

An explicit global lower certificate for a weighted Weil operator

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Abstract

We prove a global weighted lower bound for the complete Weil form of the Riemann xi-function. For every complex compact smooth test f whose Laplace transform vanishes at $\pm 1/2$, the form satisfies

$$Q[f] \geq -\frac{155299736}{10^{14}} \int_{\mathbb{R}} \frac{|f(u)|^2 \cosh(u/2)}{\psi(u)} du + \frac{33}{25000000} \|f\|_2^2,$$

where ψ is the normalized Riemann theta kernel. In particular, the specified minimal weighted realization has $T \geq 0.99999844700264 I$. Its endpoint $T \geq I$ is exactly Weil's criterion for the Riemann hypothesis. The construction replaces an unevaluated reference resolvent by a known positive multiplication operator. A full exterior-frequency floor, a paired arithmetic remainder, local Legendre approximation and two exact global constraint rows reduce the bound to signed Gram inequalities. The bound includes all omitted prime powers, spatial and polynomial tails, quadrature errors and arithmetic errors. Reflection reduces the executed certificate to two matrices of order 897; directed ball arithmetic verifies both. Portable programs reconstruct the certificate from the displayed formulas.

1 The positivity target and the global bound

The reference and its error account The known multiplication reference makes the signed receiving inequality evaluable. Its proof has five obligations: identify the original constrained domain, establish the reference comparison, bound the complete arithmetic and approximation errors, accept the finite signed matrices, and return to the original form. Source-enclosure padding already included in a certified matrix is not added again as a scalar loss. The retained positive potential and the reciprocal-theta weighted deficit are separate terms of the main conclusion.

Weil's criterion makes positivity of an explicit quadratic form on compact smooth tests equivalent to the Riemann hypothesis. This gives a concrete quantitative question: how small a *global* negative lower bound can be certified in a fixed norm, while keeping the complete form and its original test class? A finite positive compression does not answer that question, since the omitted directions can contain the negative spectrum. We answer it here by a signed finite certificate with complete infinite-dimensional error bounds.

For $\mathcal{M}f(z) = \int_{\mathbb{R}} f(u) e^{zu} du$, put

$$\mathcal{F}_0 = \{f \in C_c^\infty(\mathbb{R}; \mathbb{C}) : \mathcal{M}f(1/2) = \mathcal{M}f(-1/2) = 0\}.$$

Our Fourier transform is $\widehat{f}(\xi) = \int f(u) e^{-iu\xi} du$, with Plancherel measure $d\xi/(2\pi)$; inner products are conjugate-linear first. Write $\Psi_0 = \Gamma'/\Gamma$ for digamma and Λ for von Mangoldt's

function. The complete physical Weil form is

$$Q[f] = \int_{\mathbb{R}} [\operatorname{Re} \Psi_0(1/4 + i\xi/2) - \log \pi] |\hat{f}(\xi)|^2 \frac{d\xi}{2\pi} - \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} (\langle f, S_{\log n} f \rangle + \langle f, S_{-\log n} f \rangle), \quad S_b f(u) = f(u + b). \quad (1)$$

Define the normalized even theta kernel and its weight by

$$\psi(u) = 2 \sum_{k \geq 1} (2\pi^2 k^4 e^{9u/2} - 3\pi k^2 e^{5u/2}) e^{-\pi k^2 e^{2u}}, \quad c(u) = \cosh(u/2), \quad w(u) = \psi(u)c(u), \quad s(u) = \sqrt{\psi(u)/c(u)}. \quad (2)$$

The theta identity makes ψ even and positive, with $\mathcal{M}\psi(z) = \xi(1/2 + z)$ for the usual completed xi-function.

Theorem 1.1 (Global weighted Weil bound). *Every $f \in \mathcal{F}_0$ satisfies*

$$Q[f] \geq -\nu \int_{\mathbb{R}} \frac{|f(u)|^2}{s(u)^2} du + \lambda \|f\|_2^2, \quad \nu = \frac{155299736}{10^{14}}, \quad \lambda = \frac{33}{25000000}. \quad (3)$$

For the minimal realization H defined in Section 2, the corresponding inequality holds on its entire original form domain:

$$q_H[h] \geq -\nu \|h\|_2^2 + \lambda \|sh\|_2^2. \quad (4)$$

In particular $T = I + H \geq (1 - \nu)I = 0.99999844700264 I$.

The norm and the coefficient-one endpoint matter. On the same compact core,

$$\text{RH} \iff Q[f] \geq 0 \ (f \in \mathcal{F}_0) \iff H \geq 0 \iff T \geq I. \quad (5)$$

Section 2.1 verifies this identification, including the two moment conditions. Thus (3) measures an explicit deficit in the original criterion, together with a retained positive potential. The reciprocal weight $1/s^2 = c/\psi$ is part of the statement.

