Finite-order positivity and growing negative inertia for averaged Riemann theta products

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Abstract

We study the contrast between finite-order positivity and all-order failure for averaged Riemann theta products. Our main new theorem proves that, for the complete upper-center-tail form at the exact cutoff $A = 23/2$, each polynomial parity through degree N has negative inertia at least $3 \log N/(8\pi) - O(\log \log N)$. A fully explicit conservative bound constructs common negative subspaces, with every theta term and approximation error retained. Consequently a correction making the full form positive semidefinite must have rank at least $3 \log N/(4\pi) - O(\log \log N)$, even if the correction changes with N or couples parities. The proof combines classical sinc concentration with gamma-reference cancellation and uniform Meixner–Pollaczek projection. To place the result in its finite-order context, we consolidate previously established positivity through degree eight, degree-dependent terminal positivity, exact rational negative witnesses of degrees 4096 and 4097, and eventual witness degree at most $\lceil A^{31/4} \rceil$ in both parities. Reused results are identified explicitly. A classical lognormal counterexample explains why finite moments and a negative density region alone are insufficient.

Contents

1 Introduction **3**

2 Moment forms, reference measure and shared tools **4**

3 Growing negative inertia at a fixed theta cutoff **7**

4 The all-order monotone cone fails **13**

5 Constructive and quantitative polynomial obstructions **18**

6 Conclusions and remaining questions **24**

A A complete closed negative-band certificate **24**

B Finite-order positivity and Schur repair in full **28**

C Global lower prefix: finite and unbounded enclosures **45**

D Whole-cell baseline and regional certificate **47**

E Complete even-sextic tail certificate **49**

F Whole-cell sextic sign and mean certificates **51**

G Complete odd-septic and all-cutoff certificates **52**

H Complete even-octic certificates **54**

I A terminating effective finite-order tail algorithm **56**

1 Introduction

Averaging products of translated theta densities gives a useful test of the relation between finite moment positivity and positivity at every order. Consider a nonnegative center weight multiplying $s^2\Phi(s+t/2)\Phi(s-t/2)$. At several low degrees its signed derivative moments form positive-definite matrices, even though the fixed-center form can be indefinite. This repair under averaging is substantial.

The main result of this paper describes the size of that failure. At the exact upper-center cutoff $A = 23/2$, the negative inertia in *each* polynomial parity grows at least as

$$\frac{3}{8\pi}\log N - O(\log\log N),$$

where N is maximum polynomial degree, not the size of a parity block. The proof produces one common negative subspace of the indicated dimension. This distinction matters: many separately negative vectors can span a space that is not negative definite. A consequence is that no uniformly bounded-rank correction can restore all-order positivity. An unrestricted repair of the full polynomial space requires rank at least $3\log N/(4\pi) - O(\log\log N)$.

The construction is quantitative. An elementary trace-defect estimate for the classical sinc concentration operator supplies a growing-dimensional band-concentrated space. Gaussian damping and true reflection preserve a common norm in both parities. The gamma reference measure removes the weight at the imaginary shifts of the Meixner–Pollaczek operator. A uniform spectral projection estimate then transfers the *whole space*, rather than separately chosen vectors, into degree-N polynomials. The signed-form comparison is controlled by the squared outside-band mass. All frequency tails are bounded and the closed negative band is certified for the complete theta density.

Previous work and the contribution here. This is a new synthesis and extension of the author's previously published *Monotone center weights and the averaged odd-quintic theta gate*, edition 6 [21]. That paper establishes positivity through degree eight, a fixed-degree Hermite terminal region and a terminating exact tail algorithm, complete-theta Fourier obstructions, exact rational witnesses of degrees 4096 and 4097, and the eventual witness-degree bound $\lceil A^{31/4}\rceil$. Those results and their supplied data are reused here with attribution; they are not presented as new results of this paper. The new mathematical center is the effective common-subspace negative-inertia theorem and its repair-rank consequence. Their precise hypotheses, limitations and proof are placed before the inherited technical developments.

Classical ingredients and the comparison boundary. Time-bandlimiting, the Meixner–Pollaczek difference operator, gamma-weight transformations and rank/inertia inequalities are classical. Bizeul–Klartag [14] treat the same convolution family of reference measures and its imaginary-shift approximation mechanism. Classical Christoffel–Darboux asymptotics and Nevai localization explain the leading logarithmic local dimension [23, 24]; they already support the qualitative logarithmic mechanism. The explicit common-space transfer, conservative effective count and $O(\log\log N)$ deficit are the distinguishing quantitative assertions proved here. General multiplication-compression spectral theory [25, 26] and prolate estimates [27] provide further context, not a substitute for the actual theta hypotheses.

Scope. The upper-tail weights and the smooth increasing counterexamples belong to a broader cone than the physical family $H_y(s) = \sinh(2ys)/(2ys)$. The explicit count has very conservative

constants and a degree domain specified by its bandwidth threshold. Reference-metric negative reserves are kept distinct from eigenvalue floors in raw monomial coordinates.

Organization. Section 2 defines the moment forms and common analytic tools. Section 3 proves the new growing-inertia theorem, its effective specialization and the repair obstruction. Sections 4 and 5 provide the complete-theta obstruction, exact witnesses and quantitative degree argument on which the synthesis rests. The closed-band input is proved in Appendix A. Appendix B and the following arithmetic appendices retain the full finite-order theory and its verification details. The final discussion separates the conclusions from the remaining physical and spectral questions.

2 Moment forms, reference measure and shared tools

The complete positive even Riemann theta density is

$$\Phi(s) = \sum_{n\geq 1} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}. \tag{1}$$

Jacobi symmetry gives evenness and positivity. For $s \geq 0$ every summand is positive. Put $X(s) = \pi e^{4s}$, $dm_0 = s^2 \Phi(s)^2\, ds$, and

$$p_s(t) = \frac{\Phi(s+t/2)\Phi(s-t/2)}{\Phi(s)^2}, \qquad h_{2j}(s) = (-1)^j p_s^{(2j)}(0), \qquad h_{2j+1} = 0.$$

For a nonzero nonnegative center weight W with the required finite moments, set $Z_W = \int W\, dm_0$ and

$$\mathcal{L}_W(z^k) = Z_W^{-1} \int W(s) h_k(s)\, dm_0(s), \qquad \mathsf{H}_N[W] = (\mathcal{L}_W(z^{j+k}))_{0\leq j,k\leq N}.$$

The polynomial variable z and the center coordinate X are distinct. Our finite positive results assume that W is finite-valued, measurable, nonnegative and nondecreasing, not zero almost everywhere, and that

$$\int_0^\infty W(s)(1 + X(s)^d)\, dm_0(s) < \infty \tag{2}$$

for the degree d in question. The physical weights are $H_y(s) = \sinh(2ys)/(2ys)$, continuously defined as one at $ys = 0$. Every finite $y \geq 0$ satisfies the needed fixed-degree conditions; normalization at $y = 0$ is by $Z_0 > 0$.

For $A \geq \pi$, let $a(A) = \frac{1}{4}\log(A/\pi)$ and

$$J_A(t) = \int_{a(A)}^\infty s^2 \Phi(s+t/2)\Phi(s-t/2)\, ds, \qquad g_A(x) = \widehat{J}_A(x) = \int_{\mathbb{R}} J_A(t) e^{ixt}\, dt.$$

Its raw moments are $m_{2j}(A) = (-1)^j J_A^{(2j)}(0)$ and $m_{2j+1} = 0$. For $P(z) = \sum p_j z^j$ write

$$\mathcal{Q}_A(P) = \sum_{j,k} p_j p_k m_{j+k}(A) = \frac{1}{2\pi} \int_{\mathbb{R}} P(x)^2 g_A(x)\, dx. \tag{3}$$

The exponential envelope below justifies the last identity for every finite polynomial. No center-dependent optimizing polynomial is interchanged with the center integral. Let $\mathcal{P}_N^\epsilon$ be real polynomials of parity $\epsilon \in \{0,1\}$ (zero for even and one for odd) and degree at most N, and let $\iota_\epsilon(N; g_A)$ be the strict negative inertia of $\mathcal{Q}_A$ restricted to that space. The dimensions are $\lfloor N/2 \rfloor + 1$ and $\lfloor (N+1)/2 \rfloor$, respectively. Inertia is invariant under positive scalar factors and invertible real polynomial-basis changes.

2.1 Results and their dependencies

The finite positive theory and the quantitative obstructions fit together as follows. The detailed inherited positive proofs are in Appendix B; their full arithmetic certificates follow it.

Result	Exact scope	Status
Finite averaged positivity	$d = 5, 6, 7, 8$ under the respective condition (2); all finite physical $y \geq 0$	Recalled from [21]
Fixed-degree terminal positivity	Every finite $N \geq 1$, $X \geq 1000N^2$; normalized Hermite reserve $99/100$	Recalled from [21]
Constructive failure	Exact rational degrees $4096, 4097$ at $A = 23/2$; smooth increasing weights can fail	Recalled from [21]
Witness degree growth	Each parity has a negative rational witness of degree at most $\lceil A^{31/4} \rceil$ eventually	Recalled from [21]
Growing negative inertia	Each parity: $3 \log N/(8\pi) - O(\log \log N)$ at $A = 23/2$, with an effective conservative bound	Main new result
Repair-rank obstruction	Full degree-N repair rank at least $3 \log N/(4\pi) - O(\log \log N)$	New theta consequence

The inherited development also retains the positive-prefix variance rigidity, two-column residual-mean results, associated-kernel finite jets for all indices, local physical positivity on fixed configurations of at most nine points, and the terminating exact finite-order tail algorithm. Their assumptions and complete proofs remain explicit in the appendices.

Finite-order positivity does not imply positivity at every order. In particular, the assertions $\forall N \, \exists A_N$ (a positive tail at degree N) and unbounded negative inertia at one fixed A have different quantifier order and are compatible.

2.2 A positive reference measure and a complete strip bound

Put

$$V(x) = \prod_{j=0}^{3} \big(x^2 + (4j+2)^2\big), \quad Z_* = 4^4 \, 8! = 10321920, \quad \varpi(x) = \frac{V(x)}{4Z_* \cosh(\pi x/4)}. \tag{4}$$

This is a positive even probability density. Its rational Meixner–Pollaczek polynomials, at parameter $\lambda = 9/2$ and scale four, are

$$q_0 = 1, \quad q_1 = x/2, \qquad (n+1)q_{n+1} = (x/2)q_n - (n+8)q_{n-1}, \qquad \|q_n\|_\varpi^2 = \binom{n+8}{8} =: v_n. \tag{5}$$

The normalization follows either from the classical gamma-weight orthogonality [13, Sections 18.19, 18.23], using the gamma recurrence, or from its generating function

$$\sum_{n \geq 0} q_n(x) t^n = (1 - it)^{-9/2 + ix/4} (1 + it)^{-9/2 - ix/4}. \tag{6}$$

The polynomial variable in this basis can equally be denoted z when used in $\mathcal{Q}_A$.

Lemma 1. *For every $A \geq 11$, $g_A = \widehat{J}_A$ is analytic on a neighborhood of the closed strip $|\operatorname{Im} z| \leq 1$, and*

$$|g_A(x + iy)| \leq 10 \max(1, |x|/2)^{15/2} e^{-\pi|x|/4} \leq M\varpi(x), \quad |y| \leq 1, \qquad M = 1612800. \tag{7}$$

Also $|(\log \varpi)'| < 2$ on $\mathbb{R}$.

Proof. By symmetry take $x \geq 0$, write $\tau = (x+iy)/2 = t + iw$, and set $T = \max(1,t)$. Shift the full-line Bessel integral by $i\theta$, where $\cos\theta = 1/T$. Since $|w| \leq 1/2$, its modulus is bounded using $K_{1/2}(2Y/T)$; the argument derivative is bounded using $K_{3/2}(2Y/T)$. The elementary formulas

$$K_{1/2}(u) = \sqrt{\pi/(2u)}e^{-u}, \qquad K_{3/2}(u) = K_{1/2}(u)(1+1/u)$$

follow as well by positive-integral domination of $\cosh(wu)$ and $\cosh u \cosh(wu)$. The scaling penalty is at most $e^{\pi/2}$: for $t \geq 1$ this uses $t \arcsin(1/t) \leq \pi/2$, and for $t \leq 1$ use $\theta = 0$.

For $Y \geq 11$, after factoring $e^{\pi/2}\sqrt{\pi}T^{3/2}Y^{3/2}e^{-Y/T}$, the bound for $\mathcal{H}_\tau$ has coefficient at most

$$e^{-Y/T}\left\{\frac{2+6/Y+(9/2)/Y^2}{T} + \frac{3}{Y^2}\right\} \leq 2 + \frac{6}{11} + \frac{15}{242} < 3.$$

For complex τ, $|D_q(\tau)| \leq d(q)\sqrt{q}$. Combining (23) with $\log^2(Y/(q\pi)) \leq Y^2/(q\pi)^2$, extending the positive majorant to $Y = 0$, and using

$$\sum_{q\geq 1}\frac{d(q)}{q^2} = \zeta(2)^2 < 3, \qquad \int_0^\infty Y^5 e^{-Y/T}dY = 120T^6$$

proves the first inequality in (7), since $360e^{\pi/2}\zeta(2)^2/(64\pi^2) < 10$. Local uniform absolute convergence also pays analyticity and the contour interchanges. A slight enlargement of any compact subset of the strip is dominated in the same way, with harmless polynomial changes in q against its exponential lower-end decay.

For $|x| \geq 2$, $V(x) \geq x^8$ gives the reference constant $40Z_*/256 = M$; for $|x| \leq 2$ use $V(x) \geq 1680^2$, giving a smaller constant. Finally

$$|(\log\varpi)'| \leq \sum_{j=0}^{3}\frac{1}{4j+2} + \frac{\pi}{4} < 2.$$

□

2.3 A common Meixner–Pollaczek projection estimate

We give the polynomial approximation step explicitly, with classical credit to [14]. The normalized polynomials $e_n = q_n/\sqrt{v_n}$ form a complete basis of $L^2(\varpi)$. To see completeness, an annihilator f defines the finite measure $f\varpi\,dx$ with an exponential moment at every rate below $\pi/8$, by Cauchy–Schwarz. Its Fourier transform is analytic in a strip, with all derivatives zero at zero; Fourier uniqueness gives $f = 0$.

The Meixner–Pollaczek difference equation [13, 18.22.17–18] is

$$(\mathcal{D}f)(x) = \tfrac{1}{2}\{(\lambda - ix/4)f(x+4i) + (\lambda + ix/4)f(x-4i)\} - \lambda f(x), \qquad \mathcal{D}e_n = ne_n, \quad \lambda = 9/2.$$

It is symmetric against polynomial tests for the packets constructed below. In $u = x/4$, the gamma weight $W(u) = \Gamma(\lambda+iu)\Gamma(\lambda-iu)$ obeys

$$(\lambda - 1 - iu)W(u-i) = (\lambda+iu)W(u), \qquad (\lambda - 1 + iu)W(u+i) = (\lambda - iu)W(u).$$

Shift contours by $\pm i$; the nearest weight poles have height $\lambda > 1$, and the packets below are analytic and decay uniformly on the required closed shift strip. Thus vertical edges vanish. If Π_N denotes orthogonal projection through degree N, Bessel's inequality and completeness yield

$$\|f - \Pi_N f\|_\varpi \leq \frac{\|\mathcal{D}f\|_\varpi}{N+1}. \tag{8}$$

2.4 Reference cancellation on the shift strip

Cancel the reference before projection. The gamma representation of ϖ is analytic and nonzero in $|\operatorname{Im} z| < 18$; it has an even analytic square root, chosen positive on the real axis. The gamma recurrence gives, for real $u = x/4$,

$$\frac{\varpi(x+4i)}{\varpi(x)} = \frac{\lambda - iu}{\lambda - 1 + iu}, \qquad \frac{\varpi(x)}{|\varpi(x \pm 4i)|} \le 1. \tag{9}$$

3 Growing negative inertia at a fixed theta cutoff

One negative polynomial does not determine how many independent obstructions remain at higher degrees. We now construct a common negative subspace whose dimension grows with degree, in each parity. The reference measure and classical Meixner–Pollaczek projection identities are those of (4)–(6) and (8). The new steps are uniform over a whole concentration subspace; separately optimized negative vectors would not suffice.

For a real even signed density g, let $\mathcal{P}_N^\epsilon$ be the real polynomials of degree at most N and parity $\epsilon \in \{0, 1\}$. Write

$$\mathfrak{q}_g(P) = \int_{\mathbb{R}} P(x)^2 g(x)\, dx, \qquad \iota_\epsilon(N; g) = i_-(\mathfrak{q}_g|_{\mathcal{P}_N^\epsilon}),$$

where i_- counts strictly negative eigenvalues with multiplicity. Inertia is independent of the real polynomial basis. For $g_A = \widehat{J}_A$, Fourier inversion gives $\mathcal{Q}_A(P) = \mathfrak{q}_{g_A}(P)/(2\pi)$, so this is exactly the negative inertia of the corresponding raw moment block.

Theorem 2 (Effective growing theta inertia)**.** *Fix the complete theta tail at $A = 23/2$. For an integer N put*

$$B_N = \frac{1}{4} \log \frac{N+1}{10^{23}}, \qquad K_N = \max\left(0, \left\lceil \frac{3B_N}{2\pi} - \frac{3 + 2\log(3B_N/2)}{\pi^2\, 10^{-14}} \right\rceil\right).$$

For every N with $B_N \ge 2/3$, both parity blocks satisfy

$$\iota_\epsilon(N; g_A) \ge K_N, \qquad \epsilon = 0, 1. \tag{10}$$

In particular, for finite fixed constants C_1, C_2,

$$\iota_\epsilon(N; g_A) \ge \frac{3}{8\pi} \log(N+1) - C_1 \log\log(N+1) - C_2 \tag{11}$$

for all sufficiently large integers N. The constructed real subspace has the common bound

$$\mathfrak{q}_{g_A}(P) \le -4.9 \cdot 10^{-7} \|P\|_\varpi^2. \tag{12}$$

Rational-coefficient negative subspaces of the same dimension exist, with a possibly slightly smaller strict reserve.

The domain condition is satisfied, for example, by $N \ge 10^{25}$. The count can nevertheless remain zero at extraordinarily large degrees: the explicit estimate is effective but conservative, and is not a practical prediction of the first failing degree. The reserve is measured in the positive reference norm, not the raw monomial coefficient norm. It must not be read as a monomial-coordinate eigenvalue floor.

3.1 A general quantitative band-to-inertia transfer

We prove the fixed-cutoff result through a statement for the same reference ϖ and an arbitrary bounded signed density ratio. This separates the actual-theta input from the classical concentration and projection mechanisms.

Theorem 3. *Let g be real even measurable with $|g/\varpi| \le M$ almost everywhere. Suppose $c > R > 0$, $0 < \eta \le M$, and*

$$g/\varpi \le -\eta \quad \text{on} \quad E = [c-R, c+R] \cup [-c-R, -c+R]$$

almost everywhere. Put $\lambda = 9/2$ and define

$$q = \frac{\eta}{M+\eta}, \qquad \varepsilon_0 = \frac{e^{-2R^2} q}{64}, \qquad \delta_0 = \frac{\sqrt{q}}{8},$$
$$A_c = \lambda + \frac{c}{4} + \frac{1}{4\sqrt{2e}}, \qquad C_0 = \lambda + 2\sqrt{2}\, e^{16+R^2} A_c.$$

For every $B \ge 0$ and integer N such that

$$\frac{C_0 e^{4B}}{N+1} \le \delta_0, \tag{13}$$

each parity has a polynomial subspace of dimension at least

$$\max\left(0, \left\lceil \frac{2RB}{\pi} - \frac{3 + 2\log_+(2RB)}{\pi^2 \varepsilon_0} \right\rceil\right) \tag{14}$$

on which $\mathfrak{q}_g(P) \le -(11\eta/27)\|P\|_\varpi^2$. Here $\log_+ u = \max(0, \log u)$ for $u > 0$, with value zero at $u = 0$. Consequently each negative index is at least

$$\frac{R}{2\pi}\log(N+1) - O(\log\log(N+1)).$$

All constants in this conclusion depend only on the displayed band data.

3.2 An elementary concentration-subspace count

Let $PW_B \subset L^2(\mathbb{R})$ denote functions with Fourier support in $[-B, B]$, with angular-frequency normalization. Its orthogonal projection has kernel

$$k_B(x-y) = \frac{\sin(B(x-y))}{\pi(x-y)}, \qquad k_B(0) = B/\pi.$$

For $I = [-R, R]$, restriction $A_B : PW_B \to L^2(I)$ is Hilbert–Schmidt. The positive trace-class contraction $T_B = A_B^* A_B$ therefore obeys

$$\operatorname{tr} T_B = \frac{2RB}{\pi}, \qquad \operatorname{tr} T_B^2 = \int_{I\times I} |k_B(x-y)|^2\, dx\, dy.$$

The second equality follows from the nonzero spectrum of $A_B A_B^*$ and its sinc kernel. The projection identity $\int_{\mathbb{R}} |k_B(x-y)|^2 dy = k_B(0)$ gives the exact defect

$$\begin{aligned} D_B := \operatorname{tr}(T_B - T_B^2) &= \frac{2}{\pi^2}\int_0^{2R} \frac{\sin^2(Bu)}{u}\, du + \frac{4R}{\pi^2}\int_{2R}^{\infty} \frac{\sin^2(Bu)}{u^2}\, du \\ &\le \frac{3 + 2\log_+(2RB)}{\pi^2}. \end{aligned} \tag{15}$$

Indeed, split the first integral at $1/B$ when that point lies inside it and use $\sin^2(Bu) \le \min(B^2u^2, 1)$. The first term is at most $(1 + 2\log(2RB))/\pi^2$ when $2RB \ge 1$, and at most $1/\pi^2$ otherwise; the second is at most $2/\pi^2$.

For any $0 < \varepsilon < 1$, take the spectral subspace U_B of T_B with eigenvalues at least $1 - \varepsilon$. Every $\psi \in U_B$ satisfies

$$\|\mathbf{1}_{\mathbb{R}\setminus I}\psi\|_2^2 \le \varepsilon\|\psi\|_2^2. \tag{16}$$

For every remaining eigenvalue $\alpha < 1 - \varepsilon$, one has $\alpha \le \alpha(1-\alpha)/\varepsilon$. Splitting the trace thus proves

$$\dim U_B \ge \frac{2RB}{\pi} - \frac{D_B}{\varepsilon}. \tag{17}$$

The kernel is real, so U_B has a real form of this same dimension. We use functions real on the real axis. Compact Fourier support makes them entire, and Plancherel gives the uniform estimate

$$\|\psi(\cdot + iy)\|_2 \le e^{B|y|}\|\psi\|_2. \tag{18}$$

This elementary trace argument is deliberately conservative in ε; no sharp prolate-eigenvalue asymptotic is imported.

3.3 Damping and reflection preserve a whole subspace

Use $\varepsilon = \varepsilon_0$ from Theorem 3, and set

$$h(v) = e^{-v^2}\psi(v), \quad \alpha_0 = e^{-2R^2}(1 - \varepsilon_0), \quad \zeta = \frac{\varepsilon_0}{\alpha_0} \le \frac{q}{32}.$$

Uniformly for every $\psi \in U_B$,

$$\|h\|_2^2 \ge \alpha_0\|\psi\|_2^2, \qquad \|\mathbf{1}_{\mathbb{R}\setminus I}h\|_2^2 \le \zeta\|h\|_2^2.$$

The first inequality uses the minimum e^{-2R^2} of the squared damping on I. It is not a norm comparison inferred from one optimized packet.

Put $u(x) = h(x - c)$, $v(x) = h(-x - c)$ and

$$F_+(x) = \frac{u(x) + v(x)}{\sqrt{\varpi(x)}}, \qquad F_-(x) = \frac{u(x) - v(x)}{\sqrt{\varpi(x)}}. \tag{19}$$

These are even and odd respectively for an arbitrary base function h. Replacing v by $h(x+c)$ would be invalid without an evenness hypothesis on h.

