

# Frequency-Source Resolvent Limits and Positive-Receiver Instability

Scott Schubr  
Luminary Oddities

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## Abstract

We classify inverse energies of fixed frequency sources in a reciprocal-theta realization of the Weil form and compare two positive receivers on identical constrained spaces. Regular profiles have an exact archimedean inverse-Gram limit given by their full Fourier-dual matrix, unchanged by any fixed finite collection of known derivative-null and polar rows. A nonzero frequency-zero trace produces exactly one divergent direction, of order  $\log \log(1/\epsilon)$  on the original polar space. For the complete positive part of every fixed arithmetic cutoff, every direction of an independent negative-band source family instead outgrows every fixed power of  $\log \log(1/\kappa)$ , even after those known rows are removed. We give a complete eight-feature application at cutoff 13. For comparison with derivative sources, we identify their fixed-family normalized leading matrix by an exact outer-Hardy and Cauchy formula and sharpen the admitted growing-family range to an explicit onset asymptotic to  $4n \log n$ . The proofs preserve complete theta tails, mixed covariances and the original minimal form domains.

## 1 Exact source limits and a change of positive receiver

A singular inverse depends on the source as well as the operator. In the reciprocal-theta realization, the archimedean symbol  $a$  vanishes only at zero, while the complete positive part  $p_M$  of a fixed arithmetic cutoff vanishes on an open frequency band. This difference persists for finite source maps after exact known-row restrictions. We compare the actual source covariances, rather than ambient inverse operators.

For a frequency profile  $g$  on  $(0, \infty)$  write

$$q_g(u) = \frac{1}{\pi} \int_0^\infty g(\xi) \sin(\xi u) d\xi, \quad E_g = Q_{\mathcal{R}}(sq_g).$$

The operators, original norm and exact row projection  $Q_{\mathcal{R}}$  are defined in Sections 2–3. The following theorem summarizes the same-source comparison.

**Theorem 1.1** (Regular sources and positive-receiver instability). *Fix  $M \geq 2$ , a finite known derivative/polar row space  $\mathcal{R}$ , and an independent finite family  $g_1, \dots, g_r \in L^2(0, \infty)$  supported where  $n_M > 0$ . Suppose  $g_i/\sqrt{a} \in L^2$ . Put  $E = [E_{g_1}, \dots, E_{g_r}]$  and*

$$D_{ij} = \frac{1}{\pi} \int_0^\infty \frac{\overline{g_i} g_j}{a} d\xi.$$

*Then, on the identical original constrained recipient,*

$$E^*(A_{\mathcal{R}} + \epsilon)^{-1} E \uparrow D$$

in Loewner order and matrix norm. For every integer  $N \geq 1$  there are fixed-data constants  $c_N, \kappa_N > 0$  such that

$$\lambda_{\min}(E^*(P_{M,\mathcal{R}} + \kappa)^{-1}E) \geq c_N[\log \log(1/\kappa)]^N \quad (0 < \kappa < \kappa_N).$$

The cutoff, source family and rows are fixed before either limit.

The regular theorem below does not require support in a negative band; that support supplies the contrast with  $P_M$ . A second classification isolates the frequency-zero endpoint: if  $g_j = v_j 1_{[0,b]} + r_j$  with  $r_j/\sqrt{a} \in L^2$ , the original polar-only inverse Gram divided by the endpoint scalar tends to  $v^*v$ . That scalar has order  $\log \log(1/\epsilon)$ . Thus its singular part has rank one, while regular profiles have a finite exact matrix.

The frequency-source conclusions extend the derivative-source geometry of the first published source paper [1]. Derivative columns are a different class: their divided profiles grow at spatial infinity and their archimedean inverse energies have powers of  $\log(1/\epsilon)$ . Sections 8–11 give their completed fixed-family leading matrix and positive-receiver comparison; the appendix records complete source estimates and the sharpened growing-family admission. These derivative statements retain the columns and impose only the polar row. Frequency regularity and known-row removal therefore are not silently imported into them.

| Source class                 | Exact constraints                    | Archimedean inverse behavior                                              |
|------------------------------|--------------------------------------|---------------------------------------------------------------------------|
| Regular frequency profiles   | Fixed known derivative/polar rows    | Exact finite matrix $D$                                                   |
| Nonzero zero-frequency trace | Original polar row                   | One singular direction, $\Theta(\log \log(1/\epsilon))$                   |
| Theta derivative columns     | Original polar row, columns retained | Normalized finite positive leading matrix; explicit growing-family bounds |

The proofs combine classical Fourier duality and variational inverse identities with complete source-specific domain returns. The regular equality requires remote exact-row energy repair. The endpoint lower uses two compact bumps with an exact polar correction. The negative-band comparison requires a joint family of zero-positive-energy spectral tests, followed by a quantitatively paid return to the original weighted norm. The fixed-family derivative limit additionally requires weighted compactness, complete moment tightness and density in the same homogeneous half-line completion. Classical Hardy outer amplification supplies the derivative positive-receiver contrast [3, 4]; it does not replace the actual negative-band argument.

## 2 The constrained reciprocal-theta form

Use  $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)e^{-iu\xi} du$  and Plancherel measure  $d\xi/(2\pi)$ . Inner products are conjugate-linear in the first variable. For  $u \geq 0$  set

$$\psi(u) = e^{u/2} \sum_{m \geq 1} (4r_m^2 - 6r_m)e^{-r_m}, \quad r_m = \pi m^2 e^{2u}, \quad s(u)^2 = \frac{\psi(u)}{\cosh(u/2)}. \quad (1)$$

Theta inversion gives the smooth even continuation of  $\psi$ . The positive smooth function  $s$  is also even. Put  $p(u) = \sinh(u/2)$  and let

$$X = \{h \in L^2(\mathbb{R}) : h \text{ is odd, } \langle sp, h \rangle = 0\}.$$

Let  $\mathcal{V}_0$  be the smooth compact odd functions satisfying this moment, and let  $V$  be its completion for

$$\|h\|_V^2 = \|h\|_2^2 + \mathfrak{a}[h], \quad \mathfrak{a}[h] = \int_{\mathbb{R}} a(\xi) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi},$$

where, with  $\Psi = \Gamma'/\Gamma$ ,

$$a(\xi) = \Re \Psi(1/4 + i\xi/2) - \Psi(1/4). \quad (2)$$

The closed nonnegative form on  $X$  determines its associated selfadjoint operator  $A$ . Throughout,  $C_\epsilon = A + \epsilon I$  is formed on this same constrained space, and  $P_X$  denotes its orthogonal projection. The representation of closed forms and variational inverse identities have their usual meaning [9].

For  $i \geq 0$  define

$$\ell_i = 2i + 3, \quad z_i = \frac{\psi^{(\ell_i)} - \psi^{(\ell_i-2)}/4}{s}. \quad (3)$$

These are the standard odd derivative-null sources of the complete weighted Weil realization [6]. Their membership in  $X$  can also be seen directly: integrate twice against  $p$ , use  $p'' = p/4$ , and use the complete theta tails to remove all boundary terms. The same tails give  $z_i \in L^2$ . The inverse-growth proof uses this source formula and the polar moment directly.

**Lemma 2.1** (Elementary source and multiplier bounds). *For  $r = \pi e^{2|u|}$ ,*

$$2r^2 e^{-r} \leq s(u)^2 \leq 169e^{4|u|-r}, \quad 0 \leq a(\xi) \leq 17\xi^2. \quad (4)$$

*For  $|\xi| \leq 1$ ,  $a(\xi) \geq (16/5)\xi^2$ .*

*Proof.* The first atom is positive and  $4r^2 - 6r \geq 2r^2$  for  $r \geq \pi$ , while  $e^{|u|/2}/\cosh(u/2) \geq 1$ . For the upper bound,  $\sum_{m \geq 1} m^4 e^{-(m^2-1)r} < 2$ : successive ratios are at most  $16e^{-3r} < 1/2$ . Since  $e^{|u|/2}/\cosh(u/2) \leq 2$ , this gives  $s^2 < 16r^2 e^{-r} < 169e^{4|u|-r}$ . The digamma difference is

$$a(\xi) = \sum_{n \geq 0} \frac{(\xi/2)^2}{(n+1/4)((n+1/4)^2 + (\xi/2)^2)}.$$

The zeroth term is  $16\xi^2/(1+4\xi^2)$ . Its upper is  $16\xi^2$ , and the other terms sum to at most  $\xi^2 \sum_{n \geq 1} n^{-3}/4 < \xi^2$ . The zeroth term also proves the stated lower bound.  $\square$

**Lemma 2.2** (A complete observation upper). *For  $0 < \epsilon \leq 1$  and  $U = \log(1/\epsilon)$ ,*

$$J_\epsilon := \|M_s C_\epsilon^{-1/2}\| \leq 4 \log(U+4). \quad (5)$$

*Proof.* We reproduce the finite observation estimate of [7]. Let  $P_R$  cut physical space to  $[-R, R]$  and  $Q_b$  cut Fourier space to  $[-b, b]$ . Their product has Hilbert–Schmidt norm  $\rho = (2Rb/\pi)^{1/2}$ . For  $f = sh$ , the identity  $f = P_R Q_b f + (I - P_R)f + P_R(I - Q_b)f$  gives, when  $\rho < 1$ ,

$$(1 - \rho)\|f\| \leq \sigma_R \|h\| + a(b)^{-1/2} \sqrt{\mathfrak{a}[h]}, \quad \sigma_R = \sup_{|u| > R} s(u).$$

The series in Lemma 2.1 makes  $a$  increasing in  $|\xi|$ , justifying the frequency-tail term. Cauchy–Schwarz against  $\mathfrak{a}[h] + \epsilon\|h\|^2$  yields

$$\|sh\|^2 \leq \frac{\sigma_R^2/\epsilon + 1/a(b)}{(1 - \rho)^2} (\mathfrak{a}[h] + \epsilon\|h\|^2).$$

Take  $R = \log(U + 4)$ ,  $b = \pi/(8R) < 1$ , so that  $\rho = 1/2$ . With  $t = U + 4 \geq 4$ , the spatial envelope gives  $\sigma_R^2/\epsilon \leq 169t^4 e^{-3t^2+t-4} < 1$ . Indeed its logarithmic derivative is  $4/t - 6t + 1 < 0$ , and its value at 4 is  $43264e^{-48} < 1$ . The frequency lower gives  $1/a(b) \leq 20R^2/\pi^2$ . Thus

$$J_\epsilon^2 \leq 4(1 + 20R^2/\pi^2) < 16R^2.$$

The argument extends from the compact polar core to  $V$ , and spectral calculus identifies the energy quotient with the displayed norm.  $\square$

### 3 Positive arithmetic receivers and exact row spaces

Fix an integer  $M \geq 2$  and let  $\Lambda$  denote the von Mangoldt function. Define the complete frequency symbols

$$m_M(\xi) = \Re \Psi(1/4 + i\xi/2) - \log \pi - 2 \sum_{2 \leq q \leq M} \frac{\Lambda(q)}{\sqrt{q}} \cos(\xi \log q), \quad (6)$$

$$p_M = (m_M)_+, \quad n_M = (-m_M)_+. \quad (7)$$

Let  $P_M$  be the operator associated to

$$\mathfrak{p}_M[h] = \int_{\mathbb{R}} p_M(\xi) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi}$$

on the original polar-only space  $X$  and original minimal form domain  $V$  of Section 2. Its symbol difference  $p_M - a$  is bounded: the finite trigonometric sum is bounded, while  $a \geq 0$  and the added scalar is fixed. Thus  $P_M$  has the same form domain as  $A$ . It is the complete positive part of the finite arithmetic symbol, not a finite positive observation window. The complete signed Weil form has additional negative observations and the arithmetic tail; it is not  $P_M$ .

Since  $m_M(0) < 0$ , there is  $b > 0$  with  $p_M = 0$  on  $[-b, b]$ . The digamma asymptotic [5] gives

$$p_M(\xi) = \log |\xi| + O_M(1) \quad (|\xi| \rightarrow \infty), \quad p_M(\xi) \leq C_M + \log(2 + |\xi|). \quad (8)$$

The negative part  $n_M$  is bounded and compactly supported. All these constants concern fixed  $M$ .

We also use recipients with finitely many additional known rows. Let  $Y = L_{\text{odd}}^2(\mathbb{R})$ , and let  $\mathcal{R}$  be the span of the polar row  $sp$  and any fixed finite collection of the complete derivative rows (3). Write

$$X_{\mathcal{R}} = Y \cap \mathcal{R}^\perp, \quad Q_{\mathcal{R}} : Y \rightarrow X_{\mathcal{R}}.$$

The corresponding  $A_{\mathcal{R}}$  and  $P_{M,\mathcal{R}}$  are the associated operators of the restricted forms with the same compact-core minimal closure. Finite linear combinations of known rows are allowed; imposing the full finite list of derivative rows occurring in them is a stronger constraint and can be used for variational lower bounds. An unknown null row with no proved asymptotic is outside this definition.

**Lemma 3.1** (Compact core and logarithmic graph norm). *On each of these exact finite-row spaces the original minimal domain is*

$$\{h \in X_{\mathcal{R}} : sh \in H_{\log}\}, \quad \|f\|_{H_{\log}}^2 = \int_{\mathbb{R}} (1 + \log(2 + |\xi|)) |\widehat{f}(\xi)|^2 \frac{d\xi}{2\pi}.$$

*Proof.* The digamma series and asymptotic make  $1 + a$  comparable to  $1 + \log(2 + |\xi|)$ ;  $p_M - a$  is bounded. Let  $w(\xi) = \sqrt{1 + \log(2 + |\xi|)}$ . The inequality  $2 + |x + y| \leq (2 + |x|)(2 + |y|)$  gives  $w(x + y) \leq Cw(x)w(y)$ . Weighted convolution and Young's inequality therefore bound multiplication by a smooth compact function  $\chi$  on  $H_{\log}$  by  $C\|w\hat{\chi}\|_1$ . For an even cutoff  $\chi_R(u) = \chi(u/R)$  this bound is uniform for  $R \geq 1$ . Approximate  $sh$  in  $H_{\log}$  by Schwartz functions; cutoffs converge on each such function even in  $H^1$ , so uniform boundedness proves  $\chi_R sh \rightarrow sh$  in  $H_{\log}$ . Also  $\chi_R h \rightarrow h$  in  $L^2$ . On a compact support, a smooth compact extension of  $1/s$  is an  $H_{\log}$  multiplier. Even mollifiers consequently approximate  $\chi_R h$  by smooth compact odd functions in the graph norm. Multiplication by  $s$  is bounded on  $H_{\log}$ : its complete theta series gives rapidly decreasing derivatives and hence  $\|w\hat{s}\|_1 < \infty$ . Restore the exact finite rows by subtracting fixed smooth compact odd functions with an invertible moment matrix. Such functions exist by  $L^2$  density and independence of the rows. Their correction coefficients tend to zero, retaining graph convergence. This proves the asserted minimal closure, without an extra boundary condition.  $\square$

## 4 The exact regular frequency-source matrix

For  $g \in L^2(0, \infty)$  the sine profile has  $\hat{q}_g(\xi) = -ig(\xi)$  for  $\xi > 0$  and  $+ig(-\xi)$  for  $\xi < 0$ . No complex conjugation is used in the odd extension. Consequently  $\|q_g\|_2^2 = \pi^{-1} \int_0^\infty |g|^2$ .

