

Finite-copy risk for polynomial energies of thermal Gaussian states

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Abstract

We study estimation of homogeneous normally ordered polynomial energies from independent copies of a centered gauge-invariant Gaussian state. For every positive copy count, passive mixing of the copies produces a positive unbiased observable whose relative variance has an exact formula in terms of reduced density matrices of a covariance-weighted coefficient tensor. Under a lower spectral bound on the occupation covariance, the formula gives a sharp dimension-independent worst-case variance, attained by a single-mode monomial. It is also the worst-polynomial minimax risk among globally unbiased quantum estimators. Allowing bias, we prove matching universal bounds in the degree, copy count and occupation floor. These bounds characterize consistency, and a further two-point argument proves that the minimax relative risk tends to one in the opposite regime. The construction uses quantum symmetry averaging, while its exact risk analysis combines bosonic Gaussian contractions with complex-sphere moments.

Keywords. Quantum estimation; thermal Gaussian states; polynomial moments; minimax risk; symmetry averaging.

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1 Introduction

Thermal Gaussian states provide a standard model of unsqueezed optical noise. A natural family of observables consists of homogeneous normally ordered energies: if f is a known holomorphic polynomial and b denotes the annihilation operators, the target is $\text{Tr } \rho f(b)^\dagger f(b)$. This target also equals a classical Gaussian polynomial moment under the positive coherent-state representation. The observation experiment, however, consists of quantum states rather than direct samples of the latent coherent amplitudes.

Our aim is a finite-copy, dimension-independent risk analysis that allows the degree to vary with the available resources. We construct a positive collective observable for every number of copies, including fewer copies than the polynomial degree. Its exact relative variance is a finite sum of weighted purities of reduced coefficient tensors. The formula yields a scalar-attained sharp worst-case variance, an exact unbiased minimax conclusion, and a degree/copy/occupation law for the minimax risk with arbitrary bias.

The ingredients have substantial precedents. Guță and Butucea introduced quantum U-statistics and their Hoeffding analysis [2]. Tsang’s quantum Rao–Blackwell theorem shows how an invariant state family permits estimator improvement by group averaging [5, Section 5.3]. Dasgupta, Warsi and Chatterjee relate marginal kernels to functional gradients and prove uniqueness, efficiency and concentration results for finite-dimensional quantum U-statistics [1]. Normally ordered moment estimation for positive coherent mixtures is developed within quantum semiparametric

estimation [6]; scalar thermal photon-count information is analyzed by Nair and Tsang [3]. The exact classical Gaussian-energy problem with direct amplitude access is treated in [4].

Here symmetry averaging is an input. The additional conclusions are an exact multimode risk identity with all contraction orders retained, sharp uniform unbiased risk for every copy count, and a uniform minimax degree frontier in the thermal-state experiment. The state family is restricted to centered gauge-invariant Gaussian states; this restriction permits repeated copy indices in the estimator. The proof handles the polynomial observables through finite photon-number sectors and their thermal moments.

2 Experiment and main results

Let d, n, N be positive integers. On d bosonic modes write $b_1, \dots, b_d$ for annihilation operators, satisfying $[b_i, b_j^\dagger] = \delta_{ij}\mathbb{I}$. For a positive definite Hermitian matrix A , define

$$\rho_A = \int_{\mathbb{C}^d} |z\rangle\langle z| d\text{CN}(0, A)(z), \quad A_{ij} = \mathbb{E}[Z_i \overline{Z_j}]. \quad (1)$$

Thus A is the occupation covariance, with vacuum noise accounted for by the commutation relations. All states in this paper have the representation (1).

Let $f \neq 0$ be homogeneous and holomorphic of degree n , with symmetric coefficients

$$f(z) = \sum_{I \in [d]^n} c_I z_{i_1} \cdots z_{i_n}. \quad (2)$$

The target and loss are

$$F_f(A) = \text{Tr } \rho_A f(b)^\dagger f(b) = \mathbb{E}|f(Z)|^2, \quad L_A(y) = \left(\frac{y}{F_f(A)} - 1 \right)^2. \quad (3)$$

The equality follows from $b_i|z\rangle = z_i|z\rangle$ and positivity. We observe $\rho_A^{\otimes N}$, allowing any collective POVM and real-valued decision. The known parameter restriction is

$$A \succeq aI_d, \quad a > 0, \quad \gamma = 1 + a^{-1}. \quad (4)$$

This is a spectral floor in every mode. The estimator may depend on f, d, N, a .