The two translated bands are disjoint. Partitioning $\langle u, v\rangle$ into the right band and its complement bounds its absolute value by $2\sqrt{\zeta}\|h\|_2^2$. Hence

$$2(1 - 2\sqrt{\zeta})\|h\|_2^2 \le \|F_\pm\|_\varpi^2 \le 2(1 + 2\sqrt{\zeta})\|h\|_2^2. \tag{20}$$

In particular the parity maps are injective and

$$\|F_\pm\|_\varpi \ge \frac{e^{-R^2}}{\sqrt{2}}\|\psi\|_2, \qquad \frac{\|\mathbf{1}_{\mathbb{R}\setminus E}F_\pm\|_\varpi^2}{\|F_\pm\|_\varpi^2} \le \frac{2\zeta}{1 - 2\sqrt{\zeta}} \le \frac{q}{8}. \tag{21}$$

The numerator outside E is at most $4\zeta\|h\|_2^2$. We used $\zeta \le q/32 \le 1/32$ in the last comparisons. All statements hold for every linear combination, since it is the image of another member of U_B.

3.4 A uniform polynomial projection and common negative form

The analytic square root of ϖ and its exact gamma quotient give

$$\frac{\varpi(x)}{|\varpi(x \pm 4i)|} \le 1;$$

see (9). The functions (19) meet the contour hypotheses behind (8): the reference is nonzero in $|\operatorname{Im} z| < 18$, the compact-band functions grow at most exponentially on each horizontal strip, and the Gaussian factor dominates every vertical edge. Thus the previously derived classical MP identity gives

$$\|F_\pm - \Pi_N F_\pm\|_\varpi \le \frac{\|\mathcal{D}F_\pm\|_\varpi}{N+1}.$$

For $w = x - c$ the multiplier estimate is

$$|(\lambda - ix/4)e^{-(w+4i)^2}| \le e^{16}(\lambda + c/4 + |w|/4)e^{-w^2} \le e^{16}A_c.$$

The reflected translate has the same bound. The reference quotient and (18) imply, for either shift,

$$\|(\lambda \mp ix/4)F_\pm(\cdot \pm 4i)\|_\varpi \le 2e^{16}A_c e^{4B}\|\psi\|_2.$$

The half-sum of the two shifts in $\mathcal{D}$ leaves the same coefficient. Divide by (21) and add the diagonal term $\lambda F_\pm$ to obtain

$$\frac{\|\mathcal{D}F_\pm\|_\varpi}{\|F_\pm\|_\varpi} \le C_0 e^{4B}. \tag{22}$$

Crucially C_0 is independent of B, dimension, and the choice of vector in the concentration subspace. A pointwise bandlimited estimate containing $\sqrt{B}$ is not used for this operator norm.

Proof of Theorem 3. Normalize an arbitrary nonzero F in either parity image to $\|F\|_\varpi = 1$, and put $P = \Pi_N F$. Under (13), $\|P - F\|_\varpi \le \delta_0 \le 1/8$. Therefore $7/8 \le \|P\|_\varpi \le 9/8$ and

$$\|\mathbf{1}_{\mathbb{R}\setminus E}P\|_\varpi \le \sqrt{q/8} + \sqrt{q}/8 < \sqrt{q}/2.$$

The regionwise bound on the signed ratio gives the common estimate

$$\mathfrak{q}_g(P) \le -\eta\|P\|_\varpi^2 + (M+\eta)\|\mathbf{1}_{\mathbb{R}\setminus E}P\|_\varpi^2 \le -\frac{33\eta}{64} \le -\frac{11\eta}{27}\|P\|_\varpi^2.$$

Projection is injective on the whole parity image because its uniform relative error is < 1. Equations (15)–(17) prove (14) by the finite-dimensional characterization of negative inertia. Simultaneous sufficiently small rational approximation of a finite basis preserves both its independence and strict negative Gram matrix.

Finally choose $B = [\log(N+1) - C_*]/4$, where $C_* = \log(C_0/\delta_0)$. It satisfies (13), and for sufficiently large N it is positive. Substitution in (14) yields the stated leading logarithmic term and $O(\log\log N)$ deficit. For example, after the domain $2RB \ge 1$ is reached, admissible coefficients of the subtracted $\log\log(N+1)$ and constant terms are

$$C_1 = \frac{2}{\pi^2\varepsilon_0}, \qquad C_2 = \frac{RC_*}{2\pi} + \frac{3 + 2\log_+(R/2)}{\pi^2\varepsilon_0}.$$

□

3.5 The effective actual-theta constants

The complete closed negative band supplied by Proposition 11 permits

$$c = 617/4, \quad R = 3/4, \quad \eta = 619 \cdot 10^{-9}, \quad M = 1612800.$$

For this specialization we can improve the conservative generic tolerance to $\varepsilon = 10^{-14}$. Before dividing by $\|h\|$, the cross estimate gives directly

$$\|F_\pm\|_\varpi^2 \geq \{2e^{-9/8}(1-\varepsilon) - 4\sqrt{\varepsilon}\}\|\psi\|_2^2 > \frac{16}{25}\|\psi\|_2^2,$$
$$\frac{\|\mathbf{1}_{\mathbb{R}\setminus E}F_\pm\|_\varpi^2}{\|F_\pm\|_\varpi^2} \leq \frac{25\varepsilon}{4}.$$

Thus (22) holds with the convenient coefficient 10^9, since

$$\lambda + \frac{5}{2}e^{16}\left(\lambda + \frac{c}{4} + \frac{1}{4\sqrt{2e}}\right) < 10^9.$$

The bandwidth B_N in Theorem 2 consequently gives relative projection error at most $\delta = 10^{-14}$. Outward arithmetic verifies the explicit comparisons

$$\begin{array}{c|c} 2e^{-9/8}(1-\varepsilon) - 4\sqrt{\varepsilon} & > 0.6493045 > 16/25 \\ \lambda + (5/2)e^{16}A_c & < 959027273 < 10^9 \\ -\eta(1-\delta)^2 + (M+\eta)(\sqrt{25\varepsilon/4} + \delta)^2 & < -5 \cdot 10^{-7}. \end{array}$$

After division by $(1+\delta)^2$, the last line is still $< -4.9 \cdot 10^{-7}$. This proves the uniform reference reserve (12). Equations (15)–(17), with $2RB_N = 3B_N/2 \geq 1$, give exactly K_N. This proves Theorem 2 with all common-subspace conditioning, reflection and projection steps paid.

The leading logarithmic scale is consistent with classical weighted Christoffel–Darboux/Nevai concentration. The proof uses the classical spectral principle. The quantitative input here is the actual complete-theta negative band and the explicit uniform transfer above, including the conservative effective count and its $O(\log\log N)$ deficit.

3.6 No bounded-rank repair can restore all orders

Corollary 4. *Let Q_N be the raw complete-theta tail moment form at $A = 23/2$, restricted either to one parity or to the full degree-N polynomial space. For any symmetric correction R_N of rank at most r_N,*

$$i_-(Q_N + R_N) \geq \max(0, i_-(Q_N) - r_N).$$

Consequently a positive-semidefinite repair in either parity requires

$$r_N \geq \frac{3}{8\pi}\log N - O(\log\log N).$$

On the full polynomial space, any positive-semidefinite repair requires

$$r_N \geq \frac{3}{4\pi}\log N - O(\log\log N).$$

The correction may depend on N and may couple the two parities.

Proof. Intersect a maximal negative subspace of Q_N with the nullspace of the matrix of R_N. The intersection loses at most r_N dimensions and remains strictly negative for the sum. This classical rank argument does not require commuting matrices or shared eigenvectors. For the full space, evenness makes the original form block diagonal, so its negative index is the sum of the two parity indices. Apply Theorem 2 before subtracting the rank of the unrestricted correction. □

For example, r nonnegative real point-evaluation squares have rank at most r; their nodes and coefficients may change with N. A bounded number cannot repair all orders. The invariant statement is the rank inequality: one must not double count the ranks of two diagonal compressions when a correction couples parities. The rank obstruction applies to corrections satisfying its stated finite-rank hypothesis.

3.7 A classical density-failure control

Negative bands, a bounded density ratio, and every finite absolute moment alone do not imply even one negative polynomial square. The following lognormal Stieltjes perturbation is a classical scope control, in the tradition of Heyde [22], with the precise signed example derived here.

For $x \neq 0$ put

$$w(x) = \frac{e^{-(\log|x|)^2/2}}{2|x|\sqrt{2\pi}}, \qquad g(x) = w(x)\{1 + 2\sin(2\pi \log|x|)\},$$

and set both values to zero at the origin. The even probability density w has all absolute moments, $|g| \leq 3w$, and $g < 0$ on open reflected bands. Nevertheless all polynomial moments of g equal those of w. Odd moments vanish by parity. For even integers $n \geq 0$, Gaussian completion with a standard normal variable Y gives

$$\int_{\mathbb{R}} x^n w(x) \sin(2\pi \log|x|) dx = \operatorname{Im} \mathbb{E} e^{(n+2\pi i)Y} = e^{(n^2-4\pi^2)/2} \sin(2\pi n) = 0.$$

Therefore $\int P^2 g = \int P^2 w > 0$ for every nonzero real polynomial, in both parities and at every degree. The bounded nonzero function $\sin(2\pi \log|x|)$ is an explicit polynomial annihilator in $L^2(w)$. The reference lacks every positive exponential moment, so the completeness and shift estimates of the actual Meixner–Pollaczek reference cannot be imported. This example does not contradict Theorem 3; it explains why its specified reference and analytic approximation structure are substantive hypotheses.

3.8 Every sufficiently large fixed cutoff

The same transfer also combines with the previously established moving-band input, without asserting uniformity in two unbounded parameters.

Corollary 5. *There is a finite A_2 such that, for every fixed real $A \geq A_2$ and each parity,*

$$\iota_\epsilon(N; g_A) \geq \frac{1}{\pi} \log(N+1) - C_1(A) \log\log(N+1) - C_2(A)$$

for all sufficiently large integers N, where the constants and the degree onset may depend on A.

Proof. The uniform-band deduction (36) in the proof of Theorem 10 gives negative bands of radius two about $x_A = 2\tau(A)$ and its reflection, with ratio bound $-\eta_A$, $\eta_A = c_1 A^{-9/2} \log^2 A > 0$, at every sufficiently large cutoff. Enlarge that existential threshold if needed so that $A \geq 11$ and $x_A > 2$.

For each such fixed A, apply Theorem 3 with $c = x_A$, $R = 2$, the same global M and that fixed positive η_A. Its coefficient $R/(2\pi)$ is $1/\pi$. The constants and projection threshold now depend on the chosen cutoff. This is a consequence of the proved theta band input and the common-subspace theorem. □

4 The all-order monotone cone fails

The finite-order positive conclusions recalled in Section 2 have a genuine quantifier boundary. The results and asymptotic argument in this section are inherited from [21]; the complete derivation is retained for self-contained use below. For every fixed degree there is a positive terminal center region. The result below concerns the *complete* theta density, not a finite-atom surrogate.

Use the raw tail kernel, Fourier convention and moment form of Section 2. The cutoff threshold below is existential; it is distinct from the explicit fixed-cutoff certificate used for growing inertia.

Theorem 6 (Uniform eventual cutoff obstruction)**.** *Put $d = \pi/\log 2$ and*

$$\tau(A) = \left(2\left\lfloor \frac{5A}{2d} \right\rfloor + 1\right) d.$$

There exists a finite A_0 such that, for every real $A \geq A_0$,

$$\widehat{J}_A(2\tau(A)) < 0.$$

For each such cutoff there is a finite nonzero polynomial with rational coefficients for which $\mathcal{Q}_A(P) < 0$. At least one parity admits such a polynomial. Any such polynomial degree N satisfies $N > \sqrt{A/1000}$.

The threshold A_0 and these polynomial coefficients are not numerically specified by this asymptotic theorem. Thus the weight $W = \mathbf{1}_{[a(A),\infty)}$ disproves the stronger all-order monotone-weight assertion, while satisfying every finite moment-integrability condition. The theorem does not contradict the degree-eight theorem or the physical-weight conclusions at any proved finite degree.

4.1 Exact grouping of all atom products

Write $X = \pi e^{4s}$. The nth theta atom is

$$\Phi_n(s) = \pi n^2 e^{5s}(2n^2 X - 3)e^{-n^2 X}.$$

For $q = nm$, $Y = qX$ and $u = 2t + \log(n/m)$, the paired exponential is $e^{-2Y\cosh u}$ and its polynomial factor is $4Y^2 - 12Y\cosh u + 9$. The standard integral representation of K [13] gives the exact Fourier factor

$$e^{-i\tau\log(n/m)}\mathcal{H}_\tau(Y), \qquad \mathcal{H}_\tau(Y) = (4Y^2+9)K_{i\tau}(2Y) + 6Y\partial_Y K_{i\tau}(2Y), \quad x = 2\tau.$$

Here $6Y\partial_Y K_{i\tau}(2Y) = 12YK'_{i\tau}(2Y)$; the prime differentiates the Bessel argument. Also

$$s^2\pi^2 e^{10s}\,ds = \frac{X^{3/2}\log^2(X/\pi)}{64\sqrt{\pi}}\,dX.$$

Define the real divisor phase sum and tail integral

$$D_q(\tau) = \sum_{nm=q} e^{-i\tau\log(n/m)}, \qquad |D_q| \leq d(q) \leq q,$$

$$I_q(\tau, b) = \int_b^\infty Y^{3/2}\log^2\left(\frac{Y}{q\pi}\right)\mathcal{H}_\tau(Y)\,dY.$$

Then

$$64\sqrt{\pi}\,\widehat{J}_A(2\tau) = \sum_{q\geq 1} q^{-1/2} D_q(\tau) I_q(\tau, qA). \tag{23}$$

All sum/integral interchanges are absolute: before oscillatory cancellation the pair transform is bounded by $(4Y^2+9)K_0(2Y) + 12YK_1(2Y)$. Its lower endpoint is qA, its other factors have polynomial/logarithmic growth, and its exponential decay sums over all q. This argument does not discard atoms whose shifted arguments are negative.

4.2 Complete Mellin mass and a uniform oscillatory subtraction

For $\mu > 0$ the classical Mellin formula [13, 10.43.19], with the $2Y$ scaling retained, is

$$M_\mu(\tau) = \int_0^\infty Y^{\mu-1} K_{i\tau}(2Y)\, dY = \tfrac{1}{4}\Gamma((\mu+i\tau)/2)\Gamma((\mu-i\tau)/2).$$

Integration by parts and the gamma recurrence give

$$F_\mu(\tau) := \int_0^\infty Y^{\mu-1}\mathcal{H}_\tau(Y)\, dY = 4M_{\mu+2} + (9-6\mu)M_\mu = [(\mu-3)^2+\tau^2]M_\mu.$$

Logarithmic differentiation under the convergent integrals yields

$$I_q(\tau,0) = (\partial_\mu - \log(q\pi))^2 F_\mu(\tau)\big|_{\mu=5/2}.$$

Set

$$B(\tau) = 2\pi(\tau/2)^{7/2} e^{-\pi\tau/2}\log^2\tau > 0.$$

Differentiated Stirling asymptotics, locally uniform near $\mu = 5/2$, imply $I_q(\tau,0)/B(\tau) = 1 + O(1/\log\tau)$ for each fixed q. In particular $F_{5/2} = (\tau^2/4 + 1/16)|\Gamma(5/4+i\tau/2)|^2$, fixing the leading sign and constant.

We require a uniform estimate for the *lower* incomplete integral. For each fixed $r < 1$, $\mu > 0$, and integer $0 \leq l \leq 2$,

$$\int_0^{r\tau/2} Y^{\mu-1}\log^l(Y/(q\pi))K_{i\tau}(2Y)\, dY = O\Big(e^{-\pi\tau/2}\tau^{\mu-3/2}(1+\log\tau)^l\Big). \tag{24}$$

The constants are uniform when the upper ratio varies in a fixed compact subset of $(0,1)$. Here and below q is fixed when this estimate is used.

For completeness we specify the classical large-order interface needed for this endpoint estimate. On $0 < z \leq r < 1$, the oscillatory branches in Dunster's Theorem 1, equations (17)–(19), (25)–(27) of [12], together with the exact I–K connection [13, 10.27.4], give

$$e^{\pi\tau/2}K_{i\tau}(\tau z) = \sqrt{2\pi/\tau}(1-z^2)^{-1/4}\cos(\tau\psi(z) - \pi/4) + O(\tau^{-3/2}), \tag{25}$$

uniformly down to $z = 0$, where $\psi(z) = \operatorname{arcosh}(1/z) - \sqrt{1-z^2}$ and $\psi'(z) = -\sqrt{1-z^2}/z$. This is the oscillatory I-branch continuation, not an application of the recessive K formula across a turning point. In the source notation the differential error functions F_s are $O(z^2)$ at zero and the error-path differential has size $O(dz/z)$; hence their products are integrable there. The integrated coefficients E_s need not vanish at zero. The source remainder integrals and connection normalizers therefore have uniform $O(1/\tau)$ relative error on $0 < z \leq r$. This yields the absolute remainder in (25) without differentiating it.

After $Y = \tau z/2$, integrate the leading cosine once by parts using ψ'. The lower endpoint vanishes because $z^\mu(1+|\log z|)^l \to 0$, and the derivative of amplitude divided by ψ' is integrable on $(0, r]$. The remainder in (25) is integrated absolutely. Both terms give (24).

For $\mathcal{H}_\tau$, integrate its derivative-K term by parts on the lower finite interval first. The resulting integrals have $\mu = 9/2, 5/2$ and logarithmic powers at most two. The retained endpoint is

$$6b^{5/2}\log^2(b/(q\pi))K_{i\tau}(2b),$$

which is $O(e^{-\pi\tau/2}\tau^2\log^2\tau)$ when $2b/\tau$ stays below turning. Consequently, uniformly for $19/100 \le A/\tau \le 21/100$,

$$I_q(\tau, qA)/B(\tau) = 1 + O(1/\log\tau) + O(\tau^{-1/2}), \qquad q = 1, 2. \tag{26}$$

Their lower ratios lie in $[.38, .42]$ and $[.76, .84]$. Only the subtracted incomplete integrals use the below-turning expansion; the exact complete Mellin expression already includes the turning region.

4.3 A uniform bound for the entire omitted product sum

Choose one fixed contour angle

$$v = \arcsin(7/8), \qquad c = \cos v = \sqrt{15}/8.$$

Shifting the defining K integral by iv gives

$$|K_{i\tau}(2Y)| \le e^{-\tau v}K_0(2cY), \qquad |\partial_Y K_{i\tau}(2Y)| \le 2e^{-\tau v}K_1(2cY).$$

The vertical sides vanish exponentially; $|\cosh(u+iv)| \le \cosh u$ pays the differentiated amplitude. For $q \ge 3$ and $A/\tau \ge 19/100$, the exponent at $Y \ge qA$ is at least $\tau(v + 19qc/50)$. At the first omitted product its gap above $\pi\tau/2$ is strictly greater than $\tau/960$. Indeed, for $\alpha = \arccos(7/8)$,

$$\cos(8/15) \le 1 - \tfrac{1}{2}(8/15)^2 + \tfrac{1}{24}(8/15)^4 = \frac{130787}{151875} < \frac{7}{8},$$

so $\alpha < 8/15$, whereas $57\sqrt{15}/400 > 171/320$ and $171/320 - 8/15 = 1/960$. Each further q gains an exponent exceeding $57\tau/320$.

The elementary Gaussian bounds for K_0, K_1 at large positive argument, together with $d(q) \le q$, now bound the *complete* $q \ge 3$ sum by a polynomial in $\tau, \log\tau$ times $e^{-(\pi/2+1/960)\tau}$. Constants are uniform for $A/\tau \in [.19, .21]$: after $Y = qA + w$, the logarithm is $\log(A/\pi) + \log(1 + w/(qA))$, and the remaining powers of q are dominated by the geometric exponent just proved. In particular the diagonal product $q = 4$ is included in this negligible remainder.

Proof of Theorem 6. The nearest odd grid point obeys $|\tau(A) - 5A| \le d$ and $\cos(\tau(A)\log 2) = -1$, ties included. Since $d < 6$, for $A \ge 26$ one has $A/\tau(A) \in [.19, .21]$. This numerical bound establishes uniformity only, not negativity already at 26.

In (23), $D_1 = 1$ and $D_2 = 2\cos(\tau\log 2)$. Equation (26) and the infinite remainder estimate imply, uniformly along this piecewise constant frequency choice,

$$\frac{64\sqrt{\pi}\,\widehat{J}_A(2\tau(A))}{B(\tau(A))} = 1 - \sqrt{2} + O(1/\log\tau(A)) + O(\tau(A)^{-1/2}) < 0$$

for every sufficiently large A. Section 4.4 proves the finite-polynomial consequence. Finally, if $A \ge 1000N^2$, the proved local degree-N theorem makes its raw integral over the entire tail strictly positive. Any negative degree therefore satisfies $A < 1000N^2$. □

4.4 Why a Fourier sign forces a finite polynomial failure

For each fixed $A \geq \pi$, J_A is real even, integrable on the real axis, and analytic in $|\operatorname{Im} t| < \pi/4$. On compact subsets of this strip, $\cos(2 \operatorname{Im} t) > 0$ gives normally convergent shifted theta sums and superexponential domination in the center integral. Evenness and theta decay also pay the real-t integral. Moreover $J_A(0) > 0$.

Suppose every finite matrix of the normalized moments $m_k(A)/J_A(0)$ were PSD. The Hamburger moment theorem supplies a positive probability measure with those moments. Cauchy's estimate for the actual analytic germ gives $m_{2k}/J_A(0) \leq C(2k)!r^{-2k}$ for some $r > 0$. Monotone convergence applied to the series for cosh gives an exponential moment at each smaller rate. The representing characteristic function is therefore analytic in a strip and has exactly the same derivatives at zero as $J_A/J_A(0)$. Local equality and analytic continuation give equality everywhere on the real axis. Thus J_A would be positive definite; its continuous Fourier transform would be nonnegative, a contradiction.

Some finite real quadratic form is therefore strictly negative. Rational coefficient vectors are dense, so rational approximation preserves strict negativity. Odd moments vanish, so splitting a polynomial into its even and odd parts leaves at least one negative parity square. The moment/positive-definite bridge is classical; the complete-theta Fourier sign is the new input. Related associated-kernel questions are formulated in [3, Sections 3–4].

4.5 An explicit certified complete-theta Fourier obstruction

Proposition 7. *For the exact cutoff $A = 23/2$,*

$$-8 \cdot 10^{-49} < \widehat{J}_A(154) < -3 \cdot 10^{-49}.$$

For every x in the closed interval $[769999/5000, 770001/5000]$,

$$-10^{-48} < \widehat{J}_A(x) < -10^{-50}.$$

The same interval conclusion holds after reflection about zero.

Here is the full certification mechanism, implemented in the two portable supplements. Scale (23) by $e^{\pi\tau/2}$ and retain $q = 1, \ldots, 8$ through $Y = 256$. After one integration by parts, the retained approximation to each integral is

$$\int_{qA}^{256} Y^{3/2}\{(4Y^2 - 6)L^2 - 12L\}K_{i\tau}(2Y)\, dY - 6(qA)^{5/2} \log^2(A/\pi) K_{i\tau}(2qA), \quad L = \log(Y/(q\pi)).$$

Outward Arb analytic quadrature at 256 bits evaluates this expression at $\tau = 77$. Its analytic callback requires $\operatorname{Re} Y > 0$, paying the logarithm, square-root and Bessel branches. A finite enclosure is used regardless of whether requested integration tolerances are met; only the strict enclosure gate accepts a sign.

