A sixth-order obstruction to total positivity of an arithmetic theta trace kernel

Scott Schubr
Luminary Oddities

October 25, 2026

Abstract

We exhibit an explicit negative sixth-order Toeplitz minor of the infinite arithmetic kernel $G(x) = \sum_{n\geq 1} \exp(-n^2 e^{-x})$. At the increasing nodes $x_i = -\log(5/2) + i/100$ and $y_j = j/100$, $0 \leq i, j \leq 5$, its determinant lies strictly between $-978 \cdot 10^{-59}$ and $-977 \cdot 10^{-59}$. A portable certificate uses only integer and rational arithmetic and bounds the complete omitted arithmetic tail; an independently constructed ball-arithmetic enclosure verifies the same full-series minor. Positive exponential gauges and translation transfer this counterexample to every kernel $Ce^{-3x/4} \sum_{m\neq 0} \exp(-\alpha m^2 e^{-x})$, $C, \alpha > 0$, disproving the total-positivity assertion for the explicit trace kernel in arXiv:2602.01248v1. We also give an elementary two-summand counterexample to positive-mixture closure, classify finite positive common-tilt mixtures of Gumbel kernels, and recover an infinite-order obstruction from the Gamma–zeta transform. Separate twelfth-order ordered and derivative certificates supply additional reproducible checks. The arithmetic trace kernel is distinct from the even Riemann density.

1 The kernel and the counterexample

A nonnegative function f is a Pólya frequency function of order r, denoted PF_r, if

$$\det[f(x_i - y_j)]_{i,j=1}^{k} \geq 0 \quad (1 \leq k \leq r)$$

for every pair of strictly increasing real node lists. The notation PF_∞ requires this at every finite order. A single negative ordered minor disproves the corresponding property.

Consider the locally uniformly convergent arithmetic series

$$G(x) = \sum_{n=1}^{\infty} e^{-n^2 e^{-x}}. \tag{1}$$

Its terms and every fixed derivative converge locally uniformly on the real axis. In particular G is real analytic. The logarithmic trace kernel explicitly defined in equation (11) of Watson [1] is

$$\Phi_{C,\alpha}(x) = Ce^{-3x/4} \sum_{m\in\mathbb{Z}\setminus\{0\}} e^{-\alpha m^2 e^{-x}}, \qquad C, \alpha > 0. \tag{2}$$

Theorems 5.3 and 8.5 of that version assert that this kernel is PF_∞. The following explicit finite counterexample shows otherwise.

Theorem 1.1 (Explicit sixth-order failure)**.** *Set*

$$a_i = -\log(5/2) + \frac{i}{100}, \qquad b_j = \frac{j}{100}, \qquad 0 \le i, j \le 5.$$

For the full infinite series (1)*,*

$$-\frac{978}{10^{59}} < \det[G(a_i - b_j)]_{i,j=0}^{5} < -\frac{977}{10^{59}}. \tag{3}$$

Consequently, for every $C, \alpha > 0$*, the kernel* (2) *fails* PF_6*. Explicit node lists for it are*

$$x_i = \log(2\alpha/5) + i/100, \qquad y_j = j/100.$$

The proof of the rational interval (3) is given in Section 2, together with its complete finite reduction. The accompanying standard-library program `verify_rational.py` performs the exact integer calculation. For the claimed transfer, observe first that

$$\Phi_{C,\alpha}(x) = 2Ce^{-3x/4}G(x - \log\alpha).$$

The indicated nodes therefore give the exact determinant identity

$$\det[\Phi_{C,\alpha}(x_i - y_j)] = (2C)^6(2\alpha/5)^{-9/2}\det[G(a_i - b_j)]. \tag{4}$$

Indeed the exponential row and column factors have product

$$\exp[-(3/4)(\sum_i x_i - \sum_j y_j)] = (2\alpha/5)^{-9/2}.$$

Every factor in (4) outside the last determinant is positive. Neither truncating the arithmetic series nor dropping its negative-index copy is part of the assertion.