For copy r , denote its annihilation operator by $b_{i,r}$. For an ordered tuple $\mathbf{r} \in [N]^n$ set

$$C_{\mathbf{r}} = \sum_I c_I \prod_{j=1}^n b_{i_j, r_j}, \quad T_f = \frac{n!}{(N)^{\overline{n}}} \sum_{\mathbf{r} \in [N]^n} C_{\mathbf{r}}^\dagger C_{\mathbf{r}}. \quad (5)$$

Here $(N)^{\overline{k}} = N(N+1) \cdots (N+k-1)$, and $(N)^{\overline{0}} = 1$. Repeated indices r_j are included.

For stating the exact formula, choose a mode basis diagonalizing A , with eigenvalues λ_i , and transform c into that basis. Define

$$t_I = c_I \prod_{j=1}^n \sqrt{\lambda_{i_j}}, \quad \psi = \frac{t}{\|t\|} \in \text{Sym}^n(\mathbb{C}^d), \quad R_k = \text{Tr}_{n-k} |\psi\rangle\langle\psi|, \quad D_A = \text{diag}(1 + \lambda_i^{-1}). \quad (6)$$

The coordinate change in (6) is used in the analysis; the observable (5) is independent of the unknown A .

Theorem 2.1 (Exact variance and sharp uniform bound). *For all $d, n, N \geq 1$, the positive observable T_f is unbiased for $F_f(A)$ and has finite second moment. Its relative variance is*

$$\frac{\text{Var}_A(T_f)}{F_f(A)^2} = \sum_{k=1}^n \binom{n}{k}^2 \frac{k!}{(N)^{\overline{k}}} \text{Tr} \left(R_k^2 D_A^{\otimes k} \right). \quad (7)$$

Under (4), put

$$S(N, n, a) = \sum_{k=1}^n \binom{n}{k}^2 \frac{k! \gamma^k}{(N)^{\overline{k}}}. \quad (8)$$

Then the relative variance is at most $S(N, n, a)$. Equality holds for $d = 1$, $f(z) = z^n$, $A = a$, so the bound is sharp simultaneously in all parameters.

Define the worst-polynomial minimax risk

$$\mathcal{R}(N, n, a) = \sup_{d \geq 1} \sup_{0 \neq f \in \text{Sym}^n(\mathbb{C}^d)} \inf_{M, Y} \sup_{A \succeq a I_d} \mathbb{E}_A L_A(Y). \quad (9)$$

The notation for f in (9) identifies polynomials with their coefficient tensors. Define $\mathcal{R}_{\text{ub}}$ by restricting the infimum to estimators unbiased at every allowed A .

Theorem 2.2 (Unbiased optimum and biased minimax order). *For all $n, N \geq 1$ and $a > 0$,*

$$\mathcal{R}_{\text{ub}}(N, n, a) = S(N, n, a). \quad (10)$$

With $x = \gamma n^2 / N$,

$$\frac{7}{2304} \min(1, x) \leq \mathcal{R}(N, n, a) \leq \frac{S}{1 + S} \leq 1 - e^{-x} \leq \min(1, x). \quad (11)$$

The upper bound $S/(1 + S)$ is attained as a risk bound by a spectral measurement of $T_f/(1 + S)$.

Theorem 2.3 (Consistency and collapse). *For any sequence of allowed triples (N_j, n_j, a_j) , write $x_j = (1 + a_j^{-1}) n_j^2 / N_j$. Then*

$$\mathcal{R}(N_j, n_j, a_j) \longrightarrow 0 \iff x_j \longrightarrow 0, \quad (12)$$

$$x_j \longrightarrow \infty \implies \mathcal{R}(N_j, n_j, a_j) \longrightarrow 1. \quad (13)$$

These results quantify the resources of collective quantum estimation in the declared state model. The exact formula also retains dependence on the polynomial through the reduced-tensor purities in (7).