**Theorem 4.1** (Exact regular limit after fixed known rows). *Let  $g_1, \dots, g_r \in L^2(0, \infty)$  and  $g_i/\sqrt{a} \in L^2$ . Impose any fixed finite collection of known derivative rows and the polar row as in Section 3, and put  $E_i = Q_{\mathcal{R}} s q_{g_i}$ . Then*

$$G_{\mathcal{R}}(\epsilon) := E^*(A_{\mathcal{R}} + \epsilon)^{-1} E \uparrow D := \left[ \frac{1}{\pi} \int_0^\infty \frac{\overline{g_i} g_j}{a} d\xi \right]_{ij}$$

*in Loewner order and matrix operator norm. In particular  $G_{\mathcal{R}}(\epsilon) \preceq D$  for every  $\epsilon > 0$ .*

*Proof.* For any common coefficient vector  $c$  and any original constrained form-domain test  $h$ , projection disappears in the pairing. Fourier Cauchy–Schwarz gives

$$|\langle Ec, h \rangle|^2 = |\langle q_{g_c}, sh \rangle|^2 \leq (c^* D c) \mathfrak{a}[h], \quad g_c = \sum_i c_i g_i.$$

Variational duality for the associated constrained inverse gives the full matrix upper. We prove the matching lower by exact energy-density repair.

The homogeneous odd Fourier dual problem has supremum

$$\sup_f \{2\Re \langle q_g, f \rangle - \mathcal{E}(f)\} = \frac{1}{\pi} \int_0^\infty |g|^2 / a.$$

The formal maximizing transform  $\hat{q}_g/a$  has finite  $a$ -weighted norm. Cut it off on compact frequency sets away from zero and smooth there in that weighted norm, preserving odd symmetry. Its inverse transforms are odd Schwartz functions. Even physical cutoffs converge in  $H^1$  for each such function, hence in  $\mathcal{E}$  because  $a \leq 17\xi^2$ . The same weighted-dual inequality gives convergence of the source functional. Thus odd smooth compact physical profiles approximate this supremum, without requiring the formal maximizer to be in unweighted  $L^2$ .

It remains to impose the exact known rows. Divide each canonical Hilbert row by  $s$ . For the polar row this is  $p(u) \sim e^{u/2}/2$ . For derivative order  $\ell_i$ , the complete theta expansion gives

$$k_i(u) = z_i(u)/s(u) = c_i e^{\beta_i u} (1 + o(1)), \quad \beta_i = 2\ell_i + 1/2, \quad c_i = (-2\pi)^{\ell_i}/2 \neq 0.$$

Indeed differentiation of the first atom gives  $\psi^{(k)}/\psi = (-2r)^k(1 + O_k(r^{-1}))$ , and the entire higher-atom quotient is  $O_k(r^{C_k}e^{-3r})$ ; Appendix A gives the complete positive majorants. The polar exponent  $1/2$  and the derivative exponents are distinct. For linear-combination constraints impose the full finite canonical list underlying them; a variational lower on this smaller space remains valid for the requested one.

Fix a nonzero nonnegative even smooth bump  $\eta$  of small compact support, spacing  $\Delta > 0$ , and  $a_t = t\Delta$ ,  $0 \leq t < q$ , where  $q$  is the number of rows. Set

$$v_{R,t}(u) = \eta(u - 2R - a_t) - \eta(-u - 2R - a_t).$$

These odd bumps have uniformly bounded  $H^1$  norms. Their exact row matrix obeys

$$e^{-2\beta_l R} \int k_l v_{R,t} \longrightarrow d_l e^{\beta_l a_t}, \quad d_l = 2c_l \int \eta(v) e^{\beta_l v} dv \neq 0.$$

This limit is a nonzero diagonal times a Vandermonde matrix on the distinct positive nodes  $e^{\beta_l \Delta}$ . Its inverse is bounded for large  $R$ .

For any fixed odd compact smooth  $f$ , let  $m_l = \int k_l f$ . Solve the actual, rather than limiting, row system  $B(R)c(R) = -m$ . Row normalization gives  $\|c(R)\| = O_f(e^{-2\beta_{\min} R})$ . Therefore

$$f_R = f + \sum_t c_t(R) v_{R,t} \longrightarrow f \quad \text{in } H^1, \quad \int k_l f_R = 0 \quad \text{exactly.}$$

The correction tends to zero in  $\mathcal{E}$  and in every regular source pairing by the complete dual inequality. Each  $h_R = f_R/s$  is an original smooth compact exact-row test;  $s$  is positive smooth on every finite compact. Its original Hilbert norm can be large, but is finite for each fixed  $R$ .

Given accuracy  $\delta > 0$ , first select a near-maximizing compact  $f$ , then a finite exact repair radius  $R$ . The fixed test  $h_R$  has

$$2\Re\langle E_g, h_R \rangle - \mathfrak{a}[h_R] \geq D_g - \delta.$$

Only after this selection let  $\epsilon \downarrow 0$ , so  $\epsilon \|h_R\|^2 \rightarrow 0$ . Variational duality gives the lower limit  $D_g - \delta$ , and then  $\delta \downarrow 0$  proves equality. There is no simultaneous unpriced repair-radius limit.

Apply the argument to every common source combination. Resolvent monotonicity gives  $0 \preceq G_{\mathcal{R}}(\epsilon) \preceq D$  and increasing convergence as  $\epsilon$  decreases. Diagonal convergence and positivity yield  $\|D - G_{\mathcal{R}}(\epsilon)\| \leq \text{Tr}(D - G_{\mathcal{R}}(\epsilon)) \rightarrow 0$ .  $\square$

Independent profiles give  $D > 0$ ; the theorem does not otherwise require independence. Its hypotheses are the actual weighted-dual integrals. By  $1/a \leq 1/4 + 1/(16\xi^2)$ , the sufficient condition  $g, g/\xi \in L^2$  is often convenient.

## 5 Frequency-zero trace and one singular direction

This section uses the original polar-only space  $X$ . Fix  $b > 0$ , put  $g_0 = 1_{[0,b]}$ , and define

$$q_0(u) = \frac{1 - \cos(bu)}{\pi u}, \quad E_0 = P_X(sq_0), \quad J_0(\epsilon) = \langle E_0, (A + \epsilon)^{-1} E_0 \rangle, \quad U = \log(1/\epsilon).$$

The continuous value at zero is understood.

**Theorem 5.1** (Endpoint order and complete rank-one asymptotic). *For each fixed  $b > 0$ ,  $J_0(\epsilon) = \Theta_b(\log U)$  as  $\epsilon \downarrow 0$ . More explicitly, for  $w > 0$ ,  $T = \pi e^{2w}$ ,*

$$J_0(\epsilon) \leq \left( \sqrt{K_0(w)} + \sqrt{\tau_0(U, w)} \right)^2, \quad K_0(w) = \frac{b}{4\pi} + \frac{w}{\pi^2}, \quad (9)$$

$$\tau_0(U, w) = \frac{64(T+1)}{\pi^2 w^2} e^{U-T}. \quad (10)$$

If  $w_- = \frac{1}{2} \log(U/\pi)$ ,  $U \geq 64$  and  $w_- \geq \max\{152/b, 8 \log 4\}$ , then

$$J_0(\epsilon) \geq \frac{w_-}{172800\pi^2}. \quad (11)$$

Let  $g_j = v_j g_0 + r_j$ , where the finite family and  $b$  are fixed,  $r_j \in L^2$ , and  $r_j/\sqrt{a} \in L^2$ . For  $E_j = P_X s q_{g_j}$ ,  $v = (v_1, \dots, v_r)$  and  $D_{ij} = \pi^{-1} \int \bar{r}_i r_j / a$ ,

$$\left\| \frac{E^*(A + \epsilon)^{-1} E}{J_0(\epsilon)} - v^* v \right\| \leq 2 \|v\| \sqrt{\frac{\|D\|}{J_0(\epsilon)}} + \frac{\|D\|}{J_0(\epsilon)} \rightarrow 0. \quad (12)$$

On  $\ker v$  the full inverse Gram is bounded by  $D$ . Thus, when  $v \neq 0$ , exactly one eigenvalue diverges and all the others stay bounded.

*Proof.* For the upper,  $\|q_0\|_2^2 = b/\pi$  and  $|q_0(u)| \leq 2/(\pi|u|)$ . Put  $f = 1_{[-w, w]} q_0$  and

$$F(u) = - \int_{|u|}^w q_0(t) dt \quad (|u| \leq w), \quad F = 0 \quad (|u| > w).$$

This compact even continuous primitive satisfies  $F' = f$  distributionally; its endpoint values are zero and  $q_0 = O(u)$  at zero. Also  $|F(u)| \leq (2/\pi) \log(w/|u|)$ , so  $\|F\|_2^2 \leq 16w/\pi^2$  including both arms. Since  $1/a \leq 1/4 + 1/(16\xi^2)$ , Parseval gives

$$\int_{\mathbb{R}} \frac{|\hat{f}|^2}{a} \frac{d\xi}{2\pi} \leq \frac{1}{4} \|f\|_2^2 + \frac{1}{16} \|F\|_2^2 \leq K_0(w).$$

Fourier-dual variational bounds therefore control the actual projected compact forcing by  $K_0$ . This is an  $L^2$  forcing factor; the hard cutoff is not asserted to be an operator-domain trial.

The complete theta estimates give

$$S_0 := \|s\|_2^2 < 3, \quad S_{\text{tail}}(w) := \int_{|u|>w} s^2 du \leq 16(T+1)e^{-T}. \quad (13)$$

Indeed  $s^2 \leq 16r^2 e^{-r}$  and both arms change  $du$  to  $dr/r$ , whence  $S_0 \leq 16(\pi+1)e^{-\pi} < 3$  and the stated tail. Contractivity of  $P_X$  and  $\|(A + \epsilon)^{-1/2}\| \leq \epsilon^{-1/2}$  give the remaining inverse energy at most  $\tau_0$ . Decompose the projected source as a whole and apply Minkowski in its inverse energy norm. This proves (9), including all mixed and polar-projection terms. Choosing  $T = 2U$  proves  $J_0 = O_b(\log U)$ .

For the lower put  $\phi(t) = (t-1)^2(2-t)^2$  on  $[1, 2]$ , zero elsewhere. Its exact integrals are

$$\int \phi = 1/30, \quad \int \phi^2 = 1/630, \quad \int |\phi'|^2 = 2/105, \quad \int_1^2 \phi(t)/t dt = 4 \log 2 - 11/4 \geq 1/60.$$

Furthermore  $\int |(\phi/t)'| \leq 1/8 + 1/30 = 19/120$ , since  $\int |\phi'| = 1/8$ . Write  $w = w_-$  and on the positive arm set

$$f_1(u) = \phi(8u/w), \quad f_2(u) = \phi(8u/w - 3), \quad c = \frac{\int_0^\infty p f_1}{\int_0^\infty p f_2}, \quad f = f_1 - c f_2,$$

extending oddly. Their supports are  $[w/8, w/4]$  and  $[w/2, 5w/8]$ . Equal masses and monotonicity of  $p$  give  $0 < c \leq \sinh(w/8)/\sinh(w/4) = [2\cosh(w/8)]^{-1} \leq e^{-w/8}$ . Thus  $h = f/s$  satisfies the exact original polar moment. The bumps are compact  $H^1$  away from zero. Smooth approximation on a fixed compact, followed by a fixed exact polar correction, converges in the original form norm because  $a \leq 17\xi^2$ . Hence  $h \in V$ .

Put  $\omega = bw/8$ . Integration by parts bounds the cosine part of the first source pairing by  $19/(120\omega)$ . For  $w \geq 152/b$  this is at most  $1/120$ , whereas the nonoscillatory integral is at least  $1/60$ . Thus  $\int_0^\infty q_0 f_1 \geq 1/(120\pi)$ . The second pairing is nonnegative and at most  $(2/\pi) \int_1^2 \phi(t)/(t+3) dt \leq 1/(60\pi)$ . With  $w \geq 8 \log 4$ ,  $c \leq 1/4$ , and both arms give

$$\langle E_0, h \rangle = \int_{\mathbb{R}} q_0 f \geq 1/(120\pi).$$

The disjoint supports give  $\|f\|_2^2 \leq w/1260$  and  $\|f'\|_2^2 \leq 64/(105w)$ . On the support  $r \leq R_* = \pi^{3/8}U^{5/8}$ , so the complete first-atom lower gives

$$\epsilon \|h\|_2^2 \leq \frac{w}{2520\pi^2} e^{-U+R_*} \leq 1/w \quad (U \geq 64).$$

Here  $R_* \leq U/2$ ,  $w \leq U$ , and  $U^2 e^{-U/2} \leq 64^2 e^{-32} < 2520\pi^2$ . Therefore  $\mathfrak{a}[h] + \epsilon \|h\|^2 \leq (1088/105 + 1)/w < 12/w$ . Variational duality proves (11).

Finally write the whole source map  $E = E_0 v + E_r$ . The regular theorem gives  $E_r^*(A + \epsilon)^{-1} E_r \preceq D$ ; inverse-energy Cauchy–Schwarz bounds its mixed vector with  $E_0$  by  $\sqrt{\|D\|J_0}$ . Expanding the full matrix proves (12). On  $\ker v$  the endpoint term vanishes exactly, so minmax bounds the second largest eigenvalue by  $\|D\|$ . The top eigenvalue divided by  $J_0$  tends to  $\|v\|^2$ .  $\square$

For piecewise smooth compact profiles with right trace  $g_j(0)$  and  $g_j(\xi) - g_j(0) = O(\xi)$  near zero, take  $v_j = g_j(0)$ . The actual hypothesis remains the finite weighted-dual integral of the remainder. Interior jumps and the upper frequency endpoint introduce no further zero-frequency singularity. The theorem gives the order and rank of the singular part, rather than a sharp scalar leading constant.

## 6 Negative-observation sources after known-null removal

Known derivative columns are removed in some signed receivers. We therefore compare the actual negative-observation columns after the same fixed known-null constraints. Let  $\mathcal{O}_M = \{\xi > 0 : n_M(\xi) > 0\}$ , a nonempty bounded open set. Take independent  $g_1, \dots, g_r \in L^2(0, \infty)$  supported in  $\mathcal{O}_M$ , and define

$$F_i(u) = \int_0^\infty g_i(\xi) \sin(\xi u) \frac{d\xi}{\pi}, \quad e_i = s F_i, \quad E_{\mathcal{R}} = Q_{\mathcal{R}}[e_1, \dots, e_r]. \quad (14)$$

Actual bounded-band negative observation profiles, including interval polynomials multiplied by  $n_M$ , have this form after dependent columns are deleted. Multiplication by  $s$  and the sine transform are injective. In what follows  $M$ , the features and every row in  $\mathcal{R}$  are fixed before the regularization tends to zero.

**Theorem 6.1** (Complete positive-inverse divergence after fixed rows). *For every integer  $N \geq 1$  there are  $c_N > 0$  and  $\kappa_N > 0$  such that*

$$\lambda_{\min} \left( E_{\mathcal{R}}^* (P_{M, \mathcal{R}} + \kappa I)^{-1} E_{\mathcal{R}} \right) \geq c_N [\log \log(1/\kappa)]^N \quad (0 < \kappa < \kappa_N).$$

In particular its least eigenvalue divided by any fixed positive power of  $\log \log(1/\kappa)$  tends to infinity. If, in addition, the complete matrix

$$D_A = \left[ \int_0^\infty \frac{\overline{g_i(\xi)} g_j(\xi)}{a(\xi)} \frac{d\xi}{\pi} \right]_{i,j} \quad (15)$$

is finite, the identical source and row recipient satisfies

$$E_{\mathcal{R}}^*(A_{\mathcal{R}} + \kappa I)^{-1} E_{\mathcal{R}} \preceq D_A \quad (\kappa > 0).$$

In fact Theorem 4.1 identifies the exact limit as  $D_A$ . The finite- $D_A$  condition holds for profiles supported away from zero, and for bounded-band bounded profiles vanishing at zero to first order.

