# Finite positivity and correction inertia for the Riemann theta form

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October 25, 2026

**Abstract**

We study the original averaged Riemann theta form and its decomposition into a positive energy and a correction. A verification of the Riemann hypothesis through height $T$ implies strict positivity, uniformly over the averaging parameter, for polynomials of degree at most $\lfloor R/24 \rfloor$ whenever $10^8 \leq R \leq 2T - 2$. The published verification through $3 \cdot 10^{12}$ therefore gives degree 200,000,000,000. For the endpoint correction, a simultaneous polynomial approximation theorem in a derivative-and-potential norm transfers disjoint inverse-square channels near any fixed finite collection of real zeros into common polynomial subspaces. The contributions of the centers add. Infinitely many simple critical-line zeros then imply, in each parity, negative correction inertia divided by $\log\log N$ tending to infinity. Opposite channels of dimension $\Omega(\log N)$ have positive physical form but vanishing physical-to-energy ratio. Together these show that a uniform strict contraction obtained by scalar centering and an arbitrary symmetric correction requires rank $\omega(\log\log N)$. All mixed entries and approximation errors are retained. These are finite-positivity and proof-route results.

## Contents

**1 Introduction** **2**
1.1 Relationship to earlier results and classical inputs . . . . . 2

**2 The physical form and its endpoint decomposition** **3**

**3 Linear verified-height protection of the physical form** **4**
3.1 The complete Fourier density and the verified slab . . . . . 5
3.2 A closed-parameter exponential envelope . . . . . 5
3.3 A separate small real-zero input . . . . . 6
3.4 A lower set common to all physical parameters . . . . . 7
3.5 One polynomial on the thick set and the entire tail . . . . . 8

**4 Quantitative correction geometry** **9**
4.1 A strip-decay class and the finite-center statement . . . . . 10
4.2 Whole-line joint-norm polynomial lifting . . . . . 10
4.3 Exact logarithmic collars and common Grams . . . . . 13
4.4 Many zeros and the physical correction . . . . . 15
4.5 A receiver with explicitly positive original form . . . . . 16

**5 A positive packet block with vanishing relative reserve** **16**
5.1 A complete overlap bound . . . . . 17
5.2 Exact mixed packet forms . . . . . 17
5.3 One degree and one whole-line approximation budget . . . . . 18
5.4 The positive-parameter metric is different . . . . . 20

**6 Scope of the finite positivity and inertia results** **20**

# 1 Introduction

The polynomial positivity formulation of the Riemann theta problem contains two different questions. One asks for positivity of its original, fully integrated form. The other asks whether a decomposition into an energy and a correction supplies a useful uniform reserve. They need not have the same answer. This paper proves a large finite positivity range for the first question and a growing-dimensional obstruction to one class of arguments for the second.

The finite positivity theorem translates a verified zero height into a linear polynomial degree. Its proof combines the classical zero-product argument inside a zero-free slab with a quantitative lower set and a whole-line polynomial tail estimate. It uses the same polynomial in the lower and upper estimates; no projection is made after discarding a signed tail. The numerical zero verification is imported from Platt and Trudgian [1], rather than repeated here.

The correction theorem concerns the original endpoint form, with its original energy. At a real zero, its potential has an inverse-square singularity. Local logarithmic collars generate negative correction directions. A joint approximation estimate transfers any fixed number of centers at once, paying every mixed Gram entry. The finite-center constants may depend on the selected zeros; this is enough to deduce a super-$\log\log N$ inertia lower bound from infinitely many simple zeros, without an effective growing-center estimate. A second construction gives positive physical-form channels whose relative reserve tends to zero. The two constructions force a super-$\log\log N$ rank cost for scalar-centered strict-contraction arguments.

**The forms and the result hierarchy** The original square form is $Q$, the positive derivative energy is $E$, and the correction is $D = E - Q$. A negative correction direction therefore satisfies $Q > E > 0$. The verified-height theorem controls a finite polynomial range of the original form; the finite-center theorem and the positive packet block describe opposite correction channels. Their combination gives the necessary rank condition for scalar-centered strict contraction. The large numerical degree is obtained by analytic transfer of the cited zero-height theorem.

## 1.1 Relationship to earlier results and classical inputs

The predecessor [10] proves finite-order positivity and growing negative inertia for a fixed upper-center cutoff form, including exact negative witnesses. That cutoff form is distinct from the full physical form studied here. The present paper neither transfers a cutoff negative witness to the physical form nor claims that a negative correction is a negative physical direction. In fact, the constructed negative correction channels have positive physical form.

The main new conclusions are the linear verified-height-to-degree bound for this form, the additive finite-center correction theorem, its super-$\log\log N$ theta consequence, and the resulting rank obstruction. The real-zero product and localized Laguerre inequalities have classical ancestry; the relevant comparisons include Sra [5] and Polson [6]. Their finite blocks and normalizations are not identified with the present polynomial form. The minimum-modulus argument uses the

classical analytic mechanism discussed in [8]. Meixner–Pollaczek generating functions, exponential-weight polynomial approximation and analytic-vector density are established tools [3, 4]; we give the derivative-and-potential estimates actually needed here. The local singularity mechanism is the classical inverse-square Hardy mechanism [9].

The later clock formulation in Polson [7, Section 8] concerns the same pointwise-sign frontier. Indeed, with $\alpha = (1+y)/2$ and away from zeros,

$$g_y(x) = \frac{|F(x+iy)|^2}{8y} \operatorname{Re} \frac{\xi'}{\xi}(\alpha - ix/2).$$

It does not supply the coefficient-uniform finite-degree integral estimate or the correction-inertia rate proved here. Comparisons with the finite Hankel blocks refer specifically to version 24; the clock comparison refers to version 26.

## 2 The physical form and its endpoint decomposition

We use the Fourier convention $\widehat{f}(x) = \int_{\mathbb{R}} f(s)e^{ixs}\,ds$. Define

$$\Phi(s) = \sum_{n\geq 1} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}, \tag{1}$$

$$F(z) = \widehat{\Phi}(z) = \tfrac{1}{4}\xi((1+iz)/2), \qquad \xi(w) = \tfrac{1}{2}w(w-1)\pi^{-w/2}\Gamma(w/2)\zeta(w). \tag{2}$$

The Jacobi theta transformation makes $\Phi$ even; it is strictly positive and decays faster than any exponential on the real line. Formula (2) follows by Mellin transformation of the theta series (or two integrations by parts in the standard theta integral for $\xi$). These identities also justify differentiation and Fourier inversion below. In particular $F$ is real and even on $\mathbb{R}$.

For $y \geq 0$ put

$$w_y(s) = \begin{cases} s\sinh(2ys)/(2y), & y > 0, \\ s^2, & y = 0, \end{cases} \tag{3}$$

$$J_y(t) = \int_0^\infty w_y(s)\Phi(s+t/2)\Phi(s-t/2)\,ds, \qquad g_y = \widehat{J}_y, \tag{4}$$

$$Q_y(P) = \frac{1}{2\pi}\int_{\mathbb{R}} |P(x)|^2 g_y(x)\,dx. \tag{5}$$

All integrals converge for a polynomial $P$ and every finite $y$. The function $J_y$ is positive, even, smooth and rapidly decreasing, but its Fourier transform is not assumed nonnegative.

**Lemma 2.1.** *For $y > 0$ and real $x$,*

$$g_y(x) = \frac{\partial_y |F(x+iy)|^2}{8y}, \qquad g_0(x) = \frac{F'(x)^2 - F(x)F''(x)}{4}.$$

*For real polynomials define*

$$E_y(P) = \int_0^\infty w_y(s)|P(i\partial_s)\Phi(s)|^2\,ds, \qquad D_y(P) = E_y(P) - Q_y(P).$$

*At $y = 0$, writing $U = PF$ and $G = F'^2 - FF''$, one has*

$$E_0(P) = \frac{1}{4\pi}\int_{\mathbb{R}} |U'|^2, \qquad Q_0(P) = \frac{1}{8\pi}\int_{\mathbb{R}} GP^2, \tag{6}$$

$$D_0(P) = \frac{1}{8\pi}\int_{\mathbb{R}} F^2(P'^2 - PP''). \tag{7}$$

*Off the real zeros of $F$, set $V = G/F^2$. Then the second formula is also $Q_0(P) = (8\pi)^{-1}\int V|U|^2$.*

*Proof.* In the product $|F(x+iy)|^2$, change variables from the two spatial variables to their sum and difference: $u = s + t/2$, $v = s - t/2$. The Jacobian is one. Evenness gives

$$|F(x+iy)|^2 = 2\int_{\mathbb{R}} e^{ixt} \int_0^\infty \cosh(2ys)\Phi(s+t/2)\Phi(s-t/2)\,ds\,dt.$$

Differentiation proves the first identity; expansion at $y = 0$ proves the second. For a real polynomial, the even derivative terms in $P(i\partial_s)\Phi$ are real and even and the odd derivative terms are imaginary and odd. Its squared modulus is even. Parseval applied to $sP(i\partial_s)\Phi$ proves the energy formula. Expanding $|(PF)'|^2$ and integrating by parts proves (7); boundary terms vanish by exponential decay. The potential expression is interpreted through $GP^2$, so real zeros introduce no undefined integral. □

The finite positivity statements allow complex polynomials in (5). The correction and inertia statements use real polynomial spaces with real polarization. Let $\mathcal{P}_N^+$ and $\mathcal{P}_N^-$ denote, respectively, the even and odd real polynomials of degree at most $N$. For a symmetric form $A$, $i_-(A|\mathcal{P}_N^\pm)$ is the maximal dimension of a subspace on which $A$ is strictly negative. A rational subspace means a real span of polynomials with rational coefficients.