The mechanism is a change of reference. Retaining the arithmetic channels $n = 2, 3, 4, 5$ yields a signed multiplier m_5 with a proved positive floor outside $[-600, 600]$. After bounding that exterior part, a known multiplication operator $D_0 = \alpha + \mu s^2$ suffices. It replaces the unknown resolvent Gram in a reference-operator approach. The two original moment constraints become explicit global rows; a negative rank-two penalty vanishes exactly on their orthogonal complement. We then certify the signed finite operator, preserving all its mixed entries.

The weighted realization, ground-state identity and complete arithmetic series have antecedents in van Hemert [4] and our null-space-compression work [5]. The latter gives quantitative whole-kernel bounds and the auxiliary floor $T \geq I/100$. Our frequency-localization predecessor [6] gives sharp fixed-reference high-frequency asymptotics and a sufficient residual-Gram condition. The complementary joint-cutoff theorem [7] bounds the entire negative-eigenvalue count and puts the full negative part H_- in every Schatten class; its approximants retain actual resolvent rows. The present construction supplies an executed known-reference lower-edge certificate on the same entire original domain. The paired-shift estimate, Toeplitz organization and parity reduction are parts of this one construction. Fourier localization, Gram congruences, Legendre expansions, Gaussian quadrature and ball arithmetic are established tools [8, 9, 10]; their complete application to this signed quadratic form is the issue here. Signed polynomial kernels and exact congruence checks also appear in the independent computational work [12, 13], whose compact-window recipients differ from ours. Recent quasi-RH zero-free half-plane results [14] concern a different arithmetic conclusion; no such bound is used as a premise of (3).

2 The exact realization and the Weil endpoint

Set

$$\mathcal{K} = \{h \in L^2(\mathbb{R}) : \langle \sqrt{w}, h \rangle = \langle \sqrt{w} \tanh(u/2), h \rangle = 0\},$$

$$a(\xi) = \operatorname{Re} \Psi_0(1/4 + i\xi/2) - \Psi_0(1/4), \quad m_0 = \Psi_0(1/4) - \log \pi.$$

The positive digamma series is

$$a(\xi) = \sum_{j \geq 0} \frac{(\xi/2)^2}{(j + 1/4)((j + 1/4)^2 + (\xi/2)^2)}. \quad (6)$$

In particular a is nonnegative and increasing for $\xi \geq 0$. On $\mathcal{C} = C_c^\infty(\mathbb{R}) \cap \mathcal{K}$ take the form

$$q_A[h] = \int a(\xi) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi}.$$

Its closure in $(\|h\|_2^2 + q_A[h])^{1/2}$ is the *specified minimal* domain $\mathcal{V}$. Closability follows by restriction of the closed graph map $h \mapsto a(D)^{1/2}(sh)$. The core is dense in $\mathcal{K}$: compact smooth approximation followed by two fixed compact bump corrections repairs its two row values. The bump matrix is invertible when their small supports are near two distinct points, since the row-profile ratio $\tanh(u/2)$ is nonconstant.

With $c_n = \Lambda(n)/\sqrt{n}$, define the bounded perturbation

$$B = P_{\mathcal{K}} \left[m_0 M_s^2 - \sum_{n \geq 2} c_n M_s (S_{\log n} + S_{-\log n}) M_s \right] \Big|_{\mathcal{K}}. \quad (7)$$

The complete norm convergence is proved below. Let A be associated with q_A on $\mathcal{V}$, $H = A + B$ and $T = I + H$. On $\mathcal{C}$, $f = sh$ belongs to $\mathcal{F}_0$ and

$$Q[f] = q_H[h], \quad \|h\|_2^2 = \int |f|^2 c / \psi. \quad (8)$$

Conversely every $f \in \mathcal{F}_0$ gives $h = f/s \in \mathcal{C}$, since s is positive and smooth on each compact set. The row identities are

$$\langle \sqrt{w}, h \rangle = \int cf, \quad \langle \sqrt{w} \tanh(u/2), h \rangle = \int \sinh(u/2) f.$$

For completeness, the inherited jump normalization can be checked directly from the explicit formula. Its ground identity is $Q(\psi, g) = -\int cg$. The integral representation

$$a(\xi) = \int_{\mathbb{R}} \frac{e^{-|b|/2}}{1 - e^{-2|b|}} (1 - \cos(\xi b)) db$$

and expansion of squared differences give

$$D[v] = \frac{1}{2} \iint \frac{e^{-|u-v'|/2}}{1 - e^{-2|u-v'|}} \psi(u) \psi(v') |v(u) - v(v')|^2 du dv'$$

$$+ \sum_{n \geq 2} c_n \int \psi(u) \psi(u + \log n) |v(u) - v(u + \log n)|^2 du,$$

$$Q[\psi v] = D[v] - \|v\|_{L^2(w)}^2.$$

For $h = \sqrt{w}v$, the two form norms are equivalent because B is bounded. The minimal positive-jump closure is therefore precisely the above realization of T . No maximal-domain identification is needed.