*Proof.* Choose  $\phi_j \in C_c^\infty(\mathcal{O}_M)$  close enough to  $g_j$  that the matrix  $B_{ij} = \int_0^\infty \overline{g_i} \phi_j d\xi / \pi$  is invertible. Such a choice follows from density and positive definiteness of the Gram of the independent  $g_i$ . Let  $f_j$  be the odd Schwartz function with positive-half Fourier transform  $-i\phi_j$ . Then  $\langle F_i, f_j \rangle = B_{ij}$  by Plancherel and  $\int p_M |\widehat{f_j}|^2 d\xi / (2\pi) = 0$  exactly. The formal quotient  $f_j/s$  need not be in  $L^2$  and is not used as an admissible test.

Instead, we construct compact exact-row approximants. The complete theta expansion gives, for every fixed derivative order  $k$ ,

$$\psi^{(k)}(u)/\psi(u) = (-2r)^k (1 + O_k(r^{-1})), \quad r = \pi e^{2u}, \quad u \rightarrow \infty. \quad (16)$$

Indeed the first atom is  $4\pi^{-1/4} r^{9/4} (1 - 3/(2r)) e^{-r}$ , whose logarithmic derivative is  $-2r + 9/2 + O(r^{-1})$ . Induction proves the first-atom assertion. Each fixed derivative of the entire higher-atom sum divided by this atom is  $O_k(r^{C_k} e^{-3r})$ , using the convergent positive majorants in Lemma A.1. Thus (16) is a complete-source assertion.

Divide the imposed Hilbert rows by  $s$ . The polar row becomes  $p(u) \sim e^{u/2}/2$ . Each distinct derivative row becomes

$$k_i(u) := z_i(u)/s(u) = c_i e^{\beta_i u} (1 + O_i(e^{-u})), \quad \beta_i = 2\ell_i + 1/2, \quad c_i = (-2\pi)^{\ell_i}/2 \neq 0.$$

Together with the polar exponent  $1/2$ , all exponents are distinct and positive. If constraints are linear combinations, impose the finite full list occurring in them. A variational lower on that smaller recipient is valid on the requested one.

Let  $q$  be the number of these rows. Choose an even smooth cutoff  $\chi$  supported in  $(-1, 1)$  and equal to one near zero, and set  $f_{j,R}^0(u) = \chi(u/R) f_j(u)$ . For every integer  $A \geq 1$ ,

$$\|f_{j,R}^0 - f_j\|_{H^1} \leq C_A R^{-A}.$$

Choose a nonzero nonnegative smooth bump  $\eta$  supported in a fixed small interval and fixed centers  $a_t = t\Delta$ ,  $0 \leq t < q$ , with  $\Delta > 0$ . Define odd remote bumps

$$v_{R,t}(u) = \eta(u - 2R - a_t) - \eta(-u - 2R - a_t).$$

Their  $H^1$  norms are independent of  $R$ . The exact row matrix is

$$A_{\ell t}(R) = \int_{\mathbb{R}} k_\ell(u) v_{R,t}(u) du = d_\ell e^{2\beta_\ell R} e^{\beta_\ell a_t} (1 + o(1)), \quad d_\ell = 2c_\ell \int \eta(v) e^{\beta_\ell v} dv \neq 0.$$

After row normalization its limit is a Vandermonde matrix on the distinct nodes  $e^{\beta_\ell \Delta}$ . Its inverse is bounded for large  $R$ . The target moments of  $f_{j,R}^0$  are at most  $C e^{\beta_\ell R}$ , since  $f_j$  is bounded and

each divided row has its global exponential upper. Therefore the exact solution of  $A(R)c_j(R) = -[f k_\ell f_{j,R}^0]_\ell$  obeys  $\|c_j(R)\| \leq Ce^{-R/2}$ . The corrected functions

$$f_{j,R} = f_{j,R}^0 + \sum_t c_{tj}(R)v_{R,t}, \quad h_{j,R} = f_{j,R}/s$$

are smooth compact odd exact-row tests in the original minimal core, with

$$\|f_{j,R} - f_j\|_{H^1} \leq C_A R^{-A} + Ce^{-R/2}. \quad (17)$$

No approximation to the row equations is used.

For fixed  $M$ ,  $p_M(\xi) \leq C'_M(1 + \xi^2)$ . The complete positive form is consequently continuous in the physical  $H^1$  norm. Since  $f_j$  has zero positive energy, (17) gives the whole positive trial Gram

$$P_R = [\mathfrak{p}_M(h_{i,R}, h_{j,R})] \preceq (C_N R^{-N} + Ce^{-R})I \quad (18)$$

for every integer  $N \geq 1$ . This controls all mixed entries and the entire frequency integral. The support lies in  $[-2R - A_0, 2R + A_0]$  for a fixed  $A_0$ . The first-atom bound  $s^2 \geq 2r^2 e^{-r}$  and the bounded  $L^2$  Gram of  $f_{j,R}$  give the original Hilbert Gram bound

$$H_R = [\langle h_{i,R}, h_{j,R} \rangle] \preceq C \exp(Ce^{4R})I. \quad (19)$$

This large cost includes both remote arms. Finally the actual source pairing matrix  $B_R = E_{\mathcal{R}}^*[h_{j,R}]$  equals  $[\langle F_i, f_{j,R} \rangle]$  and tends to the invertible  $B$ . Its least singular value is at least a fixed  $b > 0$  for large  $R$ .

Put  $U = \log(1/\kappa)$  and  $R = \frac{1}{8} \log U$ . Equations (18)–(19) give, for each fixed  $N$  and all sufficiently large  $U$ ,

$$P_R + \kappa H_R \preceq C_N (\log U)^{-N} I,$$

because  $\kappa H_R \preceq C \exp(-U + C\sqrt{U})I$  and  $e^{-R} = U^{-1/8}$ . Finite-column variational duality gives

$$E_{\mathcal{R}}^*(P_{M,\mathcal{R}} + \kappa I)^{-1} E_{\mathcal{R}} \succeq B_R (P_R + \kappa H_R)^{-1} B_R^*,$$

which proves the asserted floor. Applying it with one higher integer power proves the stronger limit assertion.

For the  $A$  comparison, Plancherel and weighted Fourier Cauchy–Schwarz give on the identical recipient

$$|\langle E_{\mathcal{R}} z, h \rangle|^2 \leq (z^* D_A z) \mathfrak{a}[h].$$

The positive-form variational principle proves the stated bounded inverse Gram. Near zero  $a(\xi) \geq (16/5)\xi^2$ ; away from zero it is positive, proving the examples of finite  $D_A$ .  $\square$

The constants are fixed-data constants: substituting a moving cutoff, growing source family or growing row set into them is not justified. For regular features the same  $A$  inverse remains bounded whereas the  $P_M$  inverse outgrows every fixed power of  $\log \log(1/\kappa)$ . Thus even after the known nulls have been removed, a bound for one positive receiver cannot be imported into the other at a fixed or polylogarithmic comparison cost. The theorem concerns  $P_M$ ; the original arithmetic tail remains part of any complete signed return.

## 7 Eight actual odd features at cutoff thirteen

We give an exact finite application of the regular theorem and the same-source comparison. Let  $n_{13} = (-m_{13})_+$ ,  $b = 17/16$ , and

$$P_i(\xi) = T_{2i+1}(\xi/b), \quad g_i(\xi) = 1_{[0,b]}(\xi)n_{13}(\xi)P_i(\xi), \quad 0 \leq i < 8.$$

Here  $T_j$  is the Chebyshev polynomial. Every profile vanishes linearly at zero and is regular. Independence follows from positivity of  $n_{13}$  near zero and polynomial independence; wherever  $n_{13} = 0$  the profile vanishes, so its support in the measurable sense is in the negative band. Theorem 1.1 therefore applies on each fixed known-row recipient.

**Proposition 7.1** (Complete first-band approximation and joint inverse budget). *With physical wall  $R = 2$  and ridge  $\rho = 10^{-28}$ , eight odd features admit complete negative-observation approximation error*

$$\tau \leq \frac{17}{3} \frac{12(17/16)^{32}}{(16!)^2} + \frac{17}{3} 2416 2^{-150} + 6 10^{-28} < 1.082 10^{-24}.$$

For the corresponding actual source columns on each fixed known-row recipient,

$$E^*(A\mathcal{R} + \epsilon)^{-1}E \preceq \frac{256}{3}C, \quad C_{ij} = \int_0^b \left\{ \frac{P_i P_j}{4} + \frac{(P_i/\xi)(P_j/\xi)}{16} \right\} d\xi.$$

The matrix  $C$  is exact rational and positive definite. The actual  $A$  inverse Gram converges to its full Fourier-dual matrix  $D$ , while the same  $P_{13}$  inverse Gram satisfies the all-direction divergence theorem.

*Proof.* The nine prime powers at cutoff thirteen are 2, 3, 4, 5, 7, 8, 9, 11, 13. The exact controls supplied with this paper prove  $\sum_{q \leq 13} \Lambda(q)/\sqrt{q} < 5$ : each logarithm upper is certified by a finite positive exponential sum and each square-root lower by an exact square. Since  $m_0 = \Psi(1/4) - \log \pi = -\gamma - \pi/2 - 3 \log 2 - \log \pi$  has  $|m_0| < 6$  and  $a \geq 0$ , the complete negative symbol obeys  $n_{13} \leq 16$ . Hence its restricted mass  $\beta = \pi^{-1} \int_0^b n_{13} < 17/3$ . The bound  $|m_0| < 6$  also follows from the elementary upper bounds  $\gamma < 3/5$ ,  $\pi/2 < 11/7$ ,  $\log 2 < 7/10$  and  $\log \pi < 6/5$ .

For completeness, choose the 16 roots  $x_j = \cos((j+1/2)\pi/16)$  and odd Chebyshev-interpolate  $\sin(bxu)$  in  $x$  by degree fifteen. Its odd extension has eight odd coefficients. The standard interpolation remainder and the monic root polynomial  $2^{-15}T_{16}$  give, uniformly for  $|u| \leq 2$ ,

$$|\sin(bxu) - I_{15}(x, u)| \leq 2^{-15}(b|u|)^{16}/16! \leq 2b^{16}/16!.$$

The discrete cosine identity gives squared coefficient norm

$$\sum_{i=0}^7 |c_i(u)|^2 = \frac{2}{16} \sum_{j=0}^{15} |\sin(bx_j u)|^2 \leq 2.$$

Use the frequency Hilbert space  $L^2((0, b), d\xi/\pi)$  and raw features

$$Z_i(\xi) = \sqrt{n_{13}(\xi)}P_i(\xi), \quad (Th)(\xi) = \sqrt{n_{13}(\xi)} \int_{\mathbb{R}} s(u)h(u) \sin(\xi u) du.$$

Then  $T^*Z_i = sq_{g_i}$  exactly, before the declared row projection. Cauchy–Schwarz in  $u$  and the complete frequency mass show that the squared approximation residual, before the ridge, is at most

$$\beta \left\{ S_0 \frac{4b^{32}}{(16!)^2} + S_{\text{tail}}(2) \right\}.$$

The coefficient map integrates the interpolation coefficients against  $sh$  on  $[-2, 2]$ ; its squared norm is at most  $2S_0 < 6$ . Thus ridge cost is at most  $6\rho$ . More explicitly, the variational ridge projection  $Z(Z^*Z + \rho I)^{-1}Z^*$  minimizes residual squared plus  $\rho$  times coefficient norm squared, so the displayed admissible coefficients give the stated operator error. All normalizations use the common  $1/\pi$  sine measure and hence the same  $\beta$ .

By (13),  $S_0 < 3$  and  $S_{\text{tail}}(2) \leq 16(T+1)e^{-T}$  with  $T = \pi e^4 > 150$ . The decreasing tail and  $e > 2$  give  $S_{\text{tail}}(2) < 2416 \cdot 2^{-150}$ . These estimates prove the complete  $\tau$  bound. The exact controls verify its strict rational endpoint.

Finally  $1/a \leq 1/4 + 1/(16\xi^2)$ ,  $n_{13}^2 \leq 256$  and  $1/\pi < 1/3$  give the matrix upper by applying the common-coefficient integral inequality, retaining every cross term. Each  $P_i/\xi$  is a polynomial, so all entries are rational. A nonzero polynomial combination has positive square integral; hence  $C > 0$ . The supplied exact program computes all entries and independently checks its eight rational LDL pivots. The regular and negative-band theorems give the final assertions.  $\square$

This first-band application prices the approximation and the complete archimedean source matrix. A receiving calculation for a signed form must use the actual positive matrix in the changed basis and own its positive windows once. The approximation error  $\tau$  is an input to that calculation.

## 8 Derivative sources and quantitative admission

The derivative-source comparison retains the original polar-only space. Its source-count dependence is explicit. Put

$$Z_d = (z_0, \dots, z_d), \quad \Gamma_{\epsilon, d} = Z_d^*(A + \epsilon I)^{-1}Z_d, \quad U = \log(1/\epsilon).$$

**Theorem 8.1** (Uniform source-resolved resolvent geometry). *For every integer  $d \geq 0$  put*

$$n = d + 2, \quad L = 2n - 1, \quad m = 4n + 3, \quad b_i = \frac{(-2)^{2i+3}}{2\pi^{1/4}}, \quad \tilde{D}_U = \text{diag}(b_i U^{2i+13/4})_{i=0}^d,$$

*and define the explicit constants*

$$\begin{aligned} h_n &= \frac{2^{-26}}{(m+1)^{275^{2m}}}, & g_n &= \frac{2^{-67}h_n^4}{n^2(m+1)^2}, \\ \bar{K}_n &= 1024(2L+5)^L(L+2)!, & C_n &= 512n\bar{K}_n^2. \end{aligned}$$

*For all real  $U \geq 64n^2$  and  $\epsilon = e^{-U}$ ,*

$$g_n I \preceq \tilde{D}_U^{-1} \Gamma_{\epsilon, d} \tilde{D}_U^{-1} \preceq C_n I. \quad (20)$$

*Thus  $d$  may vary with  $U$  on the explicit schedule  $d+2 \leq \sqrt{U/64}$ . For each fixed  $d$ , writing*

$$D_U = \text{diag}(U^{2i+13/4})_{i=0}^d, \quad (21)$$

*there are positive constants  $c_d, C'_d$  such that  $c_d I \preceq D_U^{-1} \Gamma_{\epsilon, d} D_U^{-1} \preceq C'_d I$  on the stated range. In particular, for every fixed  $i \geq 0$ ,*

$$\lim_{\epsilon \downarrow 0} \frac{\log \langle z_i, C_\epsilon^{-1} z_i \rangle}{\log \log(1/\epsilon)} = 4i + \frac{13}{2}. \quad (22)$$

The quadratic theorem and its complete proof are retained from the source predecessor [1] to specify the normalization and the stronger lower floor on this range. Appendix C sharpens the broad-degree admission while retaining the same complete Gram constants. The next theorem completes the fixed-family leading matrix.