## 3 Linear verified-height protection of the physical form

We first prove a finite-degree theorem for the original physical form. The center weight is $s\sinh(2ys)/(2y)$, with value $s^2$ at $y = 0$; no upper-center cutoff or independently chosen centerwise polynomial is introduced. Throughout this section log denotes the natural logarithm.

**Theorem 3.1** (Verified height and physical polynomial degree)**.** *Suppose that every nontrivial zero $\sigma + i\gamma$ of the Riemann zeta function with $|\gamma| \le T$ has $\sigma = 1/2$. For every real $R$ with*

$$10^8 \le R \le 2T - 2,$$

*every nonzero complex polynomial $P$ of degree at most $\lfloor R/24 \rfloor$ satisfies*

$$Q_y(P) > 0 \qquad \text{for every finite } y \ge 0.$$

*The same strict positivity holds after division by $J_y(0) > 0$.*

**Corollary 3.2.** *For the complete Riemann theta density, $Q_y(P) > 0$ for every finite $y \ge 0$ and every nonzero complex polynomial of degree at most $2 \cdot 10^{11}$.*

*Proof.* Platt–Trudgian's verified-height theorem [1, Theorem 1] permits $T = 3 \cdot 10^{12}$. Take $R = 5 \cdot 10^{12}$ and use $24(2 \cdot 10^{11}) \le R \le 2T - 2$. □

The large zero verification is an imported theorem, not a computation performed here. The zero-localized sign mechanism is classical; compare Sra's stronger localized generalized-Laguerre statement and paired-product proof [5, Theorem 1.3 and Section 4]. Verified-height protection of other finite Hankel forms also appears in Polson [6, Sections 7.3 and 12.2]. Those heat-zero forms differ from $Q_y$. Here the additional task is to pay the entire potentially negative frequency complement with the *same* polynomial, uniformly in $0 \le y \le 1$.

### 3.1 The complete Fourier density and the verified slab

We record the normalization used in the argument. Absolute convergence of the complete theta series, its fixed derivatives and its exponential moments justifies the following changes of variables and differentiations. For $A_x(y) = |F(x+iy)|^2$, evenness of $\Phi$ gives

$$A_x(y) = 2\int_0^\infty \cosh(2ys)\int_{\mathbb{R}} \Phi(s+t/2)\Phi(s-t/2)e^{ixt}\,dt\,ds.$$

Indeed, in the product $F(x+iy)F(x-iy)$ use the coordinates $u = s+t/2$, $v = -s+t/2$, whose Jacobian has modulus one. Consequently

$$g_y(x) = \frac{A_x'(y)}{8y} \quad (y>0), \qquad g_0(x) = \frac{F'(x)^2 - F(x)F''(x)}{4}. \tag{8}$$

All real-frequency moments are finite under the bound proved below, so Fourier inversion gives the original polynomial form, with factor $(2\pi)^{-1}$. These are the same physical normalizations as in [10].

The classical even order-one xi product (see [5, Section 4]) has no nonconstant exponential gauge. In the $F$ variable it can be grouped into real pairs and nonreal quartets; the differentiated grouped sums converge normally on compact sets avoiding zeros. One way to see the convergence is to write the even factors as $1 - z^2/\rho^2$ and use $\sum_\rho |\rho|^{-2} < \infty$. A linear exponential factor vanishes by evenness. There are no purely imaginary zeros, since $F(iv) = \int \Phi(s)e^{-vs}\,ds > 0$ for real $v$. Every zero $a+ib$ of $F$ has $|b|<1$, and the verified-height hypothesis places every nonreal zero at $|a|>2T$.

For a real zero $r$, one factor contributes $2y/((x-r)^2+y^2)$ to $\partial_y \log A_x(y)$. A conjugate pair $a \pm ib$ contributes

$$\frac{4y\{(x-a)^2+y^2-b^2\}}{\{(x-a)^2+(y-b)^2\}\{(x-a)^2+(y+b)^2\}}. \tag{9}$$

Multiplicities repeat these contributions. The exact paired numerator, including its $-b^2$ term, is retained. Thus, under the hypothesis of Theorem 3.1,

$$g_y(x) \ge 0 \quad (|x| \le R,\ y \ge 0), \qquad g_y(x) > 0 \quad (x \in \mathbb{R},\ y \ge 1). \tag{10}$$

For the first statement $|x-a| > 2$ for every nonreal pair; for the second $y \ge 1 > |b|$. At $y=0$ use continuity in (8); at zeros use the undivided analytic expression rather than its logarithm. The global $y \ge 1$ conclusion is also the inherited physical statement recalled in [10, Section 4.6]. Once a real zero $r_*$ is fixed, retaining just one of its factors also gives, throughout the verified slab,

$$g_y(x) \ge \frac{|F(x+iy)|^2}{4\{(x-r_*)^2+y^2\}} \qquad (y>0). \tag{11}$$

This inequality has its continuous interpretation at $y=0$: if $F(z) = (z-r_*)^m B(z)$, the relevant quotient equals $|z-r_*|^{2m-2}|B(z)|^2/4$.

### 3.2 A closed-parameter exponential envelope

**Lemma 3.3.** *For every real $x$ and every $0 \le y \le 1$,*

$$|g_y(x)| \le 21200(1+|x|/2)^7 e^{-a|x|}, \qquad a = \pi/4. \tag{12}$$

*Proof.* Put $h(s) = (s-1)\zeta(s)$ and $u = 1+|t|$. Euler continuation on $\operatorname{Re} s > 0$ gives

$$h(s) = s - s(s-1)\int_1^\infty \{v\}v^{-s-1}\,dv, \qquad |h(s)| \le |s|\left(1 + \frac{|s-1|}{\operatorname{Re} s}\right).$$

On $3/8 \le \sigma \le 9/8$, this is at most $8u^2$, while $|s| \le 2u$. For $0 < q \le 1$, Euler's Gamma product and the product at $q = 1$ give

$$\frac{|\Gamma(q+iv)|}{\Gamma(q)} \le \sqrt{\frac{\pi|v|}{\sinh(\pi|v|)}} \le \sqrt{1+2\pi|v|}\, e^{-\pi|v|/2}.$$

The last inequality follows from $w/(1-e^{-w}) \le 1+w$. For $3/16 \le q \le 9/16$, splitting the Gamma integral at one gives $\Gamma(q) < 7$. Also $\sqrt{1+\pi|t|} \le 2\sqrt{u}$. Substitution in

$$\xi(s) = \frac{s}{2}\pi^{-s/2}\Gamma(s/2)h(s)$$

therefore yields

$$|\xi(\sigma+it)| \le 112u^{7/2}e^{-\pi|t|/4} \quad (3/8 \le \sigma \le 9/8). \tag{13}$$

A radius-1/8 Cauchy circle centered at $1/2 \le \sigma \le 1$ stays in this rectangle. Its common majorant is less than $200u^{7/2}e^{-\pi|t|/4}$, because $e^{\pi/32} < 8/7$ and $128^2(9/8)^7 < 200^2$. Cauchy's inequalities give constants 1600 and 25600 for $\xi'$ and $\xi''$, respectively. Functional symmetry allows us to use $s = (1+y)/2 - ix/2$ when estimating $F(x+iy)$. With $U = 1+|x|/2$, the resulting constants for $F, F', F''$ are $28, 200, 1600$, each multiplying $U^{7/2}e^{-\pi|x|/8}$.

