Exponential growth of sparse Euler products and normal graded domains

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October 23, 2026

Abstract

Let d_i be strictly increasing positive integer degrees and let M_i be positive integers with $\log M_i/d_i \to \delta \in (0,\infty)$. For the Euler product with M_i independent polynomial generators of degree d_i and one degree-one generator, we prove that the upper logarithmic coefficient rate is δ and the lower rate is $\max\{\delta-\theta,0\}$, where $\theta = \limsup_i \log(d_{i+1})/d_i$. This gives an exact convergence criterion with no regularity assumption on the support gaps. We contrast it with normal connected locally finite graded domains of Krull dimension two, obtained by adapting an infinite-pole construction of Maclean. Their lower rate is δ/R, where $R = \limsup_i d_{i+1}/d_i$, and their upper rate is δ. In particular, normality, finite dimension and a degree-one nonzerodivisor do not jointly force convergence of the exponential Hilbert rate.

1 Introduction

The support example $d_i = 2^i$ makes the comparison concrete. Under the common hypothesis $\log M_i/d_i \to \delta > 0$, independent polynomial generators have logarithmic Hilbert rate converging to δ. The normal two-dimensional pole algebra on the same support instead has lower rate $\delta/2$ and upper rate δ. The multiplication structures, not the support and block entropy alone, select the relevant gap scale.

The two exact criteria are consequently different: logarithmic gaps govern the polynomial model, while degree ratios govern the pole model. Both statements concern the indicated graded algebras. The non-approximability calculation explains why their exponential growth lies outside the hypotheses of the cited volume theorems.

For a connected locally finite graded algebra $A = \bigoplus_{n\geq 0} A_n$ over a field k, put $H_A(n) = \dim_k A_n$. We study the lower and upper logarithmic rates

$$\underline{h}(A) = \liminf_{n\to\infty} \frac{\log H_A(n)}{n}, \qquad \overline{h}(A) = \limsup_{n\to\infty} \frac{\log H_A(n)}{n}$$

in examples where $H_A(n) \geq 1$. All logarithms are natural. Equal finite rates are equivalent to existence of the corresponding limiting nth-root rate.

Weighted polynomial Hilbert series are classical Euler transforms, or generating functions for colored multisets; see Cameron [1, Section 6]. Much sharper coefficient asymptotics are known under regular component-density or analytic hypotheses, including expansive multisets [2] and Khintchine–Meinardus models [3]. Additive arithmetical semigroups supply another established formulation [4]. Vakhitov and Minenkov [5] study logarithmic restricted-partition asymptotics in the exponential case. The present note gives a direct elementary matching-bound calculation for

arbitrary degree supports under a uniform block-entropy assumption. The construction uses the classical Euler transform and exponential partition framework.

Two different support scales occur. For independent polynomial generators the decisive quantity is $\log(d_{i+1})/d_i$. For the normal finite-transcendence-degree examples below it is d_{i+1}/d_i. The latter construction is an adaptation of Maclean's explicit infinite-pole spaces [6, Section 3]. The change to exponentially many poles per block, its exact growth law and the normality argument are given in full. These algebras are not approximable, so their exponential behavior is outside the approximation hypothesis in [6, 7].

2 The sharp Euler-product law

Theorem 1. *Let k be any field, let $2 \le d_1 < d_2 < \cdots$ be integers, and let $M_i \in \mathbb{N}$ satisfy*

$$\frac{\log M_i}{d_i} \longrightarrow \delta, \qquad 0 < \delta < \infty.$$

Give

$$P = k[x, y_{i,j} : i \ge 1,\ 1 \le j \le M_i]$$

the grading $\deg x = 1$, $\deg y_{i,j} = d_i$, *and set*

$$\theta = \limsup_{i\to\infty} \frac{\log d_{i+1}}{d_i} \in [0, \infty].$$

Then P is a connected locally finite graded domain, and

$$\overline{h}(P) = \delta, \qquad \underline{h}(P) = \max\{\delta - \theta, 0\},$$

where the second expression is zero for $\theta = \infty$. Thus $H_P(n)^{1/n}$ converges if and only if $\theta = 0$.