2.1 Why the constrained endpoint is exactly RH

The complete explicit formula [1, 2] gives

$$\begin{aligned} Q(f, g) &= Z(f, g) - \mathcal{M}g(1/2)\overline{\mathcal{M}f(-1/2)} - \mathcal{M}g(-1/2)\overline{\mathcal{M}f(1/2)}, \\ Z(f, g) &= \sum_{\rho} \mathcal{M}g(\rho - 1/2)\overline{\mathcal{M}f(1/2 - \bar{\rho})}. \end{aligned} \quad (9)$$

Zeros retain multiplicity. These identities and continuity hold also when all derivatives decrease faster than every exponential. Integration by parts supplies rapid transform decay on fixed vertical strips; the classical zero count $O(t \log(t+2))$ gives absolute convergence of Z . The physical derivative bounds give convergence of (1).

Proposition 2.1. *The equivalences (5) hold for the realization just defined.*

Proof. If RH holds, each term of $Z(f, f)$ on $\mathcal{F}_0$ is $|\mathcal{M}f(i\gamma)|^2$, so it is nonnegative. Conversely, let g be any compact smooth test, without moment conditions, and set $m_{\pm} = \mathcal{M}g(\pm 1/2)$. Since

$$\mathcal{M}\psi(z) = \xi(1/2 + z), \quad \mathcal{M}\psi'(z) = -z\xi(1/2 + z), \quad \mathcal{M}\psi(\pm 1/2) = 1/2,$$

the moment matrix of ψ, ψ' is $\begin{pmatrix} 1/2 & -1/4 \\ 1/2 & 1/4 \end{pmatrix}$. Thus $F = g - (m_+ + m_-)\psi - 2(m_- - m_+)\psi'$ has both moments zero. Both subtracted profiles have transform zero at every argument in (9); symmetry of the xi zeros gives this for both factors. Hence $Z(F, F) = Z(g, g)$.

To apply compact-test positivity, approximate F by smooth compact cutoffs and repair the two small moment errors using two fixed compact bumps with invertible moment matrix. The repair coefficients tend to zero. These corrected tests belong to $\mathcal{F}_0$ and converge to F in every seminorm $\sup_u e^{A|u|}|F^{(k)}(u)|$, $A > 0$. Continuity of (9) gives $Z(g, g) = Q[F] \geq 0$. The classical full-test Weil criterion now gives RH. This is also the restricted criterion of [2, Appendix C, Proposition C.1], after $g(x) = x^{-1/2}f(\log x)$, whose Mellin transform is $\mathcal{M}f(z - 1/2)$.

Finally (8) is a bijection of the two compact cores. Positivity on $\mathcal{C}$ extends to its chosen form closure because B is bounded. The representation theorem and $T = I + H$ prove the remaining equivalences. $\square$

3 A computable reference with exact constraints

Let B_M retain the exact m_0 term and every $n \leq M$ in (7). Suppose $\mu \geq 0$ and

$$\|B - B_M\| \leq \delta, \quad m_M(\xi) := a(\xi) + m_0 - 2 \sum_{2 \leq n \leq M} c_n \cos(\xi \log n) \geq \mu \quad (|\xi| > L).$$

For $\alpha > 0$, set

$$\epsilon = \alpha + \delta, \quad b = 1_{[-L, L]}(\mu - m_M), \quad D_0 = \alpha + \mu s^2, \quad t = s/\sqrt{D_0}. \quad (10)$$

The function b remains signed. Deleting only the nonnegative exterior form gives on the original core

$$q_H[h] + \epsilon \|h\|^2 \geq \|D_0^{1/2} h\|^2 - \langle sh, b(D)sh \rangle. \quad (11)$$

This is a lower relaxation, not an identity replacing the high band. With $x = D_0^{1/2} h$, its constraints are exactly

$$\langle \ell_0, x \rangle = \langle \ell_1, x \rangle = 0, \quad \ell_0 = t \cosh(u/2), \quad \ell_1 = t \sinh(u/2). \quad (12)$$

Both are complete global L^2 functions. We require no resolvent column and no projection onto an unknown full null space.