The function $A_x$ is even in $y$ and $A_x'(0) = 0$. Its exact second derivative satisfies

$$|A_x''(v)| \le 2\big(|F'(x+iv)|^2 + |F(x+iv)F''(x+iv)|\big) \le 169600U^7e^{-\pi|x|/4}.$$

For $y > 0$, $A_x'(y)/y = \int_0^1 A_x''(ty)\,dt$; the same formula holds continuously at zero. Division by eight proves (12), without a $1/y$ loss. □

### 3.3 A separate small real-zero input

**Lemma 3.4.** *The function $F$ has a positive real zero $r_*$ in $(28, 30)$.*

*Proof.* The functional equation and conjugation make $\xi(1/2+it)$ real and continuous for real $t$. The following rational enclosures suffice:

$$\frac{2012944}{10^{10}} < \xi(1/2+14i) < \frac{2012945}{10^{10}}, \qquad -\frac{7056980}{10^{10}} < \xi(1/2+15i) < -\frac{7056979}{10^{10}}.$$

They are finite complex-ball evaluations of

$$\xi(s) = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

at the two exact rational arguments. The accompanying `verify_physical_positivity.py` evaluates this displayed formula with Arb/Acb at 256-bit precision and accepts only balls strictly inside these rational intervals. Reality follows from the exact functional equation, not numerical rounding of an imaginary part. The intermediate value theorem gives a critical-line zero with ordinate in $(14, 15)$; the change of variable in $F$ gives $r_* = 2\gamma \in (28, 30)$. No simplicity assertion is needed. This small sign certificate is separate from the imported high-zero verification. □

### 3.4 A lower set common to all physical parameters

The next estimate is the arithmetic input that distinguishes an expanding core from one fixed interval. Its complex-analytic ingredients are classical minimum-modulus tools; compare [8, Sections 1.1–1.4].

**Lemma 3.5.** *For $R \geq 10^8$ put $L = R/2$ and $K_R = \lceil 32 \log R \rceil$. There is a measurable set $E_R \subset [-L, L]$, independent of $y$, such that $|E_R| \geq 3L/2$ and, throughout the verified slab of Theorem 3.1,*

$$g_y(x) \geq \delta_R e^{-a|x|} \quad (x \in E_R,\ 0 \leq y \leq 1), \qquad \delta_R = 2^{-27} R^{-4} (1024R)^{-256} (256K_R)^{-2K_R}. \tag{14}$$

*In particular $\log \delta_R^{-1} = O(\log R \log\log R)$.*

*Proof.* Fix $t_0 \in \mathbb{R}$ and set $u = 1 + |t_0|$. On the disk centered at $2 + it_0$ of radius $7/4$, Euler continuation gives

$$|h(s)| \leq 256u^2 =: M,$$

since $\operatorname{Re} s \geq 1/4$, $|s| \leq 4u$ and $|s - 1| \leq 3u$. At the center the reciprocal Dirichlet series gives $|\zeta(2 + it_0)| \geq 1/\zeta(2) > 1/2$ and $|1 + it_0| \geq u/2$, hence $|h(2 + it_0)| \geq u/4 =: m$.

Translate the center to zero. Jensen's formula bounds the number $K$ of zeros in the closed radius-13/8 disk, counted with multiplicity:

$$K \log(14/13) \leq \log(1024u), \qquad K \leq 14 \log(1024u).$$

Here $\log(14/13) \geq 1/14$; a limiting radius includes boundary zeros. Divide by the radius-7/4 disk Blaschke factors

$$B_j(z) = \frac{(7/4)(z - z_j)}{(7/4)^2 - \overline{z_j} z}.$$

The quotient $H = h/\prod_j B_j$ is bounded by $M$ on the outer disk, has no zeros in the radius-13/8 disk, and has $|H(0)| \geq m$. Harnack's inequality for $\log(M/|H|)$, with a vanishing positive addition if necessary, has ratio

$$\frac{13/8 + 8/5}{13/8 - 8/5} = 129$$

on the radius-8/5 disk. Thus

$$|H(z)| \geq M(1024u)^{-129} = \frac{u}{4}(1024u)^{-128}.$$

If $u \leq R$ and $R \geq 1024$, then $K \leq K_R$. Remove radius-$d$ disks about these zeros, where $d = 1/(64K_R)$. Their total projected length on the imaginary coordinate is at most $2Kd \leq 1/32$. Outside these projections every point of the target disk satisfies $|B_j(z)| \geq d/4$, because $7/4 + 8/5 < 4$. Consequently

$$|h(s)| \geq h_R := \frac{1}{4}(1024R)^{-128}(256K_R)^{-K_R}. \tag{15}$$

The rectangle $1/2 \leq \operatorname{Re} s \leq 1$, $|\operatorname{Im} s - t_0| \leq 1/2$ fits in the target disk, since $5/2 < (8/5)^2$. The excluded imaginary-coordinate intervals therefore work for *all* real parts in this rectangle.

Under $t = -x/2$, cover $|x| \leq L$ by unit $t$ intervals centered at integers. At most $L + 2$ intervals are needed, and their centers have $1 + |t_0| \leq R$. Remove all the projected exceptional intervals. In the $x$ coordinate their combined length is at most $(L + 2)/16$. The remaining set $E_R$ satisfies

$$|E_R| \geq 2L - (L + 2)/16 \geq 3L/2.$$

It is the *same* set for the entire range $\operatorname{Re} s = (1+y)/2 \in [1/2, 1]$.

We also need a lower Gamma estimate. For $q \geq 1$, Euler's product gives

$$\frac{|\Gamma(q+iv)|}{\Gamma(q)} \geq \sqrt{\frac{\pi|v|}{\sinh(\pi|v|)}} \geq e^{-\pi|v|/2}.$$

For $q = q_0 + 1$, $1/4 \leq q_0 \leq 1/2$, the Gamma integral on $[1, 2]$ is greater than $e^{-2} > 1/9$. The recurrence therefore yields

$$|\Gamma(q_0 + iv)| \geq \frac{e^{-\pi|v|/2}}{9(1+|v|)}.$$

Use this at $s = (1+y)/2 - ix/2$ in $F(x+iy) = \xi(s)/4$. Since $|s| \geq 1/2$, $\pi^{-\operatorname{Re} s/2} > 1/2$ and $1 + |x|/4 \leq R$ on the core, the conservative bound

$$|F(x+iy)| \geq \frac{h_R}{576R} e^{-a|x|/2} \quad (x \in E_R,\ 0 \leq y \leq 1) \tag{16}$$

follows. The factor $e^{-a|x|/2}$ comes from $\operatorname{Im}(s/2) = -x/4$.

Retain the real zero from Lemma 3.4 in (11). On $|x| \leq L$, $0 \leq y \leq 1$, $4\{(x - r_*)^2 + y^2\} \leq 16R^2$. Squaring (16) and using (15) proves (14), since $16 \cdot 576^2 \cdot 16 = 84934656 < 2^{27}$. Continuity covers $y = 0$. The construction removes projections of zeros. □

### 3.5 One polynomial on the thick set and the entire tail

**Lemma 3.6.** *Let $E \subset [-L, L]$ be measurable with $|E| \geq 3L/2$. If $P$ is a complex polynomial of degree at most $N \geq 1$ and $\int_E |P|^2 = 1$, then, when $|x| \geq 2L$,*

$$|P(x)| \leq \sqrt{\frac{4}{3L}} \left(\frac{11|x|}{L}\right)^N. \tag{17}$$

*Proof.* The set $E' = \{x \in E : |P(x)|^2 \leq 2/|E|\}$ has measure $\mu \geq |E|/2 \geq 3L/4$, and $|P| \leq \sqrt{4/(3L)}$ there. Choose $N+1$ ordered Lebesgue-quantile nodes in its closure at mass parameters $j\mu/N$, $0 \leq j \leq N$, taking the first and last support points at the endpoints. Their distances are at least $|i-j|\mu/N$, because Lebesgue mass in an interval does not exceed its length. Continuity preserves the polynomial bound at the closure nodes. Lagrange interpolation therefore gives

$$|P(x)| \leq \sqrt{\frac{4}{3L}} (|x| + L)^N \left(\frac{N}{\mu}\right)^N \frac{2^N}{N!} \leq \sqrt{\frac{4}{3L}} \left(\frac{2e(|x|+L)}{\mu}\right)^N.$$

Use $N! \geq (N/e)^N$, $|x| + L \leq 3|x|/2$ and $4e < 11$. The argument works for disconnected measurable sets and complex coefficients, and uses no sign of the density at the interpolation nodes. □

*Proof of Theorem 3.1.* It suffices by (10) to handle $0 \leq y \leq 1$. For $N \geq 1$ normalize the nonzero polynomial by $\int_{E_R} |P|^2 = 1$; this is a genuine norm because $|E_R| > 0$. With $L = R/2$, Lemmas 3.3 and 3.6 bound the *entire* potentially negative complement:

$$\begin{aligned}\int_{|x|>R} |P(x)|^2 |g_y(x)|\, dx &\leq \frac{169600}{3L} \left(\frac{11}{L}\right)^{2N} \int_R^\infty x^{2N+7} e^{-ax}\, dx \\ &\leq \frac{339200}{3aL} R^7 \left(\frac{11R}{L}\right)^{2N} e^{-aR} < 2^{19} R^6 22^{2N} e^{-aR} =: B_N. \end{aligned} \tag{18}$$

Here $(1+x/2)^7 \le x^7$ for $x \ge R \ge 2$. The improper-integral bound used in the second line is

$$\int_R^\infty x^m e^{-ax}\,dx \le \frac{2}{a} R^m e^{-aR} \quad \text{if } m \le aR/2. \tag{19}$$

Indeed $\log(x/R) \le (x-R)/R$ bounds the integrand by $R^m e^{-aR} e^{-a(x-R)/2}$. The slope condition holds for $m = 2N+7$, $N \le R/24$, $R \ge 10^8$, since $a > 3/4$. The last scalar inequality in (18) uses $678400/(3a) < 2^{19}$.