3 Symmetry averaging and the exact risk formula

3.1 Construction, domains and first moment

Let v be uniform on the complex unit sphere in $\mathbb{C}^N$ and write

$$b_i(v) = \sum_{r=1}^N v_r b_{i,r}, \quad H_v = f(b(v))^\dagger f(b(v)). \quad (14)$$

The combined modes have canonical commutators, and their state in $\rho_A^{\otimes N}$ is again ρ_A . The sphere identity

$$\mathbb{E}_v[v_{r_1} \cdots v_{r_n} \overline{v_{s_1} \cdots v_{s_n}}] = \frac{1}{(N)^{\overline{n}}} \sum_{\pi \in S_n} \prod_{j=1}^n \delta_{r_j, s_{\pi(j)}} \quad (15)$$

follows by writing a standard complex Gaussian vector as its independent radius and direction. Its squared radius has n th moment $(N)^{\overline{n}}$, while the Gaussian numerator is the Wick sum in (15). Symmetry of c and commutation of annihilators give

$$T_f = \mathbb{E}_v H_v. \quad (16)$$

In particular $\mathbb{E}_A T_f = F_f(A)$, since this is true for every H_v .

For a precise spectral construction, first work on finite total-photon sectors in the dN modes. Each $C_r^\dagger C_r$ preserves the total number in each copy. Every sector with bounded total photon number is finite dimensional, and (5) defines a positive matrix there. The direct sum of the fixed-total-number matrices defines a positive self-adjoint operator with the finite-particle vectors as a core. Its spectral measurement is the estimator used here. Thermal states on finitely many modes have all polynomial number moments finite, which pays its second moment and justifies passage from the sector identities to the full state. Formula (16) refers to an averaged observable.

3.2 All contraction orders

We prove Theorem 2.1 by computing $\mathbb{E}_A(H_v H_w)$ for two unit vectors v, w . In an eigenbasis of A , the nonzero two-point contractions are

$$\begin{aligned} \mathbb{E}_A[b_i(v)^\dagger b_j(w)] &= \delta_{ij} \lambda_i \sum_r \overline{v_r} w_r, \\ \mathbb{E}_A[b_i(v) b_j(w)^\dagger] &= \delta_{ij} (1 + \lambda_i) \sum_r v_r \overline{w_r}. \end{aligned}$$

The product $H_v H_w$ has four groups, each of size n : creators at v , annihilators at v , creators at w , and annihilators at w . In a Wick pairing let k be the number of contractions from the first annihilator group to the second creator group. Balance forces k contractions in the reverse direction between the other two groups. The remaining $n - k$ contractions on each side are internal. The number of labeled pairings in this class is

$$\binom{n}{k}^4 (k!)^2 ((n-k)!)^2 = (n!)^2 \binom{n}{k}^2. \quad (17)$$

Symmetry of c makes their coefficient contractions identical.

After whitening by (6), the internal contractions yield the k -body reduction R_k . More explicitly, for ordered multi-indices $I, J \in [d]^k$,

$$(R_k)_{I,J} = \sum_{L \in [d]^{n-k}} \psi_{I,L} \overline{\psi_{J,L}}.$$

The coefficient contraction for the k th class, divided by $\|t\|^4$, is

$$\sum_{I,J} \left(\prod_{\ell=1}^k (1 + \lambda_{i_\ell}^{-1}) \right) |(R_k)_{I,J}|^2 = \text{Tr}(R_k^2 D_A^{\otimes k}). \quad (18)$$

The cross-copy contractions contribute $|\langle v, w \rangle|^{2k}$. Since $F_f(A) = n! \|t\|^2$, equations (17)–(18) give

$$\frac{\mathbb{E}_A(H_v H_w)}{F_f(A)^2} = \sum_{k=0}^n \binom{n}{k}^2 |\langle v, w \rangle|^{2k} \text{Tr}(R_k^2 D_A^{\otimes k}), \quad (19)$$

where the $k = 0$ term is one. All terms are nonnegative real numbers. Finite thermal fourth moments of $f(b(v))$ also bound the ordered product by state Cauchy–Schwarz, uniformly over the compact sphere, so the integrals in (19) are legitimate.