Let Y have orthonormal local Legendre columns on $[-R, R]$, $P_N = YY^*$, and let Q_L be Fourier projection onto $[-L, L]$. If

$$\|Q_L M_t - Q_L P_N M_t\| \leq \eta, \quad \|Q_L M_t\|, \|Q_L P_N M_t\| \leq j,$$

the two product differences imply

$$\|M_t b(D) M_t - M_t Y C Y^* M_t\| \leq \Gamma := 2\|b\|_\infty j \eta, \quad C = Y^* b(D) Y. \quad (13)$$

Append the rows and define

$$W = [M_t Y, \ell_0, \ell_1], \quad C_{\text{aug}} = \text{diag}(C, -5, -5).$$

The penalty vanishes identically on (12). Thus

$$W C_{\text{aug}} W^* \leq \beta I, \quad \beta + \Gamma < 1 \implies q_H[h] + \epsilon \|h\|^2 \geq (1 - \beta - \Gamma) \|D_0^{1/2} h\|^2. \quad (14)$$

The conclusion extends through the chosen form closure. The rest of the proof supplies every hypothesis of (14) with explicit constants.

4 Complete arithmetic and exterior frequency control

The known log-concavity of the Riemann theta kernel [3, Theorems 4.2 and 4.5], with the scaling (2), makes s even and log-concave: $\log \cosh(u/2)$ is convex. Hence $\|M_s S_b M_s\| = s(b/2)^2$ for $b \geq 0$.

Lemma 4.1 (Norm of a complete paired shift). *For a bounded positive even log-concave s , set $U = M_s S_b M_s$, $P = s(b/2)^2$, $q = s(b)^2$ and $s_0 = s(0)$. Then*

$$\|U + U^*\| \leq \sqrt{P^2 + 3s_0^2 q}.$$

Proof. The logarithm of $s(u)s(u+b)^2s(u+2b)$ is concave and symmetric about $-b$. Thus $\|U^2\| = s_0^2 q$. Further, $UU^* + U^*U$ multiplies by $s(u)^2(s(u+b)^2 + s(u-b)^2)$. For $u \geq 0$ its first term is at most $s_0^2 q$ by monotonicity, and its second is at most P^2 by midpoint concavity. Reflection treats $u \leq 0$. Expand $(U + U^*)^2$ and use selfadjointness to obtain the stated squared bound. $\square$

The complete positive theta series gives, for $u \geq 0$,

$$\psi(u) \leq 8\pi^2 e^{9u/2 - \pi e^{2u}}, \quad s(u)^2 \leq 16\pi^2 e^{4u - \pi e^{2u}}, \quad s(u) \leq 13e^{2u - (3/2)e^{2u}}. \quad (15)$$

Indeed, with $r_0 = \pi e^{2u}$, $k^2 - 1 \geq 3(k-1)$ and $q_0 = e^{-3\pi} < 1/100$,

$$\sum_{k \geq 1} k^4 e^{-r_0 k^2} \leq e^{-r_0} \frac{1 + 11q_0 + 11q_0^2 + q_0^3}{(1 - q_0)^5} < 2e^{-r_0}.$$

Reflection handles negative u . Consequently

$$\|B - B_M\| < \delta_M := \frac{640(M+1)^2}{20^{M+1}}. \quad (16)$$

To verify the constants, $c_n \leq 1$, the paired triangle bound is at most $32\pi^2 n^2 e^{-\pi n}$, and, for $k = M+1 \geq 2$, the exact geometric-series formula gives $\sum_{n \geq k} n^2 e^{-\pi n} < 2k^2 20^{-k}$. Since $32\pi^2 < 320$, (16) follows before compression to $\mathcal{K}$. In particular the full B is bounded and selfadjoint.

For $M = 5$, apply Lemma 4.1 to the exact remaining prefix $E = \{7, 8, 9, 11, 13, 16\}$ and use (16) only beyond 16:

$$\|B - B_5\| \leq \sum_{n \in E} c_n \sqrt{s(\frac{1}{2} \log n)^4 + 3s_0^2 s(\log n)^2} + \delta_{16} < \frac{553}{10^9} =: \delta. \quad (17)$$

The actual upper enclosure is less than $5.528504724 \cdot 10^{-7}$. Each value uses atoms 1 through 8 in (2) and a complete outward remainder 2^{-250} . For $k \geq 9$ the omitted positive atom is at most $40k^4 e^{-3k^2}$; consecutive majorants have ratio less than 1/2. Thus their full sum is bounded by

$$524880e^{-243} < 524880(3/8)^{243} < 2^{-250}.$$

If a tiny far-tail theta ball contains zero, an absolute upper endpoint is used; no tiny center is assumed positive.