For both discarded tails, shift by $i\pi/3$. For $Y \geq 1$,

$$e^{\pi\tau/2}|K_{i\tau}(2Y)| \leq e^{\pi\tau/6}\sqrt{\pi/(2Y)}e^{-Y},$$

and for $Y \geq 2$ its Y derivative is at most four times this envelope. The full $\mathcal{H}$ amplitude is then at most $37Y^2$ times the envelope. Using $L^2 \leq (Y/(q\pi))^2$, $d(q) \leq q$, $\int_b^\infty Y^5 e^{-Y} dY \leq 2b^5 e^{-b}$ for $b \geq 11$,

and the geometric ratio $(10/9)^4 e^{-A} < 1/2$, the total scaled error is at most

$$E_q = \frac{148\sqrt{\pi/2}}{\pi^2} A^5\, 9^4 e^{\pi\tau/6-9A},$$
$$E_Y = \frac{222\sqrt{\pi/2}}{\pi^2} 256^5 e^{\pi\tau/6-256}.$$

The second bound also pays the integrated-by-parts integrand, whose smaller amplitude constant is 22; $\sum_{q=1}^{8} q^{-3/2} < 3$. At 77, $E_q < 10^{-17}$ and $E_Y < 10^{-80}$. The resulting unscaled endpoints are approximately $-7.351 \cdot 10^{-49}$ and $-3.166 \cdot 10^{-49}$, proving the point bounds. A distinct implementation uses the original $\mathcal{H}$ integrand and $\partial_Y K_{i\tau}(2Y) = -K_{i\tau-1}(2Y) - K_{i\tau+1}(2Y)$, without the endpoint integration by parts, and independently certifies strict negativity.

To pay the interval statement rather than assume a bandwidth from plots, put $S(\tau) = 64\sqrt{\pi} e^{\pi\tau/2} \widehat{J}_A(2\tau)$ and let T be its retained finite-Y, finite-q part. Fix $v = 3/2$, $c = \cos v$, $\delta = \pi/2 - v$. Shifting the full-line K integral and differentiating the scaled transform give, for $76 \le \tau \le 78$,

$$|e^{\pi\tau/2} K_{i\tau}(2Y)| \le e^{78\delta - 2cY} \sqrt{\pi/(4cY)},$$
$$|\partial_\tau (e^{\pi\tau/2} K_{i\tau}(2Y))| \le e^{78\delta - 2cY} \{\delta \sqrt{\pi/(4cY)} + (2cY)^{-1}\}.$$

These follow from the Gaussian zeroth and first absolute moments on the shifted contour, not from differentiating a Bessel asymptotic. Also $|D_q'| \le q \log q$. Let $b = qA$, $d_0 = \log(A/\pi) - 1 > 0$,

$$C_q = (\log q + \delta)\sqrt{\pi/(4c)} + (2c)^{-1}, \qquad G_n(b) = e^{-2cb} \sum_{j=0}^{n} \frac{n!}{j!} \frac{b^j}{(2c)^{n+1-j}}.$$

Concavity gives $L \le Y/(qA) + d_0$. The absolute derivative of each retained integral is bounded by

$$22\sqrt{q} e^{78\delta} C_q \{G_5(b)/(qA)^2 + 2d_0 G_4(b)/(qA) + d_0^2 G_3(b)\}.$$

Its endpoint derivative is at most

$$6b^{5/2} \log^2(A/\pi) \sqrt{q} e^{78\delta - 2cb} \{(\log q + \delta)\sqrt{\pi/(4cb)} + (2cb)^{-1}\}.$$

Summing these positive bounds gives $|T'| < 10045000000$. The program checks this strict comparison with outward arithmetic. For $h = 1/10000$, add $h \sup |T'| + 2(E_q(78) + E_Y(78))$ to the point enclosure of $S(77)$, then multiply by the positive factor $e^{-\pi\tau/2}/(64\sqrt{\pi})$ over $\tau \in [77-h, 77+h]$. The resulting endpoints lie strictly between -10^{-48} and -10^{-50}. This proves the stated closed interval with every infinite remainder paid.

Proposition 7 proves finite rational polynomial existence at a specific cutoff via Section 4.4. Section 5 goes further: it supplies exact coefficient witnesses in both parities and bounds the eventual failing-degree growth. Neither result identifies a first failing degree at a specified cutoff. A rigorous finite-atom prefix check or a numerical no-hit at a large degree must not be substituted for complete-theta positivity.

4.6 Comparison and the remaining physical question

The classical associated-kernel program [3] and correlation identities [19] concern different global kernels or Wronskians. Planat's phase-aligned decomposition [18] proves negative individual blocks and cancellation between sectors of the physical growth derivative. Its blocks are finite phase periods

in that decomposition; neither their negativity nor a fixed-center negative optimizer proves the total center-tail sign in Theorem 6. The imaginary-order Bessel asymptotics, Mellin transforms, contour shifts and moment bridge are classical ingredients. The contribution is their complete-theta product grouping, uniform cutoff synchronization and actual total-tail obstruction, including the certified explicit point.

For physical weights, if $F(z) = \int_{\mathbb{R}} \Phi(s) e^{izs} ds = \xi((1+iz)/2)/4$, then

$$\widehat{\mathcal{J}}_y(x) = \frac{\partial_y |F(x+iy)|^2}{8y}, \qquad y > 0.$$

The known zero strip and paired canonical product give strict positivity for every $y \geq 1$, a classical zero-free-half-plane consequence [20]. Hence the remaining physical all-order parameter window is $0 \leq y < 1$, with the continuous normalized endpoint. The negative step weights of this section do not belong to that family.

5 Constructive and quantitative polynomial obstructions

The Fourier obstruction recalled from [21] can be made constructive in the polynomial variable. The witnesses and degree-growth theorem in this section were established there; they are included with complete proofs to make the new negative-inertia development self-contained. We first give exact rational witnesses at a specified cutoff, with a complete continuous-integral certificate. We then bound the first failing degree in both parities for every sufficiently large cutoff. The positive reference measure, Meixner–Pollaczek identities and analytic approximation mechanisms used below are classical. In particular, Bizeul and Klartag [14, Section 1.1, Lemma 7; Section 2, Lemma 9] treat the same convolution family of reference measures and the imaginary-shift approximation mechanism, including its cost for narrow Gaussians. Our assertions concern the complete theta functional to which these tools are applied.

5.1 Exact rational witnesses at one finite cutoff

Let $B_0, \ldots, B_{4096}$ be the entire integer coefficient list in the public supplement `theta_witness_even.json`, and define the ordinary rational polynomials

$$P_e(x) = 10^{-100} \sum_{n=0}^{4096} B_n q_n(x), \qquad P_o(x) = x P_e(x). \tag{27}$$

All odd B_n vanish and $B_{4096} \neq 0$. Thus these degrees and parities are exact, not numerical degree estimates. The file `theta_witness_odd.json` also gives the full rational-basis coefficient list of P_o.

Theorem 8 (Explicit complete-theta polynomial witnesses)**.** *At $A = 23/2$, the polynomials* (27) *have degrees* 4096 *and* 4097*, respectively, and*

$$-\frac{3}{10^7} < \frac{\int_{\mathbb{R}} P_\epsilon(x)^2 g_A(x)\, dx}{\int_{\mathbb{R}} P_\epsilon(x)^2 \varpi(x)\, dx} < -\frac{2}{10^7}, \qquad \epsilon = e, o. \tag{28}$$

In particular $\mathcal{Q}_A(P_e) < 0$ and $\mathcal{Q}_A(P_o) < 0$ for the complete theta density.

The coefficients are specified in a rational polynomial basis to avoid the conditioning and size of an unnecessary monomial expansion. Their finite lists define the polynomials exactly; a floating eigenvector or numerical selection procedure is not a hypothesis of the theorem. We give the entire continuous-integral certification below. The portable verifier `verify_theta_polynomial.py` checks the finite ball data and exact integer identities; its recomputation option evaluates the underlying grid afresh using `theta_witness_fourier.py`.

Complete Fourier values from Mellin mass minus a lower series

At the positive grid points below, $\tau = x/2 > 0$. Define

$$\mathsf{K}_\tau(Y) = e^{\pi\tau/2} K_{i\tau}(2Y), \quad G_\tau = e^{\pi\tau/2}\Gamma(i\tau), \quad \mathsf{M}_\mu = \tfrac{1}{4} e^{\pi\tau/2} |\Gamma((\mu + i\tau)/2)|^2.$$

With $z = (\mu + i\tau)/2$, the full first and second logarithmic moments are

$$\mathsf{M}^{\text{full}}_{\mu,1} = \mathsf{M}_\mu \{\operatorname{Re}\psi(z) - \log(q\pi)\}, \quad \mathsf{M}^{\text{full}}_{\mu,2} = \mathsf{M}_\mu \left[\{\operatorname{Re}\psi(z) - \log(q\pi)\}^2 + \tfrac{1}{2}\operatorname{Re}\psi_1(z)\right]. \tag{29}$$

These are derivatives of the classical complete Mellin formula, so their gamma and scale factors are fixed exactly.

Put $b = qA$, $L_0 = \log(A/\pi)$, $c_j = b^{2j}/[j!(1 - i\tau)_j]$, and $d_{\mu,j} = \mu + 2j - i\tau$. The nonintegral-order Bessel connection formula [13, Section 10.27] gives the convergent expansion

$$\mathsf{K}_\tau(Y) = \operatorname{Re}\left[G_\tau Y^{-i\tau} \sum_{j\geq 0} \frac{Y^{2j}}{j!(1 - i\tau)_j}\right].$$

The lower logarithmic moments are consequently

$$\mathsf{M}^{<b}_{\mu,1} = b^\mu \operatorname{Re}\left[G_\tau b^{-i\tau} \sum_{j\geq 0} c_j \left(\frac{L_0}{d_{\mu,j}} - \frac{1}{d^2_{\mu,j}}\right)\right], \tag{30}$$

$$\mathsf{M}^{<b}_{\mu,2} = b^\mu \operatorname{Re}\left[G_\tau b^{-i\tau} \sum_{j\geq 0} c_j \left(\frac{L_0^2}{d_{\mu,j}} - \frac{2L_0}{d^2_{\mu,j}} + \frac{2}{d^3_{\mu,j}}\right)\right].$$

Absolute convergence on each finite interval justifies the integrations. Write $U_{\mu,l} = \mathsf{M}^{\text{full}}_{\mu,l} - \mathsf{M}^{<b}_{\mu,l}$. The exact scaled group integral is

$$e^{\pi\tau/2} I_q(\tau, qA) = 4U_{9/2,2} - 6U_{5/2,2} - 12U_{5/2,1} - 6b^{5/2} L_0^2 \mathsf{K}_\tau(b). \tag{31}$$

The last sign is the lower boundary from integration by parts, not an omitted endpoint correction.

Here is an explicit absolute remainder for every series used. After retaining $j = 0, \ldots, J$, require $r = b^2/(J + 2)^2 < 1/2$ and put

$$R_J = \frac{|c_{J+1}|}{1 - r}, \qquad v_\mu = \mu + 2(J + 1).$$

All subsequent coefficient ratios are at most r. The tails in (30) are bounded by

$$E_{\mu,1} = |G_\tau| b^\mu R_J (L_0/v_\mu + 1/v_\mu^2),$$
$$E_{\mu,2} = |G_\tau| b^\mu R_J (L_0^2/v_\mu + 2L_0/v_\mu^2 + 2/v_\mu^3), \qquad E_K = |G_\tau| R_J.$$

Thus the total series error in (31) is at most

$$4E_{9/2,2} + 6E_{5/2,2} + 12E_{5/2,1} + 6b^{5/2} L_0^2 E_K.$$

The evaluator makes this $< 10^{-14}$ for each retained group. Outward arithmetic radii are retained in addition to this analytic error.

For a retained cutoff Q, let $q_0 = Q + 1$. The complete omitted product sum has scaled absolute bound

$$E_{>Q} = \frac{148\sqrt{\pi/2}}{\pi^2} A^5 q_0^4 e^{\pi\tau/6 - q_0 A}, \qquad \left(\frac{q_0+1}{q_0}\right)^4 e^{-A} < \frac{1}{2}. \tag{32}$$

This is the same $i\pi/3$ contour and incomplete-gamma bound used in Section 4.5, with its starting group left variable. The grid evaluator chooses an adaptive retained cutoff near $\pi\tau/(6A) + 13$; all acceptance bounds are computed for the actual retained integer, so a choice of cutoff is never substituted for a tail proof. Equations (23) and (32) give the complete Fourier enclosure without an unbounded-Y quadrature omission.

A strip rule with a paid infinite-sample error

For a real polynomial P of degree at most N, put $H = \|P\|_\varpi^2 > 0$ and $F(z) = P(z)^2 g_A(z)/H$. From (6),

$$q_n(x+iy) = \sum_{k=0}^{n} a_k(y) q_{n-k}(x), \qquad \sum_{k\geq 0} a_k(y) t^k = \left(\frac{1+it}{1-it}\right)^{y/4}.$$

For $|y| \leq 1$, the absolute numerator coefficient sum is at most two and each denominator coefficient is at most one. Hence $\sum_{k\leq N} |a_k(y)| \leq 2(N+1)$. Increasing v_n makes each lowering coefficient shift a contraction in the weighted coefficient norm. Therefore

$$\|P(\cdot + iy)\|_\varpi \leq 2(N+1)\|P\|_\varpi, \qquad \int_{\mathbb{R}} |F(x+iy)|dx \leq L_N := 4M(N+1)^2.$$

The strip trapezoidal bound [15, Theorem 5.1], or directly the shifted Fourier contours and their geometric alias sum, now gives

$$\left| \int_{\mathbb{R}} F(x)dx - h\sum_{k\in\mathbb{Z}} F(h(k+1/2)) \right| \leq \frac{2L_N}{e^{2\pi/h} - 1}, \qquad h = 1/10. \tag{33}$$

The half-grid contributes only unit-modulus alias phases. The vertical contour ends vanish because a fixed polynomial times (7) decays uniformly in the strip. Thus neither a finite-frequency plot nor an unproved sampling accuracy assumption enters the certificate.

A paid tail for the omitted discrete samples

We need only a conservative remote bound. For $t = |x|/2 \geq 1$ and $A \geq 11$, define

$$I_A = \frac{\log^2(A/\pi) + 2\log(A/\pi) + 2}{A}.$$

A complete positive majorant gives

$$\left|\frac{g_A(x)}{\varpi(x)}\right| \leq \frac{Z_* e^{\pi/2} I_A}{8192}(1 + e^{-\pi t}) \left\{ 6t^{-5/2} + \frac{96}{e\sqrt{2}} t^{-7/2} + \frac{9}{2} t^{-9/2} \right\}. \tag{34}$$

For completeness, keep the distinct exponential rates in the contour estimates:

$$\begin{aligned} e^{\pi t/2}|\mathcal{H}_t(Y)| \leq e^{\pi/2}\sqrt{\pi}\big[&(2Y^{3/2} + (9/2)Y^{-1/2})t^{1/2}e^{-2Y/t} \\ &+ 12(e\sqrt{2})^{-1} t^{3/2} Y^{-1/2} e^{-Y/t}\big]. \end{aligned}$$

They follow from $K_0(u) \le \sqrt{\pi/(2u)}e^{-u}$ and $K_1(u) \le 2(eu)^{-1}K_0(u/2)$. The elementary lattice bounds

$$\sum_{q\ge1} q^4 e^{-qz} \le 48z^{-5}, \qquad \sum_{q\ge1} q^2 e^{-qz} \le 4z^{-3} \quad (z>0)$$

can be obtained as follows. A nonnegative unimodal f satisfies $\sum_{q\ge1} f(q) \le \int_0^\infty f + \sup f$, by counting integer points in its interval superlevel sets. Applied to $u^m e^{-zu}$ this proves the stated constants for $z \le 4$ when $m = 4$, and $z \le 2$ when $m = 2$, using $e > 8/3$. For the remaining ranges use the exact sums $r(1+11r+11r^2+r^3)/(1-r)^5$ and $r(1+r)/(1-r)^3$, respectively, with $r < 1/50$ or $r < 9/64$, and $z^{m+1}e^{-z} \le (m+1)^{m+1}e^{-(m+1)}$. The resulting rational comparisons are strictly below48 and4.

With $d(q) \le 2\sqrt{q}$, these imply

$$\sum d(q)q^{7/2}e^{-qz} \le 96z^{-5}, \qquad \sum d(q)q^{3/2}e^{-qz} \le 8z^{-3}.$$

Substitute $Y = qX$ in the full nonnegative Fourier majorant and keep $\log^2(X/\pi)$ exactly. Every remaining integral is $\int_A^\infty X^{-2}\log^2(X/\pi)dX = I_A$. Dividing the resulting bound by (4), using $V(2t) \ge 256t^8$, proves (34). Its right side decreases with t; at $A = 23/2$, $|x| \ge 320$, it is $< 0.062918 < 0.063$.

Let $w = P^2\varpi/H$, $B = 7999/20$, and let the omitted samples have $|h(k+1/2)| > 400$. Their cells lie in $|x| \ge B$. Averaging the fundamental-theorem-of-calculus bound on each cell gives

$$h \sum_{\text{omitted}} w(h(k+1/2)) \le \int_{|x|\ge B} w + h\int_{|x|\ge B} |w'|.$$

Put

$$a_{32} = \frac{\|x^{16}P\|_\varpi^2}{H}, \quad a_{30} = \frac{\|x^{15}P\|_\varpi^2}{H}, \quad D = \frac{1+\log(N+16)}{2}, \quad t_0 = \frac{a_{32}}{B^{32}}, \quad s_1 = \frac{D\sqrt{a_{32}} + 16\sqrt{a_{30}}}{B^{16}}.$$

Differentiation of (6) yields odd lowering coefficients $(-1)^{(k-1)/2}/(2k)$, so $\|R'\|_\varpi \le (1+\log d)\|R\|_\varpi/2$ for degree at most $d \ge 1$. Apply this to $R = x^{16}P$ and use $x^{16}P' = R' - 16x^{15}P$. Weighted Cauchy–Schwarz and $|(\log\varpi)'| < 2$ then pay the complete omitted actual sample sum by

$$E_{\text{sample}} = 0.063\{(1+2h)t_0 + 2h\sqrt{t_0}s_1\}. \tag{35}$$

All norms needed here are exact integer coefficient calculations: orthogonality gives $\|\sum b_n q_n\|^2 = \sum v_n b_n^2$, and multiplication by x sends each coefficient b_n to $2(n+1)b_n$ in position $n+1$ and $2(n+8)b_n$ in position $n-1$. This reconstructs the degree-30 and degree-32 moment norms without a numerical integration assumption.

Completion of Theorem 8. For $k = 0, \ldots, 3999$ use the exact nodes $x_k = (2k+1)/20$. The public grid data enclose the terms

$$2h\,\frac{P_e(x_k)^2\varpi(x_k)}{\|P_e\|_\varpi^2}\,\frac{g_A(x_k)}{\varpi(x_k)}.$$

The polynomial recurrence is evaluated outward at 2048 bits and the complete Mellin evaluator at 1024 bits. Four disjoint data ranges cover every index exactly once. Outward serialization preserves every term enclosure. The verifier sums the actual term balls, recomputes all exact integer norms

and powers of the Jacobi action, and adds (33) and (35). The following deliberately rounded bounds suffice:

	P_e	$P_o = xP_e$
finite symmetric sum	$[-2.559513, -2.559511]10^{-7}$	$[-2.609432, -2.609430]10^{-7}$
E_{sample}	$< 6.874\,10^{-11}$	$< 3.835\,10^{-10}$
E_{alias}	$< 1.118\,10^{-13}$	$< 1.118\,10^{-13}$

For the odd companion, multiply each even summand by x_k^2 and by $\|P_e\|^2/\|P_o\|^2$, and recompute its own tail moments and degree-4097 alias bound. Both resulting continuous-integral enclosures satisfy (28). Fourier inversion and the exponential envelope give

$$\mathcal{Q}_A(P) = \frac{1}{2\pi}\int_{\mathbb{R}} P(x)^2 g_A(x)dx,$$

so the moment-functional signs follow. Every theta atom, real-frequency tail, discrete sampling tail and rounding error has been included. □

Corollary 9 (Positive analytic monotone weights also fail)**.** *There are bounded, strictly positive, strictly increasing real-analytic center weights satisfying every finite moment-integrability condition for which both P_e and P_o have negative averaged moment squares.*

Proof. Let $a = \frac{1}{4}\log((23/2)/\pi)$ and $W_\kappa(s) = [1 + e^{-\kappa(s-a)}]^{-1}$. Then $0 < W_\kappa < 1$ and $W_\kappa \to \mathbf{1}_{(a,\infty)}$ away from one point. For either fixed polynomial the complete raw signed center integrand is absolutely integrable: compact smoothness and the superexponential theta decay dominate its finite derivative order. Dominated convergence therefore retains each strictly negative tail value for all sufficiently large κ. Choose one κ working for both polynomials. Dividing by the positive center normalization changes no sign. □

5.2 Polynomial degree growth in both parities

For $\epsilon = 0, 1$ let $N_\epsilon(A)$ be the least degree of a negative real polynomial of parity ϵ for $\mathcal{Q}_A$, when one exists.

Theorem 10. *There is a finite A_1 such that, for every real $A \geq A_1$, both parities have rational negative witnesses and*

$$\sqrt{A/1000} < N_\epsilon(A) \leq \lceil A^{31/4} \rceil, \qquad \epsilon = 0, 1.$$

The onset A_1 is the threshold in the stated asymptotic bound.

Proof. Write $\tau_0 = \tau(A)$ from Theorem 6 and $x_A = 2\tau_0 \sim 10A$. Uniformly for $|v| \leq 1$, the two retained leading products in (23) have coefficient

$$1 - \sqrt{2}\cos(v\log 2) < -57/1000.$$

Indeed $\log 2 < 7/10$ and $\sqrt{2} > 7/5$ give $\sqrt{2}\cos(v\log 2) > (7/5)(151/200) = 1057/1000$. For sufficiently large A, all ratios $A/(\tau_0 + v)$ remain in $[.19, .21]$. The incomplete-Mellin and complete omitted-group bounds used in Theorem 6 are therefore uniform on this whole band. On

$$E_A = [x_A - 2, x_A + 2] \cup [-x_A - 2, -x_A + 2]$$

they give, for constants independent of A,

$$g_A(x)/\varpi(x) \leq -\eta_A, \qquad \eta_A = cA^{-9/2}\log^2 A > 0. \tag{36}$$

Here the Mellin scale is $A^{7/2}\log^2 A\, e^{-\pi|x|/4}$, whereas $V(x) = O(A^8)$ on the band. Globally the same ratio has absolute value at most M by Lemma 1.

Use the complete Meixner–Pollaczek basis and the projection estimate (8) from Section 2. The packets below satisfy its analytic-strip and contour-decay hypotheses.

We use a classical Kaiser–Bessel packet [16, 17], deriving the needed bounds rather than importing a concentration asymptotic. Fix $R = 2$. For $B \geq 1$ set

$$\psi_B(v) = \frac{\sinh(B\sqrt{R^2 - v^2})}{\sqrt{R^2 - v^2}} = \frac{1}{2}\int_{-B}^{B} I_0(R\sqrt{B^2 - w^2})e^{ivw}\, dw.$$

The apparent singularities at $\pm R$ are removable, and the entire series is $\sum_{j\geq 0} B^{2j+1}(R^2 - v^2)^j/(2j+1)!$. To prove the integral formula, integrate $e^{BR\xi_1 + iBv\xi_3}$ on the unit sphere. Azimuthal integration gives the I_0 integral; the spherical exponential integral is $4\pi\sinh(B\sqrt{R^2 - v^2})/(B\sqrt{R^2 - v^2})$. Both identities extend from their power series. Hence ψ_B is bandlimited to $[-B, B]$ and

$$\|\psi_B(\cdot + iy)\|_2 \leq e^{B|y|}\|\psi_B\|_2. \tag{37}$$

For $|v| \geq R$, $|\psi_B(v)|^2 \leq \min(B^2, (v^2 - R^2)^{-1})$. Splitting the outside integral at $R + B^{-2}$ and $R + 1$ bounds its energy by $C_R(1 + \log B)$. On $|v| \leq B^{-1/2}$, $\sqrt{R^2 - v^2} \geq R - v^2/R$ gives $|\psi_B(v)| \geq c_R e^{BR}$ for large B. Therefore

$$\|\psi_B\|_2^2 \geq c_R B^{-1/2} e^{2BR}, \qquad \frac{\|\mathbf{1}_{|v|\geq R}\psi_B\|_2^2}{\|\psi_B\|_2^2} \leq C_R\sqrt{B}(1 + \log B)e^{-2BR}.$$

The damped packet $\phi_B(v) = e^{-v^2}\psi_B(v)$ has comparable norm and the same outside-fraction bound: inside $[-R, R]$ the squared damping is at least e^{-2R^2}, and the original inside fraction tends one. The Gaussian factor also pays all polynomial moments and contour edges.