Scope and predecessors. Positive-mixture nonclosure is classical; convolution closure is a different statement. Our contribution is a concrete finite-order counterexample for the explicit infinite trace kernel, with a fully paid portable certificate. Earlier finite-Toeplitz and interval-certificate methods include Michałowski's order-five obstruction for the *different*, even de Bruijn–Newman density [5].

The correction here concerns the literal definitions and assertions of [1], version 1. The introduction of the companion paper [2] refers to revisions of Paper I. Unseen revised definitions and that companion's conditional divisor-identification theorems are outside the present claim. In particular, (2) is not the even original Riemann density, and its negative minor is not a negative value of the original physical theta form.

2 A complete rational certificate

Write

$$M_{ij} = G(a_i - b_j) = \sum_{n\ge1} e^{-n^2 u_{i-j}}, \qquad u_d = (5/2)e^{-d/100}, \quad -5 \le d \le 5.$$

The logarithm in the node specification disappears from the entries. It is enough to enclose 11 positive exponential sums and then their 6×6 determinant.

2.1 The whole arithmetic tail

For every relevant d,

$$u_d \geq (5/2)e^{-5/100} > (5/2)(1 - 5/100) > 2.$$

Retain exactly $n = 1, \ldots, 21$. The omitted positive tail is bounded uniformly by

$$\begin{aligned} 0 < \sum_{n=22}^{\infty} e^{-n^2 u_d} < \sum_{n=22}^{\infty} e^{-2n^2} &\leq \frac{e^{-968}}{1 - e^{-90}} < 2e^{-968} \\ &< 2(3/8)^{968} < 10^{-360}. \end{aligned} \tag{5}$$

The first exponent gap is $2(23^2 - 22^2) = 90$, and the subsequent gaps increase. The inequality $e > 8/3$ follows from the first four terms of its positive series and the remaining positive terms. The final inequality in (5) is the integer comparison $2 \cdot 3^{968} \cdot 10^{360} < 8^{968}$. No unspecified remote tail remains.

2.2 Outward exponential enclosures

Set $B = 10^{360}$. Represent a real interval by integer endpoints (l, u), meaning $[l/B, u/B]$. Addition is exact at this scale. For products, take the minimum and maximum of the four endpoint products, divide by B, and round the lower endpoint down and the upper endpoint up. For a positive interval its reciprocal is enclosed by

$$\left[\frac{\lfloor B^2/u \rfloor}{B}, \frac{\lceil B^2/l \rceil}{B}\right].$$

These rules include negative Taylor coefficients with their correct directed rounding.

To enclose e^{-a}, $a \geq 0$, repeatedly halve an enclosing interval until its upper endpoint is at most $1/8$. Evaluate

$$T_{300}(z) = \sum_{j=0}^{300} \frac{(-z)^j}{j!}$$

by outward-rounded Horner arithmetic, with outward coefficient enclosures. The alternating-series theorem gives, uniformly for $0 \leq z \leq 1/8$,

$$|e^{-z} - T_{300}(z)| \leq \frac{1}{8^{301} 301!} < B^{-1}.$$

The last inequality is checked by exact integers. Widen by one unit at each end, then square the positive interval outward to undo every halving. This also encloses $e^{-d/100}$, using a reciprocal when $d < 0$. All input rationals are enclosed outward before use. Summing the twenty-one terms and adding $[0, B^{-1}]$ for (5) yields a rigorous interval for every M_{ij}.

2.3 An exact integer determinant and its error

Let the resulting integer bounds for entry d be (l_d, u_d^{int}), and choose the integer center

$$c_d = \left\lfloor \frac{l_d + u_d^{\text{int}}}{2} \right\rfloor, \qquad E = \max_d \max(c_d - l_d, u_d^{\text{int}} - c_d).$$

Thus the rational center matrix $A_{ij} = c_{i-j}/B$ obeys $|M_{ij} - A_{ij}| \leq E/B$. The program verifies that both actual entry intervals and their centers lie strictly between zero and one.