Take $\mu = 1/2$, $L = 600$. Cancellation of constants gives

$$m_5(\xi) = \operatorname{Re} \Psi_0(1/4 + i\xi/2) - \log \pi - 2 \sum_{n=2}^5 c_n \cos(\xi \log n).$$

The checker evaluates this expression outward on all 5750 closed intervals of width 1/25 covering [600, 830]. Every interval has lower bound greater than 0.5385283118. For $d_5 = -m_0 + 2 \sum_{n=2}^5 c_n < 10$, a separate enclosure gives $a(830) - d_5 > 0.5020500175$. Monotonicity from (6) therefore covers the entire remaining half-line, and evenness covers negative frequencies. This proves the full exterior floor $m_5 \geq 1/2$. Also

$$a(L) < 4 + \frac{1}{2} \log(1 + 4L^2) < 11.5, \quad \|b\|_\infty < \frac{1}{2} + 11.5 + 10 < 24.$$

The first inequality follows by summing the first digamma-series term plus the integral of the decreasing remaining summand. The last scalar bounds, (17) and the closed interval cover are reproduced by `verify_bounds.py` with rigorous balls and exact rational comparisons.

5 Full spatial and Legendre error

Use $R = 8/5$, 32 cells of half-width $r = 1/20$, and degree $0 \leq j < p = 56$. At center $c_a = -R + (2a + 1)r$, define

$$\phi_{a,j}(u) = 1_{\text{cell } a}(u) \sqrt{\frac{2j+1}{2r}} P_j((u - c_a)/r).$$

These 1792 functions are orthonormal. Their Fourier transforms are

$$\hat{\phi}_{a,j}(\xi) = e^{-ic_a \xi} (-i)^j A_j(\xi), \quad A_j(\xi) = \sqrt{2r(2j+1)} j_j(r\xi), \quad (18)$$

where j_j is spherical Bessel.

Lemma 5.1. *With P_R spatial restriction to $[-R, R]$,*

$$e_N^2 := \|Q_L(P_R - P_N)\|^2 \leq \frac{2RL}{\pi} \frac{(rL)^{2p}}{[(2p+1)!!]^2(1-q)}, \quad q = \frac{(rL)^2}{(2p+1)(2p+3)} < 1.$$

For the present parameters $q = 180/2599$ and $e_N < 3.2 \cdot 10^{-9}$.

Proof. Rodrigues' formula, integration by parts and the beta integral give $|j_j(x)| \leq |x|^j / (2j+1)!!$ for real x . The ratio of consecutive squared coefficient majorants $(2j+1)x^{2j} / [(2j+1)!!]^2$ is $x^2 / ((2j+1)(2j+3)) \leq q$ for $j \geq p$. Sum this geometric tail, integrate over $[-L, L]$ with measure $d\xi/(2\pi)$ and sum the cells. The factor $2p+1$ cancels in that integration. This gives the displayed Hilbert–Schmidt bound, hence the operator bound. Substitution, with $\pi > 3$, gives a squared bound less than $9.931 \cdot 10^{-18} < (3.2 \cdot 10^{-9})^2$. $\square$

Now choose $\alpha = 10^{-6}$ in (10). Globally $j = \|t\|_\infty \leq \sqrt{2} < 3/2$. By (15) and $e^{3.2} > 24$,

$$j_R^2 := \sup_{|u| > R} t(u)^2 \leq 169e^{4R-3e^{2R}}/\alpha < 169(3/8)^{65}/\alpha < (2 \cdot 10^{-10})^2.$$

Split $Q_L(I - P_N)M_t$ into the exterior spatial term and $Q_L(P_R - P_N)M_t$. Thus

$$\eta \leq j_R + j_{e_N} < 5 \cdot 10^{-9}, \quad \Gamma = 2 \cdot 24 \cdot \frac{3}{2} \cdot 5 \cdot 10^{-9} = \frac{9}{25000000}.$$

Both map norms in (13) are at most j , since spatial and Fourier projections are contractions. All degrees above 55 and the entire exterior spatial interval are included. The local functions need not lie in $\mathcal{V}$: they approximate a bounded band kernel, rather than serve as unbounded-operator trial vectors.

6 The signed Gram certificate and exact symmetry

Let $G = W^*W$. Its phase/phase entries are same-cell integrals of $t^2\phi_i\phi_j$, different-cell entries are zero, and its mixed entries integrate $t^2\phi_i c$ or $t^2\phi_i \sinh(u/2)$. The two appended norms are over the *whole* real line. None of these mixed covariances is omitted.

By (18), the phase matrix has displacement blocks

$$B_d(j, k) = \int_0^L b(\xi) A_j(\xi) A_k(\xi) \operatorname{Re}\{i^j(-i)^k e^{i2r d \xi}\} \frac{d\xi}{\pi}, \quad d = a - b. \quad (19)$$

Only 32 blocks need integration, since $B_{-d} = B_d^T$. This Toeplitz structure belongs to C , not to the weighted G .