The lower set contributes at least $\delta_R e^{-aL}$, while the rest of the verified slab contributes nonnegatively. It remains to compare the tail with this positive contribution. Expanding every factor in $\delta_R$, using $\log 22 < 4$ and $N \le R/24$, gives

$$\log \frac{B_N}{\delta_R e^{-aL}} < -\frac{R}{24} + 2606 \log 2 + 266 \log R + 2K_R \log(256 K_R). \tag{20}$$

For all $R \ge 10^8$,

$$K_R \le 64 \log R, \qquad \log R \le R^{1/4}, \qquad \log(256 K_R) \le 2 \log R.$$

For the middle inequality, at $R = 10^8$ use $\log R < 27 \log 2 < 27 < 100$, and then differentiate $R^{1/4} - \log R$. The last inequality follows from $256 K_R \le 16384 R^{1/4} \le R^2$ and its endpoint check. Consequently the positive loss in (20) is at most

$$2606 + 266 R^{1/4} + 256\sqrt{R} < R/32.$$

At $R = 10^8$ the left side is $2589206 < 3125000$; after division by $R$ every term decreases. Thus the logarithm in (20) is strictly less than $-R/96$. In particular $B_N < \delta_R e^{-aL}$, and the original whole-line integral of $|P|^2 g_y$ is strictly positive. No polynomial was changed between the positive lower set and either frequency tail.

Constant polynomials are positive because $\int g_y = 2\pi J_y(0) > 0$. Finally (10) covers $y \ge 1$, and division by the positive $J_y(0)$ changes no sign. □

**What the height transfer uses.** The arithmetic minimum-modulus argument supplies a set of proportional length with only $\exp[-O(\log R \log\log R)]$ loss, *common to every* $0 \le y \le 1$. Quantile interpolation and exponential-tail domination then protect degree proportional to $R$. Neither the classical slab sign alone nor an assumed degree-independent energy margin supplies this payment. The theorem retains all mixed derivative terms through the complete density (8).

## 4 Quantitative correction geometry

The endpoint correction has a structure which is different from the sign of the original form. We prove a finite-section statement retaining that distinction. The local inverse-square mechanism is classical Hardy theory [9]; analytic polynomial-orbit arguments and Gamma-reference approximation also have substantial precedents [4, 3]. The point here is the common-degree transfer of complete finite Grams at many zeros. The upper-center cutoff functional studied in [10] is a different functional.

### 4.1 A strip-decay class and the finite-center statement

Let $f$ be nonzero, even, real on $\mathbb{R}$, and holomorphic on $|\operatorname{Im} z| < b$, where $b > 0$. Assume that on every closed substrip there are constants $C, a > 0$ and $p \geq 0$ such that

$$|f(x+iv)| \leq C(1+|x|)^p e^{-a|x|}. \tag{H}$$

The constants may depend on the substrip. Define

$$\begin{aligned} G_f = f'^2 - ff'', \qquad V_f = \frac{G_f}{f^2}, \qquad \mathcal{E}_f(P) = \frac{1}{4\pi}\int_{\mathbb{R}} |(fP)'|^2, \\ \mathcal{Q}_f(P) = \frac{1}{8\pi}\int_{\mathbb{R}} G_f P^2, \qquad \mathcal{D}_f = \mathcal{E}_f - \mathcal{Q}_f. \end{aligned} \tag{21}$$

The quotient $V_f$ is used away from the isolated real zeros; its values at those points are immaterial. All polynomials in this section are real unless specified otherwise. Cauchy's estimate and (H) pay every polynomial tail in (21). Moreover, $\mathcal{E}_f$ is positive definite: $(fP)' = 0$ would make $fP$ constant, and its decay at both infinities would make it zero.

Write

$$\mathcal{P}_N^\epsilon = \{P \in \mathbb{R}[x] : \deg P \leq N,\ P(-x) = (-1)^\epsilon P(x)\}, \qquad \epsilon = 0, 1,$$

and let $i_-(\mathcal{D}_f; \mathcal{P}_N^\epsilon)$ count strictly negative eigenvalues, with multiplicity, of the finite quadratic form. This count is independent of a choice of basis.

**Theorem 4.1** (Finite-center correction inertia)**.** *Suppose* (H) *holds, and fix distinct positive real zeros $r_1, \ldots, r_M$ of $f$, with multiplicities $\nu_1, \ldots, \nu_M \geq 1$. For each $\epsilon \in \{0, 1\}$,*

$$i_-(\mathcal{D}_f; \mathcal{P}_N^\epsilon) \geq a_M \log\log N - O_{f, r_1, \ldots, r_M}(\log\log\log N), \qquad a_M = \sum_{j=1}^{M} \frac{1}{16(\nu_j + 1)}, \tag{22}$$

*for every sufficiently large integer $N$. The negative subspaces can be chosen with bases in $\mathbb{Q}[x]$. Constants and onset may depend on the selected finite set of zeros and its local data.*

Only positive centers are counted in $a_M$. Their reflections supply the parity construction and do not double that coefficient. We will also prove a version arbitrarily near the local Hardy edge. Neither statement asserts a bounded global relative operator or a global essential-spectrum formula.

### 4.2 Whole-line joint-norm polynomial lifting

For the targets used below, put

$$\|u\|_f^2 = \int_{\mathbb{R}} (|u'|^2 + (1 + |V_f|)|u|^2). \tag{23}$$

We use this norm for compact functions supported away from real zeros and for polynomial multiples of $f$. We do not need a description of its entire completion. For such targets the polarized forms are $\mathcal{E}_f(u, v) = \int u'v'/(4\pi)$ and $\mathcal{Q}_f(u, v) = \int V_f uv/(8\pi)$.

**Lemma 4.2** (Quantitative joint lifting)**.** *Under* (H)*, there is a constant $C_f$ with the following property. Let $u$ be real, of one parity, compactly supported away from the zeros of $f$, and suppose $h = u/f$, extended by zero, belongs to $H^2(\mathbb{R})$. If $B \geq \max(1, \|h\|_{H^2})$ and $0 < \eta \leq 1$, a polynomial $P$ of the same parity, with rational coefficients, can be chosen so that*

$$\|fP - u\|_f < \eta, \qquad \log(\deg P + 2) \leq C_f \left[\frac{B}{\eta} + \log\left(\frac{B}{\eta} + 2\right)\right]. \tag{24}$$

*The constant is independent of the target and its support.*

*For the physical transform $F$ of this paper, take the Meixner–Pollaczek polynomials with scale four and any fixed integer $\lambda \geq 9$. With*

$$T_{N,k}(x) = (1-r^2)^\lambda \sum_{n=0}^{N} (ir)^n q_n(x), \qquad r = \tanh(2k),$$

*one additionally has, for $H \geq 1$,*

$$\sup_{|k|\leq H} \|F(T_{N,k} - e^{ik\cdot})\|_F \leq Ce^{7H} \exp\left(-\frac{Ne^{-4H}}{2}\right). \tag{25}$$

*This assertion is independent of the packet construction.*

*Proof.* Fix $0 < \beta < b$ and a slightly larger closed substrip. Cauchy's estimate there bounds $f, f', f''$ by polynomial multiples of the same exponential on $|\operatorname{Im} z| \leq \beta$. In particular, real-axis $f, f', G_f$ are bounded. Expanding $(fv)'$ gives

$$\|fv\|_f \leq C\|v\|_{H^1}, \qquad |V_f||fv|^2 = |G_f||v|^2. \tag{26}$$

Thus the absolute signed-form part is retained, as well as the derivative energy.

