefficient argument. □

We now prove Theorem 1.1. At trace one, (2) implies $\varepsilon \le 1/40$ and $R \succeq (7/8)R_0$. The actual omission values at R_0 are $\delta_5(R_0) = 28/375$ and $\delta_4(R_0) = 24/25$. Lemma 6.1 therefore gives

$$\delta_5(R) \ge (7/8)^4(28/375), \qquad \delta_4(R) \ge (7/8)^3(24/25), \qquad \mathrm{tr}(R^2) \ge 1/5.$$

From $K, \overline{K} \succeq 0$, one has $-R \preceq iS \preceq R$, whence $\|S\|_{\mathrm{op}} \le 9/40$. Centered antisymmetric S has rank at most four, so

$$q \le 4(9/40)^2 = 81/400.$$

Use $\sqrt{2} < 99/70$ to obtain $20160 - 464\sqrt{2} > 19500$ and $24744 - 10368\sqrt{2} > 10000$. Combining (8), (13), $\kappa q \ge 0$ and Lemma 5.2 gives a strict lower bound for $F_4(K)$ of

$$\begin{aligned} &1456(7/8)^4 + 14208(7/8)^3 + 2000 \\ &\quad - 1772500(1/40)(81/400) - 23000000(1/1600)(81/400) = \frac{124803}{256} > 487. \end{aligned}$$

The inequalities remain valid at every spectral and feasible phase endpoint. Homogeneity restores r^4 and proves the theorem.

7 Singular endpoints and unbounded conditioning

The cap in the isotropic fiber is genuinely attained. Let $C_0 = I_3 - J_3/3$ and

$$S = \frac{1}{5}\begin{pmatrix} 0 & C_0 \\ -C_0 & 0 \end{pmatrix}.$$

This centered skew matrix has two singular values $1/5$, each repeated twice, and $q = 4/25$. On the two saturated skew blocks $R_0 + iS$ has eigenvalues $2/5, 0$; its remaining centered eigenvalue is $1/5$. Thus it is positive semidefinite of rank three and lies on the full phase boundary.

For a nonflat endpoint take

$$R = R_0 + \tfrac{1}{80}(uu^T - vv^T), \quad u = (1, -1, 0, 0, 0, 0)^T, \quad v = (0, 0, 1, -1, 0, 0)^T.$$

Its centered eigenvalues are $7/40, 1/5, 1/5, 1/5, 9/40$. On the orthogonal normalized directions

$$w = (1, 1, -1, -1, 0, 0)^T/2, \qquad z = (0, 0, 0, 0, 1, -1)^T/\sqrt{2},$$

set $S = (wz^T - zw^T)/5$. Then $K = R + iS \succeq 0$ has rank four, with nonzero eigenvalues $7/40, 1/5, 9/40, 2/5$. It hits both spectral endpoints and a singular phase endpoint. The original coefficient extraction gives

$$F_4(K) = \frac{101897799}{4000} - \frac{2627031}{20000}\sqrt{2}.$$

This exact value is corroboration; the universal theorem does not depend on checking that example.

Replace S by $(1-\epsilon)S$, $0 < \epsilon \le 1$. The small centered eigenvalue becomes $\epsilon/5$, so K has rank five and its condition number can diverge while (2) remains unchanged. Adding sJ, $s > 0$, supplies a rank-six positive anchor; the derivative is still the functional of the centered residual.

8 An inherited obstruction to independent-factor certificates

The proof keeps the same Gram state in all terms. There is a precise reason why a stronger independent-factor tensor relaxation cannot prove the global statement. The earlier trine formula [1, Section6], credited there to [3], has centered trace-one projectors U, V onto $(1, \omega, \omega^2, 0, 0, 0)$ and $(0, 0, 0, 1, \omega, \omega^2)$, with $\omega^3 = 1$, $\omega \neq 1$, such that

$$[\lambda^2\mu^2]F_4(\lambda U + \mu V) = -8832 - 1600\sqrt{2} < 0.$$

If a Hermitian fourfold tensor coefficient H represented $F_4(X) = \mathrm{tr}(HX^{\otimes 4})$ and were nonnegative on every independent positive semidefinite product, this mixed coefficient would be a sum of six nonnegative expectations, a contradiction. For a slot-symmetrized H, each expectation is $-1472 - (800/3)\sqrt{2}$.

In particular no representation $H = \sum_T \Gamma_T(G_T)$, $G_T \succeq 0$, can hold, where Γ_T transposes any chosen tensor slots. Each product built from two U and two V remains positive semidefinite under every such partial transpose, so all those expectations would be nonnegative. This is the elementary product-state implication of the decomposable-witness framework [6]. Adding a tensor with zero repeated-state polynomial cannot remove the contradiction: the same negative mixed coefficient remains.

This observation repurposes an inherited coefficient in the stated compensated-derivative setting. Positivity on $X^{\otimes 4}$ is weaker than positivity on arbitrary independent factors. The remaining global problem requires actual repeated-state or common-Gram constraints.

A Original-coefficient reconstruction and verification scope

For $A = K + \overline{K}$, $B = K \circ \overline{K}$, let $N_{ij}(t) = 2 + \sqrt{2}tA_{ij} + t^2 B_{ij}$. The actual paired coefficient is

$$T_4 = [t^4] \sum_{\pi \in S_6} \sum_i (2\sqrt{2}W_{i,\pi(i)} + tW_{i,\pi(i)}A_{i,\pi(i)}) \prod_{j \neq i} N_{j,\pi(j)}(t).$$

An independent subset recurrence extends a column set S of size i by an unused column j with $P_{S\cup\{j\}} \mathrel{+}= P_S n_{ij}$ and $\dot{P}_{S\cup\{j\}} \mathrel{+}= \dot{P}_S n_{ij} + P_S \dot{n}_{ij}$, starting with $P_\varnothing = 1$, $\dot{P}_\varnothing = 0$. Ordinary slots are $1 + tK_{ij}$ with direction W_{ij}; paired slots and directions are the displayed N_{ij} and its derivative. Truncation at degree four is exact for these coefficients.

For symbolic reconstruction, an equivalent residual-slot rule retains m rows I and columns J. The remaining differentiated W slot has total weight $6|I \cap J| - m^2$, and the constant slots contribute $(5-m)!$. Expanding the four retained entries of K gives the whole polynomial for δ_5. The paired degree-five part has the three slot patterns A^5, A^3B, AB^2; the same rule reconstructs its entire quadratic phase coefficient.

The supplementary exact programs parametrize a centered real symmetric matrix by the 15 upper-triangle coordinates in its first five rows and columns, with row sums determining the sixth row and column. The skew matrix has the ten coordinates used in the SOS proof. The unnormalized bidegree-$(2,2)$ formulas in these 25 coordinates match all 4530 nonzero omission coefficients and 3630 nonzero paired coefficients of the original permanent. Dictionary equality also checks vanishing coefficients. This is a complete polynomial identity check, unlike interpolation at selected matrices.

The quartic identity has a separate symbolic reconstruction in ten skew coordinates. The exact rational Gram checker proves the SOS polynomial identity and positive definiteness; its data are essential to the computer-assisted bound (6). The printed analytic variation, monotonicity and scalar reserve arguments prove the universal domain theorem from those identities. Additional exact feasible controls, including the singular examples, corroborate normalizations and implementation. Floating searches were used only for discovery and are not an acceptance condition. Rational checks use explicit exceptions and operate under Python's optimization flag.

Computational assistance

AI assistance was used in research, drafting and checking. The author is responsible for the mathematical statements and presentation.

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