## 9 The fixed-family leading matrix

The constant-factor bounds can be sharpened to an actual matrix limit at each fixed source size. Its entries first have a half-line variational description; Section 10 identifies them explicitly and certifies their scalar factors. Put  $q_j = j + 1/4$  and define the full-line homogeneous energy

$$\mathcal{E}(g) = \frac{1}{2} \iint_{\mathbb{R}^2} k(x-y) |g(x) - g(y)|^2 dx dy, \quad k(t) = \frac{e^{-|t|/2}}{1 - e^{-2|t|}}. \quad (23)$$

For  $g \in L^2$ , this is  $\int a(\xi) |\hat{g}|^2 d\xi / (2\pi)$ . Indeed the digamma series is  $a(\xi) = \sum_j q_j^{-1} (1 - 4q_j^2 / (4q_j^2 + \xi^2))$ , and  $q_j e^{-2q_j|t|}$  has the indicated rational Fourier transform. First apply this identity to finite positive sums, then use monotone convergence. The kernel has finite second moment and behaves as  $1/(2|t|)$  at zero.

Let  $\mathcal{H}_-$  be the energy completion of  $C_c^\infty((-\infty, 0))$ , extended by zero to  $\mathbb{R}$ . For such supported functions the interaction with  $y > 0$  gives

$$\mathcal{E}(g) \geq 2 \int_{x < 0} e^{x/2} |g(x)|^2 dx. \quad (24)$$

Thus this completion has an injective weighted  $L^2$  realization. For  $\beta > 1/4$ , define  $L_\beta(g) = \int_{-\infty}^0 e^{\beta x} g(x) dx$ . Weighted Cauchy–Schwarz gives

$$|L_\beta(g)|^2 \leq \frac{\mathcal{E}(g)}{2(2\beta - 1/2)}.$$

Set  $\mathcal{H}_-^0 = \ker L_{1/2}$ ,  $\beta_i = 2\ell_i + 1/2$ , and let  $v_i \in \mathcal{H}_-^0$  be the energy Riesz representative  $L_{\beta_i}(g) = \langle v_i, g \rangle_{\mathcal{E}}$ .

**Theorem 9.1** (Leading normalized Gram). *For every fixed finite  $d$ ,*

$$\tilde{D}_U^{-1} Z_d^* (A + e^{-U} I)^{-1} Z_d \tilde{D}_U^{-1} \longrightarrow G_\infty := [2\langle v_i, v_j \rangle_{\mathcal{E}}]_{i,j=0}^d \quad (25)$$

*in matrix operator norm as  $U \rightarrow \infty$ . The matrix  $G_\infty$  is finite and positive definite. No convergence rate or uniformity in growing  $d$  is asserted.*

*Proof.* Write  $R = \frac{1}{2} \log(U/\pi)$  and  $\delta = e^{-R}$ . For an original admissible  $h$  let  $f = sh$ , and set  $g_U(x) = f(R+x)$  for  $x \geq -R$ , zero otherwise. Expansion of the two disjoint odd arms, first for each finite digamma sum, gives the exact identity

$$\mathcal{E}(f) = 2\mathcal{E}(g_U) + 2 \sum_{j \geq 0} e^{-4q_j R} \left| \int e^{-2q_j x} g_U(x) dx \right|^2. \quad (26)$$

For a finite sum the identity term has zero cross contribution; the convolution terms have negative multiplier sign and opposite arm signs, giving the positive squares. Monotone convergence proves the full identity and finiteness of the one-arm energy; no bounded half-line multiplier is assumed. All moments exist by theta confinement and the original  $h \in L^2$  norm. The exact polar condition is

$$L_+(g_U) = e^{-R} L_-(g_U), \quad L_\pm(g) = \int e^{\pm x/2} g(x) dx,$$

where here the moments extend over the whole support  $[-R, \infty)$ . The  $j = 0$  square in (26) consequently yields the global penalty  $2e^R |L_+(g_U)|^2$ .

We first prove the lower limit by compact recovery. For  $g \in C_c^\infty(-\infty, 0)$  with  $L_{1/2}(g) = 0$ , choose fixed  $\phi \in C_c^\infty((-\infty, 0))$  with  $L_{1/2}(\phi) \neq 0$ , and put

$$g_U = g + t_U \phi, \quad t_U = \frac{\delta L_-(g)}{L_+(\phi) - \delta L_-(\phi)} = O(\delta).$$

Then  $L_+(g_U) = \delta L_-(g_U)$  exactly. Form  $f_U$  from its translated odd arms and  $h_U = f_U/s$ . These are original compact polar-core tests. The complete fixed-compact theta limits give

$$\langle z_i/(b_i U^{\ell_i+1/4}), h_U \rangle \rightarrow 2L_{\beta_i}(g).$$

The cross-arm term in the jump energy is  $O(e^{-R})$  on the fixed compact supports, so  $\mathcal{E}(f_U) \rightarrow 2\mathcal{E}(g)$ . If those supports lie in  $x \leq -\eta_0 < 0$ , the complete first atom gives

$$e^{-U}/s(R+x)^2 \leq CU^{-2}e^{-U(1-e^{-2\eta_0})} \rightarrow 0.$$

Hence the complete trial energy tends to  $2\mathcal{E}(g)$ . Compact zero-moment functions are dense in  $\mathcal{H}_-^0$ : approximate in energy and correct the small polar moment with fixed  $\phi$ , using continuity of  $L_{1/2}$ . Approximate the finitely many  $v_i$  simultaneously by such tests. The completed-square variational identity gives, for their recovered trial map  $T_U$  and normalized source map  $E_U$ ,

$$E_U^*(A + e^{-U})^{-1}E_U \succeq E_U^*T_U + T_U^*E_U - T_U^*(A + e^{-U})T_U.$$

Its limit is  $G_\infty - 2(V - G)^*(V - G)$  in energy notation. Thus for every  $\eta > 0$  the left side is eventually at least  $G_\infty - \eta I$ . Positive definiteness follows because a vanishing combination of  $L_{\beta_i}$  on  $\ker L_{1/2}$  would make the corresponding combination of distinct exponentials a multiple of  $e^{x/2}$  on every compact negative interval. Distributional uniqueness of distinct exponentials forces all coefficients to vanish.

For the matching upper, consider any sequence of original admissible trials with total energy

$$Q_U(h) = \mathfrak{a}[h] + e^{-U}\|h\|_2^2 \leq 1.$$

With  $V_U(x) = e^{-U}/s(R+x)^2$  on  $x \geq -R$ ,

$$\mathcal{E}(g_U) \leq \frac{1}{2}, \quad \int V_U |g_U|^2 \leq \frac{1}{2}, \quad |L_+(g_U)|^2 \leq e^{-R}/2. \quad (27)$$

The complete upper envelope for  $s$  implies the actual wall bound

$$V_U(x) \geq \frac{\pi^2}{169} U^{-2} e^{-4x} e^{U(e^{2x}-1)}. \quad (28)$$

Thus its infimum on each fixed  $[\eta, B] \subset (0, \infty)$  tends to infinity; in particular  $\int_1^2 |g_U|^2 \rightarrow 0$ . The  $g_0$  kernel is  $e^{-|x-y|/2} \leq k(x-y)$ . Restricting the jump integral to  $x \leq 1$ ,  $y \in [1, 2]$ , and using  $|v-w|^2 \geq |v|^2/2 - |w|^2$ , gives

$$\mathcal{E}(g_U) \geq \frac{c}{2} \int_{x \leq 1} e^{x/2} |g_U|^2 dx - 2e^{1/2} \int_1^2 e^{-y/2} |g_U(y)|^2 dy, \quad c = \int_1^2 e^{-y/2} dy > 0.$$

Consequently

$$\int_{x \leq 1} e^{x/2} |g_U|^2 dx \leq C \quad (29)$$

uniformly. This weighted anchor pays the receding center near  $-R$ ; it does not assert unweighted  $L^2$  coercivity. Extract a weakly convergent subsequence in this weighted space. The wall bound forces strong convergence to zero on every fixed positive interval away from zero. Thus the weak limit  $g$  vanishes for  $x > 0$ , and convergence is weak in  $L^2$  on every bounded interval.

We must pass all source and polar moments, not only local integrals. For  $\beta > 1/4$ , (29) gives uniformly

$$\int_{x < -L} e^{\beta x} |g_U| dx \leq \sqrt{C} \left\{ \frac{e^{-(2\beta-1/2)L}}{2\beta-1/2} \right\}^{1/2}. \quad (30)$$

The actual divided normalized source profiles  $w_{i,U}(x) = (z_i/s)(R+x)/(b_i U^{\ell_i+1/4})$  obey  $|w_{i,U}| \leq C_i e^{\beta_i x}$  everywhere on  $x \geq -R$ , by the complete source envelope; they converge uniformly on fixed compacts to  $e^{\beta_i x}$ . Thus (30) pays their entire left tails, as well as the polar tail ( $\beta = 1/2$ ). On  $[0, \eta]$ , the same anchor gives  $\int_0^\eta |g_U| \leq \sqrt{C\eta}$ ; every source profile and  $e^{x/2}$  is uniformly bounded there. This pays the wall boundary layer without a trace assumption.

For the unbounded right exterior  $x \geq \eta > 0$  set  $T = Ue^{2\eta}$ . The complete source tail in (48) gives, with  $\alpha_i = \ell_i + 1/4$ ,

$$\int_\eta^\infty \frac{|w_{i,U}|^2}{V_U} dx = \frac{e^U}{b_i^2 U^{2\alpha_i}} \int_{u \geq R+\eta} |z_i(u)|^2 du \leq C'_i U^{-2\alpha_i} T^{2\alpha_i+1} e^{U-T} \rightarrow 0. \quad (31)$$

Using the two-arm bound for the one-arm integral is conservative. For the polar exterior the full  $s$  upper envelope gives independently

$$\int_\eta^\infty \frac{e^x}{V_U(x)} dx \leq \frac{169}{\pi^2 \sqrt{U}} T^{3/2} e^{U-T} \rightarrow 0. \quad (32)$$

To verify its factor, the exact integral is  $e^{U-R} \int_{u > R+\eta} e^u s(u)^2 du$ ; substitute  $r = \pi e^{2u}$  in the complete envelope and integrate  $r^{3/2} e^{-r}$  using its logarithmic slope at most  $-1/2$  for  $T \geq 3$ . Cauchy-Schwarz against the potential energy in (27) pays both whole exterior pairings. Combining weak local convergence, compact uniform source convergence, (30) and the boundary and right-tail bounds proves

$$\int \overline{w_{i,U}} g_U dx \rightarrow L_{\beta_i}(g), \quad L_+(g_U) \rightarrow L_{1/2}(g) = 0.$$

Here one first takes the subsequence limit, then  $L \rightarrow \infty$  and  $\eta \downarrow 0$ . The total polar moment has therefore been paid.

Restrict both jump variables to  $[-N, N]$  and replace  $k$  by  $\min(k, N)$ . The resulting positive quadratic form is a bounded linear difference-map norm squared on  $L^2([-N, N])$ , hence is weakly lower semicontinuous. Increasing  $N$  and monotone convergence give

$$\mathcal{E}(g) \leq \liminf \mathcal{E}(g_U), \quad 2\mathcal{E}(g) \leq \liminf Q_U(h). \quad (33)$$

It remains essential to show that this finite-energy, negative-supported limit is in the same homogeneous completion  $\mathcal{H}_-$ .

For completeness, this density holds for every measurable finite-valued negative-supported  $g$  of finite jump energy. Clip its complex amplitude by radial projection  $P_K$  onto the disk of radius  $K$ . Both  $P_K$  and  $I - P_K$  are 1-Lipschitz on  $\mathbb{C}$  as a real Hilbert space (the latter follows from firm nonexpansivity of orthogonal convex projection). Hence the full jump integrand of  $g - P_K g$  is dominated by that of  $g$  and tends pointwise to zero. Dominated convergence gives energy convergence of bounded amplitude clippings. For a bounded negative-supported function  $v$ , choose

smooth  $0 \leq \phi \leq 1$  equal to zero on  $(-\infty, -2]$  and one on  $[-1, \infty)$ , and put  $\phi_L(x) = \phi(x/L)$ . The finite second moment of  $k$  and the fundamental theorem of calculus give

$$\mathcal{E}(\phi_L) \leq \frac{1}{2} \left( \int t^2 k(t) dt \right) \|\phi'_L\|_2^2 = O(L^{-1}).$$

This difference-integral estimate does not require  $\phi_L \in L^2$ . Expanding a product difference for  $r_L = (1 - \phi_L)v$  yields

$$\mathcal{E}(r_L) \leq \iint k(x-y) |1 - \phi_L(x)|^2 |v(x) - v(y)|^2 dx dy + 2\|v\|_\infty^2 \mathcal{E}(\phi_L) \rightarrow 0.$$

The first term tends to zero by dominated convergence. Thus the bounded compactly supported  $\phi_L v$  approximate  $v$  in energy and are in  $L^2$ . Translate each slightly left, then mollify with compact kernels of radius smaller than its gap from zero. Fourier dominated convergence in the finite  $a$ -weighted norm pays both operations, producing  $C_c^\infty(-\infty, 0)$  approximants. This proves the homogeneous density assertion even for non- $L^2$  plateaux. The weighted embedding (24) identifies the energy limit with the actual function  $g$ . The passed polar moment places it in  $\mathcal{H}_-^0$ .

Finally fix a common complex source coefficient vector  $a$ . Choose  $U$  along a sequence realizing the limsup of its normalized inverse energy, and, when nonzero, take the original constrained trial

$$h_U = \frac{(A + e^{-U})^{-1} E_U a}{\sqrt{a^* E_U^* (A + e^{-U})^{-1} E_U a}}.$$

It has  $Q_U(h_U) = 1$  and source pairing equal to that square root. The established fixed-family upper ensures boundedness. The preceding compactness and moment arguments apply to a subsequence; its full odd source pairing tends to  $2\langle v_a, g \rangle_{\mathcal{E}}$ , where  $v_a = \sum_i a_i v_i$  and  $2\mathcal{E}(g) \leq 1$ . Riesz Cauchy-Schwarz gives

$$\limsup a^* E_U^* (A + e^{-U})^{-1} E_U a \leq 4\|v_a\|_{\mathcal{E}}^2 \mathcal{E}(g) \leq 2\|v_a\|_{\mathcal{E}}^2 = a^* G_\infty a.$$

Zero limsup cases are immediate. The lower limit proved earlier matches this bound for every fixed  $a$ . Polarization on the finitely many  $e_i, e_i + e_j, e_i + ie_j$  gives entrywise and therefore operator-norm matrix convergence. No growing-dimensional net or interchange of source-count and regularization limits has been used.  $\square$

## 10 An explicit outer-Hardy and Cauchy formula

The leading matrix in Theorem 9.1 admits a scalar-integral formula, an exact inverse, and a sharper condition budget. Throughout this section let  $r = d + 1$  denote the actual number of derivative columns,  $\beta_i = 4i + 13/2$ , and  $\beta_p = 1/2$ . This  $r$  differs from the appendix shell count  $n = d + 2$ .

**Theorem 10.1** (Exact same-domain factorization and leading matrix). *For  $\Re z > 0$  define the positive-real normalized outer factor*

$$O(z) = \exp \left\{ \frac{1}{\pi} \int_0^\infty \frac{z \log a(t)}{z^2 + t^2} dt \right\}. \quad (34)$$

*The map, initially defined for the negative compact smooth core by*

$$(Jg)(z) = O(z) \int_{-\infty}^0 e^{zx} g(x) dx,$$

extends to a unitary map from the same energy completion  $\mathcal{H}_-$  onto  $H^2(\Re z > 0)$ , with boundary norm  $\int |F(i\xi)|^2 d\xi/(2\pi)$ . For  $\beta > 1/4$ ,

$$L_\beta(g) = (Jg)(\beta)/O(\beta).$$

Consequently, putting  $B(z) = (z - 1/2)/(z + 1/2)$  and  $D = \text{diag}(B(\beta_i)/O(\beta_i))$ ,

$$G_\infty = 2DCD, \quad C_{ij} = \frac{1}{\beta_i + \beta_j} = \frac{1}{4(i + j + 13/4)}. \quad (35)$$

In particular every entry of the actual fixed-family leading matrix is an explicit convergent scalar integral times a rational Cauchy factor.