Reflection sends the local feature (a, j) to $(-1)^j(31 - a, j)$, fixes ℓ_0 and negates ℓ_1 . Let J be this exact signed permutation on the 1794 feature coordinates. Then $WJ = \mathcal{R}W$ for physical reflection $\mathcal{R}$, and both C_{aug} and G commute with J . In parity $\sigma = \pm 1$, use the integer columns

$$e_{a,j} + \sigma(-1)^j e_{31-a,j}, \quad a < 16, \quad j < 56,$$

together with the corresponding global row coordinate. Write the resulting matrix as E_σ . Its metric and normalized isometry are

$$D = E_\sigma^* E_\sigma = \operatorname{diag}(2I_{896}, 1), \quad U_\sigma = E_\sigma D^{-1/2}.$$

The two physical ranges WU_+ and WU_- are orthogonal. Direct folded entries are

$$\begin{aligned} (C_f^\sigma)_{aj,bk} &= 2[B_{a-b}(j, k) + \sigma(-1)^k B_{a+b-31}(j, k)], \\ (G_f^\sigma)_{aj,ak} &= 2 \int_{\text{cell } a} t^2 \phi_{a,j} \phi_{a,k}, \\ (G_f^\sigma)_{aj,*} &= 2 \int_{\text{cell } a} t^2 \phi_{a,j} r_\sigma, \quad r_+ = c, \quad r_- = \sinh(u/2), \end{aligned} \quad (20)$$

with row norm $\int_{\mathbb{R}} t^2 r_\sigma^2$, penalty entry -5 and zero other coefficient/row entries.

Lemma 6.1 (Signed upper certificate). *Suppose symmetric dyadic centers C_0^σ, G_0^σ for the folded matrices have operator error less than e in normalized coordinates. Put*

$$\overline{C}_\sigma = C_0^\sigma + eD, \quad \overline{G}_\sigma = G_0^\sigma + 2eD.$$

If $\beta \geq 0$ and both exact matrices

$$F_\sigma = \beta \overline{G}_\sigma - \overline{G}_\sigma D^{-1} \overline{C}_\sigma D^{-1} \overline{G}_\sigma \quad (21)$$

are positive definite, then $WC_{\text{aug}}W^* \leq \beta I$.

Proof. Normalize by $D^{-1/2}$ and write $C_n = D^{-1/2} \overline{C}_\sigma D^{-1/2}$, $G_n = D^{-1/2} \overline{G}_\sigma D^{-1/2}$. The error bounds give $C_n \geq U_\sigma^* C_{\text{aug}} U_\sigma$ and $G_n \geq U_\sigma^* G U_\sigma + eI > 0$. Equation (21) is the congruence of $\beta G_n - G_n C_n G_n$ by $D^{1/2}$, so $C_n < \beta G_n^{-1}$. With $Z = WU_\sigma$, $Z^* Z \leq G_n$, and hence

$$Z(U_\sigma^* C_{\text{aug}} U_\sigma) Z^* \leq Z C_n Z^* \leq \beta Z G_n^{-1} Z^* \leq \beta I.$$

This uses no positivity of the signed C_n . The parity ranges are orthogonal, so the two bounds combine by their maximum, not their sum. $\square$

Padding by D , rather than by the identity in integer coordinates, is essential. All executed inverses of D are the exact diagonal rationals $1/2$ and 1 . Exact invariance of the true matrices also permits pinching any proposed center by $(A + JAJ)/2$, a norm contraction; its diagonal parity folds equal $E_\sigma^* A E_\sigma$. Thus no numerical orthogonality assumption is hidden in the reduction.

7 Whole quadrature errors

We use exact Gauss–Legendre quadrature; arithmetic enclosures of its nodes and weights are additional, as described in Section 8. If an integrand has modulus at most M on a Bernstein ellipse of parameter $\rho > 1$, truncating its Chebyshev series through degree $2n - 1$, followed by Gaussian exactness and positive weights, gives on a cell of half-width r

$$E(M, n) \leq \frac{8rM\rho^{-2n}}{1 - \rho^{-1}}. \quad (22)$$

Indeed the uniform polynomial remainder is at most $2M\rho^{-2n}/(1 - \rho^{-1})$, and both integral and quadrature have total mass 2 on $[-1, 1]$.

7.1 Signed frequency integrals

For (19), use 24 points on each of 16 panels of width $1/4$ in $[0, 4]$ and 298 panels of width 2 in $[4, 600]$, a total of 7536 positive-frequency nodes. The $\rho = 4$ ellipses of small panels have $|\Im z| \leq 15/64 < 1/2$. Later panels have $\text{Re } z \geq 23/8$, $|\Im z| \leq 15/8$. They avoid all poles $z = \pm i(2k + 1/2)$ of the holomorphic continuation

$$a(z) = \frac{1}{2} [\Psi_0(1/4 + iz/2) + \Psi_0(1/4 - iz/2)] - \Psi_0(1/4).$$

There is no analytic continuation of a real-part operation or of the band indicator.