Proof. Only finitely many variables have degree at most any fixed n, so each component is finite dimensional. Multiplication by x is injective; hence $H_P(n) \ge 1$ and H_P is nondecreasing.

For every $\eta > 0$ there is $C_\eta \ge 1$ with $M_i \le C_\eta e^{(\delta+\eta)d_i}$ for all i. If $s > \delta$, choose $\eta < s - \delta$. The Euler product

$$F(e^{-s}) = (1 - e^{-s})^{-1} \prod_i (1 - e^{-sd_i})^{-M_i}$$

converges, since

$$-\log(1 - e^{-sd}) \le \frac{e^{-sd}}{1 - e^{-2s}} \quad (d \ge 2)$$

and the distinct integer degrees make $\sum_i M_i e^{-sd_i}$ finite. Positivity gives $H_P(n) \le F(e^{-s})e^{sn}$, proving $\overline{h}(P) \le \delta$. The reverse bound follows from $H_P(d_i) \ge M_i$.

Suppose first that $\theta < \delta$. Fix $\eta > 0$ with $c = \delta - \theta - 2\eta > 0$, and choose I so that for $i \ge I$,

$$\log M_i \ge (\delta - \eta)d_i, \qquad \log(d_{i+1}/d_i) \le (\theta + \eta)d_i.$$

For $n \ge d_I$, greedy division down the available degree list gives

$$n = r + \sum_{i=I}^{j} k_i d_i, \qquad 0 \le r < d_I, \qquad 0 \le k_i < d_{i+1}/d_i.$$

The top digit obeys this bound because $n < d_{j+1}$; each subsequent entering remainder is below the next degree. No divisibility between successive degrees is needed.

Choosing k_i factors from the ith block gives $\binom{M_i+k_i-1}{k_i}$ different monomials. Independence of the variables and a factor x^r therefore give

$$H_P(n) \geq \prod_i \binom{M_i + k_i - 1}{k_i}.$$

For $k \geq 1$, $\binom{M+k-1}{k} \geq M^k/k! \geq \exp(k \log M - k \log k)$. Omitting zero digits, we obtain

$$\log H_P(n) \geq \sum_i k_i(\log M_i - \log k_i) \geq c \sum_i k_i d_i \geq c(n - d_I).$$

Letting $\eta \downarrow 0$ proves $\underline{h}(P) \geq \delta - \theta$. If $\theta \geq \delta$, positivity gives the required zero lower bound.

For the opposite inequality assume $\theta > 0$ and fix $s > \max\{\delta - \theta, 0\}$. Choose $b > 0$ so that

$$\max\{\delta - s, 0\} < b < \theta,$$

with any suitable finite b if $\theta = \infty$. Choose $\eta > 0$ small enough that $a = \max\{\delta + \eta - s, 0\} < b$. Infinitely many i satisfy $\log d_{i+1} > bd_i$. Put $n_i = d_{i+1} - 1$ on that subsequence. Only the first i blocks contribute to this coefficient, so the *finite* product

$$F_i(t) = (1-t)^{-1} \prod_{j \leq i} (1 - t^{d_j})^{-M_j}$$

satisfies $H_P(n_i)e^{-sn_i} \leq F_i(e^{-s})$. There are at most d_i available distinct degrees, and consequently

$$\log F_i(e^{-s}) \leq C_s + C_{s,\eta} d_i e^{ad_i}.$$

Meanwhile $n_i \geq e^{bd_i} - 1$, so $\log F_i(e^{-s})/n_i \to 0$. It follows that $\underline{h}(P) \leq s$. Decreasing s gives the claimed endpoint. For $\theta = 0$, the already proved upper rate supplies the matching bound. This argument includes $\theta = \delta$, $\theta > \delta$ and $\theta = \infty$. □

Corollary 2. *Adjoining finitely many independent positive-degree polynomial variables to P preserves both rates. The same holds for any connected graded algebra A containing P that is finitely generated as a graded P-module.*