*Proof.* The complete digamma bounds give  $a(t) = ct^2 + O(t^4)$  near zero for  $c > 0$  and  $a(t) = \log |t| + O(1)$  at infinity. Thus  $\int_{\mathbb{R}} |\log a(t)|/(1+t^2) dt < \infty$ . The regularized analytic outer integral is

$$\log O(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ \frac{1}{z - it} - \frac{it}{1 + t^2} \right\} \log a(t) dt.$$

Its subtraction cancels the leading imaginary term at infinity; even pairing gives (34). Its real part is half the Poisson extension of  $\log a$ . Therefore  $|O(i\xi)|^2 = a(\xi)$  almost everywhere and  $O(\beta) > 0$  for positive real  $\beta$ . For every  $\sigma > 0$ ,  $a(\xi) \leq C_\sigma(1 + \xi^2)^\sigma$ , and Poisson integration gives

$$|O(z)| \leq \sqrt{C_\sigma} |1 + z|^\sigma.$$

This follows from the actual outer integral, which contains no additional singular exponential factor at infinity. We use the classical outer construction and conformal Hardy correspondence [2, 3]; the range argument is supplied here.

For a compact smooth negative-supported  $g$  with support distance  $\eta > 0$  from zero, repeated integration by parts gives  $|F_g(z)| \leq C_N e^{-\eta \Re z} (1 + |z|)^{-N}$  for its Laplace transform. The preceding bound makes  $OF_g$  an actual  $H^2$  function. Its boundary norm is

$$\|Jg\|_{H^2}^2 = \int_{\mathbb{R}} a(\xi) |F_g(i\xi)|^2 \frac{d\xi}{2\pi} = \mathcal{E}(g),$$

since  $F_g(i\xi) = \widehat{g}(-\xi)$  and  $a$  is even. Hence  $J$  extends isometrically from the original core completion with closed range  $\mathcal{K}$ . The anchor (24) makes the actual weighted representative unique and makes  $L_\beta$  continuous for  $\beta > 1/4$ . Hardy evaluation is continuous too, so the displayed evaluation identity passes to this same completion. No maximal weighted boundary space has been substituted.

To prove onto, let  $g_*(x) = e^x$  for  $x < 0$ , zero otherwise. It has finite complete energy  $\int a(\xi)/(1 + \xi^2) d\xi/(2\pi)$ . The clipping, slow-cutoff and mollification density argument in the proof of Theorem 9.1 puts it in the same  $\mathcal{H}_-$ . Evaluation continuity and the analytic identity theorem give the genuine range vector

$$H(z) = Jg_*(z) = O(z)/(z + 1) \in \mathcal{K}.$$

Left translation  $S_t g(x) = g(x + t)$ ,  $t \geq 0$ , preserves the negative core and is an energy isometry. Fourier dominated convergence on the core and then density give its strongly continuous extension, and  $JS_t = e^{-tz}J$ . Thus  $\mathcal{K}$  is invariant under all these Hardy semigroup multipliers. For  $F \in \mathcal{K}$ , the norm-convergent Bochner integral

$$\int_0^\infty e^{-t} e^{-tz} F dt = F/(z + 1)$$

remains in  $\mathcal{K}$ . Hence  $\mathcal{K}$  is invariant under  $w(z) = (z - 1)/(z + 1)$ .

The unitary disk-to-half-plane transform in this normalization is

$$(Uf)(z) = \frac{\sqrt{2}}{z+1} f\left(\frac{z-1}{z+1}\right).$$

The boundary Jacobian is  $|d\theta| = 2d\xi/(1+\xi^2)$ , verifying the factor  $\sqrt{2}$ . The vector  $H$  corresponds to  $A_0(w) = O((1+w)/(1-w))/\sqrt{2}$ . Its boundary modulus squared is  $a(\xi)/2$ , integrable in disk measure. Conformal substitution in the actual Poisson integral shows that its log modulus is the Poisson extension of the integrable boundary logarithm, with no singular mass at  $w = 1$ . Thus  $A_0$  is a disk  $H^2$  outer function, not merely zero-free.

Here is the required cyclicity argument. Let  $A_N$  be outer, positive at zero, with boundary modulus  $\min(1/|A_0|, N)$ . It is bounded by  $N$ . The product  $A_0 A_N$  has modulus at most one and positive value at zero tending to one by dominated convergence of its logarithm. Hence

$$\|A_0 A_N - 1\|_2^2 \leq 2 - 2(A_0 A_N)(0) \rightarrow 0.$$

For each fixed  $N$ , radial dilation  $A_N(\rho w)$  has uniformly convergent Taylor truncations on the closed disk. As  $\rho \uparrow 1$ , boundedness by  $N$  and dominated convergence against  $|A_0|^2$  put  $A_0 A_N$  in the closure of polynomial multiples of  $A_0$ . That closure contains one, is shift invariant, and contains every polynomial, so it is all disk  $H^2$ . Since  $\mathcal{K}$  contains  $H$  and is Cayley invariant, its disk image contains this dense set. Its range is closed, hence  $\mathcal{K} = H^2$ .

For the stated half-plane norm the evaluation kernel is  $k_\beta(z) = 1/(z + \beta)$ , with Gram  $1/(\beta + \gamma)$ . The Laplace model  $e^{-\beta t}$  independently checks its squared norm  $1/(2\beta)$ . Dividing by the outer values gives the energy evaluation kernel  $1/[(\beta + \gamma)O(\beta)O(\gamma)]$ . The exact polar kernel  $L_{1/2} = 0$  maps onto the Hardy subspace vanishing at  $1/2$ , namely  $BH^2$ . Its kernel is the exact short

$$\frac{1}{\beta + \gamma} - \frac{2\beta_p}{(\beta + \beta_p)(\gamma + \beta_p)} = \frac{B(\beta)B(\gamma)}{\beta + \gamma}.$$

Returning through  $J$  and multiplying by the odd-arm factor two of Theorem 9.1 proves (35).  $\square$

**Corollary 10.2** (Literal inverse, determinant and condition budget). *Put  $d_i = B(\beta_i)/O(\beta_i)$  and  $\alpha_i = \prod_k(\beta_i + \beta_k)/\prod_{k \neq i}(\beta_i - \beta_k)$ , retaining its alternating sign. Then*

$$(G_\infty^{-1})_{ij} = \frac{\alpha_i \alpha_j}{2d_i d_j (\beta_i + \beta_j)}, \quad (36)$$

$$\det G_\infty = 2^r \prod_i d_i^2 \frac{\prod_{i < j} (\beta_j - \beta_i)^2}{\prod_{i,j} (\beta_i + \beta_j)}. \quad (37)$$

For every  $r \geq 1$ ,

$$\lambda_{\min}(G_\infty) \geq \frac{36}{140777r^2(r+2)^2 2^{8(r+2)}}, \quad \lambda_{\max}(G_\infty) \leq r/26, \quad (38)$$

$$\text{cond } G_\infty \leq \frac{140777}{936} r^3 (r+2)^2 2^{8(r+2)}. \quad (39)$$

Thus  $\log \text{cond } G_\infty = O(d+1)$  and  $\log \|G_\infty^{-1}\| = O(d+1)$ .

*Proof.* For  $R(z) = \prod_k (z - \beta_k)/\prod_k (z + \beta_k)$ , residues give  $R(z) = 1 - \sum_j \alpha_j/(z + \beta_j)$ . Evaluation at  $\beta_i$ , subtraction at two nodes, and differentiation at a node give respectively the Cauchy inverse

identities, including  $R'(\beta_i) = 1/\alpha_i$ . Hence  $(C^{-1})_{ij} = \alpha_i \alpha_j / (\beta_i + \beta_j)$ ; congruence gives (36). The determinant is the Cauchy alternant, obtained by its simple poles and squared zero factors.

Poisson integration of  $4t^2/(t^2 + 1/4) \leq a(t) \leq 17t^2$  gives

$$\frac{2\beta}{\beta + 1/2} \leq O(\beta) \leq \sqrt{17}\beta, \quad \frac{12}{91\sqrt{17}r} \leq d_i \leq \frac{1}{2}.$$

Here  $B(\beta_i) \geq 6/7$  and  $\beta_{r-1} = 4r + 5/2 \leq 13r/2$ . In the outer convention  $\pi^{-1} \int_0^\infty \beta \log(t^2 + c^2)/(\beta^2 + t^2) dt = \log(\beta + c)$ , which checks the half-Poisson normalization.

Let  $H_\alpha = [1/(i + j + \alpha)]_{0 \leq i, j < r}$ . Its monomial weight  $x^{9/4}$  gives  $H_{13/4} \succeq H_5$ , while  $H_5$  is the principal submatrix at indices  $2, \dots, r + 1$  of the ordinary Hilbert matrix of size  $m = r + 2$ . The literal inverse of that ordinary matrix is

$$(-1)^{i+j}(i+j+1) \binom{m+i}{m-j-1} \binom{m+j}{m-i-1} \binom{i+j}{i}^2,$$

a specialization of the checked Cauchy formula. Each binomial is at most two to its upper index, so every inverse entry is at most  $2m2^{8m}$  in absolute value, and the inverse norm is at most  $2m^22^{8m}$ . Now  $G_\infty = \frac{1}{2}DH_{13/4}D$  and  $91^2 \cdot 17 = 140777$  give the lower bound. The trace upper  $\text{Tr } H_{13/4} \leq 4r/13$  gives the upper bound, and division gives the condition estimate.  $\square$

These are estimates for the leading matrix. The convergence theorem fixes  $d$  before  $U \rightarrow \infty$ , so it transfers them only beyond a  $d$ -dependent threshold; no effective growing- $d$  finite- $U$  estimate is inferred.

## 10.1 Certified scalar factors and complete eight-column matrices

The scalar integrals can be enclosed over all real frequencies. Set

$$\mathfrak{r}(t) = \frac{a(t)(t^2 + 1/4)}{4t^2} = 1 + \sum_{j \geq 1} \frac{t^2 + 1/4}{4q_j(4q_j^2 + t^2)}, \quad q_j = j + 1/4.$$

Each summand increases in  $t^2$ , so  $\mathfrak{r} \geq 1$  is increasing. Its removable zero limit is  $1 + \sum_{j \geq 1} (4j + 1)^{-3}$ . A partial sum through  $J$  is a lower and its tail is at most  $1/[8(4J + 1)^2]$ . Write

$$O(\beta) = \frac{2\beta}{\beta + 1/2} e^{I_\beta}, \quad I_\beta = \frac{1}{\pi} \int_0^\infty \frac{\beta \log \mathfrak{r}(t)}{\beta^2 + t^2} dt.$$

On every cell  $[u, v]$ , multiply the exact weight  $\arctan(v/\beta) - \arctan(u/\beta)$  by  $\log \mathfrak{r}(u)$  and  $\log \mathfrak{r}(v)$  for lower and upper payments. The zero cell uses the lower partial sum and the complete right-endpoint value. This encloses the removable endpoint without subtracting singular special-function values there.

For the unbounded exterior, the decreasing digamma summands give  $a(t) \leq 4 + \frac{1}{2} \log(1 + 4t^2) \leq 4 + \log(3t)$  for  $t \geq 1$ : retain the zeroth term below four and bound the others by their integral from  $q = 1/4$  to infinity. For  $T \geq 1$  put  $A_T = 4 + \log(3T)$ . Since  $(t^2 + 1/4)/(4t^2) \leq 5/16$  and the logarithmic slope of  $\log(4 + \log(3t))$  is at most  $1/(tA_T)$ ,

$$0 \leq \int_T^\infty \frac{\beta \log \mathfrak{r}(t)}{\beta^2 + t^2} dt \leq \frac{\beta}{T} \left\{ \log \left( \frac{5}{16} A_T \right) + \frac{1}{A_T} \right\}.$$

The two integrals of  $t^{-2}$  and  $t^{-2} \log(t/T)$  are both  $1/T$ . Thus the exterior is paid, rather than replaced by a last finite mesh cell.

The supplied program uses 384-bit Arb/Acb,  $J = 256$ , and exact transcendental mesh points  $t_k = \beta 2^{k/256}$ ,  $-4096 \leq k \leq 4096$ . Complete digamma values enclose  $a(t)$  at each point. All arithmetic, cell weights and accumulated endpoints retain their rounding. The following outward rounded balls enclose the eight factors; their decimal centers are not exact values.

| $i$ | $O(4i + 13/2)$       | $i$ | $O(4i + 13/2)$       |
|-----|----------------------|-----|----------------------|
| 0   | $2.179 \pm 0.000316$ | 4   | $2.507 \pm 0.000675$ |
| 1   | $2.317 \pm 0.000518$ | 5   | $2.545 \pm 0.000594$ |
| 2   | $2.402 \pm 0.000657$ | 6   | $2.576 \pm 0.000423$ |
| 3   | $2.461 \pm 0.000651$ | 7   | $2.603 \pm 0.000671$ |

The reader data contain all 64 leading entries and all 64 inverse entries, obtained through the exact identities above. The code checks exact rational complete Cauchy inverse identities through dimension twelve and rejects the wrong odd-arm normalization. Numerical acceptance consists of 131764 explicit controls with byte-identical normal and optimized mathematical reports. These are certified limit matrices, not finite-regularization inverse evaluations.

## 10.2 Exact fixed-row constraint prices

**Corollary 10.3** (Fixed canonical rows and a prefix price). *Let  $I, J$  be disjoint fixed finite derivative-index sets. On  $X_J = X \cap \text{span}\{z_j : j \in J\}^\perp$  let  $Q_J$  be the original Hilbert projection and  $A_J$  the same minimal restricted form operator. Put  $E_J = Q_J[z_i : i \in I]$  and  $S_I(U) = \text{diag}(b_i U^{2i+13/4})_{i \in I}$ . Then*

$$S_I(U)^{-1} E_J^* (A_J + e^{-U})^{-1} E_J S_I(U)^{-1} \longrightarrow 2 \text{diag}(\delta_i) \left[ \frac{1}{\beta_i + \beta_l} \right]_{i,l \in I} \text{diag}(\delta_l),$$

$$\delta_i = \frac{B(\beta_i)}{O(\beta_i)} \prod_{j \in J} \frac{\beta_i - \beta_j}{\beta_i + \beta_j}.$$

The limit is positive definite; the nonzero real  $\delta_i$  may be signed. A source also in  $J$  is exactly zero after projection and is deleted before asserting positive definiteness. For source  $m$  and removed prefix  $J = \{0, \dots, m-1\}$ , define its scalar limit  $L_m$ . Then

$$P_m = \frac{m! \Gamma(m + 13/4)}{\Gamma(2m + 13/4)}, \quad L_m = \frac{B(\beta_m)^2 P_m^2}{O(\beta_m)^2 \beta_m}, \quad \lim_{m \rightarrow \infty} \frac{-\log L_m}{m} = \log 16.$$

The last limit is iterated: first  $U \rightarrow \infty$  at each fixed  $m$ .