For $\text{Re } z \geq a_0 > 0$, the digamma series gives $|\Psi_0(z)| < 1 + 1/a_0 + 2|z|$. On large panels use one recurrence to put both arguments in $\text{Re}(z+1) \geq 5/16$, $|z+1| < 304$, with $|1/z| < 1$. Thus digamma is bounded by 614 there, and by 15 on the small panels. It follows that $|a(z)| < 620$. Since $|m_0| < 6$, $c_n < 1$ and $|\cos(z \log n)| < 25$ for $n \leq 5$, the continued signed multiplier has modulus less than 827.

Cauchy–Schwarz gives $|\hat{\phi}_i(\pm z)| \leq \sqrt{2r} e^{R|\Im z|}$. Write $b_0(z) = \mu - m_5(z)$ for the holomorphic symbol without the band indicator. The holomorphic integrand $b_0(z)\hat{\phi}_i(-z)\hat{\phi}_j(z)$ is bounded by $827(1/10)e^6 < 10^5$. Its real part on the real axis is the positive-frequency integrand in (19). Bound the complex quadrature error first and then take its real part. Apply (22), sum panel half-widths to 300, divide by $\pi > 3$, and multiply the entry error by 1792. The entire phase-matrix operator error is less than

$$1792 \frac{8 \cdot 300 \cdot 10^5}{9} 4^{-47} < 3 \cdot 10^{-18}. \quad (23)$$

The penalty coefficients are exact.

7.2 The complex theta denominator

For Gram quadrature the analytic weight is

$$W_t(z) = \frac{\psi(z)}{\alpha \cosh(z/2) + (1/2)\psi(z)}.$$

We prove $|W_t| < 4 \cdot 10^7$ throughout $|\Im z| \leq h = 1/200$, including zero-freeness of the denominator at the actual $\alpha = 10^{-6}$. Evenness reduces to $z = x + iy$, $x \geq 0$. For $x \leq 6/5$, factor the first theta atom as

$$\psi_1(z) = 4\pi^2 e^{9z/2} (1 - q_x e^{-2iy}) e^{-\pi e^{2z}}, \quad q_x = 3e^{-2x}/(2\pi) \leq 1/2.$$

The bracket has real part at least $1/2$ and imaginary part of modulus at most $|y|$, so its argument is at most $2|y|$. The first atom's unwrapped argument has modulus at most $(13/2)h + 2\pi e^{12/5}h < 1$. Thus $\operatorname{Re} \psi_1 > |\psi_1|/2$. For $r_1 = \pi e^{2z}$, $\operatorname{Re} r_1 > 2.99$, $|r_1| > 3$, and direct prefactor bounds give

$$\frac{|\psi_k|}{|\psi_1|} \leq 3k^4 e^{-(\operatorname{Re} r_1)(k^2-1)}.$$

Since $e^{3\operatorname{Re} r_1} > 1000$, the full sum for $k \geq 2$ is less than $3 \sum_{k \geq 2} k^4 1000^{-(k-1)} < 1/10$. Therefore $\operatorname{Re} \psi > 0$ in this central rectangle.

For all $x \geq 0$, summing the absolute theta series as in (15) gives

$$|\psi(x + iy)| \leq 120 e^{(9/2)x - \pi e^{2x} \cos(2y)}.$$

Its exponent decreases. For $x \geq 6/5$, use $e^{12/5} > 11$, $e^{27/5} < 729$ and $\pi e^{12/5} \cos(2h) > 32$ to get $|\psi| < 120 \cdot 729(3/8)^{32} < 2.047 \cdot 10^{-9}$. Hence everywhere in the strip the denominator has real part greater than $\alpha/2$. The same envelope gives $|\psi| < 17$, so $|W_t| < 34 \cdot 10^6 < 4 \cdot 10^7$. Normal convergence of the theta series on closed substrips justifies these holomorphic operations. No complex square root is used.

7.3 Complete physical Gram and global rows

Use 340 points on each physical cell, with ellipse parameter $\rho = 11/10$. Its imaginary half-height is $r(\rho - \rho^{-1})/2 < h$ and real part stays inside $[-2, 2]$. Laplace's integral for Legendre gives $|P_j| \leq \rho^j$ on that ellipse. Normalization products are at most 1110, each factor is less than 34, and $|\cosh(z/2)|, |\sinh(z/2)| < 2$. Valid integrand bounds are therefore

$$M_p = 4 \cdot 10^7 \cdot 1110 \rho^{110}, \quad M_m = 4 \cdot 10^7 \cdot 68 \rho^{55}, \quad M_r = 16 \cdot 10^7.$$

Let $E_p = E(M_p, 340)$, $E_m = E(M_m, 340)$ and $E_r = 32E(M_r, 340)$. The complete global-row exterior obeys

$$\int_{|u| > R} w(u) du < 16000(3/8)^{72}, \quad E_{\text{tail}} := 16000(3/8)^{72}/\alpha < 4 \cdot 10^{-21}.$$

For example (15), $c(u) \leq e^{|u|/2}$ and $x = e^{2|u|} > 24$ reduce the first integral to

$$80 \int_{24}^{\infty} x^2 e^{-3x} dx < 16000e^{-72}.$$

Since $t^2 c^2 \leq w/\alpha$, this pays both row norms and absolute cross tails. The cross row is exactly zero by parity.