For independent uniform v, w , unitary invariance and (15) imply

$$\mathbb{E}|\langle v, w \rangle|^{2k} = \frac{k!}{(N)^k}.$$

This includes $N = 1$, when both sides equal one. Integrating (19) and subtracting its $k = 0$ term proves (7). Finally $D_A \preceq \gamma I$ and $\text{Tr} R_k^2 \leq 1$ prove (8). A single-mode coefficient tensor is pure at every reduction and $A = a$ makes every weight equal γ , proving simultaneous equality. This completes the proof of Theorem 2.1.

4 Sharp unbiased risk

In one mode the thermal state is diagonal in the photon basis, with geometric probabilities

$$p_t(k) = \frac{t^k}{(1+t)^{k+1}}, \quad k \geq 0. \quad (20)$$

For N copies, the total count K has negative-binomial probabilities

$$\Pr_t(K = k) = \binom{N+k-1}{k} \frac{t^k}{(1+t)^{N+k}}.$$

Conditioned on K , the individual counts are uniform over compositions of K into N nonnegative parts. This distribution is independent of t . Since the quantum family is commuting, every quantum measurement is a classical randomization of the full count vector. Conditional averaging given K therefore reduces the variance of any unbiased estimator.

For $f(z) = z^n$, formula (5) simplifies to

$$T_f = \frac{n!(K)_n}{(N)^n}, \quad F_f(t) = n!t^n. \quad (21)$$

The identity follows by counting ordered choices of n photons across the N modes, or by the sphere average. The factorial moment $\mathbb{E}(K)_n = (N)^n t^n$ verifies unbiasedness directly.

The negative-binomial family is complete even after restriction to $t \geq a$. To see this, put $q = t/(1+t)$. If an integrable function $g(K)$ has expectation zero for every allowed t , its power series

$$\sum_{k \geq 0} g(k) \binom{N+k-1}{k} q^k$$

vanishes on $q \in (a/(1+a), 1)$. Integrability at every larger $q < 1$ gives convergence throughout the unit disk. Analytic uniqueness forces all coefficients to vanish. Consequently (21) is the unique unbiased function of K , and conditional averaging proves that it has minimum variance among all scalar unbiased measurements and decisions.

At $t = a$ its relative variance is $S(N, n, a)$ by Theorem 2.1. This proves the lower bound in (10); the estimator of Theorem 2.1 proves the upper bound for every d, f . Thus (10) holds with the displayed order of optimization.

For comparison, the factorial-product identity gives the scalar relative second moment in the alternative form

$$\mathbb{E}_t \left(\frac{T_f}{F_f(t)} \right)^2 = \sum_{j=0}^n \binom{n}{j}^2 j! \frac{(N)^{\overline{2n-j}}}{((N)^{\overline{n}})^2} t^{-j}. \quad (22)$$

Equating (22) with one plus (7) supplies a scalar identity between the falling-factorial and contraction expansions.

5 Biased minimax bounds and asymptotic phases

5.1 Upper bound

If T is unbiased and has relative variance at most S , its rescaling $T/(1+S)$ has relative risk at most

$$\frac{S}{(1+S)^2} + \left(\frac{1}{1+S} - 1 \right)^2 = \frac{S}{1+S}.$$

Moreover $\binom{n}{k}^2 k! \leq n^{2k}/k!$ and $(N)^{\overline{k}} \geq N^k$. Summing the exponential series gives $S \leq e^x - 1$ and proves all upper bounds in (11).