Choose a scale $\ell > 0$ such that $\pi/\ell < 2a$, where $a$ is the chosen substrip exponent, and an integer $\lambda \geq 1$. The MP probability density and generating function are

$$w(x) = \frac{2^{2\lambda-1}}{\pi\ell\Gamma(2\lambda)} |\Gamma(\lambda + ix/\ell)|^2, \tag{27}$$

$$\sum_{n\geq 0} q_n(x)t^n = (1-it)^{-\lambda+ix/\ell}(1+it)^{-\lambda-ix/\ell}, \qquad |t| < 1. \tag{28}$$

Their classical squared norms are $v_n = (2\lambda)_n/n!$; see also the Gamma-reference methods in [3]. The density is positive and has tail a positive polynomial factor times $e^{-\pi|x|/\ell}$. For integer $\lambda$ this also follows from the gamma recurrence and $|\Gamma(1+it)|^2 = \pi t/\sinh(\pi t)$. The strict exponential gap therefore gives a common bound

$$|f(x)|^2,\ |G_f(x)|,\ |f(x+i\beta)|^2,\ |f(x-i\beta)|^2 \leq Cw(x). \tag{29}$$

The shifted generating function differs from (28) by $(1-it)^{-\beta/\ell}(1+it)^{\beta/\ell}$, or its reciprocal. Cauchy's coefficient estimate at radius $j/(j+1)$ gives a bound $C(j+1)^{\beta/\ell}$ for its $j$th coefficient; the constant term is handled separately. Since $\sqrt{v_n} \leq C(n+1)^{\lambda-1/2}$, convolution and Minkowski give

$$\|q_n(\cdot + iv)\|_{L^2(w)} \leq C(n+1)^p, \qquad p = \lambda + \beta/\ell + \tfrac{1}{2}, \quad v \in \{0, \beta, -\beta\}. \tag{30}$$

For $r = \tanh(\ell k/2)$, substitution $t = ir$ in the generating function gives exactly

$$e^{ikz} = (1-r^2)^\lambda \sum_{n\geq 0} (ir)^n q_n(z). \tag{31}$$

The series converges in each shifted $L^2(w)$ by (30), and the pointwise generating identity identifies its limit. For $\delta = 1 - |r| \in (0, 1]$,

$$\sum_{n>N} (n+1)^p |r|^n \leq C_p \delta^{-p-1} e^{-N\delta/2}.$$

For example, use $|r|^n \le e^{-\delta n}$, bound $(n+1)^p e^{-\delta n/2}$ by $C_p \delta^{-p}$, and sum the remaining geometric series. Retaining $(1-r^2)^\lambda \le 2^\lambda \delta^\lambda$ is essential. The degree-$N$ truncation $T_{N,k}$ consequently satisfies

$$\|T_{N,k} - e^{ik\cdot}\|_{L^2(w),v} \le C\delta^{-\sigma} e^{-N\delta/2}, \qquad \sigma = \beta/\ell + \tfrac{3}{2}, \quad v \in \{0, \beta, -\beta\}. \tag{32}$$

Here the norm evaluates the function at $x + iv$. At $k = 0$ the remainder is zero. For $|k| \le H$ one has $\delta \ge e^{-\ell H}$.

To pay the derivative without introducing a degree factor, set $U(z) = f(z)(T_{N,k}(z) - e^{ikz})$. The strip decay justifies horizontal Fourier contour shifts and vanishing vertical edges. If $\widehat{U}$ uses the minus-sign Fourier convention, Plancherel on the two shifted lines and $s^2 \le \cosh(2\beta s)/(2\beta^2)$ yield

$$\int_{\mathbb{R}} |U'|^2 \le \frac{1}{4\beta^2} \int_{\mathbb{R}} (|U(x+i\beta)|^2 + |U(x-i\beta)|^2)\, dx. \tag{33}$$

The real-line $|f|^2$ and $|G_f|$ parts of (23) are controlled by (29). Hence

$$\sup_{|k|\le H} \|f(T_{N,k} - e^{ik\cdot})\|_f \le C e^{\ell\sigma H} \exp(-N e^{-\ell H}/2). \tag{34}$$

All differentiated and signed mixed terms are paid by this norm.

Let $\widehat{h}(k) = \int h(x) e^{-ikx}\, dx$, and cut its Fourier integral at $|k| \le H$. For $H \ge 1$, Plancherel and Cauchy–Schwarz give

$$\|h - h_H\|_{H^1} \le H^{-1} \|h\|_{H^2}, \qquad \frac{1}{2\pi} \int_{-H}^{H} |\widehat{h}(k)|\, dk \le \sqrt{H/\pi}\, \|h\|_2. \tag{35}$$

One may use the Bessel $H^2$ norm, whose square is $\int (|h|^2 + 2|h'|^2 + |h''|^2)$. Take $H = \max(1, CB/\eta)$, with the fixed constant large enough that (26) pays the cutoff error by $\eta/4$. Integrate $T_{N,k}$ against $\widehat{h}(k)\, dk/(2\pi)$ on $[-H, H]$. This produces a polynomial; Minkowski, (34) and (35) pay its joint approximation error by $\eta/4$ whenever

$$N \ge 2e^{\ell H} \left[ \ell\sigma H + \log_+\left( \frac{C\sqrt{H}B}{\eta} \right) + 1 \right]. \tag{36}$$

Here and below a sufficient integer ceiling is understood and $\log_+ s = \max(0, \log s)$. Constants can be increased once for the fixed $f$ and reference. The constructed degree in (36) satisfies (24).

Conjugate Fourier symmetry makes the polynomial real. Projection to the desired parity is a contraction in (23), because $f, V_f$ are even. Finally, each of the finitely many norms $\|x^n f\|_f$ is finite. Rational approximation of the coefficients within, for example,

$$\frac{\eta}{4(N+1)\max(1, \max_{n\le N} \|x^n f\|_f)}$$

pays the remaining joint error by $\eta/4$ and does not increase degree. Coefficients of the other parity are set exactly to zero.

We verify the physical specialization, including its critical MP scale, directly. For $s = (1+iz)/2$ and $|\operatorname{Im} z| \le 1/2$, one has $1/4 \le \operatorname{Re} s \le 3/4$. The continuation formula

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \{u\} u^{-s-1}\, du, \qquad \operatorname{Re} s > 0,$$

gives $|\zeta(s)| \le C(1 + |\operatorname{Im} s|)$ on this strip. The defining xi product and uniform vertical Stirling therefore give

$$|F(x+iv)| \le C(1+|x|)^3 e^{-\pi|x|/8}, \qquad |v| \le 1/2. \tag{37}$$

Bounded $x$ is handled by continuity. Cauchy estimates on circles inside this strip give the corresponding real-axis bounds for $F'$ and $F''$. The scale-four MP density with integer $\lambda \geq 9$ has tail $c|x|^{2\lambda-1}e^{-\pi|x|/4}$, so it dominates $F^2, |G_F|, |F(x+i/2)|^2, |F(x-i/2)|^2$ on the whole line. Use $\beta = 1/2$, $\ell = 4$ in (33). The shifted MP norm exponent can be enlarged to $\lambda + 3/4$; retaining the prefactor in (31) gives $\delta^{-7/4}e^{-N\delta/2}$, with $\delta \geq e^{-4H}$. This proves (25) in the original physical joint norm. □

### 4.3 Exact logarithmic collars and common Grams

At the fixed positive zero $r_j$, choose a sufficiently short right-hand neighborhood and write

$$f(r_j + t) = t^{\nu_j} A_j(t), \quad |A_j(t)| \geq a_j > 0, \qquad V_f(r_j + t) = \frac{\nu_j}{t^2} + R_j(t), \quad |R_j(t)| \leq C_j. \tag{38}$$

The neighborhoods and their reflections can be made pairwise disjoint. The reciprocal of $A_j$ and its first two derivatives are bounded there. The expression $R_j = -(\log|A_j|)''$ has no simple-pole term.

For $t = e^{-s}$ and $u(r_j + t) = \sqrt{t}\, v(s)$ with compact support in the collar,

$$4\pi\mathcal{E}_f(u) = \int (|v'|^2 + |v|^2/4), \tag{39}$$

$$8\pi\mathcal{Q}_f(u) = \int [\nu_j + e^{-2s}R_j(e^{-s})]|v(s)|^2\, ds. \tag{40}$$

The energy cross term integrates to zero at the two endpoints. These are compact test identities, not an assumed operator extension.

Use the zero-extended function

$$\chi(z) = z^2(1-z)^2 \quad (0 \leq z \leq 1), \qquad \chi(z) = 0 \quad \text{otherwise.}$$

It is $C^1 \cap H^2(\mathbb{R})$: both $\chi$ and $\chi'$ vanish at the endpoints, so the second weak derivative has no point masses. Direct integration gives

$$A_0 = \int \chi^2 = \frac{1}{630}, \qquad A_1 = \int |\chi'|^2 = \frac{2}{105}, \qquad A_2 = \int |\chi''|^2 = \frac{4}{5}. \tag{41}$$

For a fixed length $L$, set

$$v_{j,l}(s) = c_L L^{-1/2}\chi((s - S_{j,l})/L), \quad S_{j,l} = S_{j,0} + L(l-1), \quad c_L^2 = \frac{4\pi}{A_1/L^2 + A_0/4}.$$

Then $\mathcal{E}_f(u_{j,l}) = 1$, and the singular quotient from (40) is

$$\frac{2\nu_j}{1 + 48/L^2}. \tag{42}$$

The log intervals may abut. Values and first derivatives vanish at their endpoints, and every divided target $u_{j,l}/f$, extended by zero, belongs to $H^2$.