Use the analytic square root and exact reference-shift identity (9) from Section 2. Define

$$f_0(x) = \frac{\phi_B(x - x_A) + \phi_B(x + x_A)}{\sqrt{\varpi(x)}}, \qquad f_1(x) = \frac{\phi_B(x - x_A) - \phi_B(x + x_A)}{\sqrt{\varpi(x)}}.$$

These functions are analytic on the needed closed shift strip and have the indicated parity. Their weighted norms are exactly the unweighted norms of their numerators. The two center bands are disjoint. If ε_B denotes the one-packet outside fraction, partitioning the cross inner product into one center band and its complement bounds its absolute value by $2\sqrt{\varepsilon_B}\|\phi_B\|_2^2$. Thus both signed norms are comparable to $2\|\phi_B\|_2^2$, including the odd difference, and the mass outside E_A is $O(\varepsilon_B)$ of the total.

Choose $B = (9/8)\log A$, normalize either f_ϵ to norm one, and write $\varepsilon_A = \|\mathbf{1}_{\mathbb{R}\setminus E_A} f_\epsilon\|_\varpi^2$. Then

$$\varepsilon_A = O\big(A^{-9/2}\sqrt{\log A}(1 + \log\log A)\big) = o(\eta_A). \tag{38}$$

To bound the operator cost, put $v = x - x_A$. The exact shifted Gaussian modulus is $e^{16 - v^2}$, and $(\lambda + |x|/4)e^{-v^2} \leq C(1 + x_A)$. Equations (37) and (9) therefore give

$$\|(\lambda \mp ix/4)f_\epsilon(\cdot \pm 4i)\|_\varpi \leq C(1 + x_A)e^{4B}\|f_\epsilon\|_\varpi, \qquad \|\mathcal{D}f_\epsilon\|_\varpi = O(A^{11/2}).$$

The inverse square root of the gamma weight has at most exponential/polynomial growth along these contours; the Gaussian damping dominates it. Thus the contour argument for (8) applies to these reference-cancelled functions, without requiring them to be entire.

Take $N = \lceil A^{31/4} \rceil$ and $P_\epsilon = \Pi_N f_\epsilon$. Parity is preserved and (8) gives

$$\delta_A := \|P_\epsilon - f_\epsilon\|_\varpi = O(A^{-9/4}) = o(\sqrt{\eta_A}).$$

It is useful to transfer concentration quadratically rather than bound the whole signed-form perturbation linearly. Since $\|P_\epsilon\| \geq 1 - \delta_A$ and $\|\mathbf{1}_{\mathbb{R}\setminus E_A} P_\epsilon\| \leq \sqrt{\varepsilon_A} + \delta_A$, the actual signed density satisfies

$$\int_{\mathbb{R}} P_\epsilon^2 g_A \leq -\eta_A(1-\delta_A)^2 + (M+\eta_A)(\sqrt{\varepsilon_A}+\delta_A)^2 < 0$$

eventually, by (38). Rational approximation in the same finite-dimensional parity space preserves the strict sign and degree. The lower degree bound is the already proved $X \geq 1000N^2$ local theorem, integrated over the complete tail, as in Theorem 6. □

The generic reference, contour-shift energy, reference cancellation and bandlimited concentration tools in this proof are established mechanisms, particularly close to [14, 16, 17]. The quantitative moving-band and both-parity consequences use the actual complete-theta obstruction.

6 Conclusions and remaining questions

The results establish a precise separation. Nonnegative monotone averaging repairs all moment directions through degree eight under the respective finite-moment hypotheses. Every fixed degree also has a positive terminal center region. Nevertheless, a fixed complete upper tail can have an unbounded number of negative directions in *both* parities. This failure survives the addition of any uniformly bounded-rank form. The full-space coefficient in the repair-rank obstruction is twice the per-parity coefficient because the original even density makes the form block diagonal; an arbitrary correction need not preserve that splitting.

The new assertions concern common subspaces and strict inertia in the stated reference norm. The supplied rational witnesses are explicit finite examples. The eventual $A^{31/4}$ bound is asymptotic, while the effective inertia formula has an explicit but potentially vacuous conservative degree domain.

Further research directions include physical-family initial-center compensation, practical first-negative-degree/inertia bounds, and sharper spectral laws for the signed Meixner–Pollaczek compressions. A physical correction may have infinite rank. A local spectral-distribution law at thresholds bounded away from zero would also not automatically count all negative eigenvalues accumulating at zero.

Reproducibility and assistance. The accompanying files provide editable sources, exact coefficient and point-enclosure data, portable arithmetic checkers, full regeneration options and a complete checksum manifest. The new closed-band and inertia constants are separate acceptance gates; a plot or a sampled sign is never used as a whole-band proof. Retained verification evidence is distinguished from fresh replay in the reader instructions. The research and manuscript used AI assistance and internal independent proof/code checks.

A A complete closed negative-band certificate

The fixed-cutoff inertia theorem needs a whole interval, not merely negative values at finitely many frequencies. We give the full analytic interpolation reserve and the numerical enclosures used for that interval. All quantities refer to the complete theta density and to the Fourier convention in (3).

Proposition 11 (Complete negative band)**.** *At $A = 23/2$, the reference density ϖ in* (4) *satisfies*

$$\frac{g_A(x)}{\varpi(x)} < -\frac{619}{10^9} \quad \textit{for every } x \in [307/2, 155] \cup [-155, -307/2].$$

Proof. Put

$$S(\tau) = 64\sqrt{\pi}\, e^{\pi\tau/2} g_A(2\tau), \qquad t_0 = \frac{617}{8}, \quad h = \frac{3}{8}, \quad R_0 = 2,$$

and choose the eight exact nodes

$$\theta_j = \frac{(2j+1)\pi}{16}, \qquad \tau_j = t_0 + h\cos\theta_j, \qquad x_j = 2\tau_j, \quad 0 \le j < 8. \tag{39}$$

Complete enclosures at the exact nodes

The grouped formula (23) fixes all scale and divisor factors. For the supplied node data, retain $q = 1, \ldots, 8$ and use the scaled integration-by-parts expression

$$e^{\pi\tau/2}\Bigg[\int_{qA}^{256} Y^{3/2}\{(4Y^2-6)L^2 - 12L\}K_{i\tau}(2Y)\,dY \\ - 6(qA)^{5/2}\log^2(A/\pi)K_{i\tau}(2qA)\Bigg], \qquad L = \log(Y/(q\pi)). \tag{40}$$

Only the lower boundary is subtracted: the missing upper integral is paid separately, not replaced by an unrecorded upper endpoint. Outward analytic quadrature at 512 bits encloses these expressions. The analytic callback requires $\operatorname{Re} Y > 0$, so the logarithm, square root and Bessel branches used in its contour evaluations are valid. The exact irrational nodes themselves are enclosed by outward π and cosine evaluations.

For completeness, the positive error radii at any of these real τ are

$$E_q(\tau) = \frac{148\sqrt{\pi/2}}{\pi^2} A^5\, 9^4 e^{\pi\tau/6 - 9A}, \\ E_Y(\tau) = \frac{222\sqrt{\pi/2}}{\pi^2} 256^5 e^{\pi\tau/6 - 256}. \tag{41}$$

They follow from the $i\pi/3$ Bessel contour: the scaled K envelope is $e^{\pi\tau/6}\sqrt{\pi/(2Y)}e^{-Y}$ and its Y derivative is at most four times that envelope for $Y \ge 2$. The resulting $\mathcal{H}_\tau$ amplitude is at most $37Y^2$ times it. Use $L^2 \le Y^2/(q\pi)^2$, $d(q) \le q$, $\int_b^\infty Y^5 e^{-Y} dY \le 2b^5 e^{-b}$ for $b \ge 11$, and $(10/9)^4 e^{-A} < 1/2$ for the omitted product groups. The finite-Y tail uses $\sum_{q\ge1} q^{-3/2} < 3$. The integration-by-parts amplitude in (40) is bounded by the same larger envelope. Thus every omitted group and every unbounded integration tail is included.

The mathematical data in `theta_negative_band_nodes.json` contain all retained group-integral enclosures, not just the following rounded summary. Recomputing the exact divisor phases, summing the groups and adding (41) gives these convenient strict integer bounds:

j	lower bound for $S(\tau_j)$	upper bound for $S(\tau_j)$
0	-1418895	-1418894
1	-1637051	-1637050
2	-1938540	-1938539
3	-2132965	-2132964
4	-2090900	-2090899
5	-1825042	-1825041
6	-1480426	-1480425
7	-1245115	-1245114

The finer supplied enclosures, rather than only these integer summaries, are used below. Independent regeneration is available through the complete Mellin-minus-lower-series formulas (29)–(31) and their explicit remainders. The public evaluator uses 1024 bits for this route; it retains additional product groups and applies (32). That route integrates to infinity analytically and therefore has no finite-Y omission.

A complete complex-disk bound

We now bound S throughout $|\tau - t_0| \le R_0$, including all product groups. Write $\tau = t + iw$ and set

$$v = \frac{3}{2}, \qquad c = \cos v > 0, \qquad \delta = \frac{\pi}{2} - v > 0.$$

On this disk $|w| \le 2$ and $t < 80$. Shifting the full-line Bessel integral by iv gives the scaled modulus factor

$$|e^{\pi\tau/2} e^{i\tau(u+iv)}| \le e^{80\delta} e^{2|u|} \le 4e^{80\delta} \cosh^2 u.$$

The argument derivative contributes at most one additional $\cosh u$, because $|\cosh(u+iv)| \le \cosh u$. For $z > 0$, $s \ge 1$ and integer $k \ge 1$,

$$s^k e^{-zs} \le \left(\frac{2k}{ez}\right)^k e^{-zs/2}, \qquad K_0(cY) \le \sqrt{\frac{\pi}{2cY}} e^{-cY} \le \sqrt{\frac{\pi}{cY}} e^{-cY}.$$

Applying these inequalities with $k = 2, 3$ yields the deliberately loose bounds

$$|e^{\pi\tau/2} K_{i\tau}(2Y)| \le e^{80\delta} C_0 Y^{-5/2} e^{-cY}, \qquad C_0 = \frac{16\sqrt{\pi}}{e^2 c^{5/2}},$$

$$|e^{\pi\tau/2} K'_{i\tau}(2Y)| \le e^{80\delta} C_1 Y^{-7/2} e^{-cY}, \qquad C_1 = \frac{108\sqrt{\pi}}{e^3 c^{7/2}}.$$

The prime differentiates the Bessel argument. Consequently

$$|Y^{3/2} e^{\pi\tau/2} \mathcal{H}_\tau(Y)| \le e^{80\delta} \left\{ 4C_0 Y + \frac{9C_0 + 12C_1}{Y} \right\} e^{-cY}.$$

The exact complex divisor factor satisfies $|D_q(\tau)| \le d(q) q^{|w|} \le q^3$. Since $\log^2(Y/(q\pi)) \le Y^2/(q\pi)^2$ for $Y \ge qA$, define

$$G_n(b) = \int_b^\infty Y^n e^{-cY} dY = e^{-cb} \sum_{j=0}^{n} \frac{n! b^j}{j! c^{n+1-j}},$$

$$T_q = \sqrt{q} \{ 4C_0 G_3(qA) + (9C_0 + 12C_1) G_1(qA) \}.$$

Absolute summation in (23) now gives

$$|S(\tau)| \le \frac{e^{80\delta}}{\pi^2} \sum_{q\ge 1} T_q.$$

Every positive term of T_q is a constant times $q^s e^{-cAq}$ with $s \in \{1/2, 3/2, 5/2, 7/2\}$. For $q \ge 40$ its successive ratio is at most

$$(41/40)^4 e^{-cA} < 0.489336 < \frac{1}{2}.$$

Thus all $q \ge 40$ contribute at most $2T_{40}$. Direct outward evaluation of the displayed positive finite expressions gives

$$|S(\tau)| \le \frac{e^{80\delta}}{\pi^2} \left(\sum_{q=1}^{39} T_q + 2T_{40} \right) < 643476598290 =: M_0. \tag{42}$$

The positive majorant proves uniform convergence on this disk. A slight enlargement of the exponents and constants gives a summable majorant on a slightly larger disk, since the lower endpoints still supply exponential decay in q. This pays the analyticity needed next, not merely a bound at real nodes.

Interpolation and exact normalization

Let p_7 interpolate the exact values $S(\tau_j)$ at (39). For the monic node polynomial w_8,

$$\max_{|\tau - t_0| \le h,\ \tau \in \mathbb{R}} |w_8(\tau)| \le 2(h/2)^8, \qquad \min_{|z - t_0| = R_0} |w_8(z)| \ge (R_0 - h)^8.$$

The Cauchy interpolation formula and (42) imply

$$\begin{aligned} \max_{\tau \in [t_0 - h, t_0 + h]} |S(\tau) - p_7(\tau)| &\le \frac{2R_0}{R_0 - h} M_0 \left(\frac{h}{2(R_0 - h)} \right)^8 \\ &= \frac{2110924980690345}{42417997492} < 49765. \end{aligned} \tag{43}$$

Writing $p_7(\tau) = \sum_{k=0}^{7} \alpha_k T_k((\tau - t_0)/h)$, its coefficients are exactly

$$\alpha_0 = \frac{1}{8} \sum_{j=0}^{7} S(\tau_j), \qquad \alpha_k = \frac{1}{4} \sum_{j=0}^{7} S(\tau_j) \cos(k\theta_j), \quad 1 \le k \le 7.$$

The full node enclosures and outward cosine arithmetic give

$$\alpha_0 + \sum_{k=1}^{7} |\alpha_k| < -1199633.$$

Since $|T_k| \le 1$ on $[-1, 1]$, (43) proves the complete whole-interval bound

$$S(\tau) < -1149868 \qquad (t_0 - h \le \tau \le t_0 + h). \tag{44}$$

The check is not an inference from sampled negativity: both the complex-disk and interpolation errors are included.

Finally the exact reference conversion is

$$\frac{g_A(2\tau)}{\varpi(2\tau)} = S(\tau)\frac{Z_*(1+e^{-\pi\tau})}{32\sqrt{\pi}\,V(2\tau)}.$$

On this interval $2\tau \le 155$, all factors of V increase, and $\sqrt{\pi} < 1773/1000$ follows from $\pi < 22/7 < (1773/1000)^2$. The negative bound (44) therefore yields

$$\begin{aligned}\frac{g_A(2\tau)}{\varpi(2\tau)} &< -\frac{1149868\cdot 1000 Z_*}{32\cdot 1773\,V(155)}\\ &= -\frac{329690152960}{532433012644407929} < -\frac{619}{10^9}.\end{aligned}$$

Here $V(155) = 337838205992644625$, so the last comparison is an exact rational inequality. Evenness supplies the reflected interval and finishes the proof. □

The reader checker `verify_theta_negative_band.py` recomputes the divisor phases, omitted-product and integration tails, the complete positive disk majorant, the interpolation reserve and the final rational comparison. Its default input is the supplied reviewed group-integral data. Its `-recompute-nodes` mode instead calls the complete public Mellin evaluator and prints replacement data; `-nodes-file` checks those replacement enclosures by the same proof. Neither mode reads private receipts, and no historical hash is used in place of an analytic bound.

B Finite-order positivity and Schur repair in full

B.1 The degree-five Schur theorem

For the initial odd Schur decomposition use

$$G_s = \begin{pmatrix} h_2 & h_4 \\ h_4 & h_6 \end{pmatrix}, \quad r_s = \begin{pmatrix} h_6 \\ h_8 \end{pmatrix}, \quad S_s = h_{10}, \quad \beta_s = G_s^{-1} r_s, \quad \delta_s = S_s - r_s^T G_s^{-1} r_s. \tag{45}$$

Global invertibility is proved below. For this degree-five result, an admissible weight is a finite-valued measurable nonnegative nondecreasing function W on $[0,\infty)$, not zero almost everywhere, such that

$$\int_0^\infty W(s)(1+X(s)^5)\,dm_0(s) < \infty. \tag{46}$$

Let $Z_W = \int W\,dm_0$, $d\mu_W = W\,dm_0/Z_W$, and use $\mathbb{E}_W$ for expectation. Define $\mathcal{L}_W(z^{2j}) = \mathbb{E}_W h_{2j}$ through $j=5$, with odd moments zero.

Theorem 12. *For every admissible W, the matrix of $\mathcal{L}_W$ on $\{z, z^3, z^5\}$ is positive definite. More precisely, with*

$$\bar{G} = \mathbb{E}_W G_s, \quad \bar{r} = \mathbb{E}_W r_s, \quad \bar{\beta} = \bar{G}^{-1}\bar{r},$$

one has

$$\begin{aligned}\Delta_W &:= \mathbb{E}_W S_s - \bar{r}^T \bar{G}^{-1}\bar{r} = \mathbb{E}_W \delta_s + V_W,\\ V_W &:= \mathbb{E}_W[(\beta_s - \bar{\beta})^T G_s(\beta_s - \bar{\beta})] > 0, \qquad \mathbb{E}_W \delta_s > 0.\end{aligned} \tag{47}$$

Let $a = \frac{1}{4}\log(a_1/\pi)$, where

$$3.64888758188562 < a_1 < 3.64888758188563$$

is the first fixed-center quintic transition of [1]. If $W(a) > 0$, then

$$\mathbb{E}_W \delta_s > \frac{2\,000\,000\, W(a)}{Z_W}. \tag{48}$$

Corollary 13. *For $H_y(s) = \sinh(2ys)/(2ys)$, continuously defined as 1 when $ys = 0$, Theorem 12 holds for every finite $y \geq 0$. Moreover*

$$\mathbb{E}_0 \delta_s > 17\,000\,000\,000, \qquad V_y > 1\,000\,000\,000 \quad (0 \leq y \leq 1).$$

Here $Z_y = \int H_y \, dm_0 > 0$ defines normalization also at $y = 0$.

For $y > 0$ the customary unnormalized physical kernel has central value $4yZ_y$; its value vanishes at $y = 0$ and is never used as a denominator there. The normalized law has the displayed continuous endpoint.

The earlier bounded-parameter odd-quintic certificate [2] gave a stronger numerical estimate for Δ_y on $[0, 1]$. The advance here is the full parameter half-line, the monotone-weight cone, and the separation of two strictly positive mechanisms.

The moment hypotheses will remain distinct. The fifth-moment assumption (46) suffices for the full degree-five conclusion in Corollary 19. The degree-six theorem in Section B.7 assumes $\int W(1 + X^6)\, dm_0 < \infty$ and adds the even polynomial z^6. Its new local sign, complete matrix tail and positive baseline are proved separately; positivity of one finite moment level is not extrapolated to the next. Section B.8 adds degree seven under its own seventh-moment condition, with a different upper-tail matrix argument.

B.2 Classical algebra and the new theta inputs

The Schur identity (47) is completion of squares. Indeed, integrate

$$S_s - 2b^T r_s + b^T G_s b = \delta_s + (b - \beta_s)^T G_s (b - \beta_s)$$

and minimize over b. This is classical matrix-fractional geometry [4]; its positive variance alone does not determine the sign of the mean local innovation. For example, two equally weighted scalar prefixes $G_1 = G_2 = 1$, columns $r_1 = 2, r_2 = -2$, and $S_1 = S_2 = 3$ have local innovations -1 but averaged innovation 3.

The following elementary single-crossing statement is also classical; stronger integral versions occur in [7].

Lemma 14. *Suppose $f \leq 0$ on $[0, a]$, $f \geq 0$ on $[a, \infty)$ and $\int f\, dm > 0$. For a nonnegative nondecreasing W with the necessary absolute integrability,*

$$\int W f \, dm \geq W(a) \int f \, dm.$$

If $W(a) = 0$, W is eventually bounded below by a positive constant, and f is strictly positive on a final half-line of positive measure, then $\int W f\, dm > 0$ as well.

Proof. The function $(W(s) - W(a)) f(s)$ is nonnegative on both sides of a. If $W(a) = 0$, monotonicity makes W vanish before a, and its eventual positivity makes a strictly positive final contribution. □

We use one authenticated published result rather than reprove the entire compact classification.

Proposition 15 (Compact fixed-center premise from [1])**.** *For the exact density* (1) *and the exact signed derivatives used here, the full fixed-center moment matrix through degree four is positive definite on* $3 \le X \le 12$*. The minimum negative-square degree is five before* a_1*, at least six after* a_1 *through* 12*, and the degree-five matrix is positive semidefinite at* a_1*.*

This is the relevant portion of the published classification theorem, including its endpoint statement. Its original full interval classification is provided by the cited source. Because parity separates odd and even blocks and $G_s > 0$, it implies

$$\delta_s < 0 \quad (\pi \le X < a_1), \quad \delta_s = 0 \quad (X = a_1), \quad \delta_s \ge 0 \quad (a_1 < X \le 12). \tag{49}$$

The new deductions that complete the argument are a global lower prefix, a positive full-theta tail beyond 12, and the independently integrated positive baseline. We provide their proofs and finite-arithmetic specifications below. Associated theta kernels and generalized Laguerre inequalities have a broader classical literature [3].

B.3 The complete-theta positive tail

Proposition 16. *For* $X \ge 12$*, the full odd matrix* $M_o = (h_{2i+2j+2})_{i,j=0}^{2}$ *satisfies*

$$M_o \succeq \frac{10}{63001}\,\mathrm{diag}(16X, 16384X^3, 67108864X^5). \tag{50}$$

Consequently $\delta_s \ge (671088640/63001)X^5 > 0$ *there.*

Proof. The common e^{5s} factor cancels in the paired derivative. Put

$$R_0(z) = 2z - 3, \quad R_{j+1}(z) = 2z(R_j'(z) - R_j(z)), \quad Q_k(X) = \sum_{j=0}^{2k} (-1)^{k+j} \binom{2k}{j} R_j(X) R_{2k-j}(X).$$

For the first atom, $h_{2k} = Q_k/(2X-3)^2$, and exact cancellation gives $\deg Q_k \le k+2$ for $k \le 5$. Its odd matrix determinant is

$$\frac{6039797760X^6 P_9(X)}{(2X-3)^6},$$

where

$$\begin{aligned} P_9(X) = {} & 1024X^9 - 12544X^8 + 81216X^7 - 331200X^6 + 902736X^5 \\ & - 1668960X^4 + 2026620X^3 - 1519020X^2 + 637875X - 127575. \end{aligned}$$

The first two leading-minor numerator polynomials, and $P_9 - 256X^9$, have strictly positive coefficients after $X = 12 + v$. Thus the head matrix is positive definite with determinant at least $24159191040X^9$. For $d = (4, 128, 8192)$ and $D = \mathrm{diag}(d_i X^{i+1/2})_{i=0}^{2}$, its congruence by D^{-1} has determinant at least $45/32768$. Its trace is at most

$$\sum_{i=0}^{2} \sum_{j} \frac{|[X^j]Q_{2i+1}| 12^{j-(2i+1)-2}}{(2-3/12)^2 d_i^2} = \frac{256401048845}{99883155456} < \frac{13}{5}.$$

Hence its least eigenvalue exceeds $1/5000$ by $\lambda_{\min} \ge \det / \mathrm{tr}^2$.