Lemma 2.1 (Determinant perturbation)**.** *If two real $r \times r$ matrices have entries of absolute value at most one and corresponding entries differ by at most δ, then their determinants differ by at most $r!r\delta$.*

Proof. For each permutation, telescope the difference of its two products, changing one factor at a time. Each of the r terms has absolute value at most δ, since the other factors have absolute value at most one. Sum over all $r!$ permutations. □

Compute the integer

$$D = \det[c_{i-j}]_{i,j=0}^{5}$$

by fraction-free Bareiss elimination, checking that every division is exact. Lemma 2.1 gives the rational enclosure

$$\frac{D - 6!\,6EB^5}{B^6} \le \det M \le \frac{D + 6!\,6EB^5}{B^6}. \tag{6}$$

The supplied program constructs these integers and verifies, by integer comparisons with rational endpoints, that (6) is strictly contained in (3). This proves Theorem 1.1. It uses no floating-point determinant, numerical logarithm, third-party numerical package, or unbounded search. The arithmetic parameters are the fixed scale 10^{360}, Taylor degree 300, twenty-one retained atoms and the stated node spacing 1/100.

2.4 Independent reconstruction

The optional program `verify_order_six_arb.py` independently constructs all eleven full-series entries using directed Arb balls, at both 512 and 1024 bits. It uses twenty-one retained atoms, the symmetric radius $2e^{-968}$ for the entire positive omitted tail, and a ball-matrix determinant. It verifies the same strict rational endpoints (3). It reads neither the integer centers nor the Bareiss determinant from the primary verifier. The symmetric tail enclosure is less sharp than positivity allows but still contains every omitted atom. The primary proof does not depend on this numerical library.

3 Why the positive-mixture inference fails

The proof of Lemma 8.2 in [1] expands a determinant of a positive sum into mixed-column determinants and assigns a nonnegative sign to each. Separate PF_∞ hypotheses control determinants of one common function. They do not control determinants whose columns use different functions.

Proposition 3.1 (Two individually totally positive summands)**.** *For $\beta > 0$ and $v > 0$, the integrable function*

$$f_{\beta,v}(x) = e^{-\beta x - v e^{-x}}$$

is PF_∞*. Nevertheless the sum*

$$f(x) = e^{-3x/4}\big(e^{-e^{-x}} + e^{-25e^{-x}}\big)$$

fails PF_2*.*

Proof. The bilateral Laplace transform of the individual function is

$$\int_{\mathbb{R}} e^{-sx} f_{\beta,v}(x)\,dx = v^{-s-\beta}\Gamma(s+\beta), \qquad \Re s > -\beta.$$

Its reciprocal is an exponential times $1/\Gamma(s+\beta)$, an entire Laguerre–Pólya function, so Schoenberg's characterization applies [3, 4]. Alternatively, the kernel $e^{-ve^{-(x-y)}}$ is an exponential kernel e^{ab} with both $a = -ve^{-x}$ and $b = e^y$ increasing; its strict total positivity and a positive exponential gauge give the same conclusion.

For the sum put $x_0 = \log(6/\log 2)$ and $d = \log 2$. Increasing nodes $(x_0, x_0 + d)$ and $(0, d)$ give

$$f(x_0)^2 - f(x_0 - d)f(x_0 + d) = \frac{e^{-3x_0/2}2^{-1/3}}{1024}(1156 - 1285\,2^{-1/12}) < 0.$$

Indeed $e^{-x_0} = \log 2/6$, so the three unweighted values are $(17/16)2^{-1/6}$, $(257/256)2^{-1/3}$ and $(5/4)2^{-1/12}$. The sign is the exact integer inequality

$$1285^{12} - 2 \cdot 1156^{12} = 887923576717993052132422561058751035 3 > 0.$$

The two rates 1 and 25 are two of the arithmetic building blocks. This example disproves the asserted general closure rule; the separate infinite-series certificate in Section 2 is needed to reject the actual infinite kernel. □

Theorem 3.2 (Finite common-tilt mixtures)**.** *Fix $\beta > 0$ and let*

$$f(x) = e^{-\beta x}\sum_{j=1}^{M} c_j e^{-v_j e^{-x}}, \qquad c_j, v_j > 0.$$

After equal rates are combined, f is PF_∞ if and only if $M = 1$.