First take $L = 16$. In this case

$$\begin{gathered} c_{16}^2 = \frac{161280\pi}{19}, \qquad \|v_{j,l}\|_2^2 = \frac{256\pi}{19}, \qquad \|v'_{j,l}\|_2^2 = \frac{12\pi}{19}, \\ q_{j,\text{sing}} = \frac{32\nu_j}{19} = 1 + 4\gamma_j, \quad \gamma_j = \frac{32\nu_j - 19}{76} > 0. \end{gathered} \tag{43}$$

Choose each $S_{j,0}$ large enough that its collar lies in (38) and $2C_j e^{-2S_{j,0}} \le \gamma_j$. Indeed the absolute regular contribution in (40) is at most $2C_j e^{-2S_{j,0}}$. Reflect each target about zero with the plus or minus sign and divide by $\sqrt{2}$. The full unapproximated Grams, for either parity, are consequently

$$\mathcal{E}_f(u_{j,l}, u_{k,m}) = \delta_{(j,l),(k,m)}, \qquad \mathcal{Q}_f(u_{j,l}, u_{k,m}) = q_{j,l}\delta_{(j,l),(k,m)}, \qquad q_{j,l} \ge 1 + 2\gamma_*, \quad \gamma_* := \min_j \gamma_j > 0. \tag{44}$$

Disjoint supports pay all mixed entries, including those between distinct centers; no sign of a regular remainder was assumed.

There is one bound $M_0$ for the joint norms of all these targets. For example $\|v\|_2^2 \le 16\pi$ follows from (39), and thus one may take

$$M_0^2 = \max_j \{4\pi + 16\pi\nu_j + 16\pi(1 + C_j)e^{-2S_{j,0}}\}.$$

The divided targets have the quantitative bound

$$\|u_{j,l}/f\|_{H^2} \le K_j e^{16(\nu_j+1)l}. \tag{45}$$

To see the exponent, differentiate $t^{1/2-\nu_j} A_j(t)^{-1} v(-\log t)$ at most twice. Every term is bounded by a constant times $t^{1/2-\nu_j-q}|v^{(m)}|$, where $q \le 2$ and $m \le q$. After squaring, $dt = t\,ds$ gives at $q = 2$ the exponent $-2(\nu_j + 1)$. Taking the norm square root and using fixed $L = 16$ proves (45). All local constants and starting points are absorbed in $K_j$, independently of $l$.

*Proof of Theorem 4.1.* Choose $d_j \ge 1$ collars at center $r_j$ and let $d = \sum_j d_j$. Set

$$\kappa = \min(1/4, 2\gamma_*/3), \qquad a_* = \min\left(1, \frac{2\pi\kappa}{2M_0 + 1}\right), \qquad \eta = \frac{a_*}{d}.$$

Apply Lemma 4.2 to all $d$ targets with common

$$B = \max\left(1, \max_j K_j e^{16(\nu_j+1)d_j}\right).$$

For two targets of joint norm at most $M_0$, replacing each by a polynomial multiple within $\eta$ changes an $\mathcal{E}_f$ entry by at most $\eta(2M_0 + \eta)/(4\pi)$, and an $\mathcal{D}_f$ entry by at most $3\eta(2M_0 + \eta)/(8\pi)$. The operator norm of a $d \times d$ error matrix is at most $d$ times its maximum absolute entry. Therefore the COMPLETE polynomial Grams satisfy

$$\mathcal{E}_{f,\text{poly}} \ge \tfrac{7}{8} I, \qquad \mathcal{D}_{f,\text{poly}} \le -\tfrac{3}{2}\gamma_* I. \tag{46}$$

In particular the polynomials are independent and their whole span is negative. This does not infer a subspace conclusion from separately negative basis vectors.

For large $X$, allocate

$$d_j = \left\lfloor \frac{X}{16(\nu_j + 1)} \right\rfloor.$$

Then $a_M X - M \le d \le a_M X$ and $B \le K_M e^X$, with $K_M$ fixed. The common sufficient degree furnished by (24) obeys

$$\log(N_X + 2) \le C_M(X + 1)e^X. \tag{47}$$

Let $Y = \log\log N$ and choose $X = Y - 2\log(Y + 2) - C_0$, with $C_0$ fixed and sufficiently large. For every sufficiently large $N$, all $d_j$ are positive and (47) is at most $\log N$. Padding the polynomial coefficient spaces now gives

$$i_-(\mathcal{D}_f; \mathcal{P}_N^\epsilon) \ge a_M Y - 2a_M \log(Y + 2) - O_M(1),$$

which proves (22). The constants can deteriorate with the fixed set of centers. No uniformity in $M$ was used. □

### 4.4 Many zeros and the physical correction

**Corollary 4.3.** *If a function satisfying* (H) *has infinitely many positive real zeros of multiplicity at most one fixed integer, then*

$$\frac{i_-(\mathcal{D}_f;\mathcal{P}_N^\epsilon)}{\log\log N}\longrightarrow\infty,\qquad \epsilon=0,1.$$

*Proof.* For every prescribed $A>0$, choose a fixed finite set of centers with $a_M>A$ and then apply Theorem 4.1. The statement holds for all sufficiently large $N$, with onset depending on that finite set. This is the asserted order of quantifiers; it does not exchange the limits $M,N$. □

**Theorem 4.4** (Original endpoint correction)**.** *For the physical transform $F$ and the original forms of this paper,*

$$\frac{i_-(D_0;\mathcal{P}_N^\epsilon)}{\log\log N}\longrightarrow\infty,\qquad \epsilon=0,1. \tag{48}$$

*Bases of the negative subspaces may be chosen rational. For every fixed $\theta\in(0,1)$, each parity also contains spaces $\mathcal{L}_N^{\epsilon,\theta}$ such that*

$$Q_0(P)\ge(2-\theta)E_0(P),\qquad D_0(P)\le(-1+\theta)E_0(P)\quad(P\in\mathcal{L}_N^{\epsilon,\theta}),\qquad \frac{\dim\mathcal{L}_N^{\epsilon,\theta}}{\log\log N}\longrightarrow\infty. \tag{49}$$

*Constants depend on $\theta$.*

*Proof.* The endpoint identities identify $(\mathcal{E}_F,\mathcal{Q}_F,\mathcal{D}_F)=(E_0,Q_0,D_0)$. Equation (37) verifies the strip hypothesis. Bui, Conrey and Young [2, Theorem 1.1] prove a positive lower proportion of simple critical-line zeros. Only their infinitude is needed here. A simple zero $1/2+i\gamma$ gives the positive simple zero $r=2\gamma$ of $F(z)=\xi((1+iz)/2)/4$; the nonzero xi prefactor and the affine change preserve multiplicity. Corollary 4.3 proves (48).

For (49), fix $\theta$ and select a common length $L=L_\theta$ with

$$L^2\ge\frac{48(8-\theta)}{\theta}.$$

At every simple center, (42) is at least $2-\theta/4$. Choose its starting collar to pay the entire regular contribution by $\theta/4$. For any fixed finite set the raw Grams are thus $\mathcal{E}=I$ and $\mathcal{Q}\ge(2-\theta/2)I$. As above, their joint norms are uniformly bounded, and their division norms grow as $K_je^{2Ll}$.

Choose the common joint errors $a_{M,\theta}/d$ so that the whole $\mathcal{E}$ Gram error is at most $\kappa=\min(1/4,\theta/16)$ and the whole $\mathcal{Q}$ error is at most $\kappa/2$. If $M_0$ is the joint-norm bound for this fixed set and length, the explicit choice

$$a_{M,\theta}=\min\left(1,\frac{4\pi\kappa}{2M_0+1}\right)$$

and the same entry bounds give these inequalities. Since

$$2-\theta/2-\kappa/2>(2-\theta)(1+\kappa),$$

the whole polynomial span satisfies the claimed relative floor. Repeating the degree allocation with exponent $2L$ gives dimension

$$\frac{M}{2L_\theta}\log\log N-O_{M,\theta}(\log\log\log N)$$

for every fixed $M$. Letting the fixed choice of $M$ be arbitrarily large proves (49), with rational bases as in Lemma 4.2. □

The conclusion concerns $D_0$, not negative inertia of $Q_0$. Indeed every nonzero vector in the spaces just constructed has $Q_0 > E_0 > 0$. There is no explicit onset uniform in the selected zeros, no specified faster rate than (48), and no improvement to a verified zero height.