We now bound every omitted atom. If $r_{jl} = [z^l]R_j$, put

$$C_j = \sum_l |r_{jl}| 12^{l-j-1}, \qquad B_j = 2^{2j+5}.$$

For $X \geq 12$, $n^2|R_j(n^2X)| \leq C_jX^{j+1}n^{2j+4}$. For $n \geq 2$ the successive majorant ratio is at most $(3/2)^{24}e^{-60} < 1/2$ for $j \leq 10$. Therefore the derivative tail after removing the common head exponential is at most $B_jC_jX^{j+1}e^{-3X}$. Both cross terms and the tail-tail term in the paired product are retained. Set $\varepsilon = 20^{-12} > e^{-36}$. The error in the kth moment after division by X^k, before the full theta denominator correction, is bounded by

$$E_k = \frac{12^k\varepsilon}{(2-3/12)^2}\sum_{j=0}^{2k}\binom{2k}{j}C_jC_{2k-j}(2B_{2k-j} + B_jB_{2k-j}\varepsilon).$$

Here X^ke^{-3X} and $(2-3/X)^{-2}$ decrease for $X \geq 12$, $k \leq 5$. Symmetry in j accounts for both cross terms. The symmetric matrix perturbation norm is at most

$$3\max_{i,j}\frac{E_{i+j+1}}{d_id_j} = \frac{5312409501805455073001503693}{1486356480000000000000000000000000} < \frac{1}{25000}.$$

Writing $\Phi = \Phi_1(1+R)$, the denominator correction is exactly $(1+R)^2$ and $0 < R < 1/250$ (Appendix C). Thus

$$(1+R)^2D^{-1}M_oD^{-1} \succeq (1/5000 - 1/25000)I = I/6250.$$

Since $(1+R)^2 < (251/250)^2$, this proves (50). Minimizing the quadratic form with its monic fifth-degree coefficient fixed to one proves the Schur bound. □

Combining (49) with Proposition 16 gives one global negative-to-nonnegative crossing at $s = a$. Later higher-degree transitions in the compact classification do not change this fixed quintic conclusion.

B.4 Positive baseline and independent variance reserve

Proposition 17. *For the full theta density,*

$$\int_0^1 \delta_s\, dm_0 > 2\,000\,000, \qquad I_0 := \int_0^\infty \delta_s\, dm_0 > 2\,000\,000, \qquad I_0/Z_0 > 17\,000\,000\,000.$$

Proof. The first assertion is a finite interval certificate with the full inverse penalty, described in Appendix D. It gives the stronger raw lower bound 2300747. The second follows by dropping the strictly positive tail $s \geq 1$, since $\pi e^4 > 12$. The third uses the independently enclosed full normalizer. This argument does not infer a positive local mean from a known positive averaged Schur complement. □

For two disjoint regions A, B, let $Q_A = \mathbb{E}_W[1_AG_s]$ and $c_A = Q_A^{-1}\mathbb{E}_W[1_Ar_s]$, and similarly for B. Completion of squares within the regions and minimization over the global coefficient give the classical parallel-sum lower bound [5, 6]

$$V_W \geq (c_A - c_B)^T(Q_A^{-1} + Q_B^{-1})^{-1}(c_A - c_B) \geq \frac{[u^T(c_A - c_B)]^2}{u^T(Q_A^{-1} + Q_B^{-1})u}. \tag{51}$$

The final inequality is matrix Cauchy–Schwarz, valid without commutativity. For the numerical estimate, use the fixed shifted basis of Appendix D, which preserves V_y; in this paragraph c_A, c_B are the corresponding coefficient vectors. For $A = [0, 3/16]$, $B = [3/16, 1]$, $u = (0,1)^T$, the certificate encloses the matrices and columns directly over eight closed y bins and proves

$$V_y > 1916442774 > 10^9 \qquad (0 \leq y \leq 1). \tag{52}$$

The complement has nonnegative variance and is discarded. This is a physical-region computation, not the difference of two separately computed total innovations.

B.5 Proof of the global weighted theorem

The prefix proof in Appendix C gives $G_s > 0$, $h_2 \geq 21$, and $\det G_s \geq 4 \cdot 10^6$ for every s. The paired polynomial formulas in Section B.3 give

$$\frac{h_{2j}(s)}{X^j} \longrightarrow (2j-1)!!\, 8^j \quad (1 \leq j \leq 5). \tag{53}$$

Indeed the head numerator has degree $j+2$, and the full higher-atom correction after division by X^j is $O(X^j e^{-3X})$. Put $v = 8X$. Then $\det G_s/v^4 \to 6$ and the explicit inverse gives

$$\frac{\beta_1}{v^2} \to \frac{225-315}{6} = -15, \qquad \frac{\beta_2}{v} \to \frac{105-45}{6} = 10.$$

Thus $\beta_1/X^2 \to -960$ and $\beta_2/X \to 80$. In particular $r_s^T G_s^{-1} r_s = O(X^5)$. Smoothness on compact intervals and (46) make all the expectations in (47) absolutely convergent. Positive averaging makes $\bar{G} > 0$.

The global single crossing and Proposition 17 allow Lemma 14 with $f = \delta_s$, proving $\mathbb{E}_W \delta_s > 0$ and (48). If $V_W = 0$, positive definiteness forces $\beta_s = \bar{\beta}$ almost everywhere for μ_W. Every admissible nonzero nondecreasing weight is bounded below by a positive constant on a half-line. Continuity then forces β_s constant on that entire tail, contradicting $\beta_2/X \to 80$. Hence $V_W > 0$. Identity (47) and the Schur criterion prove Theorem 12, also for complex polynomial coefficient vectors.

Finally H_y is nondecreasing because $z\cosh z - \sinh z$ vanishes at zero and has derivative $z \sinh z \geq 0$. Its growth is at most e^{2ys}; the theta density has superexponential decay. Thus (46) holds for every fixed finite y. Propositions 17 and (52) finish Corollary 13.

B.6 The even block and full degree-five consequence

Proposition 18. *The complete even matrix $M_e = (h_{2i+2j})_{i,j=0}^2$ is positive definite for every center. On $X \geq 12$,*

$$M_e \succeq \frac{1}{4000} \operatorname{diag}(1, 1024X^2, 1048576X^4).$$

Proof. Use the recurrence of Section B.3, adding $Q_0 = (2X-3)^2$. The first-atom even determinant numerator is $393216X^3\widehat{P}_9(X)$, where

$$\begin{aligned}\widehat{P}_9 = {} & 2048X^9 - 22016X^8 + 120960X^7 - 407232X^6 + 875808X^5 \\ & - 1201680X^4 + 986904X^3 - 424980X^2 + 80190X - 8505.\end{aligned}$$

The first two leading-minor numerators have positive coefficients after $X = 12 + v$, as does

$$393216X^3\widehat{P}_9 - 3145728X^6(2X-3)^6.$$

With $D_e = \operatorname{diag}(1, 32X, 1024X^2)$, the head congruence has determinant at least $3/1024$ and trace at most $1420942925/520224768$. Its least eigenvalue is therefore at least

$$\frac{792872488009728}{2019078796107555625} > 1/3000.$$

The complete head-tail calculation of Section B.3, now for $k = 0, \ldots, 4$ with $d = (1, 32, 1024)$, gives perturbation norm at most

$$\frac{5415087591445761969320387}{6193152000000000000000000000000} < 1/1000000.$$

The zeroth entry is included: before dividing by $(1+R)^2$, it changes from 1 to $(1+R)^2$. Thus the tail bound follows from $(1/3000 - 1/1000000)(250/251)^2 > 1/4000$. For $\pi \le X \le 12$, the even block is a principal submatrix of the positive M_4 in Proposition 15. Even reflection covers negative centers. □

Corollary 19. *For every admissible W and every nonzero complex polynomial P of degree at most five, $\mathcal{L}_W(|P|^2) > 0$. In particular this holds for every physical weight H_y, $0 \le y < \infty$.*

Proof. Positive averaging preserves the pointwise strictly positive even block. Odd moments vanish, so a permutation of the full degree-five matrix is the direct sum of that even block and the odd block in Theorem 12. □

The even block on $\{1, z^2, z^4\}$ alone permits arbitrary nonnegative center weights with finite required moments; this does not extend the odd or even-sextic conclusions to arbitrary weights.

B.7 The even sextic and full degree-six theorem

Suppose, in addition to the stated measurability, sign, monotonicity and nonzero conditions on W, that

$$\int_0^\infty W(s)(1 + X(s)^6)\, dm_0(s) < \infty. \tag{54}$$

Extend $\mathcal{L}_W$ by $\mathcal{L}_W(z^{12}) = \mathbb{E}_W h_{12}$, retaining zero odd moments. Use a distinct notation for the even prefix and its next column:

$$A_s = (h_{2i+2j})_{i,j=0}^2, \qquad c_s = (h_6, h_8, h_{10})^T, \qquad b_s = A_s^{-1} c_s, \qquad \eta_s = h_{12} - c_s^T A_s^{-1} c_s.$$

Proposition 18 proves $A_s > 0$ at every center.

Theorem 20. *For every W satisfying (54), the full moment matrix $H_6 = (\mathcal{L}_W(z^{i+j}))_{i,j=0}^6$ is positive definite. With $\bar{b} = (\mathbb{E}_W A_s)^{-1} \mathbb{E}_W c_s$, its even Schur complement satisfies*

$$\Delta_{6,W} = \mathbb{E}_W \eta_s + \mathbb{E}_W[(b_s - \bar{b})^T A_s (b_s - \bar{b})], \tag{55}$$

and both terms on the right are strictly positive. In particular the theorem holds for every fixed finite physical parameter $y \ge 0$, including the normalized endpoint $y = 0$. Moreover

$$\int_0^\infty \eta_s\, dm_0 > 250000000, \qquad \mathbb{E}_0 \eta_s > 2000000000000. \tag{56}$$

Let $a = \frac{1}{4}\log(a_2/\pi)$, where

$$4.11304745500471102656 < a_2 < 4.11304745500471102658.$$

If $W(a) > 0$, then

$$\mathbb{E}_W \eta_s > \frac{250000000\, W(a)}{Z_W}. \tag{57}$$

B.7.1 The local sign and the initial obstruction

The compact classification in [1], with the normalization of Proposition 15, has minimum negative-square degree six on (a_1, a_2), at least seven on $(a_2, 12]$, and a positive-semidefinite degree-six matrix at a_2. The degree-six crossing has zero even Schur complement. Before a_1, minimum negative degree five alone does not settle the sign of the even sextic. We supply that missing interval directly.

Proposition 21. *For the actual fixed-center functional, the monic polynomial*

$$P(z) = z^6 - 167z^4 - 7243z^2 + 303062$$

satisfies $\mathcal{L}_X(P^2) < 0$ *for every* $157/50 \le X \le 73/20$.

Proof. The exact whole-cell construction in Appendix F encloses this contraction on 256 closed cells, retaining all theta atoms, and gives a strictly negative upper endpoint on each cell. No point-sample or first-atom approximation is used. □

Since $\pi > 157/50$ and $a_1 < 73/20$, Schur minimization with $A_s > 0$ gives $\eta_s \le \mathcal{L}_X(P^2) < 0$ on $[\pi, a_1]$. On (a_1, a_2), every square of degree at most five is nonnegative by the compact classification. Thus its negative degree-six square must have a negative even component: the odd component has degree at most five, and cross parity vanishes. The positive even prefix then implies $\eta_s < 0$. At a_2 one has $\eta_s = 0$, and after a_2 through 12 the degree-six matrix is positive semidefinite, so $\eta_s \ge 0$.

Proposition 22. *For every* $X \ge 12$*, the complete even matrix* $E_6 = (h_{2i+2j})_{i,j=0}^{3}$ *satisfies*

$$E_6 \succeq \frac{1}{25000}\,\mathrm{diag}(1, 1024X^2, 1048576X^4, 1073741824X^6). \tag{58}$$

In particular $\eta_s \ge (1073741824/25000)X^6 > 0$ *there.*

Proof. Appendix E proves the full half-line bound by a first-atom determinant reserve, exact second-atom paired cancellation and a geometric enclosure of every remaining atom. Minimizing (58) with the coefficient of z^6 equal to one gives the innovation bound. □

Consequently η_s has the negative-to-nonnegative crossing required by Lemma 14, at $s = a$, and is strictly positive on a final half-line. The fixed-center obstruction of Proposition 21 is not a counterexample to the averaged theorem.

B.7.2 Integrability, baseline and proof of Theorem 20

For each fixed $k \le 6$, paired differentiation gives $h_{2k} = O(X^k)$ as $X \to \infty$. One direct justification is to factor the leading atom's paired product as

$$e^{-2X(\cosh(2t)-1)}\,\frac{(2Xe^{2t}-3)(2Xe^{-2t}-3)}{(2X-3)^2}.$$

Its $2k$th coefficient is a polynomial of degree at most k in X, times bounded amplitude derivatives. The complete remaining paired atoms are $O(X^{2k}e^{-3X})$ after head-square normalization; their differentiated series is uniformly summable for $X \ge \pi$. Thus Proposition 18 gives

$$c_s^T A_s^{-1} c_s \le 4000\left(h_6^2 + \frac{h_8^2}{1024X^2} + \frac{h_{10}^2}{1048576X^4}\right) = O(X^6).$$

Compact continuity and positive definiteness give a bounded inverse on every bounded center interval. Therefore all terms of (55) are absolutely integrable under (54).

The direct local-innovation integration in Appendix F gives a raw lower bound greater than 262225959 on $0 \le s \le 1$. The remaining innovation is positive by Proposition 22; the complete normalizer upper bound then gives (56). This computation retains the whole inverse penalty; it does not obtain the local mean by subtracting a variance from an averaged innovation.

Apply Lemma 14 to η_s. If $W(a) > 0$, it gives (57). If $W(a) = 0$, monotonicity makes W vanish before a and bounds it below by a positive constant on a final half-line, which supplies a strictly positive integral. Completion of squares proves (55), and its positive mean proves the averaged even block positive definite. The sixth-moment hypothesis implies (46); Theorem 12 supplies the odd block. Parity therefore proves $H_6 > 0$, also for complex coefficient vectors.

For strict variance, the leading moments satisfy $h_{2k}/X^k \to (2k-1)!!8^k$. In scaled coordinates, the regression problem converges to the positive Gaussian moment problem of variance 8. Its monic sixth Hermite polynomial gives

$$b_0/X^3 \to 7680, \qquad b_1/X^2 \to -2880, \qquad b_2/X \to 120.$$

If the variance vanished, positive definiteness of A_s would give $b_s = \bar{b}$ almost everywhere on the final half-line where W is bounded below. Continuity would make b_s constant there, contradicting the last limit. Finally, $H_y \le e^{2ys}$ and the superexponential theta decay imply (54) for every fixed finite y. This completes the theorem.

B.8 The odd septic and full degree-seven theorem

Assume now

$$\int_0^\infty W(s)(1 + X(s)^7)\, dm_0(s) < \infty, \tag{59}$$

with the same sign, measurability, monotonicity and nonzero conditions as before. Extend $\mathcal{L}_W$ through h_{14}. Put

$$M_7(s) = (h_{2i+2j+2}(s))_{i,j=0}^3, \qquad T_7(a) = \int_a^\infty M_7(s)\, dm_0(s).$$

Let C have columns equal to the coefficient vectors of

$$\begin{aligned} &q_0(z) = z/8, && q_1(z) = (z^3 - 139z)/1024, \\ &q_2(z) = (z^5 - 461z^3 + 32200z)/262144, \\ &q_3(z) = (z^7 - 1015z^5 + 242645z^3 - 11576832z)/67108864 \end{aligned}$$

in the monomial basis (z, z^3, z^5, z^7). Thus $q(z) = C^T(z, z^3, z^5, z^7)^T$ and C is invertible.

Theorem 23. *Every finite $a \ge 0$ satisfies $T_7(a) > 0$. More quantitatively,*

$$C^T \left(\int_a^1 M_7(s)\, dm_0(s) \right) C \succeq \frac{1}{50000} I_4 \qquad (0 \le a \le 7/100). \tag{60}$$

For every W satisfying (59), the full moment matrix $H_7 = (\mathcal{L}_W(z^{i+j}))_{i,j=0}^7$ is positive definite. This includes every fixed finite physical $y \ge 0$, with the normalized endpoint $y = 0$.

The new ingredient is the odd block on four monomials. The corresponding pointwise three-monomial odd prefix is indefinite near the initial center, so its inverse cannot serve as a positive local metric. The proof uses integrated upper tails instead.

Proposition 24. *For $X \geq 12$ the complete theta matrix satisfies*

$$M_7(s) \succeq \frac{1}{150000}\,\mathrm{diag}(16X, 16384X^3, 67108864X^5, 1099511627776X^7). \tag{61}$$

Appendix G proves this whole-halfline estimate and (60), with all atoms and derivatives through order fourteen. The compact fixed-center classification [1] has minimum negative-square degree at least nine on $[a_3, 12]$, including its third transition, where

$$a_3 < 41415765848945270755 3/10^{20}.$$

This implies $M_7 \succeq 0$ there, without claiming strict pointwise positivity at a transition. The exact bridge

$$\pi e^{28/100} > 41415765848945270755 3/10^{20}$$

is checked by the compact script. Consequently $M_7(s) \succeq 0$ for every $s \geq 7/100$, and it is strictly positive definite on a terminal halfline.

Proof of Theorem 23. For $a \leq 7/100$, add the nonnegative integral over $s \geq 1$ to (60). For $a \geq 7/100$, the entire tail is nonnegative, and (61) makes its quadratic form strictly positive on a terminal interval of positive dm_0 measure for every nonzero vector. Thus $T_7(a) > 0$ for every finite cutoff.

For every fixed k, the paired first-atom argument used in Section B.7 gives $h_{2k} = O(X^k)$: its amplitude derivatives remain bounded after division by $(2X-3)^2$, and its opposite-shift exponential has coefficient of degree at most k. Every remaining paired atom has polynomial growth of fixed order and a relative factor at most e^{-3X}; its differentiated all-atom series is uniformly summable for $X \geq \pi$. Hence all entries through h_{14} are absolutely integrable under (59).

For a nonzero real vector u, use the classical layer-cake identity

$$W(s) = \int_0^\infty \mathbf{1}_{\{W(s)>t\}}\,dt.$$

A nonempty superlevel set of a nondecreasing function is a terminal interval, up to an irrelevant endpoint. Its quadratic-form integral equals $u^T T_7(a) u > 0$ at a finite cutoff. Absolute integrability justifies signed Fubini. Since W is not zero almost everywhere, a positive-measure set of levels has such a nonempty tail. Therefore $\int W u^T M_7 u\,dm_0 > 0$. This is the usual directional upper-tail criterion for the positive-semidefinite matrix cone. Theorem 20 supplies the even block under the stronger seventh-moment assumption. Parity proves $H_7 > 0$, also for complex coefficient vectors. Finally $H_y \leq e^{2ys}$ and the theta decay pay (59) for every fixed finite y. □

B.8.1 A stronger two-column local-residual theorem

Retain the genuinely positive pointwise prefix G_s on $U = (z, z^3)$, and add $V = (z^5, z^7)$. Write

$$M_7 = \begin{pmatrix} G_s & B_s \\ B_s^T & D_s \end{pmatrix}, \quad B_s = \begin{pmatrix} h_6 & h_8 \\ h_8 & h_{10} \end{pmatrix}, \quad D_s = \begin{pmatrix} h_{10} & h_{12} \\ h_{12} & h_{14} \end{pmatrix}, \quad R_s = D_s - B_s^T G_s^{-1} B_s.$$

Proposition 25. *Every finite cutoff satisfies $\int_a^\infty R_s\,dm_0 > 0$. Consequently $\mathbb{E}_W R_s > 0$ for every weight in Theorem 23.*

Proof. Modulo the prefix U, the last two q columns have output transformation

$$S = \begin{pmatrix} 1/262144 & -1015/67108864 \\ 0 & 1/67108864 \end{pmatrix}.$$

The Schur residual in the q basis is exactly $S^T R_s S$: invertible prefix changes and addition of prefix columns cancel in a residual. The direct whole-cell checker in Appendix G encloses, uniformly for $0 \le a \le 7/100$, the raw matrix

$$S^T \left(\int_a^1 R_s \, dm_0 \right) S.$$

Its first diagonal exceeds $3/100000$ and its determinant exceeds $2399/10^{12}$, so it is positive definite. For $s \ge 7/100$, the pointwise PSD classification and (61), together with $G_s > 0$, imply $R_s \succeq 0$ and strict positivity on a terminal halfline. This joins the compact certificate to every finite tail. The growth $B_s^T G_s^{-1} B_s = O(X^7)$ entrywise follows from the scaled Gaussian limit below; it also pays absolute integrability of the residual. The same layer-cake argument completes the claim. □

The integrated odd Schur complement therefore has two positive mechanisms. With $\beta_s = G_s^{-1} B_s$ and $\bar{\beta} = (\mathbb{E}_W G_s)^{-1} \mathbb{E}_W B_s$, completion of squares gives

$$\mathbb{E}_W D_s - (\mathbb{E}_W B_s)^T (\mathbb{E}_W G_s)^{-1} \mathbb{E}_W B_s = \mathbb{E}_W R_s + \mathbb{E}_W [(\beta_s - \bar{\beta})^T G_s (\beta_s - \bar{\beta})]. \tag{62}$$

This variable-metric least-squares identity is classical; see [4] and the explicit averaging formulation in [9]. Positivity of the old scalar diagonal alone would not control the new coupling. Proposition 25 pays the entire residual matrix independently.

B.8.2 Strict variance for any finite higher odd output list

The next statement concerns variance only and does not require a monotone density weight.

Proposition 26. *Fix distinct integers $2 \le k_1 < \cdots < k_m$, take outputs z^{2k_j+1} against the prefix $U = (z, z^3)$, and form B_s and $\beta_s = G_s^{-1} B_s$ from their actual theta moments. Let W be any nonnegative measurable density weight, positive on a set of positive dm_0 measure, with*

$$\int W(s)(1 + X(s)^{2k_m+1}) \, dm_0(s) < \infty.$$

Then the m by m regression variance

$$\mathcal{V}_W = \mathbb{E}_W [(\beta_s - \bar{\beta})^T G_s (\beta_s - \bar{\beta})]$$

is strictly positive definite.

Proof. For $\tau = 8X$, the complete paired first-atom expression gives $h_{2k}/\tau^k \to (2k-1)!!$ for every fixed k. Equivalently, after $t = u/\sqrt{8X}$ the normalized paired expression tends locally uniformly on complex compact u sets to $e^{-u^2/2}$. The remaining atoms and their fixed derivatives are exponentially small; Cauchy's formula gives the moment limit. This is a fixed-order asymptotic.

The leading Gaussian prefix is

$$G_0 = \begin{pmatrix} \tau & 3\tau^2 \\ 3\tau^2 & 15\tau^3 \end{pmatrix}.$$

For $c_k = (2k+1)!!$, the cross column is $(c_k \tau^{k+1}, (2k+3) c_k \tau^{k+2})^T$. Inversion gives the actual leading limits

$$\frac{\beta_{1,k}}{\tau^k} \longrightarrow (1-k) c_k, \qquad \frac{\beta_{2,k}}{\tau^{k-1}} \longrightarrow k c_k / 3. \tag{63}$$

Indeed congruence by $\mathrm{diag}(\tau^{1/2}, \tau^{3/2})$ makes the prefix converge to $\left(\begin{smallmatrix} 1 & 3 \\ 3 & 15 \end{smallmatrix}\right) > 0$, so inversion is justified. The inverse-energy entry for outputs i, j is $O(X^{k_i + k_j + 1})$; the highest displayed moment pays every entry and the square identity. Compact continuity handles bounded centers.

The complete theta sum is real analytic locally around each positive real center, by normal convergence on a sufficiently small complex neighborhood. Positivity of Φ and $\det G_s$ makes β_s real analytic on $(0,\infty)$. If $x^T\mathcal{V}_W x = 0$, the positive metric forces $\beta_s x = \bar{\beta} x$ almost everywhere where $W > 0$. Some finite interior interval contains a positive-measure subset of this support. The real-analytic identity theorem extends this equality to every $s > 0$. Divide its second row by τ^{k_m-1} and pass to infinity in (63); the nonzero leading coefficient gives $x_m = 0$. Descending through the distinct powers gives every $x_j = 0$. Thus $\mathcal{V}_W > 0$. □

For a single atomic center the variance is zero; a density-support hypothesis cannot be omitted. The proposition concerns the stated finite-order variance. For the two outputs z^5, z^7, the Gaussian leading regression is

$$\beta_0(\tau) = \begin{pmatrix} -15\tau^2 & -210\tau^3 \\ 10\tau & 105\tau^2 \end{pmatrix}, \quad \det(\beta_0(\tau_2) - \beta_0(\tau_1)) = 525(\tau_2 - \tau_1)^4.$$

Thus actual full-rank contrasts occur at sufficiently large separated centers. A useful quantitative regional bound still requires paid regional masses and errors. In particular a nonmonotone compactly supported density need not charge such distant regions; its strictness above follows from analyticity instead. Applied to (62), Proposition 26 makes the second term strictly positive as well as the first.