*Proof.* For  $R_\epsilon = (A + \epsilon)^{-1}$ ,  $Z_J = [z_j]_{j \in J}$  and  $e = Z_I c$ , the original-domain vector

$$u = R_\epsilon e - R_\epsilon Z_J (Z_J^* R_\epsilon Z_J)^{-1} Z_J^* R_\epsilon e$$

has exact rows zero and solves the restricted coercive form equation against every  $v \in V \cap X_J$ . The row Gram is positive definite. Uniqueness identifies  $u = (A_J + \epsilon)^{-1} Q_J e$ , giving the exact full mixed Schur complement. The compact-core identification follows by fixed finite core corrections as in Lemma 3.1. Signed diagonals  $S_J(U)$  cancel exactly before taking the limit. Apply Theorem 9.1 on one fixed prefix containing  $I \cup J$ ; continuity of inversion at its positive definite limiting  $J$  block gives the limiting Schur complement of  $2DCD$ .

For a single removed node  $t > 0$  the Cauchy identity is

$$\frac{1}{x+y} - \frac{2t}{(x+t)(y+t)} = \frac{(x-t)(y-t)}{(x+t)(y+t)(x+y)}.$$

Successive elimination proves the stated finite Blaschke product, with all signs retained. Overlap yields the exact projected zero column. For the prefix, its product equals  $P_m = \prod_{j < m} (m-j)/(m+j+13/4)$ . Stirling's formula gives  $P_m \sim 2^{-9/4} \sqrt{\pi m} 4^{-m}$ : split off the inverse central binomial coefficient and the fixed-shift Gamma ratios. The outer bounds in Corollary 10.2 leave only polynomial factors in  $L_m/P_m^2$ , so  $-\log L_m = m \log 16 + O(\log m)$ . This takes no simultaneous growing-row limit.  $\square$

## 11 Derivative-source instability for the positive arithmetic part

The bounded difference of two forms does not preserve their singular source-resolvent scale. The following result uses precisely the derivative columns retained in Theorem 8.1.

**Theorem 11.1** (Fixed-family arithmetic-receiver instability). *Fix  $M \geq 2$  and  $d \geq 0$ . On the original polar-only space  $X$ ,*

$$\lambda_{\min} \left( \tilde{D}_U^{-1} Z_d^* (P_M + e^{-U} I)^{-1} Z_d \tilde{D}_U^{-1} \right) \longrightarrow \infty.$$

*The limits hold with  $M, d$  fixed. There is no claimed rate or uniformity in changing either parameter. In particular no constant upper comparison with the bounded normalized A Gram can hold for this same source family and recipient.*

*Proof.* We first construct half-line test functions using the classical Laplace–Hardy correspondence and an outer factor [3, 4]. For  $g$  supported in  $(-\infty, 0]$ , write

$$\mathcal{E}_P(g) = \int_{\mathbb{R}} p_M(\xi) |\hat{g}(\xi)|^2 \frac{d\xi}{2\pi}, \quad L_\beta(g) = \int_{-\infty}^0 e^{\beta x} g(x) dx.$$

Take the gap  $[-b, b]$  from Section 3, any  $a_0 > 0$ , and  $\beta_p = 1/2$ . In  $\Re z > 0$  put

$$h_b(z) = \frac{1}{\pi} \int_{-b}^b \frac{d\xi}{z - i\xi}, \quad F_t(z) = \frac{z - \beta_p}{(z + \beta_p)(z + a_0)} e^{th_b(z)} \quad (t \geq 0).$$

The real part of  $h_b$  is harmonic measure of the gap, between zero and one. Its boundary value is one on the gap and zero outside, almost everywhere. The rational factor is the product of a Blaschke factor and an  $H^2$  function. Thus  $F_t \in H^2$  for each finite  $t$ . The half-line Laplace isometry supplies  $g_t \in L^2(-\infty, 0)$  with  $L_\beta(g_t) = F_t(\beta)$  and  $\hat{g}_t(\xi) = F_t(-i\xi)$ . Its boundary modulus is

$$|\hat{g}_t(\xi)|^2 = \begin{cases} e^{2t}/(a_0^2 + \xi^2), & |\xi| < b, \\ 1/(a_0^2 + \xi^2), & |\xi| > b. \end{cases}$$

Consequently  $g_t \in H_{\log}$  and

$$\mathcal{E}_P(g_t) = c_0 := \int_{\mathbb{R}} \frac{p_M(\xi)}{a_0^2 + \xi^2} \frac{d\xi}{2\pi}, \quad 0 < c_0 < \infty, \quad (40)$$

independently of  $t$ , while  $L_{1/2}(g_t) = 0$  exactly. For  $\beta_i = 2\ell_i + 1/2$ , the source evaluations are

$$L_{\beta_i}(g_t) = c_i e^{th_i}, \quad c_i = \frac{\beta_i - \beta_p}{(\beta_i + \beta_p)(\beta_i + a_0)} > 0, \quad h_i = \frac{2}{\pi} \arctan(b/\beta_i) > 0.$$

The  $h_i$  are distinct. Let  $r = d + 1$  and use the  $r$  tests  $g_t, \dots, g_{t+r-1}$ . Their evaluation matrix is

$$V_t = \text{diag}(c_i e^{th_i}) [e^{jh_i}]_{0 \leq i, j < r}.$$

The last factor is an invertible Vandermonde matrix, so  $\sigma_{\min}(V_t) \geq c_* e^{th_*}$  for fixed positive constants  $c_*, h_*$ . The complete energy Gram is at most  $rc_0 I$  by (40) and Cauchy–Schwarz. This joint statement retains all cross terms; separate diagonal amplification would not prove the theorem.

Fix a finite  $t$ . Approximate this tuple in  $H_{\log}$  and  $L^2$  by smooth compact functions supported strictly in  $(-\infty, 0)$ . To see the support detail, first translate a small distance to the left, then cut off far to the left and mollify within the strict gap from zero. Translation is strongly continuous; the cutoff multiplier argument in Lemma 3.1 proves convergence in  $H_{\log}$ . The functionals  $L_\beta$  are  $L^2$  continuous, with norm  $(2\beta)^{-1/2}$ . Subtract the small polar error times one fixed compact test with  $L_{1/2} = 1$ . Thus we obtain compact tests  $k_j \in C_c^\infty(-\infty, 0)$ , each with  $L_{1/2}(k_j) = 0$ , whose evaluation matrix  $V$  and energy Gram  $Q$  obey

$$\sigma_{\min}(V) \geq \frac{1}{2}\sigma_{\min}(V_t), \quad Q \preceq 2rc_0 I. \quad (41)$$

The  $H_{\log}$  convergence pays the whole  $p_M$  energy by (8).

We return these tests to the actual theta domain. Put  $R = \frac{1}{2}\log(U/\pi)$  and  $\delta = e^{-R}$ , and define  $L_-(k) = \int e^{-x/2} k(x) dx$ . Choose one compact negative-half-line  $\phi$  with  $L_{1/2}(\phi) \neq 0$ . Replace  $k_j$  by

$$k_{j,U} = k_j + t_{j,U}\phi, \quad t_{j,U} = \frac{\delta L_-(k_j)}{L_{1/2}(\phi) - \delta L_-(\phi)} = O(\delta).$$

Then  $L_{1/2}(k_{j,U}) = \delta L_-(k_{j,U})$  exactly. Define  $f_{j,U}(R+x) = k_{j,U}(x)$  on the positive arm and extend oddly. For large  $U$  the supports stay away from zero, and  $h_{j,U} = f_{j,U}/s$  is in the original smooth compact polar core. Indeed the displayed moment identity is exactly its polar condition. The complete fixed-compact profile limits of Lemma A.1 give the actual normalized pairing matrix

$$W_U = (Z_d \tilde{D}_U^{-1})^* [h_{j,U}] \longrightarrow 2V. \quad (42)$$

The factor two retains both odd arms.

The complete positive energy Gram tends to  $2Q$ . Here are the opposite-arm payments. The digamma integral gives

$$a(\xi) = \int_0^\infty (1 - \cos(\xi t)) \frac{2e^{-t/2}}{1 - e^{-2t}} dt.$$

For two fixed compact envelopes separated by  $2R$ , its cross kernel is  $O(e^{-R})$ , since the displayed density is bounded by  $Ce^{-t/2}$  for  $t \geq 1$ . This proves decay of the entire  $a$  cross term. Write  $p_M = a + c_M - 2\sum_{q \leq M} \Lambda(q)q^{-1/2} \cos(\xi \log q) + n_M$ . The scalar term has disjoint supports, and each of the finitely many shifts cannot connect the receding arms for large  $R$ . Finally  $n_M \in L^1$  is compactly supported, so its convolution kernel tends to zero at infinity by Riemann–Lebesgue; dominated convergence on the fixed compact supports pays its full mixed term. The small exact-polar corrections tend to zero in  $H_{\log}$ . Hence the claimed joint energy limit holds without assuming a nonnegative physical jump kernel for  $p_M$ .

Each fixed compact support lies in  $x \leq -\eta_0 < 0$ . The complete first theta atom gives there

$$\frac{e^{-U}}{s(R+x)^2} \leq CU^{-2}e^{-U(1-e^{-2\eta_0})} \longrightarrow 0.$$

Thus the full trial Gram for  $P_M + e^{-U}I$  tends to  $2Q$ . For large  $U$  it is at most  $5rc_0 I$ , and  $\sigma_{\min}(W_U) \geq \sigma_{\min}(V)$ . Finite-column variational duality gives the normalized inverse Gram at least

$$\frac{W_U W_U^*}{5rc_0} \succeq \frac{\sigma_{\min}(V_t)^2}{20rc_0} I.$$

Given any desired lower constant, first choose finite  $t$  making this quantity large, then choose the compact approximation, and only then take  $U$  sufficiently large. All limits hold for every subsequent  $U$ , proving the theorem. This order supplies no estimate uniform in  $t, M, d$ , and therefore no divergence rate.  $\square$

The homogeneous Hardy amplification is classical; the source-specific content is its joint return to the complete theta sources, exact polar constraint and original two-arm minimal domain. This theorem retains the derivative columns. A recipient that projects out those columns would make their source Gram zero and is outside its statement.

## 12 Reproducible data and proof dependencies

The portable code uses exact rational arithmetic for the cutoff-thirteen prime-power prices, all eight polynomial source rows, the complete  $8 \times 8$  upper matrix, the approximation budget and its LDL positivity controls. Separate controls check the endpoint bump integrals and source-count onset arithmetic. Execute the programs normally and with Python optimization; explicit exceptions retain every control under optimization. The scripts emit JSON only to caller-selected paths, and their README identifies the analytical inputs which remain proofs rather than finite tests.

The source-independent theta and observation estimates, the quadratic derivative geometry and its slower growing-family corollary belong to the source and observation predecessors [1, 7]. They are reproduced where the current proof uses their constants. The new source classification combines exact regular equality after fixed known-row repair, endpoint order and rank-one shape, and all-direction complete-positive divergence for actual negative-band columns. The fixed-family derivative matrix is now an actual limit rather than its earlier lower-limit candidate. Its exact original-domain outer-Hardy representation gives a Cauchy formula, literal inverse and determinant, and an  $O(d + 1)$  logarithmic condition budget. Directed all-frequency integration certifies the eight-column limit matrix and its inverse. The literal  $F(n)$  admission is a refinement of the broad polynomial argument, with the quadratic lower retained on overlap. No finite calculation substitutes for any of these limiting or domain arguments.

## A Complete source geometry

The following source geometry concerns prescribed derivative rows rather than arbitrary vectors in the kernel. Their complete theta expansion makes both the upper estimate and the matching logarithmic power accessible. Write  $p(u) = \sinh(u/2)$  and  $r = \pi e^{2u}$  on  $u \geq 0$ .

**Lemma A.1** (Derivative atoms and scaled profiles). *Define polynomials by*

$$P_0(x) = 4x^2 - 6x, \quad P_{n+1}(x) = \left(\frac{1}{2} - 2x\right)P_n(x) + 2xP'_n(x). \quad (43)$$

*Then*

$$\psi^{(n)}(u) = e^{u/2} \sum_{m \geq 1} P_n(rm^2) e^{-rm^2}, \quad \deg P_n = n + 2,$$

*and the leading coefficient of  $P_n$  is  $4(-2)^n$ . For  $\ell_i = 2i + 3$  put*

$$R_i = P_{2i+3} - \frac{1}{4}P_{2i+1}, \quad k_i = z_i/s, \quad D_i = \deg R_i = \ell_i + 2.$$

*The exact divided source is*

$$k_i(u) = \cosh(u/2) \frac{\sum_{m \geq 1} R_i(rm^2) e^{-rm^2}}{\sum_{m \geq 1} P_0(rm^2) e^{-rm^2}}. \quad (44)$$

For each fixed  $i$ , uniformly in  $C^1$  on every compact subinterval of  $(0, \infty)$ ,

$$U^{-\ell_i-1/4} k_i(\tfrac{1}{2} \log(Ux/\pi)) \longrightarrow b_i x^{\ell_i+1/4}, \quad b_i = \frac{(-2)^{\ell_i}}{2\pi^{1/4}}, \quad (45)$$

$$U^{-1/4} p(\tfrac{1}{2} \log(Ux/\pi)) \longrightarrow b_p x^{1/4}, \quad b_p = \frac{1}{2\pi^{1/4}}. \quad (46)$$

*Proof.* Differentiating one atom uses  $r' = 2r$  and gives (43). The series and any fixed number of its derivatives converge locally uniformly, so the differentiation is valid. The leading term in the recurrence multiplies the previous leading coefficient by  $-2$ . Substituting  $s^2 = \psi/\cosh(u/2)$  gives (44).

For  $r \geq \pi$  all denominator atoms are positive and  $P_0(r) \geq 2r^2$ . If  $R$  has fixed degree  $D$  and coefficient absolute sum  $B$ , then

$$\sum_{m \geq 2} |R(rm^2)| e^{-rm^2} \leq Br^D e^{-4r} \sum_{m \geq 2} m^{2D} e^{-\pi(m^2-4)}.$$

The last sum is finite. One  $r$  derivative has a bound of the same form with a larger fixed polynomial power. Hence, relative to the positive first denominator atom, the remaining atoms and one fixed derivative are bounded by  $O_i(r^{C_i} e^{-3r})$ . Polynomial division gives

$$\frac{R_i(r)}{P_0(r)} = (-2)^{\ell_i} r^{\ell_i} \{1 + O_i(r^{-1})\}.$$

The expansion and its derivative can be used uniformly when  $r = Ux$  and  $x$  stays in the stated compact interval. Finally

$$\cosh(u/2) = \tfrac{1}{2} \pi^{-1/4} r^{1/4} (1 + \sqrt{\pi/r}), \quad \sinh(u/2) = \tfrac{1}{2} \pi^{-1/4} r^{1/4} (1 - \sqrt{\pi/r}),$$

which prove both  $C^1$  limits.  $\square$

In particular  $k_i$  grows as a nonzero power of  $r$  on each arm, and  $k_i \notin L^2(\mathbb{R})$ . The whole-line factorization  $z_i = M_s k_i$  therefore cannot directly use an  $L^2$  observation-factor bound. The half-line cancellation and complete exterior return in Section B resolve this source-specific obstruction.