### 4.5 A receiver with explicitly positive original form

The strip theorem has concrete non-arithmetic receivers. Take

$$f(z) = e^{-z^2/2} \cos z.$$

It is the Fourier transform of a positive even mixture of two translated Gaussian densities, satisfies (H), and has infinitely many positive simple zeros. Direct computation gives

$$G_f(x) = e^{-x^2}(1 + \cos^2 x) > 0, \qquad V_f(x) = 1 + \sec^2 x$$

off its zeros. For the corresponding physical product family,

$$|f(x + iy)|^2 = e^{-x^2+y^2}(\cos^2 x + \sinh^2 y),$$

whose $y$ derivative is

$$e^{-x^2+y^2}[2y(\cos^2 x + \sinh^2 y) + \sinh(2y)] > 0 \qquad (y > 0).$$

The continuous endpoint density is $G_f/4 > 0$ as well. Thus its original polynomial-square forms are nonnegative at every physical parameter, while Corollary 4.3 still gives growing negative correction inertia. This illustrates the precise distinction between correction geometry and failure of the original form.

For a polynomial-times-Gaussian example, $f(x) = (1 - x^2)e^{-x^2/2}$ has one positive simple zero and $G_f(x) = e^{-x^2}(x^4 + 3) > 0$, so Theorem 4.1 applies. The general lower rate need not be sharp for Gaussian receivers. These examples concern polynomial-times-Gaussian receivers and use the classical Hardy principle.

## 5 A positive packet block with vanishing relative reserve

The negative correction directions constructed near real zeros have an opposite counterpart. Translations of the original theta density produce many directions on which $Q_0$ is positive but small relative to $E_0$. The following statement concerns the complete forms on one common polynomial subspace, including their mixed entries.

**Theorem 5.1.** *There are constants $c, C, C_1 > 0$ and $N_0$ such that, for every integer $N \geq N_0$, each parity sector contains a subspace $\mathcal{S}_N$ spanned by real rational polynomials of degree at most $N$, with*

$$\dim \mathcal{S}_N \geq c \log N - C_1, \qquad 0 < \frac{Q_0(P)}{E_0(P)} \leq \frac{C}{(\log N)^2} \quad (0 \neq P \in \mathcal{S}_N).$$

*Consequently, for every fixed $\varepsilon > 0$ and all sufficiently large $N$, the same spaces satisfy*

$$(1 - \varepsilon)E_0(P) \leq D_0(P) < E_0(P) \quad (0 \neq P \in \mathcal{S}_N).$$

*No sign hypothesis on the full density $g_0$ is imposed.*

We give the construction and the simultaneous approximation estimates. Write

$$Z = J_0(0) > 0, \qquad C_0 = \int_{\mathbb{R}} \Phi(s)^2\, ds > 0, \qquad B = \frac{1}{2\pi}\int_{\mathbb{R}} |g_0(x)|\, dx < \infty.$$

Their finiteness follows from the theta estimates used in Lemma 4.2. For a real trigonometric test $p$, set $U = pF$ and extend the forms by their convergent integral formulas:

$$E_0(p) = \frac{1}{4\pi}\int_{\mathbb{R}} |U'(x)|^2\, dx, \qquad Q_0(p) = \frac{1}{2\pi}\int_{\mathbb{R}} p(x)^2 g_0(x)\, dx. \tag{50}$$

These agree with the original forms for polynomial $p$.

### 5.1 A complete overlap bound

There is a fixed $D > 0$ such that, for every $d \geq 0$,

$$0 < J_0(2d) \leq D\exp\{-\tfrac{1}{2}e^{4d}\}, \qquad 0 < T(d) := \int_{\mathbb{R}} \Phi(r+d)\Phi(r-d)\, dr \leq D\exp\{-\tfrac{1}{2}e^{4d}\}. \tag{51}$$

Here and below inequalities involving $J_0$ retain the full theta series. For completeness, the elementary bound

$$0 < \Phi(u) \leq 1000\exp\{-e^{4|u|}\} \qquad (u \in \mathbb{R}) \tag{52}$$

suffices. To verify the upper bound for $u \geq 0$, put $X = e^{4u} \geq 1$, drop the negative term of the theta series, and use $3 < \pi < 4$:

$$\Phi(u) \leq 32X^3 \sum_{n\geq 1} n^4 e^{-3n^2X} \leq 864 \sum_{n\geq 1} n^{-2} e^{-2n^2X} < 1000e^{-X}.$$

The middle step uses $v^3e^{-v} \leq 27$ at $v = n^2X$; the last follows from $2n^2 - 1 \geq n$ and $\sum_{n\geq 1} e^{-n} < 1$. Evenness handles $u < 0$. Positivity is immediate termwise for $u \geq 0$ and then by evenness.

For $r, d \geq 0$, put $U = e^{4r}$ and $V = e^{4d}$. Since $UV \geq (U+V)/2$, (52) gives

$$\Phi(r+d)\Phi(r-d) \leq 10^6 \exp\{-\tfrac{1}{2}e^{4r}\}\exp\{-\tfrac{1}{2}e^{4d}\}.$$

Thus (51) holds, for example, with the finite constant

$$D = 2\cdot 10^6 \int_0^\infty (1+r^2)\exp\{-\tfrac{1}{2}e^{4r}\}\, dr.$$

### 5.2 Exact mixed packet forms

For $k > 0$ let $p_k^+(x) = \cos(kx)$ and $p_k^-(x) = \sin(kx)$. Fourier inversion and product-to-sum give the exact identities

$$Q_0(p_k^\pm, p_l^\pm) = \frac{1}{2}\{J_0(k-l) \pm J_0(k+l)\}. \tag{53}$$

The reflected term has a minus sign in the odd sector and is not omitted. For the energy put $m = (k+l)/2$ and $d = (k-l)/2$. The spatial inverse transforms of $Fp_k^\pm$ are $(\Phi(s-k)+\Phi(s+k))/2$ and $(\Phi(s-k)-\Phi(s+k))/(2i)$, respectively. Parseval, followed by the change of variable to the midpoint of each pair, gives

$$E_0(p_k^\pm, p_l^\pm) = \frac{1}{4}\{2J_0(2|d|) + m^2T(|d|) \pm [2J_0(2m) + d^2T(m)]\}. \tag{54}$$

Odd midpoint terms integrate to zero because the product of the two translated densities is even. In particular,

$$E_0(p_k^{\pm}) = \frac{Z}{2} + \frac{C_0 k^2}{4} \pm \frac{J_0(2k)}{2}. \tag{55}$$

Choose $K \geq 1$ so large that, for every $k \geq K$,

$$De^{-e^{4k}/2} \leq \min\{Z/4, C_0 k^2/4\}.$$

Then $E_0(p_k^{\pm}) \geq C_0 k^2/8$ and $3Z/8 \leq Q_0(p_k^{\pm}) \leq 5Z/8$. If $k, l \in [K', 3K']$ for some $K' \geq K$, then $m^2/(kl) \leq 2$ and $d^2/(kl) \leq 1$. Equations (51) and (54) therefore imply, with $H = 14D/C_0$,

$$\frac{|E_0(p_k^{\pm}, p_l^{\pm})|}{\sqrt{E_0(p_k^{\pm})E_0(p_l^{\pm})}} \leq H \exp\{-\tfrac{1}{2}e^{2|k-l|}\} \qquad (k \neq l). \tag{56}$$

Indeed the numerator is at most $D[4 + m^2 + d^2]e^{-e^{2|k-l|}/2}/4 \leq (7Dkl/4)e^{-e^{2|k-l|}/2}$, and the diagonal square root is at least $C_0 kl/8$.

Choose a fixed $L \geq 1$ so that, writing $q = \exp\{-e^{2L}/2\}$,

$$q \leq \min\{(16H + 1)^{-1}, Z/(16D + Z)\}. \tag{57}$$

For $h \geq 1$, $e^{2Lh} \geq he^{2L}$, so the relevant overlap sums are bounded by $q/(1-q)$. Any finite sequence of centers with $|k_i - k_j| \geq L|i - j|$, all in $[K', 3K']$, consequently has normalized energy Gram at least $3I/4$. Its raw $Q_0$ Gram lies between $ZI/4$ and $3ZI/4$: the diagonal bounds above apply, and the complete off-diagonal row sum in (53) is at most $2Dq/(1-q) \leq Z/8$. These estimates hold in either parity and are independent of the number of packets.