B.8.3 All associated indices at a fixed finite moment degree

For an integer $n \geq 1$ define the associated kernel in the normalization of [3] by

$$K_n(t) = \int_{\mathbb{R}} \Phi(s+t)\Phi(s-t)s^{2n}\, ds, \qquad J_n(t) = \int_0^\infty s^{2n}\Phi(s+t/2)\Phi(s-t/2)\, ds.$$

Corollary 27. *Every $n \geq 1$ has a positive-definite normalized derivative-moment matrix of degree seven at the origin for K_n. If $O_n(J_n)$ denotes the raw odd-four moment block of J_n, then*

$$C^T O_n(J_n) C \succeq \frac{(7/100)^{2n-2}}{50000} I_4. \tag{64}$$

Proof. Take $W(s) = s^{2n-2}$, with $W \equiv 1$ for $n = 1$. Every such weight is admissible under (59). Evenness gives exactly $J_n(t) = K_n(t/2)/2$. Normalized moments differ by the positive diagonal congruence $\mathrm{diag}(2^{-j})_{j=0}^7$, so Theorem 23 proves the jet claim. For $n = 1$, (64) is the cutoff-zero reserve. For $p = 2n-2 > 0$, signed Fubini and the positive Stieltjes representation of s^p give

$$O_n(J_n) = \int_0^\infty p a^{p-1} T_7(a)\, da.$$

Insert the congruent reserve for $0 \leq a \leq 7/100$ and drop the PSD remaining tails. Their early scalar mass is $(7/100)^p$. □

Here n ranges over associated indices, not derivative orders. The case $n = 0$ would require $W = s^{-2}$ and is not a substitution into this theorem; its classical autocorrelation positivity has a different proof. Csordas's Theorem 4.5 gives admissibility of the associated theta kernels, while Theorem 4.6 and Open Problem 4.7 address global positive definiteness and Fourier/Laguerre inequalities. The positive-K_1, negative-K_2 example there uses a different Gaussian-polynomial density.

B.9 The even octic and full degree-eight theorem

Assume the same sign, monotonicity, measurability and nonzero conditions on W, now with

$$\int_0^\infty W(s)(1+X(s)^8)\,dm_0(s) < \infty. \tag{65}$$

Extend $\mathcal{L}_W$ through h_{16} and put

$$E_8(s) = (h_{2i+2j}(s))_{i,j=0}^4, \qquad T_8(a) = \int_a^\infty E_8(s)\,dm_0(s).$$

Let C_8 have columns the coefficient vectors, in $(1, z^2, z^4, z^6, z^8)$, of

$$\begin{aligned}
&q_0 = 1, && q_1 = (z^2-47)/64,\\
&q_2 = (z^4 - 275z^2 + 6419)/8192,\\
&q_3 = (z^6 - 703z^4 + 98976z^2 - 1557845)/2097152,\\
&q_4 = (z^8 - 1399z^6 + 512772z^4 - 49595891z^2 + 598408111)/1073741824.
\end{aligned}$$

Theorem 28. *For every finite $a \ge 0$, $T_8(a) > 0$. Uniformly on $0 \le a \le 7/100$,*

$$C_8^T \left(\int_a^1 E_8(s)\,dm_0(s) \right) C_8 \succeq I_5/100000. \tag{66}$$

On that interval one also has the raw monomial bound $T_8(a) \succeq I_5/400000$. For every W satisfying (65), the full degree-eight moment matrix $H_8 = (\mathcal{L}_W(z^{i+j}))_{i,j=0}^8$ is positive definite. Every fixed finite physical $y \ge 0$ is included, with the normalized endpoint $y = 0$.

Proposition 29. *For $X \ge 12$ the complete even-five matrix satisfies*

$$E_8(s) \succeq \frac{1}{2500000}\,\mathrm{diag}(1, 1024X^2, 1048576X^4, 4294967296X^6, 70368744177664X^8). \tag{67}$$

Proof of Theorem 28. Appendix H proves (66) by whole-cell integration of all fifteen symmetric entries, retaining every theta atom through derivative sixteen. Proposition 29 is proved there by a new strict head determinant reserve and complete error budget. The compact classification [1] gives $E_8 \succeq 0$ on $a_3 \le X \le 12$, because the minimum negative-square degree is at least nine. The exact bridge past a_3 in Section B.8 therefore gives $E_8(s) \succeq 0$ for every $s \ge 7/100$, with strict positivity on a terminal halfline. For $a \le 7/100$, add the nonnegative $s \ge 1$ tail to (66); for larger a, integrate the PSD matrix with its strictly positive terminal part. This proves every finite upper tail PD.

The exact basis has

$$\det C_8 = 2^{-70}, \qquad \|C_8\|_F^2 = \frac{870866245054201579}{288230376151711744} < 4.$$

From $C_8^T T C_8 \succeq \eta I$ one obtains $T \succeq \eta (C_8 C_8^T)^{-1} \succeq \eta I / \|C_8\|_{\mathrm{op}}^2$, proving the conservative raw $1/400000$ floor. A normalized weighted floor would require division by Z_W.

The fixed-order bound $h_{2k} = O(X^k)$ also holds for $k = 8$. Thus (65) pays entrywise absolute integrability through h_{16}. The signed layer-cake argument in Theorem 23, applied now to T_8, gives a strictly positive weighted even-five matrix for every admitted W. Its odd-four block is already PD by Theorem 23, since the eighth-moment hypothesis implies the seventh. Parity proves full $H_8 > 0$, also for complex coefficients. The superexponential theta tail pays (65) for every finite physical parameter. □

B.9.1 Two added even columns and a positive residual mean

Let A_s be the globally PD even-three prefix on $U = (1, z^2, z^4)$ from Proposition 18, and take $V = (z^6, z^8)$. Write

$$E_8 = \begin{pmatrix} A_s & B_s \\ B_s^T & D_s \end{pmatrix}, \qquad R_{8,s} = D_s - B_s^T A_s^{-1} B_s.$$

Proposition 30. *Every finite $a \geq 0$ satisfies $\int_a^\infty R_{8,s}\, dm_0(s) > 0$. Hence $\mathbb{E}_W R_{8,s} > 0$ for every weight in Theorem 28.*

Proof. Modulo U, the last two q columns have output transformation

$$S_8 = \begin{pmatrix} 1/2097152 & -1399/1073741824 \\ 0 & 1/1073741824 \end{pmatrix}.$$

The transformed local Schur residual is $S_8^T R_{8,s} S_8$. The separate complete-theta whole-cell certificate in Appendix H directly bounds its integral from a to 1 for every $0 \leq a \leq 7/100$. Its first diagonal exceeds $1/10000$ and its determinant exceeds $1/100000000$. It retains all three-by-three inverse-prefix terms and the output cross coupling. There is no subtraction of an estimated variance to infer this mean. For $s \geq 7/100$, the full matrix PSD statement and $A_s > 0$ imply $R_{8,s} \succeq 0$, strict on a terminal halfline. Add this tail and apply layer cake. Gaussian scaling gives inverse-energy entries $O(X^6), O(X^7), O(X^8)$; compact inverse continuity, including $s = 0$, and (65) pay absolute integrability. □

With $\beta_s = A_s^{-1} B_s$ and $\bar{\beta} = (\mathbb{E}_W A_s)^{-1} \mathbb{E}_W B_s$, the integrated even Schur complement is

$$\mathbb{E}_W R_{8,s} + \mathbb{E}_W[(\beta_s - \bar{\beta})^T A_s (\beta_s - \bar{\beta})].$$

The first term is PD by Proposition 30. The second is PD by the following common rigidity mechanism.

Proposition 31 (Parity-prefix regression rigidity)**.** *Let $\epsilon \in \{0, 1\}$ and an integer $p \geq 0$, and suppose the actual pointwise moment matrix G_s on $U = (z^\epsilon, z^{2+\epsilon}, \ldots, z^{2p+\epsilon})$ is PD for every $s \geq 0$, including the endpoint. For distinct integers $k_j > p$, take outputs $z^{2k_j+\epsilon}$ and their regression $\beta_s = G_s^{-1} B_s$. If $W \geq 0$ is any measurable density weight positive on a set of positive dm_0 measure, and*

$$\int W(s)(1 + X(s)^{2k_{\max}+\epsilon})\, dm_0(s) < \infty,$$

then its integrated regression variance is PD. Monotonicity is not required for this variance statement.

Proof. Put $\tau = 8X$. The scaled prefix converges to the positive Gaussian parity Gram, so its inverse converges. Gaussian integration by parts gives

$$\mathbb{E}[x^d \mathrm{He}_r(x)] = \frac{d!}{2^{(d-r)/2}((d-r)/2)!}, \qquad \mathbb{E}[\mathrm{He}_r^2] = r!$$

when $d \geq r$ has the same parity. Only the highest retained Hermite polynomial contributes to the highest monomial coefficient. Therefore the actual last regression row satisfies

$$\frac{\beta_{p,k}}{\tau^{k-p}} \longrightarrow \frac{(2k+\epsilon)!}{(2p+\epsilon)!\, 2^{k-p}(k-p)!} > 0.$$

Its inverse-energy entry for columns i, j is $O(X^{k_i+k_j+\epsilon})$. The displayed moment pays integrability at infinity. The assumed PD at $s = 0$, as well as all positive centers, bounds the inverse on each closed compact interval. Positivity merely on the open halfline would not by itself supply that endpoint bound.

As in Proposition 26, a variance-null vector makes $\beta_s x$ constant on a positive-measure set and hence everywhere on $(0, \infty)$ by real analyticity. The last-row asymptotics, divided successively by the largest remaining distinct power, eliminate every output coefficient. This proves strictness. A single atomic center is excluded, under the stated density-support hypothesis. □

The verified even-three prefix gives last-row limits $15\tau, 210\tau^2$ for outputs z^6, z^8. The odd-two special case gives $10\tau, 105\tau^2$ for z^5, z^7.

Corollary 32. *For every integer $n \geq 1$, the associated kernel K_n in Corollary 27 has a positive-definite normalized degree-eight derivative-moment matrix at the origin. Its raw even-five block for $J_n(t) = K_n(t/2)/2$ obeys*

$$C_8^T O_{\mathrm{even},n}(J_n) C_8 \succeq \frac{(7/100)^{2n-2}}{100000} I_5.$$

Proof. Use $W = s^{2n-2}$, with $W = 1$ for $n = 1$, exactly as in Corollary 27. The eighth-moment integrability follows from theta decay. The early Stieltjes mass on $0 \leq a \leq 7/100$ is $(7/100)^{2n-2}$; the remaining tails are PSD. The identity $J_n(t) = K_n(t/2)/2$ pays both the half-line factor and the positive diagonal dilation of derivative jets. □

This covers every associated index at one finite derivative degree. It does not replace the all-derivative-order premise in a global positive-definiteness statement.

B.10 An explicit degree-dependent theorem at every finite order

Define the pointwise signed functional $\ell_s(z^j) = h_j(s)$. No positive measure representing all its moments is assumed. Write He_j for the monic probabilists' Hermite polynomials, generated by $e^{uz-u^2/2} = \sum_j \mathrm{He}_j(z) u^j/j!$.

Theorem 33. *For every integer $N \geq 1$ and every complete-theta center $X \geq 1000N^2$, set*

$$P_j(z) = \frac{\mathrm{He}_j(z/\sqrt{8X})}{\sqrt{j!}}, \qquad 0 \leq j \leq N.$$

Then

$$(\ell_s(P_i P_j))_{i,j=0}^N \succeq \frac{99}{100} I. \tag{68}$$

Thus the full local monomial moment matrix through degree N is PD on this explicit final halfline. Every upper-tail integral beginning there is PD. The basis depends on the center and the threshold on the degree; this is not positivity of every order at any one finite center.

The theorem gives a common quantitative formula beyond unspecified fixed-order Gaussian convergence. Hermite approximations in the Jensen-polynomial setting [10, 11] concern a different global coefficient sequence and do not imply the present raw center-moment estimate.

Lemma 34 (Hermite coefficient row bound)**.** *Let $M(t) = \sum_j m_j t^j/j!$ be analytic on a neighborhood of $|t| \leq 4\sqrt{N}$, with $m_0 = 1$ and real coefficients. For its signed polynomial functional put $f(t) = M(t)e^{-t^2/2}$ and $B = \max_{|t|=4\sqrt{N}} |f(t) - 1|$. Its normalized Hermite Gram matrix G_N satisfies $\|G_N - I\|_{\mathrm{op}} < B$ if $B > 0$, and $G_N = I$ if $B = 0$.*

Proof. The product generating function is $e^{uv}f(u+v)$. Write $f = 1 + \sum_{m\geq 1} a_m t^m$. For $r+s=m$ and $l = i-r = j-s \geq 0$, the contribution to entry i,j is

$$a_m \binom{m}{r} \frac{\sqrt{i!j!}}{l!} = a_m \binom{m}{r} \sqrt{(i)_r (j)_s}.$$

For a fixed row and m, each admissible r determines at most one j, and the factorial factor is at most $N^{m/2}$. Terms $m > 2N$ cannot enter. Hence the absolute row sum of $G_N - I$ is bounded by

$$\sum_{m=1}^{2N} |a_m| (2\sqrt{N})^m.$$

Cauchy's estimate gives $|a_m| \leq B/(4\sqrt{N})^m$, and the resulting geometric sum is strictly less than B. Real symmetry bounds the operator norm by this row norm. This is coefficient algebra and Cauchy's inequality, with no positivity premise on the signed functional. □

Proof of Theorem 33. Take $M_X(t) = p_s(it/\sqrt{8X})$, whose moments are $h_{2k}/(8X)^k$, and set

$$w = \frac{t}{\sqrt{2X}}, \qquad \delta = \sqrt{8N/X} < 1/10.$$

On $|t| \leq 4\sqrt{N}$, the shifted atom arguments are $Xe^{\pm iw}$, and the elementary exponential series gives

$$|e^{\pm iw} - 1| < 1/9, \quad \Re e^{\pm iw} > 8/9, \quad |e^{\pm iw}| < 10/9.$$

The theta series therefore converges normally on a neighborhood of the closed disc; the first-atom amplitudes have no zero there.

The first-atom generating function is exactly

$$M_1(t) = A(t) e^{2X(1-\cos w)}, \quad A(t) = \frac{4X^2 - 12X\cos w + 9}{(2X-3)^2}.$$

For $f_1 = M_1 e^{-t^2/2} = Ae^E$, one has $E = 2X(1 - \cos w - w^2/2)$. The complex cosine remainder and $\cosh\delta < 10/9$ give

$$|E| \leq \frac{|t|^4 \cosh\delta}{48X} \leq \frac{4}{675}, \qquad |A-1| \leq \frac{160N}{3X^2} < \frac{1}{10000}.$$

Since $e^r \leq (1-r)^{-1}$ for $0 \leq r < 1$,

$$|f_1 - 1| \leq \frac{1627}{268400}, \qquad |f_1| \leq \frac{270027}{268400}.$$

To pay every higher atom, put $Z = Xe^z$, $|z| \leq \delta$. For $n \geq 2$,

$$\left| \frac{n^2(2n^2 Z - 3)}{2Z - 3} \right| \leq 2n^4, \qquad \Re Z > X/2.$$

Thus the complete relative remainder obeys

$$|R(z)| \leq 2 \sum_{n\geq 2} n^4 e^{-(n^2-1)X/2} \leq 64 e^{-3X/2} < 1/100000.$$

The consecutive-term ratio is at most $(3/2)^4 e^{-5X/2} < 1/2$; the conservative last bound already follows for $X \geq 20$ from $e > 2$. It applies to both shifted remainders and the real central $R_0 \geq 0$.

Full normalization is $f_X = f_1 Q$, where

$$Q = \frac{(1+R_+)(1+R_-)}{(1+R_0)^2}, \qquad |Q-1| \leq 4b + 2b^2 = \frac{200001}{5000000000}, \quad b = 1/100000.$$

Consequently, throughout the disc,

$$|f_X - 1| \leq \frac{1627}{268400} + \frac{270027}{268400}\frac{200001}{5000000000} = \frac{8189005670027}{1342000000000000} < \frac{1}{100}.$$

Lemma 34 gives (68). The exact rational constants and a separate finite generic-coefficient reconstruction are reproducible with `verify_hermite_tail.py`; those bounded checks do not replace this all-N analytic proof. □

For each fixed N, positive averaging also allows any nonnegative density weight supported in $X \geq 1000N^2$, positive on a set of positive measure, with the necessary Nth moment. This qualitative averaged conclusion does not preserve a 99/100 floor in one fixed monomial basis. The tail threshold depends on the order. A complementary terminating finite-order certificate algorithm is given in Appendix I; it often supplies much smaller low-order thresholds than the uniform formula.

B.11 Local positivity for at most nine physical kernel points

Define, only for the physical weights and each fixed finite $y \geq 0$,

$$k_y(t) = \frac{1}{Z_y}\int_0^\infty s^2 H_y(s)\Phi(s+t/2)\Phi(s-t/2)\,ds.$$

For $y > 0$ this is the customary physical kernel divided by its nonzero central value. At $y = 0$ it is $Q_1(t)/Q_1(0)$, where $Q_1(t) = 2\int s^2\Phi(s+t/2)\Phi(s-t/2)\,ds$.

Corollary 35. *Fix $1 \leq n \leq 9$ and distinct real numbers $r_0, \ldots, r_{n-1}$. For every fixed finite $y \geq 0$ there exists $\epsilon_0 > 0$ such that*

$$[k_y(\epsilon(r_i - r_j))]_{i,j=0}^{n-1} > 0 \qquad (0 < |\epsilon| < \epsilon_0).$$

With $\mathcal{V}(r) = \prod_{i<j}(r_j - r_i)$ and $H_{n-1} = (\mathcal{L}_y(z^{i+j}))_{i,j=0}^{n-1}$,

$$\det[k_y(\epsilon(r_i - r_j))] = \frac{\epsilon^{n(n-1)}\mathcal{V}(r)^2 \det H_{n-1}}{\prod_{j=0}^{n-1}(j!)^2} + o(\epsilon^{n(n-1)}).$$

The denominator is 1194393600 *for $n = 6$ and* 619173642240000 *for $n = 7$.*

Proof. For every fixed derivative order j and $u \geq 1$, the complete derivative-polynomial series gives $|\Phi^{(j)}(u)| \leq C_j e^{(4j+9)u} e^{-\pi e^{4u}}$. For $|t| \leq T$ and $s \geq T/2 + 1$, the differentiated paired product is therefore bounded by $C_{j,T} e^{C_j s} e^{-c_T e^{4s}}$, with $c_T = 2\pi e^{-2T} > 0$. The factor $s^2 H_y(s) \leq s^2 e^{2ys}$ preserves integrability. On the remaining compact interval smooth theta convergence supplies domination. Consequently k_y is smooth, real and even, $k_y(0) = 1$, and $k_y^{(2j)}(0) = (-1)^j \mathcal{L}_y(z^{2j})$.

Put $m = n - 1 \leq 8$. We give the classical Taylor-confluence argument using the leading moment matrix H_m. Let $V_{ia} = r_i^a$ for $0 \leq i, a \leq m$ and $C_\epsilon = \mathrm{diag}(1, \epsilon^{-1}, \ldots, \epsilon^{-m})V^{-1}$. Taylor expansion through order $2m$ has remainder $o(\epsilon^{2m})$ and bivariate terms

$$\epsilon^{p+q} r_i^p r_j^q \frac{(-1)^q k_y^{(p+q)}(0)}{p!q!}, \qquad p + q \leq 2m.$$

Multiplication by V^{-1} maps the evaluation vector of degree $p \le m$ to its coordinate vector. For $p > m$, its coordinate $a \le m$ acquires a factor ϵ^{p-a} tending to zero; p, q cannot both exceed m. The scaled remainder is $o(1)$ since $a + b \le 2m$. Hence the real congruence of the kernel matrix tends entrywise to

$$\left(\frac{(-1)^b k_y^{(a+b)}(0)}{a!b!}\right)_{a,b=0}^{m} = JH_mJ^*, \qquad J = \operatorname{diag}(i^a/a!)_{a=0}^{m}.$$

This is positive definite by Theorem 28; the complex phases express a real matrix because cross-parity moments vanish. Openness of positive definiteness and invertibility of C_ϵ prove the claim. Finally $\det C_\epsilon = \epsilon^{-m(m+1)/2}/\mathcal{V}(r)$, proving the determinant asymptotic. □

The assertion applies to the physical family, whose locally uniform shifted-derivative domination through order sixteen was just proved.

B.12 Why generic log-concavity is insufficient

Arbitrary nonnegative weights are excluded: the known negative fixed-center square near $X = 13/4$ remains negative under a smooth bump supported in a sufficiently small neighborhood. More fundamentally, even smooth positive even log-concavity of a density does not imply the averaged odd-quintic gate.

Proposition 36. *The density $\varphi(s) = e^{-s^2}(1+\cos(4s)/10)$ is positive, even, Schwartz and uniformly strictly log-concave. At $y = 0$ its actual normalized averaged paired-moment functional has a positive odd two-by-two prefix but gives a negative square to*

$$P(z) = z^5 - 45z^3 + 59z.$$

Proof. For $c = \cos(4s)$ and $\epsilon = 1/10$,

$$(\log\varphi)'' = -2 - 16\epsilon\frac{c+\epsilon}{(1+\epsilon c)^2} \le -2/9.$$

If $c + \epsilon \ge 0$ this is immediate; otherwise the positive correction is at most $16\epsilon/(1-\epsilon) = 16/9$. Let $J(t) = \int_{\mathbb{R}} s^2\varphi(s+t/2)\varphi(s-t/2)\,ds$ and $F(\xi) = \int \varphi(s)e^{i\xi s}\,ds$. The substitution $a = s+t/2, b = s-t/2$ has Jacobian one and gives

$$\widehat{J}(\xi) = \tfrac{1}{2}(F'(\xi)^2 - F(\xi)F''(\xi)), \qquad F(\xi) = \sqrt{\pi}\left(e^{-\xi^2/4} + \frac{e^{-(\xi-4)^2/4} + e^{-(\xi+4)^2/4}}{20}\right).$$

All operations are justified for Schwartz functions. Up to a common positive factor the signed Fourier density is

$$\begin{aligned}\nu(\xi) = {} & (\tfrac{1}{2} - \tfrac{3}{80}e^{-8})e^{-\xi^2/2} + \tfrac{1}{800}\{e^{-(\xi-4)^2/2} + e^{-(\xi+4)^2/2}\} \\ & - \tfrac{3}{20}e^{-2}\{e^{-(\xi-2)^2/2} + e^{-(\xi+2)^2/2}\}.\end{aligned}$$

For $J/J(0)$ the polynomial moment functional equals $\int P^2\nu/\int\nu$. Evenness makes this the same normalized half-line $y = 0$ functional. Variance-one Gaussians with means $0, 2, 4$ give respectively

$$\mathbb{E}P^2 = 11191, \quad 334171, \quad 2544391.$$

Thus the numerator in Gaussian units, with $q = e^{-2}$, is

$$N(q) = \frac{4782591}{400} - \frac{1002513}{10}q - \frac{33573}{80}q^4 < -1600.$$

The positive Taylor series for e^2 through degree 20, with omitted-term ratio at most 2/22, proves $1353/10000 < q < 677/5000$. Substitution of these rational endpoints proves the strict displayed inequality. The same exact Gaussian moment calculation gives mass $> 2/5$, unnormalized second moment $> 1/3$, and odd prefix determinant $> 5/2$. Division by the positive mass preserves its definiteness and the negative square. These explicit checks are included in the companion arithmetic script. □

This example does not refute the actual theta result or stronger curvature criteria for three-point positivity. The associated-kernel Fourier identities are classical [3]; the example serves to distinguish their hypotheses from generic log-concavity.