**Lemma A.2** (Two complete source budgets). *For each  $i$  there is an effectively specified  $K_i > 0$  such that*

$$\left| \sum_{m \geq 1} R_i(rm^2) e^{-rm^2} \right| \leq K_i r^{D_i} e^{-r} \quad (r \geq \pi).$$

For  $T \geq \pi$ , both physical arms together satisfy

$$\int_{r \leq T} |k_i(u)|^2 du \leq \frac{K_i^2 T^{2D_i-7/2}}{4\sqrt{\pi}(2D_i-7/2)}. \quad (47)$$

For  $T \geq 4D_i$ ,

$$\int_{r > T} |z_i(u)|^2 du \leq \frac{K_i^2}{2\sqrt{\pi}} \int_T^\infty r^{2D_i-5/2} e^{-r} dr \leq \frac{K_i^2}{\sqrt{\pi}} T^{2D_i-5/2} e^{-T}. \quad (48)$$

*Proof.* If  $B_i$  is the coefficient absolute sum of  $R_i$ , one possible choice is

$$K_i = B_i \sum_{m \geq 1} m^{2D_i} e^{-\pi(m^2-1)}.$$

This convergent scalar sum admits finite enclosures: after sufficiently many terms its successive ratios are bounded by a geometric ratio strictly below one. Since  $r \geq \pi > 1$ , it bounds every lower polynomial power as well as the leading one.

The positive first denominator atom in (44) and  $\cosh(u/2) \leq e^{u/2}$  give

$$|k_i(u)| \leq \frac{1}{2} K_i e^{u/2} r^{D_i-2}.$$

Here  $e^u = \sqrt{r/\pi}$ , and integration on both arms changes  $du$  to  $dr/r$ . Extending the lower integration endpoint from  $\pi$  to zero proves (47). Alternatively, the numerator of  $z_i$  is bounded by  $K_i e^{u/2} r^{D_i} e^{-r}$  and the source lower bound is  $s^2 \geq 2r^2 e^{-r}$ . Consequently

$$|z_i(u)|^2 \leq \frac{K_i^2}{2\sqrt{\pi}} r^{2D_i-3/2} e^{-r},$$

which gives the first inequality in (48). For  $\nu = 2D_i - 5/2$  and  $r \geq T \geq 4D_i$ ,  $\frac{d}{dr} \log(r^\nu e^{-r}) = \nu/r - 1 \leq -1/2$ . Integrating this exponential majorant bounds the remaining integral by  $2T^\nu e^{-T}$ .  $\square$

## B Uniform two-sided source geometry

We prove Theorem 8.1 with explicit dependence on the source count. Throughout this section  $n = d + 2$ ,  $L = 2n - 1$ ,  $m = 4n + 3$  and  $U \geq 64n^2$ . The normalization includes the signed leading coefficients:

$$\tilde{D}_U = \text{diag}(b_i U^{\ell_i+1/4})_{i=0}^d, \quad E = Z_d \tilde{D}_U^{-1}.$$

### B.1 Half-line cancellation and the complete upper

Choose a common wall  $T \geq \max\{4(L+2), \pi e^2\}$ , put  $w = \frac{1}{2} \log(T/\pi)$  and  $\chi = 1_{[-w, w]}$ . Write

$$\alpha_i = \ell_i + \frac{1}{4}, \quad \beta_i = 2\alpha_i, \quad A_i = \frac{K_i}{2^{\ell_i}} (T/U)^{\alpha_i}, \quad H_w = (1 - e^{-w/2})^{-2}.$$

Here  $K_i$  is any complete source coefficient from Lemma A.2. For  $k_i = E_i/s$ , that lemma gives

$$|k_i(u)| \leq A_i e^{-\beta_i(w-u)} \quad (0 \leq u \leq w). \quad (49)$$

Set  $I_i(u) = \int_0^u k_i(v) dv$ ,  $I_p(u) = 2(\cosh(u/2) - 1)$  and

$$c_i = I_i(w)/I_p(w), \quad f_i = \chi(k_i - c_i p), \quad F_i(u) = \int_{-\infty}^u f_i(v) dv.$$

Oddness makes the integral of  $f_i$  on each half-line vanish exactly. Thus  $F_i$  vanishes outside  $[-w, w]$  and at zero; there is no central primitive plateau. For  $0 \leq u \leq w$ , elementary hyperbolic identities give

$$\frac{I_p(u)}{I_p(w)} \leq H_w e^{-(w-u)/2}, \quad \frac{p(u)}{I_p(w)} \leq \frac{1}{2} H_w e^{-(w-u)/2}.$$

Also  $|I_i(u)| \leq (A_i/\beta_i)e^{-\beta_i(w-u)}$ . Integrating these bounds over both arms, and using the squared triangle inequality, yields

$$\begin{aligned}\|f_i\|_2^2 &\leq 2A_i^2/\beta_i + H_w^2 A_i^2/\beta_i^2, \\ \|F_i\|_2^2 &\leq 2A_i^2(1 + H_w)^2/\beta_i^2.\end{aligned}$$

Consequently

$$\frac{1}{4}\|f_i\|_2^2 + \frac{1}{16}\|F_i\|_2^2 \leq \mathcal{K}_i := A_i^2 \left\{ \frac{1}{2\beta_i} + \frac{3H_w^2 + 2H_w + 1}{8\beta_i^2} \right\}. \quad (50)$$

The zeroth positive digamma summand in Lemma 2.1 gives  $1/a(\xi) \leq 1/4 + 1/(16\xi^2)$  for  $\xi \neq 0$ . For a compact  $L^2$  function  $f$  with  $L^2$  primitive  $F$ , Parseval and  $\hat{f} = i\xi\hat{F}$  therefore imply

$$\int_{\mathbb{R}} \frac{|\hat{f}|^2}{a} \frac{d\xi}{2\pi} \leq \frac{1}{4}\|f\|_2^2 + \frac{1}{16}\|F\|_2^2.$$

Weighted Fourier Cauchy–Schwarz proves, for every  $h \in V$ ,

$$|\langle P_X M_s f_i, h \rangle|^2 \leq \mathcal{K}_i \mathbf{a}[h]. \quad (51)$$

This extends from the chosen compact core by its form closure; the hard cutoff is an  $L^2$  forcing factor, never an operator-domain trial.

The exact exterior return is

$$E_i = P_X M_s f_i + P_X V_i, \quad V_i = (1 - \chi)(E_i - c_i sp). \quad (52)$$

Indeed  $P_X E_i = E_i$  and  $P_X(sp) = 0$ . In particular the exterior polar term cannot be dropped. The complete atom envelope also gives  $s^2 p^2 \leq \psi \cosh(u/2) \leq 8r^{5/2}e^{-r}/\sqrt{\pi}$ . Integrating both arms and using the logarithmic slope of  $r^{3/2}e^{-r}$  gives

$$\|1_{r>T} sp\|_2^2 \leq 16T^{3/2}e^{-T}/\sqrt{\pi} \quad (T \geq 3).$$

For completeness,  $\sum_{a \geq 1} a^4 e^{-r(a^2-1)} < 2$  for  $r \geq \pi$ , so the envelope follows directly by bounding  $P_0(ra^2) \leq 4r^2 a^4$  and  $e^{u/2} \cosh(u/2) \leq e^u = \sqrt{r/\pi}$ . Since  $|c_i| \leq H_w A_i e^{-w/2}/\beta_i$ , the source tail in (48) and the polar tail give

$$\epsilon^{-1}\|V_i\|_2^2 \leq \mathcal{T}_i := A_i^2(8 + 32H_w^2/\beta_i^2)Te^{U-T}. \quad (53)$$

Before applying the squared triangle inequality, the two payments are

$$4A_i^2 Te^{U-T} \quad \text{and} \quad 16H_w^2 A_i^2 Te^{U-T}/\beta_i^2.$$

Thus their mixed term has also been paid.

For one common complex coefficient vector  $a$ , combine the columns  $f_i, F_i, V_i$  before estimating. Cauchy–Schwarz in the column index, (51), variational duality and Minkowski in the inverse energy norm give the joint inequality

$$E^* C_\epsilon^{-1} E \preceq \left( \sqrt{\sum_i \mathcal{K}_i} + \sqrt{\sum_i \mathcal{T}_i} \right)^2 I. \quad (54)$$

Every mixed source entry is included; there is no diagonal substitution. In a single coordinate, cancellation of the normalization gives the useful literal finite bound

$$\langle z_i, C_\epsilon^{-1} z_i \rangle \leq \frac{K_i^2 T^{2\alpha_i}}{4\sqrt{\pi}} \left[ \sqrt{\kappa_i(H_w)} + \sqrt{(8 + 32H_w^2/\beta_i^2) T e^{U-T}} \right]^2, \quad (55)$$

where  $\kappa_i(H) = 1/(2\beta_i) + (3H^2 + 2H + 1)/(8\beta_i^2)$ . The wall assumptions are the same as above.

We make the order dependence explicit. If  $V_j$  is the coefficient  $\ell^1$  norm of  $P_j$ , the recurrence gives

$$V_0 = 10, \quad V_{j+1} \leq (2j + 13/2)V_j, \quad V_j \leq 10(2j + 5)^j.$$

Hence  $\|\text{coeff } R_i\|_1 \leq 13(2\ell_i + 5)^{\ell_i}$ . Writing  $D = \ell_i + 2$ , the inequality  $x^D \leq D!e^x$  gives

$$\sum_{a \geq 1} a^{2D} e^{-\pi(a^2-1)} \leq e^\pi D! (2/\pi)^D \sum_{a \geq 1} e^{-\pi a^2/2} < 64D!.$$

The last sum is less than 1 by  $a^2 \geq a$  and  $\pi > 3$ . Thus all  $K_i$  may be bounded by

$$\bar{K}_n = 1024(2L + 5)^L (L + 2)!. \quad (56)$$

Take  $T = 2U$ . All wall assumptions hold,  $H_w \leq 7$  and  $A_i^2 = \sqrt{2}K_i^2$ . Since  $\beta_i \geq 13/2$ ,

$$\frac{1}{2\beta_i} + \frac{3 \cdot 49 + 14 + 1}{8\beta_i^2} < 1, \quad 16 + 64 \cdot 49/\beta_i^2 < 128.$$

Using  $Ue^{-U} \leq 1$  and fewer than  $n$  source columns gives  $\sum_i \mathcal{K}_i \leq 2n\bar{K}_n^2$  and  $\sum_i \mathcal{T}_i \leq 256n\bar{K}_n^2$ . Equation (54) proves the upper bound of Theorem 8.1 with  $C_n = 512n\bar{K}_n^2$ .

## B.2 Exact polynomial geometry on the outward shell

Use the even cutoff

$$\chi_0(x) = (2x - 1)^4(1 - x)^4 \quad (1/2 \leq x \leq 1), \quad \chi_0(x) = 0 \quad \text{otherwise.} \quad (57)$$

It is  $C^1$ , with zero endpoint derivatives and  $|\chi'_0| \leq 16$ . On the positive shell put  $r = Ut^2$ ,  $1/\sqrt{2} \leq t \leq 1$  and  $\delta = \sqrt{\pi/U}$ . Define

$$A_k(r) = \frac{P_k(r)}{4(-2)^k}, \quad a_k(t) = \frac{A_k(Ut^2)}{U^{k+2}}, \quad C_k = k^2/2 + 7k/4 + 3/2.$$

The exact recurrence becomes

$$a_0 = t^4 - 3t^2/(2U), \quad a_{k+1} = (t^2 - 1/(4U))a_k - (t/(2U))a'_k, \quad \|\text{coeff } a_k\|_1 \leq e^{C_k/U}. \quad (58)$$

The norm bound follows by multiplying  $1 + 3/(2U)$  by the step factors  $1 + (k + 9/4)/U$ . Put

$$\begin{aligned} Q_p &= 4a_0, & Q_i &= 4(a_{\ell_i} - a_{\ell_i-2}/(16U^2)), \\ T_p &= 4H_0, & T_i &= 4(H_{\ell_i} - H_{\ell_i-2}/(16U^2)), \\ H_k(t) &= \sum_{a \geq 2} a_k(at) e^{-Ut^2(a^2-1)}. \end{aligned}$$

The denominator  $D = Q_p + T_p$  is positive. With  $F = t^{-1/2}/D$  set

$$p_p = (t - \delta)Q_p, \quad p_i = (t + \delta)Q_i, \quad e_p = (t - \delta)T_p, \quad e_i = (t + \delta)T_i.$$

The actual normalized divided profiles, including every atom, are exactly

$$v_p = \frac{p(u)}{b_p U^{1/4}} = F(p_p + e_p), \quad v_i = \frac{z_i(u)/s(u)}{b_i U^{\ell_i+1/4}} = F(p_i + e_i). \quad (59)$$

The common factor is retained in every subsequent covariance.

We first prove a coefficient floor for the polynomial rows. Writing  $A_k(r) = \sum_j (-1)^j a_{k,j} r^{k+2-j}$ , the recurrence gives

$$a_{k+1,j} = a_{k,j} + (k + 13/4 - j)a_{k,j-1}, \quad 0 \leq a_{k,j} \leq C_k^j/j!.$$

Terms outside the polynomial degree are zero. The bound follows coefficientwise from  $(1 + 3z/2) \prod_{q=0}^{k-1} (1 + (q+9/4)z)$ , whose node sum is  $C_k$ . Select the odd coefficients of  $p_p/4, p_0/4, \dots, p_d/4$  at degrees  $5, 11, 15, \dots, 4d + 11$ . The resulting square matrix  $B$  has diagonal one. The terms containing  $\delta$  have even degree and disappear exactly. Let  $x = C_L/U$  and  $\delta_0 = 1/(16U^2)$ . A source-to-source off-diagonal coefficient loses an even number  $j \geq 2$  of powers of  $r$ ; a source-to-polar coefficient loses an odd  $j \geq 3$ . Its absolute value is at most

$$x^j/j! + \delta_0 x^{j-2}/(j-2)!.$$

Summing the even and odd losses bounds both row and column sums of  $B - I$  by  $e^x - 1 - x + \delta_0 e^x$ . Since  $C_L \leq 3n^2$ , even the weaker condition  $U \geq 4n^2$  gives  $x \leq 3/4$  and  $U \geq 16$ . The elementary series bound  $e^{3/4} < 17/8$  shows that this sum is at most  $12305/32768 < 1/2$ . Thus  $\sigma_{\min}(B) \geq 1/2$ , and, for every  $\alpha \in \mathbb{C}^n$ ,

$$\left\| \text{coeff} \left( \alpha_p p_p + \sum_i \alpha_i p_i \right) \right\|_2 \geq 2 \|\alpha\|_2. \quad (60)$$

This is an exact finite polynomial estimate, not proximity to limiting monomials.

### B.3 All-atom errors and the complete shorted covariance

For  $k \leq L$ ,  $e^{C_k/U} \leq 2$ . Differentiating the absolutely convergent positive majorant for  $H_k$  gives

$$\begin{aligned} |H_k| + |H'_k| &\leq 2 \sum_{a \geq 2} a^{2k+4} (1 + 2k + 4 + 2Ua^2) e^{-U(a^2-1)/2} \\ &\leq 6U \sum_{a \geq 2} a^{2L+6} e^{-U(a^2-1)/2} \leq 12U e^{-21U/16}. \end{aligned}$$

Here  $2L + 5 \leq U$ ,  $\log a \leq (a^2 - 1)/4$ , and  $U \geq 16(n + 1)$  imply that each last summand is at most  $e^{-7U(a^2-1)/16}$ . Consecutive  $a^2 - 1$  differ by at least 5, so the geometric factor is at most 2. It follows that

$$\max_j (|T_j| + |T'_j|) \leq 52U e^{-21U/16} \leq e^{-U}. \quad (61)$$

The final scalar inequality holds for  $U \geq 64$  by logarithms and monotonicity; the same summable majorants justify termwise differentiation. Since  $Q_p = t^4(4 - 6/(Ut^2))$  and  $T_p \geq 0$ , these bounds imply

$$3/4 \leq D \leq 5, \quad 1/5 \leq F \leq 3, \quad |F'| \leq 70. \quad (62)$$

For the derivative bound use  $|Q'_p| \leq 17$ ,  $|T'_p| \leq 1$  and  $t^{-1/2}, t^{-3/2} < 2$ .