Proof. For the first assertion, if B_j counts monomials in the adjoined variables, then

$$H_P(n) \leq H_{P \otimes B}(n) = \sum_{j=0}^{n} H_P(n-j)B_j \leq H_P(n) \sum_{j=0}^{n} B_j.$$

The last factor grows at most polynomially. For the second assertion, homogeneous module generators of nonnegative degrees $e_1, \ldots, e_r$ give

$$H_P(n) \leq H_A(n) \leq \sum_{\ell=1}^{r} H_P(n - e_\ell) \leq rH_P(n),$$

with negative indices interpreted as zero. □

For every $0 \leq \alpha \leq \beta < \infty$ with $\beta > 0$, Theorem 1 realizes the endpoint pair (α, β). Take $M_i = \lceil e^{\beta d_i} \rceil$. If $\alpha < \beta$, choose $d_{i+1} = \lceil \exp((\beta - \alpha)d_i) \rceil$ starting sufficiently large; then $\theta = \beta - \alpha$. If $\alpha = \beta$, take $d_i = 2^i$. The case $(0, 0)$ is realized by $k[x]$. The rings P are UFDs and hence integrally closed: any finite collection of elements lies in a finite-variable polynomial subring. The next section retains normality while reducing the fraction-field transcendence degree to two.

3 Normal domains with a different gap scale

Theorem 3. *Let k be infinite and let $(d_i), (M_i), \delta$ satisfy Theorem 1. There is a connected locally finite normal graded k-domain C, with a nonzero element of degree one, such that*

$$\mathrm{Frac}(C) = k(z,t), \qquad \dim C = 2, \qquad H_C(n) = 1 + \sum_i M_i \left\lfloor \frac{n}{d_i} \right\rfloor.$$

Writing $R = \limsup_i d_{i+1}/d_i \in [1,\infty]$, it satisfies

$$\overline{h}(C) = \delta, \qquad \underline{h}(C) = \frac{\delta}{R},$$

where $\delta/\infty = 0$.

Proof. Choose pairwise distinct $a_{i,j} \in k$, $i \geq 1$, $1 \leq j \leq M_i$. Let $V_n \subset k(z)$ consist of rational functions with no pole at infinity or at any unselected finite point, and pole order at most $\lfloor n/d_i \rfloor$ at $a_{i,j}$. Set $C = \bigoplus_{n\geq 0} V_n t^n \subset k(z)[t]$.

Partial fractions give the basis

$$1, \quad (z - a_{i,j})^{-r} \quad (d_i \leq n,\ 1 \leq j \leq M_i,\ 1 \leq r \leq \lfloor n/d_i \rfloor)$$

of V_n. Principal parts prove independence, and subtracting all principal parts leaves a rational function without poles on $\mathbb{P}^1$, hence a constant. This proves the dimension formula and local finiteness. The floor inequality proves $V_m V_n \subset V_{m+n}$, and $V_0 = k$. The algebra is explicitly generated by t and

$$g_{i,j} = \frac{t^{d_i}}{z - a_{i,j}},$$

since each basis vector in degree n equals $t^{n-rd_i} g_{i,j}^r$. It follows that $\mathrm{Frac}(C) = k(z,t)$.

We prove normality inside this fraction field. For a selected point $p = a_{i,j}$ use the weighted Gauss valuation

$$w_p\left(\sum_n f_n(z) t^n\right) = \min_n \{d_i \operatorname{ord}_p(f_n) + n\}.$$

For every other closed point of $\mathbb{P}^1_k$, including infinity, use the Gauss valuation with weight zero,

$$w_p\left(\sum_n f_n(z) t^n\right) = \min_n \operatorname{ord}_p(f_n).$$

The minimum rule is multiplicative: initial polynomials multiply in a polynomial ring over the associated graded coefficient domain, so their product cannot vanish. These rules thus extend to valuations of $k(z,t)$. Coefficientwise pole bounds now give exactly

$$C = k(z)[t] \ \cap \bigcap_{p \in \mathbb{P}^1_k} \{h \in k(z,t) : w_p(h) \geq 0\},$$

where the intersection ranges over closed points. Each valuation ring and the polynomial ring over $k(z)$ is integrally closed. Their intersection is therefore integrally closed in $k(z,t)$.