C Global lower prefix: finite and unbounded enclosures

We give the analytic bounds behind the lower-prefix certificate. All derivatives in this appendix are physical s derivatives unless polynomial arguments are displayed. Let $b = 3/2$, $\tau_n = n^2 - 1$, $\Phi = \Phi_1(1 + R)$, and $\ell = \log \Phi$. Algebra gives the positive Laplace representation

$$R = \int_3^\infty e^{-vX}\, d\sigma(v), \quad d\sigma = \sum_{n\geq 2} \left(n^4 \delta_{\tau_n} + bn^2 \tau_n e^{b(v-\tau_n)} 1_{v\geq \tau_n}\, dv \right).$$

Define $P_0 = 1$, $P_{j+1}(z) = z(P_j - P_j')$. The physical derivative of e^{-vX} is $(-4)^j P_j(vX)e^{-vX}$. For $j \leq 6$, both $P_j(9 + w)$ and $[(z^j - P_j(z))]_{z=9+w}$ have nonnegative coefficients; so do the shifted ordinary derivatives of P_4, P_6. Hence the derivatives R_j alternate in sign and are bounded by the highest Laplace moments.

For every $n \geq 2$ their majorants are

$$M_{n,j}(X) = 4^j X^j e^{-\tau_n X} \left[n^4 \tau_n^j + bn^2 \tau_n \sum_{l=0}^{j} \binom{j}{l} \frac{\tau_n^{j-l} l!}{(X-b)^{l+1}} \right].$$

They decrease on $X \geq 3$ for $j \leq 6$. Their complete $n \geq 3$ tails are bounded by

$$\mathcal{T}_j = \frac{C_j 3^{4+2j}}{20^8(1-q_j)}, \quad C_j = 12^j \left(1 + \sum_{l=0}^{j} \binom{j}{l} l!(2/27)^l \right), \quad q_j = (4/3)^{4+2j}/20^7 < 1.$$

The ratio bound retains every atom. In particular

$$\begin{aligned} |R_1| &\leq 1104/8000 + (100/99)24 \cdot 3^6/2^{24} < 1/4, \\ 0 < R_2 &\leq 44736/8000 + (100/99)288 \cdot 3^8/2^{24} < 6, \\ |R_3| &\leq M_{2,3}(3)e^9/8000 + \mathcal{T}_3 = 34736311628179317/147622497952000 < 256, \\ R &\leq 28/8000 + 2 \cdot 3^4/[20^8(1 - (4/3)^4/20^7)] < 1/250. \end{aligned}$$

Here $e^3 > 20$ follows from its Taylor polynomial through degree eight. The first two weaker tails use $e^{-24} < 2^{-24}$ and ratio $< 1/100$.

Set $U = 1 + R$. For the leading atom

$$\ell_1'' = -16[X + T_2(b/X)], \quad \ell_1'''' = -256[X + T_4(b/X)], \quad \ell_1^{(6)} = -4096[X + T_6(b/X)],$$

where

$$T_2(r) = \frac{r}{(1-r)^2}, \quad T_4(r) = \frac{r(1+4r+r^2)}{(1-r)^4}, \quad T_6(r) = \frac{r(1+26r+66r^2+26r^3+r^4)}{(1-r)^6}.$$

The fourth logarithmic correction is

$$(\log U)'''' = R_4/U - (4R_1R_3 + 3R_2^2)/U^2 + 12R_1^2R_2/U^3 - 6R_1^4/U^4.$$

It is at most $R_4 + 9/2$. The qualitative sign $\ell'''' < 0$ is also proved in Umihara [8]; we include the quantitative independent bounds needed here. The positive decreasing $n = 2$ fourth contribution is $C_4(X)e^{-3X}$, where

$$C_j(X) = 4^j \left[16P_j(3X) + \frac{18}{X-b} \sum_{l=0}^{j} P_j^{(l)}(3X) \left(\frac{X}{X-b} \right)^l \right] \quad (j = 4, 6).$$

Indeed $(P_4 - P_4')(9+w) = 1162 + 921w + 241w^2 + 26w^3 + w^4 > 0$, so its integrated fourth contribution decreases. Its $n \geq 3$ fourth tail is less than

$$\frac{100}{99} \frac{62208 \cdot 3^{12}}{20^8} = \frac{14348907}{11000000} < 2.$$

The exact endpoint coefficients are $C_4(3) = 43140096$, $C_4(16/5) = 42636828818522112/887410625$, $C_4(7/2) = 59421432$ and $C_4(4) = 278436163584/3125$. Direct substitution yields

X interval	upper for $C_4(X)e^{-3X}$	lower for $256[X + T_4(b/X)]$
$[3, 16/5]$	5393	5430
$[16/5, 7/2]$	3337	3801
$[7/2, 4]$	1748	2557
$[4, \infty)$	557	1024

The exponential lower bounds are respectively $8000, 14400, 34000, 160000$. They follow from $e^3 > 20$, $e^{3/5} > 9/5$, $e^{3/2} > 17/4$. Each final column exceeds the middle column plus $2 + 9/2$.

For the determinant, put $A = -\ell''$, $d = -\ell'''' > 0$, $e = \ell^{(6)}$. Direct paired differentiation gives

$$h_2 = -\ell''/2, \quad h_4 = (\ell'''' + 6\ell''^2)/8, \quad h_6 = -(\ell^{(6)} + 30\ell''\ell'''' + 60\ell''^3)/32,$$

$$64 \det G = 24A^4 - 18A^2d - d^2 - Ae. \tag{69}$$

The fourth logarithmic error budget is $46595/128$. Expanding $\log(1 + R)$ through order six and dropping its negative groups gives

$$(\log U)^{(6)} \leq R_6 + (15/8)R_4 + 841095/16.$$

The constants follow respectively from $4(1/4)256 + 3 \cdot 6^2 + 6(1/4)^4$ and $120(1/4)6 \cdot 256 + 30 \cdot 6^3 + 360(1/4)^4 6$. For a cell $[L, H]$, bound R_4 below by $C_4(H)e^{-3H}$ and above by $C_4(L)e^{-3L} + \mathcal{T}_4$. Bound R_6 above by $C_6^+(L, H)e^{-3L} + \mathcal{T}_6$, where C_6^+ replaces polynomial arguments by $3H$, denominators

by $L-b$, and $X/(X-b)$ by $L/(L-b)$. Positive shifted derivative coefficients justify the enclosure. With these bounds $r_{4,-}, r_{4,+}, r_{6,+}$ define

$$A_- = 16[L + T_2(b/H)] - 6, \qquad A_+ = 16[H + T_2(b/L)] + 1/16,$$
$$d_+ = 256[H + T_4(b/L)] - r_{4,-}/(251/250) + 46595/128,$$
$$e_+ = -4096[L + T_6(b/H)] + r_{6,+} + (15/8)r_{4,+} + 841095/16.$$

On the 96 closed cells $L = 3 + j/32$, $H = L + 1/32$, rational arithmetic proves $A_- > 0$, $d_+ > 0$, $A_-^2 \geq 3d_+/8$ and

$$24A_-^4 - 18A_-^2 d_+ - d_+^2 - (A_+ \text{ if } e_+ \geq 0 \text{ else } A_-)e_+ > 256000000.$$

The minimum lower floor is 264806350, and the minimum guard floor is 3928. The guard justifies the monotonic substitution in A; the other signs justify every remaining substitution. Exponential enclosures here use $\sum_{j=0}^{80} q^j/j!$ and its remainder $(q^{81}/81!)/(1-q/82)$ for $0 \leq q \leq 18$. For $X \geq 6$, the sixth envelope at 6 gives $e < 150000$; also $16X - 6 \leq A \leq 16X + 33$ and $d \leq 256X + 782$. The resulting lower polynomial in $v = X - 6$ has coefficients

$$(1211952476, 958667264, 274580480, 34209792, 1572864),$$

and its monotonicity guard has coefficients $(28923/4, 2784, 256)$. These are positive. Equation (69) proves globally $\det G \geq 4 \cdot 10^6$ and $h_2 \geq 21$. Thus both the finite interval and the remaining unbounded domain are covered.

D Whole-cell baseline and regional certificate

The separate baseline uses 1024 closed cells $[j/1024, (j+1)/1024]$, not point evaluations. Work in the unit-triangular basis

$$U = z, \quad V = z^3 - 139z, \quad R = z^5 - 461z^3 + 32200z.$$

The scalar monic innovation and matrix-weighted residual variance are invariant under these fixed lower-degree basis shifts. Let $b(s) = \pi e^{5s-X}$ and use tildes for moment numerators multiplied by Φ^2/b^2. Then

$$s^2\Phi^2\delta_s = s^2 b^2(\widetilde{S} - \widetilde{r}^T\widetilde{G}^{-1}\widetilde{r}). \tag{70}$$

This exact cancellation avoids division by a small density.

To specify the complete derivative enclosures, use $q_0(z) = 2z - 3$, $q_{j+1}(z) = (5 + 4z\partial_z - 4z)q_j(z)$ through $j = 10$. The head derivative is

$$d_j = \sum_{n=1}^{3} n^2 q_j(n^2 X) e^{-(n^2-1)X}.$$

For a whole cell with $X \in [X_-, X_+]$, the omitted derivative tail is bounded by the sum over monomials $q_{jl}z^l$ of

$$|q_{jl}| X_+^l 4^{2l+2} e^{-15X_-} \Big/ \left[1 - (5/4)^{2l+2} e^{-9X_-}\right].$$

Every denominator is positive. The paired head moment polynomial is formed before interval evaluation:

$$\widetilde{m}_{2k}^{(n,m)} = \frac{(-1)^k}{4^k} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} [n^2 q_j(n^2 X)][m^2 q_{2k-j}(m^2 X)] e^{-(n^2+m^2-2)X},$$

summed for $1 \le n \le m \le 3$ with multiplicity two off the diagonal. With head and tail derivative bounds, the absolute moment error is

$$4^{-k} \sum_{j=0}^{2k} \binom{2k}{j} (|d_j| t_{2k-j} + t_j |d_{2k-j}| + t_j t_{2k-j}).$$

All cross terms are retained. The six fields are the linear combinations

$$\begin{aligned}
a &= m_2, \\
b &= m_4 - 139 m_2, \\
c &= m_6 - 278 m_4 + 19321 m_2, \\
r_1 &= m_6 - 461 m_4 + 32200 m_2, \\
r_2 &= m_8 - 600 m_6 + 96279 m_4 - 4475800 m_2, \\
S &= m_{10} - 922 m_8 + 276921 m_6 - 29688400 m_4 + 1036840000 m_2.
\end{aligned}$$

All are formed at numerator level before enclosure. Absolute errors are combined using absolute coefficients. On every cell the lower bounds for a and $ac - b^2$ are positive. The inverse penalty is at most the upper bound of $(cr_1^2 - 2br_1r_2 + ar_2^2)/(ac - b^2)$, and also at most $2(cr_1^2 + ar_2^2)/(ac - b^2)$ with absolute upper bounds. Use the smaller of these valid upper bounds, and subtract it from the lower endpoint for S in (70). Integrating the resulting interval times s^2b^2 proves the baseline. The recorded rational lower exceeds 2300747; no adaptive subdivision is needed.

For normalization, $B_0 = 81/40$ and the positive theta series give, for $s \ge 1$,

$$\Phi(s) \le 2B_0 \pi X e^{5s-X}, \qquad H_y(s) \le e^{2s} \quad (0 \le y \le 1).$$

Indeed $|2n^2X - 3| \le B_0 n^2 X$ for $X > 120$, and $\sum n^4 e^{-(n^2-1)X} < 2$ by the successive ratio $16e^{-3X} < 1/2$. Therefore

$$s^2 H_y \Phi^2 \le 4B_0^2\, 4^4 s^2 e^{20s - 6e^{4s}}.$$

Since $e^4 > 40$ and $e^3 > 20$, $20s - 6e^{4s} \le -220 - 940(s-1)$ and $e^{-220} < 20^{-73}$. The full normalizer tail is at most

$$\frac{4B_0^2\, 4^4}{20^{73}} \left(\frac{1}{940} + \frac{2}{940^2} + \frac{2}{940^3} \right). \tag{71}$$

The compact denominator interval for the whole bin $0 \le y \le 1/8$ contains Z_0 after adding this tail. Interval division of the baseline by this full denominator proves the 17 billion bound.

For the variance, integrate the five fields of G, r separately on $A = [0, 3/16]$ and $B = [3/16, 1]$, with the common factor $s^2b^2H_y$. Use the eight closed bins $j/8 \le y \le (j+1)/8$, $j = 0, \ldots, 7$. Regional interval inverses have positive determinant. Enclose the second-coordinate contrast in (51), square its minimum absolute value, and divide by upper bounds for the inverse sum and full Z_y. The binwise gain floors exceed, respectively,

$$\begin{aligned}
&1966012227,\ 1955643969,\ 1946391148,\ 1938242732, \\
&1931186825,\ 1925210688,\ 1920300777,\ 1916442774.
\end{aligned}$$

This includes the $1/Z_y$ scaling of raw regional matrices. It certifies (52) independently of a total averaged innovation.

The interval script uses exact rational operations with outward rounding at 80 decimal places; the other finite checks use exact rational polynomial arithmetic. Pi is enclosed by the Machin

alternating series; exponentials use a positive Taylor series after range reduction, with an explicit geometric remainder; the hyperbolic quotient uses its positive series. They reconstruct polynomials and tails instead of reading stored numerical enclosures. Explicit exception gates, active also under optimized Python, require raw baseline $> 2 \cdot 10^6$, normalized baseline $> 17 \cdot 10^9$, and every variance bound $> 10^9$. The public scripts are `verify_prefix.py` (lower-order controls and the 96-cell prefix), `verify_tail.py` (odd complete tail), `verify_even.py` (even complete tail), `verify_separated.py` (baseline and regional variance), and `verify_counterexample.py` (signed Gaussian control). The accompanying README specifies exact commands. Arithmetic checking supports the displayed analytic proof and does not replace the inherited compact classification theorem.

E Complete even-sextic tail certificate

Use R_j, Q_k from Section B.3, now through $j = 12$ and $k = 6$, and $Q_0 = (2X-3)^2$. The first-atom even determinant numerator is

$$\det(Q_{i+j})_{i,j=0}^{3} = 4638564679680X^6 P_{14}(X),$$

where

$$\begin{aligned} P_{14} = {} & 131072X^{14} - 2293760X^{13} + 20570112X^{12} - 120004608X^{11} \\ & + 493764096X^{10} - 1486503936X^9 + 3312458496X^8 - 5440255488X^7 \\ & + 6483101760X^6 - 5448021120X^5 + 3089651040X^4 - 1112123520X^3 \\ & + 247531950X^2 - 39803400X + 3444525. \end{aligned}$$

After $X = 12 + v$, each of its first three leading-minor numerators has nonnegative coefficients with a positive constant coefficient. The reserve polynomial

$$\det(Q_{i+j})_{i,j=0}^{3} - 2^{50}X^{12}(2X-3)^8$$

also has nonnegative shifted coefficients. These exact coefficient checks prove the head matrix positive definite and its determinant at least $2^{50}X^{12}$ on the whole half-line. They do not infer definiteness from a determinant alone.

Let $D = \operatorname{diag}(1, 32X, 1024X^2, 32768X^3)$, with constant factors $d = (1, 32, 1024, 32768)$. For the preconditioned head, the determinant is at least $1/1024$ and the trace is at most

$$T = \sum_{i=0}^{3} \sum_{l} \frac{|[X^l]Q_{2i}|12^{l-2i-2}}{(2-3/12)^2 d_i^2} < 8.093.$$

The product of the other three positive eigenvalues is at most $(T/3)^3$. Hence its least eigenvalue is at least

$$\frac{27}{1024T^3} = \frac{2905790377975168091802036781455507456}{5839460756163561398590735440327365157422 9} > \frac{1}{21000}.$$

Here and below T denotes the displayed exact rational sum, before its decimal upper bound.

To retain enough cancellation at $X = 12$, treat the second atom by its paired polynomials rather than by absolute derivative products. Put

$$\begin{aligned} R_{j,n}(X) &= n^2 R_j(n^2 X), \\ C_k(X) &= 2\sum_{j=0}^{2k} (-1)^{k+j} \binom{2k}{j} R_{j,1}(X) R_{2k-j,2}(X), \\ B_k(X) &= 16 Q_k(4X). \end{aligned}$$

The additional retained numerator is exactly $C_k(X)e^{-3X} + B_k(X)e^{-6X}$: both ordered cross pairs and the diagonal second-atom pair are included once. Write

$$\varepsilon = (3/8)^{36}, \quad \varepsilon_8 = (3/8)^{96}, \quad c_j = \sum_l |[X^l]R_j|12^{l-j-1},$$

$$H_j = \sum_l |[X^l]R_j|(1 + 2^{2l+2})12^{l-j-1}, \qquad b_j = 2 \cdot 3^{2j+4}.$$

The elementary inequality $e > 8/3$ gives $e^{-3X} \le \varepsilon$ and $e^{-8X} \le \varepsilon_8$ for $X \ge 12$. The retained two-atom derivative head is bounded by $H_j X^{j+1}$. For the remaining atoms $n \ge 3$, the ratio of successive $n^{2j+4}e^{-(n^2-1)X}$ terms is at most

$$(4/3)^{28}e^{-7X} \le (4/3)^{28}(3/8)^{84} < 1/2 \qquad (j \le 12).$$

Thus their complete derivative tail is at most $b_j c_j X^{j+1} e^{-8X}$.

For $k = 0, \dots, 6$, define the rational numerator-error bound

$$\begin{aligned} e_k = \frac{1}{(2-3/12)^2} \Bigg\{ & \varepsilon \sum_l |[X^l]C_k|12^{l-k-2} + \varepsilon^2 \sum_l |[X^l]B_k|12^{l-k-2} \\ & + 12^k \varepsilon_8 \sum_{j=0}^{2k} \binom{2k}{j} \big[H_j b_{2k-j} c_{2k-j} + b_j c_j H_{2k-j} \\ & + b_j b_{2k-j} c_j c_{2k-j} \varepsilon_8 \big] \Bigg\}. \end{aligned}$$

This bounds the omitted moment numerator after division by $X^k(2X-3)^2$. The first line retains the exact second-atom cancellation; the remaining lines retain both head-tail terms and the tail-tail term. Every normalized monomial $X^a e^{-3X}$ arising in the first line decreases from 12, since $a \le k \le 6$; the terms with e^{-6X} also decrease. The other terms use the decreasing functions $X^k e^{-8X}$ and $X^k e^{-16X}$. These facts justify the uniform substitution $X = 12$, rather than merely an endpoint test.

The symmetric preconditioned perturbation consequently has norm at most

$$N = 4 \max_{0 \le i,j \le 3} \frac{e_{i+j}}{d_i d_j} < \frac{1}{180000}.$$

The exact value is approximately 0.00000529965257745. The zeroth moment is included in this bound. As in the earlier tails, before the full denominator is restored its zeroth entry is $(1+R)^2$, not one. For the denominator itself,

$$0 < R \le \frac{2X}{2X-3} 32e^{-3X} < 1/250.$$

The geometric ratio here is bounded by $(3/2)^{28}e^{-5X} < 1/2$; the exponent 28 safely covers the zeroth derivative as well. Both rational endpoint ratio checks use $e > 8/3$. Therefore

$$D^{-1}E_6D^{-1} \succeq \left(\frac{1}{21000} - \frac{1}{180000}\right)\left(\frac{250}{251}\right)^2 I \succ \frac{1}{25000} I,$$

which proves Proposition 22. The standalone checker `verify_sextic_tail.py` reconstructs the polynomials, leading minors, determinant reserve and every rational bound in this appendix.

F Whole-cell sextic sign and mean certificates

The arithmetic uses the interval method and paired-polynomial construction of Appendix D, extended through physical derivative twelve and moments $m_0, \dots, m_{12}$. In particular $q_{j+1} = (5 + 4z\partial_z - 4z)q_j$ is now continued through $j = 12$; the same geometric formula encloses every $n \geq 4$ derivative. Every head-tail and tail-tail product remains in the moment error. The zeroth moment is treated by the same calculation.

For Proposition 21, divide $[157/50, 73/20]$ into 256 equal closed X intervals. The coefficients of the square of the displayed monic polynomial, in ascending even powers, are

$$91846575844,\ -4390156132,\ -48761659,\ 3025286,$$
$$13403,\ -334,\ 1.$$

Combine these with the paired moment polynomials before enclosure. On every whole cell, the sum of its head upper bound and its full absolute tail error is strictly negative. Dividing by the positive complete density square proves the stated fixed-center sign. The script `verify_sextic_initial.py` supplies the exact upper bound and rejects empty partitions.

For the mean, use the fixed scaled basis

$$\begin{aligned}
V_0 &= 1,\\
V_1 &= z^2/64 - 1,\\
V_2 &= z^4/4096 - 6z^2/64 + 3,\\
V_3 &= z^6/262144 - 15z^4/4096 + 45z^2/64 - 15.
\end{aligned}$$

The residual innovation for V_3 is $\eta_s/262144^2$. Lower-degree shifts do not affect this residual, and the scaling is restored at the end. Work in raw paired numerators as in (70). If their prefix is

$$\widetilde{A} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}, \qquad \widetilde{c} = (u, v, w)^T,$$

its six adjugate entries are

$$df - e^2, \quad ce - bf, \quad be - cd, \quad af - c^2, \quad bc - ae, \quad ad - b^2.$$

The whole-cell lower endpoints of a, $ad - b^2$ and $\det \widetilde{A}$ are positive. Contract the interval adjugate with (u, v, w) on both sides to obtain a numerator upper bound U. If $D_- > 0$ is the determinant lower bound, then

$$\widetilde{c}^T \widetilde{A}^{-1} \widetilde{c} \leq \max(0, U)/D_-.$$

Subtract this entire inverse penalty from the lower top-square endpoint, multiply by the interval for $s^2\pi^2 e^{10s-2X}$ and the cell width, and sum. Exact outward arithmetic on the 1024 closed intervals $[j/1024, (j+1)/1024]$ gives, after multiplication by 262144^2,

$$\int_0^1 \eta_s \, dm_0 > 262225959 > 250000000.$$

The positive remainder for $s \geq 1$ follows from Proposition 22. An independent compact normalizer enclosure plus the full tail (71) gives

$$\frac{\int_0^1 \eta_s \, dm_0}{Z_0} > 2049676440449 > 2000000000000.$$

The checker `verify_sextic_mean.py`, run with argument `1024`, reconstructs these enclosures with the same outward 80-decimal rational arithmetic, Machin bound for pi and Taylor/geometric bounds for exponentials. Explicit exception gates retain all prefix signs and require the conservative raw and normalized thresholds in (56). All three new scripts may be run normally and under optimized Python. The paired recurrences, infinite tails, sign transfer and inverse-energy arguments above are the analytic justification for their finite checks; the scripts do not reprove the full inherited compact classification.

G Complete odd-septic and all-cutoff certificates

G.1 The entire local tail beyond twelve

Cancel the e^{5s} factors in the opposite paired shifts. Set

$$R_0(X) = 2X - 3, \quad R_{j+1} = 2X(R_j' - R_j), \quad Q_k = \sum_{j=0}^{2k} (-1)^{k+j} \binom{2k}{j} R_j R_{2k-j} \quad (0 \le k \le 7).$$

The first-atom matrix has entries $Q_{i+j+1}/(2X-3)^2$, with $\deg Q_k \le k+2$. Exact arithmetic reconstructs

$$\det(Q_{i+j+1}) = 1994953897436774400 X^{10} P_{14}(X),$$

where the ascending coefficients of P_{14} are

$$\begin{gathered} 442047375, -3150592200, 11194876350, -25569432000, \\ 39943380960, -44288622720, 35956699200, -21886364160, \\ 10134653184, -3589152768, 967872000, -195207168, \\ 28237824, -2686976, 131072. \end{gathered}$$

After $X = 12+v$, the first three leading-minor numerators have nonnegative coefficients and positive constants. The polynomial

$$\det(Q_{i+j+1}) - 2^{68} X^{16} (2X-3)^8$$

also has nonnegative shifted coefficients. Sylvester's criterion and this reserve give head PD and normalized determinant at least $2^{68} X^{16}$.