Proof. For $M = 1$ use Proposition 3.1. In general the bilateral transform is

$$\Gamma(s+\beta)A(s+\beta), \qquad A(z) = \sum_j c_j e^{-z\log v_j}.$$

The exponential polynomial A is entire of order at most one and is positive on the real axis. If it were zero-free, Hadamard factorization would give $A(z) = e^{az+b}$. Positivity on the real axis makes a real and $e^b > 0$. Comparing the largest real exponent as $z \to +\infty$ would identify this exponential with the largest summand. Subtraction leaves a sum of strictly positive terms equal to zero on the real axis, unless $M = 1$. For $M \geq 2$, A therefore has a nonreal zero.

Gamma is finite and nonzero at that zero. The reciprocal transform has a genuine nonreal pole, whereas Schoenberg's necessary condition for an integrable PF_∞ function is an entire reciprocal. Uniqueness of meromorphic continuation gives a contradiction. This proves necessity. □

This statement has one common tilt β. It is not a classification of arbitrary mixtures. Nor does failure of all finite partial sums alone imply failure of an infinite limit.

4 An independent transform obstruction

The full infinite kernel (2) admits a short analytic obstruction as well. Its negative tail is super-exponentially decreasing, and its positive tail is $O(e^{-x/4})$. For the latter, compare the decreasing Gaussian sum with its integral:

$$\sum_{n\geq 1} e^{-tn^2} \leq \int_0^\infty e^{-tu^2}\,du = \frac{\sqrt{\pi}}{2\sqrt{t}}.$$

Thus its bilateral Laplace transform converges for $\Re s > -1/4$. Absolute convergence, followed by the substitution $t = e^{-x}$, gives

$$\mathcal{B}\Phi_{C,\alpha}(s) = 2C\,\Gamma(s+3/4)\alpha^{-s-3/4}\zeta(2s+3/2). \tag{7}$$

For any nontrivial zeta zero ρ, the meromorphic continuation in (7) vanishes at $s = \rho/2 - 3/4$. This point is nonreal; the Gamma and exponential factors there are finite and nonzero. Hence the reciprocal transform has a pole and cannot be entire. Nontrivial zeta zeros exist unconditionally; no off-critical-line zero is needed. Schoenberg's necessary condition again disproves PF_∞.

The defining Laplace integral is not evaluated outside its convergence half-plane. The argument uses uniqueness of continuation of the transform and of the hypothetical entire reciprocal. Trivial zeta zeros, which can interact with Gamma poles, are not used. Theorem 1.1 supplies the finite failing order.

A An additional ordered-node certificate

The supplied `verify_order_twelve.py` applies precisely the fixed-point and determinant method of Section 2 at $a_i = -\log 4 + i/100$, $b_j = j/100$, $0 \le i, j \le 11$. There $u_{i-j} = 4e^{-(i-j)/100} > 3$; seventeen retained atoms leave a tail bounded by $2e^{-972} < 10^{-360}$. The program verifies

$$-6 \cdot 10^{-206} < \det[G(a_i - b_j)] < -5 \cdot 10^{-206}.$$

The sixth-order main theorem is stronger as an order obstruction. This twelfth-order witness is included as a separate reproducible instance of the same kernel failure, rather than as an independent contribution.