Evaluation of coefficients at infinity gives a surjection $C \to k[t]$. Its kernel $\mathfrak{p}$ contains all $g_{i,j}$, but not t, and the preimage of (t) is C_+. Hence

$$(0) \subsetneq \mathfrak{p} \subsetneq C_+$$

is a prime chain. A k-domain of finite fraction-field transcendence degree has Krull dimension at most that degree, without a finite-generation hypothesis. Indeed, for a nonzero prime, lift an algebraically independent set from the residue domain and choose a nonzero element of the prime. If that element were algebraic over the field generated by the lifts, a minimal polynomial would have nonzero constant term; clearing denominators and reducing modulo the prime contradicts independence. Every strict prime-chain step thus lowers transcendence degree. This gives $\dim C \leq 2$, and the displayed chain gives equality.

Let $d(n)$ denote the largest $d_i \leq n$. For every $\eta > 0$, the dimension formula gives

$$M_{i(n)} \leq H_C(n) \leq 1 + n \sum_{d_i \leq n} M_i \leq 1 + C_\eta n^2 e^{(\delta+\eta)d(n)}.$$

Together with $\log M_{i(n)} = (\delta + o(1))d(n)$, these bounds imply

$$\frac{\log H_C(n)}{n} = \delta \frac{d(n)}{n} + o(1).$$

For completeness, the lower error is bounded by $|\log M_{i(n)}/d(n) - \delta|$, and the upper error by $\eta + O_\eta(\log n/n)$; then let $\eta \downarrow 0$. The ratio $d(n)/n$ has limsup one, attained at $n = d_i$, and liminf $1/R$, attained at the ends $d_{i+1} - 1$ of the degree intervals. This proves the rates. □

The construction uses the pole-space mechanism of Maclean [6, Section 3]. In her explicit example there is one point per degree 2^i and the total divisor mass is finite; her algebra is approximable and has linear growth. Here the exponentially large blocks create a different growth regime. Directly, for any fixed p with $V_p \neq 0$, products of q elements of V_p have pole divisor bounded by the fixed divisor $q\lfloor pD \rfloor$, where $D = \sum_{i,j} d_i^{-1}[a_{i,j}]$. Their span has dimension at most $1 + q \deg \lfloor pD \rfloor$. On the subsequence $q_i = \lceil d_i/p \rceil$, the full degree-$q_i p$ component has dimension at least M_i, exponential in d_i. The approximation ratio has liminf zero for each fixed p. Thus these examples fail approximability.

Taking $d_i = 2^i$ gives rates $\delta/2$ and δ in a normal dimension-two domain. More generally every finite nth-root endpoint pair $1 \leq L \leq U$, $U > 1$, occurs: set $\delta = \log U$ and choose degree ratios tending to $\delta/\log L$ if $1 < L < U$, unbounded ratios if $L = 1$, or consecutive degrees if $L = U$. Integer rounding preserves the required ratios. For $L = U = 1$, a polynomial ring suffices. The dimension formula also gives

$$H_C(m + n) \geq H_C(m) + H_C(n) - 1.$$

These examples distinguish this additive lower bound from multiplicative entropy control.

Proposition 4. *If A is a connected normal graded k-domain with* $\operatorname{trdeg}_k \operatorname{Frac}(A) = 1$ *and* $0 \neq x \in A_1$*, then* $A = k[x]$.

Proof. For homogeneous $f \in A_n$, put $h = f/x^n$, a fraction of degree zero. If h were transcendental over k, then h and x would be algebraically independent: after clearing a power of x, any putative relation splits by homogeneous degree, forcing a polynomial relation for h. Thus h is algebraic over k. It is integral over k and hence over A, so normality gives $h \in A$. Comparing degrees after multiplication by x^n yields $h \in A_0 = k$. Therefore $A_n = kx^n$ for every n. □

This minimality statement uses normality, connectedness and the degree-one element. It asserts no analogous result after removing any of those hypotheses.

Use of AI tools. AI tools assisted in developing and checking the arguments and preparing the manuscript. All proofs are presented in full; no computational result is required.

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