### 5.3 One degree and one whole-line approximation budget

Fix an integer $A \geq e^{4L}$. For large $N$ set

$$M_0 = \lceil N^{1/4} \rceil, \qquad M_j = M_0 A^j \leq \lfloor \sqrt{N} \rfloor, \qquad k_j = \frac{1}{4}\log(2M_j - 1),$$

using every admissible $j \geq 0$, and denote their number by $m_N$. Then

$$m_N = 1 + \left\lfloor \frac{\log(\lfloor \sqrt{N} \rfloor / M_0)}{\log A} \right\rfloor \geq \frac{\log N}{4 \log A} - \frac{\log 4}{\log A}. \tag{58}$$

The centers have spacings greater than $L$, and lie in $[k_0, 3k_0]$. For the latter assertion use $\lfloor \sqrt{N} \rfloor \leq M_0^2$ and $2M_0^2 - 1 \leq (2M_0 - 1)^3$ when $M_0 \geq 2$. Also

$$k_0 \geq \frac{\log N}{16}, \qquad k_j \leq \frac{\log N}{8} + \frac{\log 2}{4}.$$

In particular the first-center condition holds for all sufficiently large $N$, and every analytic packet energy is at least

$$e_N := \frac{C_0 (\log N)^2}{2048}. \tag{59}$$

We now use the uniform plane-wave estimate of Lemma 4.2. In its Meixner–Pollaczek notation choose the rational parameters

$$r_j = 1 - M_j^{-1} = \tanh(2k_j), \qquad T_{N,j}(x) = (1 - r_j^2)^{\lambda} \sum_{n=0}^{N} (ir_j)^n q_n(x).$$

Let $P^+_{N,j}$ and $P^-_{N,j}$ be its real and imaginary parts. They have real rational coefficients, the indicated even or odd parity, and degree at most $N$. The exact generating identity has limit $e^{ik_jx}$. With $H_N = (\log N)/8 + (\log 2)/4$, that lemma bounds the error in the original joint norm by

$$Ce^{7H_N} \exp\{-\tfrac{1}{2}Ne^{-4H_N}\} \le C'N^{7/8}e^{-\sqrt{N}/4}.$$

The derivative part of that norm controls the energy in (50); its absolute-potential part controls the $L^2(|g_0|/(2\pi))$ norm, since $g_0 = (F'^2 - FF'')/4$. Taking real or imaginary parts is contractive. Thus, after enlarging $C'$, one common quantity

$$\delta_N = C'N^{7/8}e^{-\sqrt{N}/4} \tag{60}$$

bounds BOTH

$$\|P^\pm_{N,j} - p^\pm_{k_j}\|_{E_0}, \qquad \|P^\pm_{N,j} - p^\pm_{k_j}\|_{L^2(|g_0|/(2\pi))}$$

uniformly over all $j$. This invokes the whole-line, shifted estimate proved in Lemma 4.2, not pointwise approximation or orthogonality on a shifted line. The underlying MP generating functions and approximation framework are classical; see [3].

Normalize each analytic packet by its own energy square root. Its normalized polynomial error is at most $\delta_N/\sqrt{e_N}$. Consequently every normalized energy-Gram perturbation entry is at most

$$2\delta_N/\sqrt{e_N} + \delta_N^2/e_N.$$

Likewise every raw signed-$Q_0$ Gram perturbation entry is at most $\delta_N(2\sqrt{B} + \delta_N)$, by Cauchy–Schwarz in the absolute density. Because $m_N = O(\log N)$ and (60) decays faster than every power, for all sufficiently large $N$ both

$$m_N(2\delta_N/\sqrt{e_N} + \delta_N^2/e_N) \le \frac{1}{4}, \qquad m_N\delta_N(2\sqrt{B} + \delta_N) \le \frac{Z}{8} \tag{61}$$

hold. The dimension factor pays all mixed approximation entries.

In the SAME raw polynomial coefficient basis, the resulting energy and $Q_0$ matrices therefore satisfy

$$\mathbf{E}_N \succeq \frac{e_N}{2}I, \qquad \frac{Z}{8}I \preceq \mathbf{Q}_N \preceq \frac{7Z}{8}I.$$

For the first inequality, undo the diagonal normalization of the energy Gram; for the second, use the raw analytic $Q_0$ bounds and the second condition in (61). Linear independence follows. On their rational span,

$$0 < \frac{Q_0(P)}{E_0(P)} \le \frac{7Z}{4e_N} = \frac{3584Z}{C_0(\log N)^2}.$$

Together with (58), this proves Theorem 5.1. The upper bound is dimension independent because the complete raw $Q_0$ rows are summable; replacing them by a maximum-entry bound would lose a factor of $\log N$.

The translations, exact Gram identities and finite-rank consequences use classical mechanisms. Their role here is to retain the original theta form and its actual energy simultaneously. The construction neither produces a negative $Q_0$ direction nor pays the mixed terms between this positive packet block and its complement. Those are distinct from the fixed-cutoff negative-form results of [10].

**Corollary 5.2** (Necessary rank for centered contraction)**.** *On the degree-$N$ real polynomial space, let $R_N$ be any symmetric correction and $\alpha_N$ any real scalar. If some fixed $c < 1$ satisfies*

$$\|E_{0,N}^{-1/2}(D_{0,N} + R_N - \alpha_N E_{0,N})E_{0,N}^{-1/2}\| \le c \tag{62}$$

*for every sufficiently large $N$, then necessarily*

$$\frac{\operatorname{rank} R_N}{\log\log N} \longrightarrow \infty. \tag{63}$$

*The correction may depend on $N$, may be indefinite, and may couple the two parities.*

*Proof.* For fixed $0 < \theta < 1 - c$, Theorem 5.1 supplies positive-$D_0$ spaces of dimension $\Omega(\log N)$ on which $D_0/E_0 \ge 1 - \theta$. Theorem 4.4 supplies negative-$D_0$ spaces satisfying (49) whose dimension divided by $\log\log N$ tends to infinity. If (63) failed, a subsequence would have $\operatorname{rank} R_N \le C \log\log N$ for one fixed $C$. Both spaces then intersect $\ker R_N$ nontrivially along that subsequence. This dimension argument uses the full rank of $R_N$, even when it mixes parities.

The generalized Rayleigh quotients of $D_{0,N} + R_N$ relative to $E_{0,N}$ therefore attain values at least $1 - \theta$ and at most $-1 + \theta$. The generalized eigenvalue spread is at least $2 - 2\theta$. Subtracting any scalar multiple of the energy cannot reduce its operator norm below half that spread, namely $1 - \theta > c$. This contradicts (62). □

This is a necessary condition. It is distinct from the rank needed to create a fixed positive $Q_0/E_0$ reserve.

### 5.4 The positive-parameter metric is different

For a real polynomial, $U = PF$ is entire and Parseval also gives, for each $y > 0$,

$$E_y(P) = \frac{1}{8\pi y}\int_{-y}^{y}\int_{\mathbb{R}} |U'(x + iv)|^2\, dx\, dv. \tag{64}$$

To verify it, the Fourier inverse of $U'$ is $isP(i\partial_s)\Phi(s)$ (the immaterial sign depends on inversion convention). Integrating $e^{-2vs}$ over $v \in [-y, y]$ gives $\sinh(2ys)/s$; evenness then gives exactly the displayed factor. For any finite real interval $I$, the submean inequality for the holomorphic function $U'$ on radius-$y$ disks yields

$$\frac{1}{4\pi}\int_I |U'(x)|^2\, dx \le \frac{2|I|}{\pi y} E_y(P).$$

The disk is contained in the strip of (64); integrating the pointwise disk bound proves the inequality. Thus concentrating unit endpoint derivative energy into intervals whose lengths tend to zero requires unbounded $E_y$ at fixed $y > 0$. The endpoint logarithmic collars cannot be transferred to that metric at no cost. No positive-parameter relative-operator conclusion is being inferred from endpoint correction inertia.

## 6 Scope of the finite positivity and inertia results

The finite positivity theorem is unconditional when used with the published finite zero verification, but its degree remains finite. The endpoint inertia theorem is unconditional as an asymptotic statement. Constants in the finite-center theorem depend on the selected zero set. The quantitative conclusion is super-$\log\log N$ growth.

The rank obstruction concerns a prescribed method: scalar centering followed by a symmetric finite-rank correction and a uniform strict contraction relative to $E_0$. It is a necessary rank condition for this architecture. Negative correction channels satisfy $Q_0 > E_0 > 0$. The packet channels also have $Q_0 > 0$.

The physical all-order problem asks whether $Q_y(P) \geq 0$ for every polynomial $P$ and every $y > 0$. In this normalization it is equivalent to the Riemann hypothesis: positivity in all polynomial degrees forces $g_y \geq 0$ by exponential-weight polynomial density; the derivative identity then makes $|F(x+iy)|^2$ nondecreasing in $y > 0$, excluding a zero there. Conversely, if all zeros of $F$ are real, its canonical product gives a nonnegative vertical logarithmic derivative. Endpoint positivity alone is a weaker statement. The present work retains the mixed terms of the physical form and delineates obstacles to a reserve argument without settling this equivalence's unresolved side.

**Reproducibility and assistance.** The supplementary source includes the complete manuscript and finite arithmetic checks of the stated constants, together with a separate interval check of the low-zero anchor. The large verified zero height is a cited input; the all-parameter conclusions have analytic proofs. AI tools assisted mathematical development, independent internal checks, literature comparison, computation and manuscript preparation. The author is responsible for the manuscript. No external research funding is declared.

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