Let $d = (4, 128, 8192, 1048576)$ and $D = \operatorname{diag}(d_i X^{i+1/2})_{i=0}^3$. The preconditioned head determinant is at least $1/65536$, and its trace is bounded by

$$T = \sum_{i=0}^{3} \frac{\sum_l |[X^l] Q_{2i+1}| 12^{l-2i-3}}{(2-3/12)^2 d_i^2} = \frac{46528533188040845}{14728370570919936}.$$

The product of the other three positive eigenvalues is at most $(T/3)^3$, so the least eigenvalue exceeds $27/(65536 T^3) > 1/80000$.

Retain the second atom in paired form. Put $R_{j,n} = n^2 R_j(n^2 X)$ and

$$C_k = 2 \sum_j (-1)^{k+j} \binom{2k}{j} R_{j,1} R_{2k-j,2}, \qquad B_k = 16 Q_k(4X).$$

The additional numerator is $C_k e^{-3X} + B_k e^{-6X}$. Its normalized degrees are at most k, so every monomial times the relevant exponential decreases for $X \geq 12$. With $\epsilon = (3/8)^{36}$ and $\epsilon_8 = (3/8)^{96}$, the exact inequality $e > 8/3$ supplies the endpoint majorant

$$A_k = \epsilon \sum_l |[X^l]C_k| 12^{l-k-2} + \epsilon^2 \sum_l |[X^l]B_k| 12^{l-k-2}.$$

For $0 \leq j \leq 14$, define

$$c_j = \sum_l |[X^l]R_j| 12^{l-j-1},$$
$$H_j = \sum_l |[X^l]R_j|(1 + 2^{2l+2}) 12^{l-j-1}, \qquad b_j = 2\,3^{2j+4}.$$

The retained derivative head is bounded by $H_j X^{j+1}$. The entire $n \geq 3$ derivative tail is bounded by $b_j c_j X^{j+1} e^{-8X}$: its successive-atom ratio is at most $(4/3)^{32} e^{-7X} < (4/3)^{32}(3/8)^{84} < 1/2$. This pays every atom through derivative fourteen. Both cross terms and the tail-tail product remain in

$$F_k = 12^k \epsilon_8 \sum_{j=0}^{2k} \binom{2k}{j} \left(H_j b_{2k-j} c_{2k-j} + b_j c_j H_{2k-j} + b_j b_{2k-j} c_j c_{2k-j} \epsilon_8 \right).$$

Set $e_k = (A_k + F_k)/(2 - 3/12)^2$. A symmetric four-by-four entry bound gives the operator error

$$N = 4 \max_{0 \leq i,j \leq 3} \frac{e_{i+j+1}}{d_i d_j} < \frac{1}{200000}.$$

All comparisons are exact rational gates, not rounded decimal tests. Finally, writing $\Phi = \Phi_1(1+R)$, the positive zeroth tail has

$$R \leq \frac{2X}{2X-3}\, 32 e^{-3X} < 1/250.$$

The geometric test $(3/2)^{32}(3/8)^{60} < 1/2$ pays that zeroth sum. Restoring $(1+R)^{-2}$ after the perturbation yields

$$\frac{1/80000 - 1/200000}{(1 + 1/250)^2} > 1/150000,$$

which proves (61). The standalone checker `verify_septic_tail.py` reconstructs these polynomials, shifted minors and rational budgets.

G.2 Every real short cutoff, rather than sampled cutoffs

For the compact integrators use the physical derivative recurrence $q_{j+1} = (5 + 4X\partial_X - 4X)q_j$ through $j = 14$, beginning at $q_0 = 2X - 3$, as in Appendix D. Retain atoms $1, 2, 3$, combine ordered pairs and the fixed $q_i(z)$ basis products exactly before evaluating intervals, and enclose every $n \geq 4$ derivative monomial using ratio

$$(5/4)^{2l+2} e^{-9X} < 1$$

on its entire cell. The exponent reaches 32 at derivative fourteen. Head-tail and tail-tail errors are retained for every moment. Removing the common factor πe^{5s-X} from each derivative leaves the raw integration weight $s^2 \pi^2 e^{10s-2X}$; the normalized Φ^2 cancels, so no small normalizer is inverted.

Write $F(s) = s^2\Phi(s)^2 C^T M_7(s) C$. On each of the 1024 closed cells $[j/1024, (j+1)/1024]$, whole-cell interval arithmetic encloses every entry f_{ij} by $[f_-, f_+]$. If that cell overlaps $[0, 7/100]$ in width w, its contribution to any removed initial segment $\int_0^a f_{ij}$, $0 \le a \le 7/100$, is contained in

$$[w \min(0, f_-),\ w \max(0, f_+)].$$

Indeed the selected portion has any length from zero to w; the signed enclosure covers all choices. Summing gives simultaneous loss intervals for every real cutoff. Subtract them from the independently enclosed entries of $\int_0^1 F$. Discarding correlation weakens the bound safely. For the resulting symmetric interval matrix, each Gershgorin row reserve exceeds $1/50000$; the smallest exceeds $33/10^6$. This proves (60). The checker `verify_septic_compact.py 1024` also verifies the exact bridge past a_3.

For the second certificate use the same basis and raw fields, but split the cell containing $7/100$ and form the local odd-two residual. If its prefix is $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, cross columns are $(u, w)^T, (z, t)^T$, and output entries are d_{11}, d_{12}, d_{22}, the three residual entries are

$$\begin{aligned} &d_{11} - \frac{cu^2 - 2buw + aw^2}{ac - b^2}, \\ &d_{12} - \frac{cuz - b(ut + wz) + awt}{ac - b^2}, \\ &d_{22} - \frac{cz^2 - 2bzt + at^2}{ac - b^2}. \end{aligned}$$

Both a and $ac - b^2$ are checked positive on every entire cell, with subdivision if needed. Apply the same signed-loss construction to the three weighted residual entries. Exact lower bounds for its first diagonal and determinant exceed $3/100000$ and $2399/10^{12}$ respectively. This proves the finite input of Proposition 25; `verify_septic_residual.py 1024` reconstructs it. These two representations share the outward rational interval engine, not independent arithmetic implementations. The complete tail, compact-classification dependency, layer-cake step and variance arguments supply the analytic lifting beyond the finite checks.

H Complete even-octic certificates

H.1 A sharper exponential envelope for the unbounded tail

Use the paired polynomials R_j, Q_k from Appendix G, now through derivative sixteen and $k \le 8$. Put

$$D = \operatorname{diag}(1, 32X, 1024X^2, 65536X^3, 8388608X^4).$$

All four preceding leading-minor numerators are strictly positive after shifting $X = 12 + v$. The exact reserve polynomial

$$\det(Q_{i+j})_{i,j=0}^4 - 2^{89} X^{20} (2X - 3)^{10}$$

has nonnegative shifted coefficients, with a positive constant. Thus the head is PD and has normalized determinant at least $2^{89} X^{20}$. Its D-congruent trace is bounded by $5.292776831925055\ldots$ (an exact rational computed by the checker). AM–GM on the other four eigenvalues gives

$$\lambda_{\min} \ge \frac{256 \det(D^{-1} E_{8,1} D^{-1})}{\operatorname{tr}(D^{-1} E_{8,1} D^{-1})^4} > 1/2000000.$$

Here $E_{8,1}$ is the first-atom block; its determinant is at least $2^{89}/(1 \cdot 32 \cdot 1024 \cdot 65536 \cdot 8388608)^2$ after congruence.

For $P(L+v)=\sum_j c_j v^j$ and $c>0$, define

$$\mathrm{Env}(P,L,c)=\sum_j |c_j|(3j/(8c))^j,$$

with the zeroth factor 1. It bounds $|P(L+v)|e^{-cv}$ for every $v\ge 0$, since $\sup v^j e^{-cv}=(j/(ce))^j$ and $e>8/3$. Shifting before taking absolute values retains useful cancellation.

For the exact second-atom increment $C_k e^{-3X}+B_k e^{-6X}$ of Appendix G, with $L=12$ and $\varepsilon=(3/8)^{36}$, use

$$\frac{\varepsilon\,\mathrm{Env}(C_k,12,3)+\varepsilon^2\,\mathrm{Env}(B_k,12,6)}{12^{k+2}}.$$

This bounds the increment divided by X^{k+2} on the whole halfline without losing the near-endpoint paired cancellation. Continue the same complete $n\ge 3$ derivative-tail construction through $j=16$. Its maximal atom power is 36, and

$$(4/3)^{36}(3/8)^{84}<1/2.$$

Both cross terms and the tail-tail product remain. The exact symmetric five-by-five preconditioned error norm is

$$5\max_{i,j}\frac{e_{i+j}}{d_i d_j}=0.000000008356726546345431\ldots<1/100000000.$$

The positive central denominator has $R<1/250$, with ratio check $(3/2)^{36}(3/8)^{60}<1/2$. Therefore

$$D^{-1}E_8D^{-1}\succeq(1/2000000-1/100000000)(250/251)^2 I\succeq I/2500000.$$

This proves Proposition 29. The exact polynomial, ratio and normalization inequalities are reconstructed by `verify_octic_tail.py`.

H.2 The compact full matrix and the separate local residual

Extend the whole-cell recurrence of Appendix G through order sixteen. There are now fifteen symmetric fields from the five even basis polynomials. Moment index zero is retained: there is no odd-basis index shift. The $n\ge 4$ monomial ratio has maximal power 36, and every complete paired error remains. On 1024 closed cells, the signed-loss construction gives simultaneous enclosures of every $\int_a^1 E_8\,dm_0$, $0\le a\le 7/100$. Each congruent Gershgorin row reserve exceeds 1/100000; the smallest exceeds 9/100000. The checker `verify_octic_compact.py 1024` also verifies the bridge past a_3. The determinant $\det C_8=2^{-70}$ and the displayed Frobenius bound follow by direct rational calculation from the stated basis.

For the residual, let the raw even-three prefix be

$$\widetilde{A}=\begin{pmatrix} a & b & c\\ b & d & e\\ c & e & f\end{pmatrix}.$$

Use all six adjugate entries $df-e^2, ce-bf, be-cd, af-c^2, bc-ae, ad-b^2$. For cross-output columns u,w and top entries d_{11},d_{12},d_{22}, subtract

$$\frac{u^T\,\mathrm{adj}(\widetilde{A})u}{\det\widetilde{A}},\quad \frac{u^T\,\mathrm{adj}(\widetilde{A})w}{\det\widetilde{A}},\quad \frac{w^T\,\mathrm{adj}(\widetilde{A})w}{\det\widetilde{A}}$$

from those three entries. Whole-cell Sylvester gates are $a > 0$, $ad - b^2 > 0$ and $\det \widetilde{A} > 0$. Homogeneous Schur scaling restores the same raw weight $s^2\pi^2 e^{10s-2X}$; it loses no Φ^2 factor. Split the cell containing $7/100$, retain full interval dependency errors and use the signed initial-loss intervals. The resulting two-by-two matrix has first diagonal $> 1/10000$ and determinant $> 1/100000000$. The checker `verify_octic_residual.py 1024` reconstructs the certificate and enforces its PD signs. These two compact representations share an interval engine, not independent arithmetic implementations. Their analytic terminal PSD and strictness premises were proved separately above.

I A terminating effective finite-order tail algorithm

For input $N \geq 0$ and parity $\epsilon \in \{0, 1\}$, omit empty blocks and set

$$r_\epsilon = \lfloor (N - \epsilon)/2 \rfloor + 1, \quad \mathcal{B}_\epsilon = (h_{2i+2j+2\epsilon})_{i,j=0}^{r_\epsilon - 1}, \quad K = r(r-1) + \epsilon r.$$

The checker `verify_finite_order_tail.py N` tests successive integers $L \geq \max(3, N+1)$ and returns rational bounds proving both parity blocks PD for every $X \geq L$. We give a computable termination bound, rather than assume eventual numerical success.

For $g_k = (2k-1)!!8^k$, $g_0 = 1$, paired cancellation gives $\deg Q_k \leq k + 2$ and leading coefficient $4g_k$. Gaussian parity Hermite norms give

$$G_{\det} = 8^K \prod_{i=0}^{r-1} (2i + \epsilon)!, \qquad \kappa = G_{\det}/4.$$

The head determinant numerator has degree $K + 2r$ and leading coefficient $4^r G_{\det} > 0$. Thus the reserve polynomial

$$P_{\text{res}} = \det(Q_{i+j+\epsilon}) - \kappa X^K (2X - 3)^{2r}$$

and every preceding leading minor have positive leading coefficient. If $P(x) = \sum_{l=0}^{d} a_l x^l$ with $a_d > 0$, then every coefficient of $P(L + v)$ is strictly positive whenever

$$L \geq B(P) := \left\lceil 1 + \frac{\sum_{l<d} |a_l|}{a_d} \right\rceil.$$

Indeed $[v^j]P(L+v) \geq \binom{d}{j} L^{d-j-1}(a_d L - \sum_{l<d} |a_l|) > 0$ for $j < d$; the leading coefficient is unchanged. Let B_{head} be the maximum over these finitely many polynomials for both parities.

Choose d_i to be the least power of two with $d_i^2 \geq g_{2i+\epsilon}$ and put $D_\epsilon = \operatorname{diag}(d_i X^{i+\epsilon/2})$. For $L \geq 3$, a trace bound is

$$T_\epsilon(L) = \sum_i \frac{\sum_l |[X^l] Q_{2i+\epsilon}| L^{l-2i-\epsilon-2}}{(2 - 3/L)^2 d_i^2} \leq T_\epsilon(3).$$

Once the shifted leading minors and determinant reserve pass, the head lower bound is

$$\gamma_\epsilon(L) = \frac{(r-1)^{r-1}\kappa}{(\prod_i d_i)^2 T_\epsilon(L)^{r-1}} \geq \gamma_0 := \min_\epsilon \gamma_\epsilon(3) > 0.$$

For $r = 1$, the omitted eigenvalue product and zeroth powers are defined as 1.

Use the exact second-atom envelope of Appendix H. For $j \leq 2N$ set

$$c_j(L) = \sum_l |[X^l] R_j| L^{l-j-1},$$

$$H_j(L) = \sum_l |[X^l] R_j| (1 + 2^{2l+2}) L^{l-j-1}, \quad b_j = 2\, 3^{2j+4}.$$

The full $n \geq 3$ derivative tail and all paired errors follow the same formula as Appendix G, with 12 replaced by L and $k \leq N$. Both needed geometric ratios are $< 1/2$ for $L \geq N+1$:

$$(3/2)^{4N+4}(3/8)^{5L} < 1/2, \qquad (4/3)^{4N+4}(3/8)^{7L} < 1/2.$$

The same inequality makes the remaining polynomial-exponential tails decrease from L. Let $e_k(L)$ be the complete normalized paired error, including the second-atom envelope and division by $(2-3/L)^2$. The block error is bounded by

$$\varepsilon_\epsilon(L) = r \max_{i,j} \frac{e_{i+j+\epsilon}(L)}{d_i d_j}.$$

The central ratio is bounded by $\rho(L) = [2L/(2L-3)]32(3/8)^{3L}$. If $\varepsilon_\epsilon(L) < \gamma_\epsilon(L)/2$, return

$$\mathcal{B}_\epsilon(X) \succeq D_\epsilon(X)^2 \frac{\gamma_\epsilon(L) - \varepsilon_\epsilon(L)}{(1+\rho(L))^2} > 0 \quad (X \geq L).$$

Every acceptance comparison uses exact rational arithmetic.

To bound when this must happen, write $c_j^* = \sum_l |[X^l]R_j|$ and $H_j^* = \sum_l |[X^l]R_j|(1+2^{2l+2})$, and define

$$\mathfrak{A}_c(P) = \sum_l |[X^l]P| \sum_{j=0}^{l} \binom{l}{j} (3j/(8c))^j.$$

The zeroth factor is 1. The shifted envelope divided by L^{k+2} is at most $L^k \mathfrak{A}_c(P)$ for the degrees occurring here. Put

$$\begin{aligned}
\mathfrak{K}_k &= \mathfrak{A}_3(C_k) + \mathfrak{A}_6(B_k) \\
&\quad + \sum_{j=0}^{2k} \binom{2k}{j} [H_j^* b_{2k-j} c_{2k-j}^* + b_j c_j^* H_{2k-j}^* + b_j b_{2k-j} c_j^* c_{2k-j}^*], \\
\mathfrak{K}_N^* &= \max_\epsilon r \max_{i,j} \mathfrak{K}_{i+j+\epsilon}/(d_i d_j).
\end{aligned}$$

Then $\varepsilon_\epsilon(L) \leq \mathfrak{K}_N^* L^N (3/8)^{3L}$. For $L \geq \max(1, N^2)$, the elementary $\log L \leq \sqrt{L}$ gives $N \log L \leq L$, and hence

$$L^N (3/8)^{3L} \leq e^L (3/8)^{3L} < (27/128)^L < 4^{-L}.$$

Choose an integer $q \geq 1$ with $4^q > 2\mathfrak{K}_N^*/\gamma_0$. This is computable without logarithms: if the latter rational has positive numerator u in lowest terms, take $q = \max(1, \lceil \mathrm{bitlength}(u)/2 \rceil)$. Therefore

$$L_{\text{safe}} = \max(3, N+1, N^2, B_{\text{head}}, q)$$

pays every head, ratio and perturbation test. The successive-integer search must halt no later than this finite bound.

For illustration, exact normal/optimized checks returned

N	0	1	4	7	8	10
L	3	4	7	9	11	17

with corresponding conservative safe bounds $12, 18, 2608, 2007039, 125672971, 298569113120$. The closed-form $1000N^2$ theorem is the cleaner symbolic estimate; this exact search can certify much smaller low-order instances. Neither the finite cases nor the terminating order-dependent search settles all orders at any fixed center.

References

[1] S. Schubr, *Minimum-Degree Obstructions in the Riemann Theta Product: Classification, transition bounds and robust Gram certificates*, edition 2, October 18, 2026, `https://luminaryoddities.com/researcharchive/minimum-degree-obstructions-riemann-theta-product/v/2`.

[2] S. Schubr, *Averaged Quintic Gate by Paired Kernel and Endpoint Conversion*, unpublished manuscript, 2026.

[3] G. Csordas, *Fourier transforms of positive definite kernels and the Riemann ξ-function*, Comput. Methods Funct. Theory **15** (2015), 373–391, arXiv:1309.0055.

[4] J. Gallier, *The Schur complement and symmetric positive semidefinite (and definite) matrices*, author notes, `https://www.cis.upenn.edu/~jean/schur-comp.pdf`.

[5] W. N. Anderson, Jr. and R. J. Duffin, *Series and parallel addition of matrices*, J. Math. Anal. Appl. **26** (1969), 576–594, doi:10.1016/0022-247X(69)90200-5.

[6] Z. Tarcsay, *On the parallel sum of positive operators, forms, and functionals*, arXiv:1501.01922, 2015.

[7] L. Smith, *Monotone comparative statics under uncertainty: interval dominance order*, author lecture notes, Integral Single Crossing Corollary, `https://www.lonessmith.com/wp-content/uploads/2025/03/1-Monotone-comparative-statics.pdf`.

[8] H. Umihara, *A Fourth-Derivative Inequality for the Riemann Xi Kernel*, preprint, August 2026, doi:10.5281/zenodo.21902312.

[9] GettysburgResearch, *Reviewer B—independent first-pass scientific review*, September 5, 2026, Section 6, repository report (variable-metric Schur averaging identity), `https://github.com/GettysburgResearch/riemann/blob/main/reviews/B/REPORT.md`.

[10] M. Griffin, K. Ono, L. Rolen and D. Zagier, *Jensen polynomials for the Riemann zeta function and other sequences*, Proc. Natl. Acad. Sci. USA **116** (2019), 11103–11110, arXiv:1902.07321.

[11] C. O'Sullivan, *Zeros of Jensen polynomials and asymptotics for the Riemann xi function*, manuscript dated November 29, 2020, `https://fsw01.bcc.cuny.edu/cormac.osullivan/Research/xi-revised.pdf`.

[12] T. M. Dunster, *Uniform asymptotic expansions for Bessel functions of imaginary order and their zeros*, arXiv:2412.12595, version 1, December 17, 2024, `https://arxiv.org/html/2412.12595v1`.

[13] F. W. J. Olver et al., eds., *NIST Digital Library of Mathematical Functions*, Sections 10.27, 10.32, 10.43, 18.19, 18.22 and 18.23, `https://dlmf.nist.gov/`.

[14] P. Bizeul and B. Klartag, *Polynomial Approximation in L^2 of the Double Exponential via Complex Analysis*, arXiv:2502.07448, 2025, `https://arxiv.org/abs/2502.07448`.

[15] L. N. Trefethen and J. A. C. Weideman, *The exponentially convergent trapezoidal rule*, SIAM Rev. **56** (2014), 385–458, doi:10.1137/130932132.

[16] J. F. Kaiser and R. W. Schafer, *On the use of the I_0-sinh window for spectrum analysis*, IEEE Trans. Acoust. Speech Signal Process. **28** (1980), 105–107, doi:10.1109/TASSP.1980.1163349.

[17] J. O. Smith III, *Spectral Audio Signal Processing*, Section "Kaiser Window," `https://www.dsprelated.com/freebooks/SASP/Kaiser_Window.html`.

[18] M. Planat, *Nonlocal Cancellation in a Theta-Kernel Decomposition of the Riemann Ξ-Growth Derivative: An Obstruction to Phase-Aligned Blockwise Positivity*, preprint, August 25, 2026, doi:10.20944/preprints202608.1692.v1. See also *A Theta-Kernel Reformulation of Riemann-Ξ Growth and the Obstruction to Blockwise Positivity*, Symmetry **18** (2026), 1283, doi:10.3390/sym18081283.

[19] D. K. Dimitrov and Y. Xu, *Wronskians of Fourier and Laplace transforms*, arXiv:1606.05011, 2016, `https://arxiv.org/abs/1606.05011`.

[20] J. Sondow and C. Dumitrescu, *A monotonicity property of Riemann's xi function and a reformulation of the Riemann hypothesis*, Period. Math. Hungar. **60** (2010), 37–40, `https://arxiv.org/abs/1005.1104`.

[21] S. Schubr, *Monotone center weights and the averaged odd-quintic theta gate*, published edition 6, October 24, 2026, Luminary Research Archive, `https://research.luminaryoddities.com/monotone-center-weights-averaged-odd-quintic-theta-gate/v/6`.

[22] C. C. Heyde, *On a property of the lognormal distribution*, J. R. Stat. Soc. B **25** (1963), 392–393, doi:10.1111/j.2517-6161.1963.tb00521.x.

[23] G. Świderski and B. Trojan, *Asymptotic behaviour of Christoffel–Darboux kernel via three-term recurrence relation I*, Constr. Approx. **54** (2021), 49–116, doi:10.1007/s00365-020-09519-w.

[24] G. Świderski, *Nevai's condition for measures with unbounded supports*, J. Approx. Theory **314** (2026), 106232; arXiv:2404.07566, version 4, `https://arxiv.org/html/2404.07566v4`.

[25] K. Morrison, *Spectral approximation of multiplication operators*, New York J. Math. **1** (1995), 75–96, `https://nyjm.albany.edu/j/1995/1-5.pdf`.

[26] S. Serra-Capizzano, *The spectral approximation of multiplication operators via asymptotic (structured) linear algebra*, Linear Algebra Appl. **424** (2007), 154–176, doi:10.1016/j.laa.2006.08.029.

[27] S. Karnik, J. Romberg and M. A. Davenport, *Improved bounds for the eigenvalues of prolate spheroidal wave functions and discrete prolate spheroidal sequences*, Appl. Comput. Harmon. Anal. **55** (2021), 97–128, doi:10.1016/j.acha.2021.04.002.