5.2 A uniform two-point lower bound

It suffices to use the scalar family (20), since it is included in (9). Compare $t_0 = a$ with $t_1 = a(1+h)$. Direct computation gives

$$\text{KL}(p_{t_0} \| p_{t_1}) = (1+a) \log \left(1 + \frac{ah}{1+a} \right) - a \log(1+h). \quad (23)$$

Its derivative in h is

$$\frac{ah}{(1+h)(1+a+ah)} \leq \frac{h}{\gamma},$$

so the one-copy divergence is at most $h^2/(2\gamma)$ for every $h \geq 0$. Set

$$h = \frac{1}{4} \min \left(\frac{1}{n}, \sqrt{\frac{\gamma}{N}} \right).$$

The N -copy divergence is at most $1/32$, and Pinsker's inequality bounds the total variation distance by $1/8$. The target ratio is $r = (1+h)^n$. Its relative separation

$$\delta = \frac{r-1}{r+1}$$

satisfies $\delta \geq nh/3$, because $nh \leq 1/4$, $r \leq e^{1/4}$ and $r-1 \geq nh$.

Threshold any estimator at the harmonic target midpoint $2F_0F_1/(F_0+F_1)$. Each classification error entails relative squared loss at least δ^2 . The sum of the testing errors is at least one minus total variation, so the maximum of the two risks is at least $\delta^2(1-1/8)/2$. Since $n^2h^2 = \min(1, x)/16$, this proves the lower bound $7 \min(1, x)/2304$. A quantum POVM cannot improve the distance of the commuting family. Theorem 2.2 follows.

5.3 Consistency and complete collapse

Equation (12) follows immediately from the upper and lower bounds in (11). To prove (13), take along the given sequence

$$h = \frac{x^{1/4}}{n}, \quad t_0 = a, \quad t_1 = a(1 + h).$$

The divergence bound above gives $N \text{KL} \leq 1/(2\sqrt{x})$, hence the total variation η_x tends to zero. Meanwhile $r = (1 + h)^n \geq 1 + x^{1/4}$ tends to infinity.

Let P_0, P_1 be the full photon-count laws and let their common subprobability measure have density $\min(dP_0, dP_1)$; its mass is $1 - \eta_x$. For every real y , direct minimization gives

$$\frac{1}{1+r} \left(\frac{y}{F_0} - 1 \right)^2 + \frac{r}{1+r} \left(\frac{y}{F_1} - 1 \right)^2 \geq \left(\frac{r-1}{r+1} \right)^2. \quad (24)$$

The minimizing y is the harmonic midpoint. Integrating (24) over the common measure bounds every estimator's maximum risk from below by

$$(1 - \eta_x) \left(\frac{r-1}{r+1} \right)^2 \rightarrow 1.$$

Classical randomization, and therefore arbitrary quantum measurement on this scalar subfamily, preserves this lower bound. The identically zero decision has risk one, proving Theorem 2.3.

6 Examples and consequences

For $n = 1$, $S = \gamma/N$ and the exact formula weights the one-body tensor purity. For $n = 2$,

$$S(N, 2, a) = \frac{4\gamma}{N} + \frac{2\gamma^2}{N(N+1)}.$$

In both cases the scalar monomial attains the uniform unbiased bound. For $N = 1$, the observable is simply $f(b)^\dagger f(b)$, and Theorem 2.1 computes all of its thermal fluctuation orders.

For fixed n, a and $N \rightarrow \infty$, the sharp unbiased worst-case variance has leading coefficient $\gamma n^2/N$. For growing degrees the exact finite sum is more informative. If $n \rightarrow \infty$ and $x = \gamma n^2/N \rightarrow \lambda \in (0, \infty)$, each fixed- k summand in (8) tends to $\lambda^k/k!$. Exponential domination then gives $S \rightarrow e^\lambda - 1$, and the rescaled estimator has a worst-case risk bound tending to $1 - e^{-\lambda}$. Theorem 2.3 separately characterizes vanishing risk and collapse for the full minimax problem.

The factor $\gamma = 1 + a^{-1}$ displays the occupation penalty of the state-only experiment. A small covariance floor increases the copies required at a fixed degree. Formula (7) simultaneously shows how coefficient tensors with mixed reductions improve upon the worst-polynomial risk. These conclusions use unrestricted collective spectral measurements and the full occupation model (4).

Computational verification. The accompanying exact-arithmetic program checks the contraction formula against an independent normal-order expansion for several nontrivial two-mode polynomials and copy counts. It also checks the scalar negative-binomial identity (22). The proofs above establish the results for arbitrary finite dimension and degree.

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