Give the shell the common positive measure

$$d\nu = 2\chi_0(t^2)F(t)^2 dt/t, \quad \nu(\mathbb{R}) < 9.$$

On  $I = [4/5, 9/10]$ , its density is at least  $2^{-24}$ . For any polynomial  $P$  of degree at most  $m$ ,

$$\int_I |P|^2 dt \geq \frac{\|\text{coeff } P\|_2^2}{10(m+1)^2 75^{2m}}. \quad (63)$$

Indeed the coefficient  $\ell^1$  norm of the shifted Legendre polynomial  $L_k(20t - 17)$  is at most  $75^k$ . The first norms are 1, 37; the three-term recurrence bounds the next by 74 times the preceding norm plus its predecessor. Orthonormalization on the interval of length  $1/10$  and Cauchy–Schwarz then give the coefficient cost in (63), also for complex coefficients. Together with (60), this proves that the full polynomial covariance in  $\nu$  is at least

$$h_n I_n, \quad h_n = \frac{2^{-26}}{(m+1)^2 75^{2m}}.$$

All polynomial rows have coefficient  $\ell^1$  norm at most 12; (61) gives  $|e_j| \leq 2e^{-U}$ . Expanding every product, the norm of the full covariance perturbation is bounded by

$$n\nu(\mathbb{R})(48e^{-U} + 4e^{-2U}) \leq 512ne^{-U} \leq h_n/2.$$

For the last inequality,  $\log(1024n/h_n) < 80n$  follows from  $m = 4n + 3$ ,  $\log 75 < 9/2$  and  $\log n \leq n$ , while  $U \geq 64n^2 > 80n$  for  $n \geq 2$ . Thus the actual joint covariance of  $v_p, v_0, \dots, v_d$ , with measure  $\chi_0 du$  on both arms, is at least  $h_n I_n/2$ . Minimizing over the actual polar coordinate leaves a shorted covariance at least  $h_n I_{n-1}/2$ .

## B.4 Original-domain tests and the lower bound

For  $a \in \mathbb{C}^{d+1}$  put  $k_a = \sum_i a_i v_i$  and

$$c_a = \frac{\int_{\mathbb{R}} \chi_0(r/U) v_p k_a du}{\int_{\mathbb{R}} \chi_0(r/U) v_p^2 du}, \quad f_a = \chi_0(r/U)(k_a - c_a v_p), \quad h_a = f_a/s.$$

Profiles and  $f_a$  extend oddly, with the cutoff even. The exact polar constraint holds because  $v_p$  is a constant multiple of  $p$ . The shell is compact and avoids zero, and  $s$  is positive smooth on it. Approximate  $f_a$  in  $H^1$  by odd smooth compact functions on a common larger compact set; subtract a multiple of one fixed compact odd function with nonzero polar moment. The multiples tend to zero. After division by  $s$ , convergence holds in the original Hilbert norm, and  $\mathfrak{a}[h] \leq 17\|(sh)'\|_2^2$  gives convergence in its form norm. Hence  $h_a \in V$ , the original minimal domain. The exact variance identity and shorted floor give

$$\langle Ea, h_a \rangle = \int_{\mathbb{R}} \chi_0(r/U) |k_a - c_a v_p|^2 du \geq (h_n/2) \|a\|_2^2. \quad (64)$$

The earlier bounds imply  $|v_j| \leq 42$ ,  $|v'_j| \leq 256(m+1)$ , where the prime here denotes  $t$  differentiation. The polar denominator is at least  $h_n/2$  and the cross vector has norm at most  $2048\sqrt{n}$ , so  $|c_a| \leq 4096\sqrt{n}h_n^{-1}\|a\|_2$ . Since  $dt/du = t$  and  $|d\chi_0(r/U)/du| \leq 32$ , product differentiation gives

$$|f_a|, |f'_a| \leq 2^{27}(m+1)\sqrt{n}h_n^{-1}\|a\|_2.$$

The two-arm shell has measure  $\log 2 < 1$ ; consequently

$$\|f_a\|_2^2 + \|f'_a\|_2^2 \leq E_n \|a\|_2^2, \quad E_n = 2^{60} n^2 (m+1)^2 h_n^{-2}. \quad (65)$$

On  $U/2 \leq r \leq U$ , the full first atom gives  $\epsilon/s^2 \leq e^{r-U}/(2r^2) \leq 2U^{-2}$ . The original constrained energy is therefore at most  $19E_n \|a\|_2^2$ . Variational duality with (64) yields

$$a^* E^* C_\epsilon^{-1} E a \geq \frac{h_n^2}{76E_n} \|a\|_2^2 \geq g_n \|a\|_2^2, \quad g_n = \frac{2^{-67} h_n^4}{n^2 (m+1)^2}.$$

This completes the lower bound and Theorem 8.1.

For each fixed  $d$ , conjugating by the fixed real diagonal  $\text{diag}(b_i)$  gives positive constant lower and upper bounds for  $D_U^{-1} \Gamma_{\epsilon,d} D_U^{-1}$ . In a fixed coordinate this is the constant-factor scale  $U^{4i+13/2}$ ; logarithms give (22). Moreover  $\log(g_n^{-1}) = O(d)$  and  $\log C_n = O(d \log(d+2))$ , so the logarithm of the normalized condition number has the latter order. These quantitative bounds alone do not identify a limiting Gram or the work required to evaluate its entries. The separate fixed-family limit theorem in Section 9 identifies the leading matrix variationally.

**Corollary B.1** (Slowly growing families). *Fix  $\eta > 0$  and let  $n = d + 2 = \lfloor \eta \log U / \log \log U \rfloor$  for sufficiently large  $U$ . Then*

$$U^{-o(1)} I \preceq \tilde{D}_U^{-1} \Gamma_{\epsilon,d} \tilde{D}_U^{-1} \preceq U^{8\eta+o(1)} I, \quad \text{cond}(\tilde{D}_U^{-1} \Gamma_{\epsilon,d} \tilde{D}_U^{-1}) \leq U^{8\eta+o(1)}.$$

*If instead  $2 \leq n = o(\log U / \log \log U)$ , the normalized Gram, its inverse and its condition number all have upper norm bounds  $U^{o(1)}$ .*

*Proof.* The explicit constants give

$$\log(g_n^{-1}) = 32(\log 75)n + O(\log n), \quad \log C_n = 8n \log n + O(n).$$

The first identity follows by substituting  $m = 4n + 3$  in  $h_n$ ; the second follows from the factorial in (56), using  $\log(k!) = k \log k - k + O(\log(k+1))$ . Both schedules eventually satisfy  $U \geq 64n^2$ . For the first,  $n \log n / \log U \rightarrow \eta$  and  $n / \log U \rightarrow 0$ ; divide the logarithms of the bounds in Theorem 8.1 by  $\log U$ . For the second schedule both logarithmic costs are  $o(\log U)$ . The norm of the inverse is at most  $g_n^{-1}$  and the condition number at most  $C_n/g_n$ , proving the claims.  $\square$

The corollary prices this specific source geometry. It does not replace a complete signed-form comparison or certify an inverse evaluation algorithm.

## C A broader source-count regime

The quadratic onset of Theorem 8.1 has a useful small lower cost, but it is not the largest source family available. A diagonal similarity of the exact coefficient map gives a second regime. Keep  $n, L, m, \bar{K}_n, C_n$  as in that theorem and put

$$\tau_n = \frac{1}{8(n+1)}, \quad B_n = 8 \cdot 4^n, \quad \hat{h}_n = h_n \tau_n^{4n-2}, \quad \hat{g}_n = \frac{2^{-47} \hat{h}_n^4}{n^2 (m+1)^2 B_n^6}. \quad (66)$$

**Theorem C.1** (Broad-degree source geometry). *For every integer  $n = d + 2 \geq 2$  and every real*

$$U \geq F(n) := \max\{64n, 68 + 54n + (4n+3) \log(n+1)\},$$

$$\hat{g}_n I \preceq \tilde{D}_U^{-1} \Gamma_{\epsilon,d} \tilde{D}_U^{-1} \preceq C_n I.$$

In particular  $U \geq 32n \log(n+1)$  suffices for  $n \geq 9$ , and  $U \geq 1024n \log(n+1)$  suffices for all  $n \geq 2$ . Both logarithmic costs are  $O(n \log(n+1))$ . The smaller floor  $\hat{g}_n$  supplements the sharper quadratic floor  $g_n$ ; neither assertion replaces the other.

*Proof.* Use the exact polynomials and selected coefficient matrix  $B$  from Section B. The selected degrees are  $5, 11, 15, \dots, 4d+11$ . Put  $q_j = 2, 5, 7, \dots, 2n+1$  and  $S = \text{diag}(\tau_n^{q_j})$ . The matrix  $B$  is upper triangular with diagonal one. For an off-diagonal entry in row  $i$ , column  $j$ , the descending degree loss  $q = q_j - q_i$  is at least 2; in a fixed row or column the losses are distinct. The coefficient bounds already proved give, with  $x = C_L/U$  and  $\delta_0 = 1/(16U^2)$ ,

$$|B_{ij}| \leq x^q/q! + \delta_0 x^{q-2}/(q-2)!.$$

Both row and column sums of  $S^{-1}(B-I)S$  are therefore at most

$$e^{\tau_n x} - 1 - \tau_n x + \delta_0 \tau_n^2 e^{\tau_n x}.$$

Each admitted wall implies  $U \geq 64n$ ,  $x \leq n$  and  $\tau_n x < 1/8$ . Since  $e^{1/8} < 8/7$ , the displayed bound is at most  $1/56 + 1/896 = 17/896 < 1/2$ . The row/column norm estimate gives  $\sigma_{\min}(S^{-1}BS) > 1/2$ , hence

$$\sigma_{\min}(B) \geq \frac{1}{2} \tau_n^{2n-1}, \quad \left\| \text{coeff} \sum_j \alpha_j p_j \right\|_2 \geq 2 \tau_n^{2n-1} \|\alpha\|_2.$$

This similarity permits a degree-dependent coefficient floor even when  $C_L/U$  grows.

We pay the changed constants throughout the complete shell proof. The coefficient norm of  $a_k$  is now at most  $e^{C_k/U} \leq e^n < 4^n$ . The same differentiated atom majorant as before gives

$$\max_j (|T_j| + |T'_j|) \leq 52U 4^n e^{-21U/16} \leq e^{-U}.$$

Indeed  $U \geq 64n \geq 128$  implies  $\log(52U) \leq U/8$  and  $n \log 4 < U/32$ , paying the comparison with slack. Also  $U \geq 16(n+1)$  and  $2L+5 \leq U$ , so all intermediate atom estimates and their termwise differentiations remain valid. The denominator and common-factor bounds (62) are unchanged. The polynomial coefficient norms are at most  $B_n$ , with derivatives bounded by  $mB_n$ . Applying (63) and the new coefficient floor gives the complete polynomial Gram floor  $\hat{h}_n$ . The all-atom perturbation is at most

$$9n(4B_n e^{-U} + 4e^{-2U}) \leq 72nB_n e^{-U}.$$

Here is an all-parameter admission, including the reduced onset. Write  $\ell = \log(n+1)$ . Direct substitution of (66),  $m = 4n+3$  and the inequalities  $\log 2 < 1$ ,  $\log 4 < 2$ ,  $\log 8 < 3$ ,  $\log 75 < 5$  gives

$$\log(144nB_n/\hat{h}_n) \leq 68 + 54n + (4n+3)\ell. \quad (67)$$

For  $n \geq 9$ ,  $\ell \geq \log 10 > 9/4$ , whence the ratio of the right side to  $n\ell$  is at most  $272/81 + 24/4 + 1/3 = 2567/81 < 32$ . For all  $n \geq 2$ , it is less than  $96n\ell$  and hence less than  $1024n\ell$ . Thus the perturbation is at most  $\hat{h}_n/2$  whenever  $U \geq F(n)$ . The first term  $64n$  also pays all preceding coefficient, wall and atom requirements:  $C_L \leq 3n^2$ ,  $L = 2n-1$ ,  $U \geq 16(n+1)$ ,  $2L+5 \leq U$ , and  $\delta = \sqrt{\pi/U} \leq 1/4$ . Thus this is an all-parameter onset, including  $2 \leq n \leq 8$ . One elementary certificate of  $\log 10 > 9/4$  is  $e^2 < 15/2$  (Taylor sum through order six and its geometric tail) and  $e^{1/4} < 4/3$ .

The actual joint shell Gram and its exact polar short are therefore at least  $\hat{h}_n/2$ . The profiles satisfy  $|v_j| \leq 4B_n$ ,  $|v'_j| \leq 16(m+1)B_n$ . The exact polar coefficient in the trial construction obeys  $|c_a| \leq 32B_n^2\sqrt{n}\hat{h}_n^{-1}\|a\|_2$ . Since  $|\mathrm{d}\chi_0(r/U)/\mathrm{d}u| \leq 32$ , the same tests give

$$|f_a|, |f'_a| < 2^{11}(m+1)B_n^3\sqrt{n}\hat{h}_n^{-1}\|a\|_2, \quad \|f_a\|_2^2 + \|f'_a\|_2^2 \leq \hat{E}_n\|a\|_2^2,$$

where  $\hat{E}_n = 2^{40}n^2(m+1)^2B_n^6\hat{h}_n^{-2}$ . For example, before multiplying the cutoff the function and derivative bounds are  $132B_n^3\sqrt{n}/\hat{h}_n$  and  $528(m+1)B_n^3\sqrt{n}/\hat{h}_n$ ; these fit the displayed constants because  $m \geq 11$ . The original compact-core/domain argument is identical: it uses compact support, exact polar correction and  $a(\xi) \leq 17\xi^2$ , not a quadratic wall. The source pairing is at least  $(\hat{h}_n/2)\|a\|_2^2$  and the full constrained energy at most  $19\hat{E}_n\|a\|_2^2$ . Variational duality gives the stated  $\hat{g}_n$ .

Finally,  $U \geq 64n$  admits every wall in the half-line upper proof, so the same  $C_n$  applies. Substituting the explicit constants proves both logarithmic cost statements.  $\square$

For sufficiently large  $U$ , the choice  $n = \lfloor U/(32\log(U+1)) \rfloor$  is admitted:  $n \geq 9$ ,  $n \leq U$  and  $32n\log(n+1) \leq U$ . Its logarithmic geometry budgets are  $O(U)$ . The theorem concerns positive-reference source geometry; it supplies no signed inverse margin.

Since  $F$  is increasing and  $F(n) \sim 4n\log n$ , its largest admitted integer satisfies  $n_{\max}(U) \sim U/(4\log U)$ . Indeed the inequalities  $(4-\eta)n\log n \leq F(n) \leq (4+\eta)n\log n$  eventually hold for each  $\eta > 0$ ; inversion and  $\log n/\log U \rightarrow 1$  give the assertion. The genuine source count is  $n-1$ ; the polar shell row is auxiliary. The broad lower has  $\log(\hat{g}_n^{-1}) = 16n\log n + O(n)$ , whereas the quadratic lower has  $\log(g_n^{-1}) = O(n)$ ; the common upper has  $\log C_n = 8n\log n + O(n)$ . The two regimes therefore provide a source-count versus lower-floor tradeoff.

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