B An independent derivative certificate

The supplement `verify_arb.py` supplies a different finite certificate. Put $\beta = 3/4$, $P_0(v) = 1$ and

$$P_{k+1}(v) = (v-\beta)P_k(v) - vP_k'(v).$$

At $x_0 = \log(\alpha/4)$ the kth derivative of (2), divided by its common positive factor $2Ce^{-\beta x_0}$, is

$$A_k = \sum_{n\ge 1} e^{-4n^2} P_k(4n^2), \qquad 0 \le k \le 22.$$

The signed confluent determinant is $(-1)^{66}\det[A_{i+j}]_{i,j=0}^{11} = \det[A_{i+j}]$. The coefficientwise absolute-value recurrence bounds

$$|P_k(v)| \le \prod_{j=0}^{k-1}(v+\beta+j) \le (2v)^k \quad (v \ge \beta + k).$$

For completeness, if R_k is the polynomial of absolute coefficients, then $R_{k+1} \le (v+\beta)R_k + vR_k'$ coefficientwise. For the product bound Q_k, the elementary inequality $vQ_k' \le kQ_k$ closes the induction.

When $n \ge 17$, $v = 4n^2 \ge 1156$ and $k \le 22$. The inequality $k\log(2v) \le v/2$ holds: at 1156, use $\log 2312 < 8$ and $176 < 578$, and then differentiate $v/2 - 22\log(2v)$. Consequently the entire omitted signed tail of each jet is bounded by

$$\sum_{n\ge 17} e^{-2n^2} \le \frac{e^{-578}}{1-e^{-70}}.$$

Using sixteen retained terms and this radius for every jet, 1024-bit Arb ball arithmetic certifies

$$-61 \cdot 10^{57} < \det[A_{i+j}] < -6 \cdot 10^{58}.$$

Its center is approximately $-6.0198638432581949286 \cdot 10^{58}$. This is a separate derivative-based check of finite-order failure, not a second theorem count or an assumption in the standard-library ordered-node proof. The derivative recurrence, Gamma–zeta argument and integer ordered-node certificate provide distinct reproducible ways to examine the same kernel obstruction. Indeed, Taylor confluence at the increasing nodes $x_i = x_0 + \varepsilon i$ and $y_j = \varepsilon j$ gives

$$\lim_{\varepsilon \downarrow 0} \frac{\det[\Phi_{C,\alpha}(x_0 + \varepsilon(i-j))]_{i,j=0}^{11}}{\varepsilon^{132}} = (2Ce^{-\beta x_0})^{12} \det[A_{i+j}] < 0.$$

The two Vandermonde factors for $0, \ldots, 11$ cancel the Taylor factorials, and the column sign is $(-1)^{66} = 1$. Thus sufficiently small positive spacings give negative ordered minors independently of the explicit spacing in Theorem 1.1.

Reproduction and assistance

The supplement contains the standard-library sixth- and twelfth-order verifiers, the independent sixth-order Arb reconstruction and the twelfth-order derivative verifier, their dependency instructions and editable manuscript source. The primary verifier and its independent reconstruction check the explicit nodes and rational interval in Theorem 1.1; the derivative verifier checks Appendix B. All analytic reductions and infinite-tail bounds are given above. OpenAI Codex assisted with literature discovery, mathematical development and checking, reproducibility code and manuscript preparation. The author is responsible for the content.

References

[1] D. F. Watson, *The Riemann Ξ-function from primitive Markovian cycles I: A canonical construction*, arXiv:2602.01248v1 (2026). https://arxiv.org/abs/2602.01248v1.

[2] D. F. Watson, *The Riemann Ξ-function from primitive Markovian cycles II: Strip rigidity and divisor identification*, arXiv:2602.06080v1 (2026). https://arxiv.org/abs/2602.06080v1.

[3] I. J. Schoenberg, On Pólya frequency functions. I. The totally positive functions and their Laplace transforms, *J. Analyse Math.* **1** (1951), 331–374.

[4] A. Belton, D. Guillot, A. Khare and M. Putinar, Preservers of totally positive kernels and Pólya frequency functions, *Math. Res. Rep.* **3** (2022), 35–56. doi:10.5802/mrr.12.

[5] W. Michałowski, *On the Pólya frequency order of the de Bruijn–Newman kernel: Certified failure at order five and the Toeplitz threshold phenomenon*, arXiv:2602.20313v1 (2026). https://arxiv.org/abs/2602.20313v1.