Maximum absolute row sum bounds the entire augmented Gram error by

$$\max\{56E_p + 2E_m, 1792E_m + 2(E_r + E_{\text{tail}})\} < 2.8 \cdot 10^{-11}. \quad (24)$$

Different-cell zeros remain exact. The normalized parity isometries do not amplify (23) or (24). These are whole operator errors, not convergence diagnostics.

8 Executed arithmetic and completion of the proof

The public `verify_certificate.py` uses python-flint 0.9.0 and Arb balls at 192-bit precision [10, 11]. Exact Gaussian nodes and weights are enclosed by the Legendre-root routine. Digamma, logarithms, Bessel functions, phases, full-theta values and polynomial inputs are evaluated outward. All 1,232,816 individual conversions to dyadic centers are checked: radius 2^{-52} suffices except for normalized Legendre values, which use 2^{-47} . The full theta remainder is the one proved in Section 4.

NumPy proposes symmetric finite centers and congruence witnesses. The code then recomputes *every quadrature sum* in Arb from enclosing primitive balls. It bounds the symmetric difference from the exact dyadic center by maximum absolute row sum in normalized parity coordinates; local/local entries divide by 2 and mixed row entries by $\sqrt{2}$. The resulting upper bounds are

	Even	Odd
C quadrature minus center	$1.2920 \cdot 10^{-11}$	$1.2921 \cdot 10^{-11}$
G quadrature minus center	$3.9430 \cdot 10^{-10}$	$1.4915 \cdot 10^{-10}$

The executed analytic allowances are 10^{-15} and 10^{-10} , larger than (23) and (24). Thus $e = 10^{-7}$ strictly pays all source errors for both matrices. No accuracy model for BLAS is assumed.

Use this e , $\beta = 999997/10^6$ and the exact D in Lemma 6.1. Interpret every binary64 center and proposed matrix V_σ as exact dyadic rationals. Arb evaluates the complete congruence $V_\sigma^T F_\sigma V_\sigma$ outward. In every row its diagonal lower endpoint exceeds the sum of absolute off-diagonal upper endpoints. The least certified margins exceed 0.9999999788 for even parity and 0.9999999994 for odd parity. Strict diagonal dominance gives positive definiteness; it also forces V_σ invertible, so $F_\sigma > 0$. Floating Cholesky only proposes the witness and is not an acceptance premise. This proves the ambient cap (14).

Combining the executed cap with the whole localization error gives

$$\kappa = 1 - \beta - \Gamma = \frac{66}{25000000} > 0, \quad \epsilon = \alpha + \delta = \frac{1553}{10^9}.$$

Equation (14) now yields on $\mathcal{C}$

$$q_H[h] + \epsilon \|h\|^2 \geq \kappa \int (\alpha + \mu s^2) |h|^2.$$

Since $\epsilon - \kappa\alpha = 155299736/10^{14}$ and $\kappa\mu = 33/25000000$, this is (4). Boundedness of B and of the lower-bound operators extends the inequality through the specified form closure. Substitution $f = sh$ proves (3), completing Theorem 1.1.

8.1 Reconstruction and the endpoint family

The supplementary source package contains the two portable verification programs, fixed dependency versions, mathematical expected-output files and build instructions. `verify_bounds.py` reconstructs the scalar interval cover, complete paired tail and rational analytic budgets. `verify_certificate.py` reconstructs all primitive inputs, both signed matrices and both congruences from formulas; no private matrix cache is an input. Its explicit exception gates remain active under Python optimization. Numerical witnesses can vary with the floating backend; only the directed acceptance inequalities determine success.

The argument also specifies what a sequence approaching the exact endpoint would require. If complete certificates of this same realization give $H \geq -\epsilon_j I$ with $\epsilon_j \downarrow 0$, applying them to each fixed core vector proves $H \geq 0$, hence RH by Proposition 2.1. In the present method both α and the complete arithmetic debit must decrease while the signed cap and all localization errors remain paid. Simply setting $\alpha = 0$ is invalid: D_0 loses its uniform inverse, t tends pointwise to $\sqrt{2}$, and the transformed global rows cease to be L^2 . Joint control of these parameters, rather than a finite-matrix eigenvalue alone, is the natural continuation of the certificate.

Assistance and responsibility

AI systems assisted with exploration, literature comparison, mathematical and code checking, and manuscript preparation. The author takes responsibility for the statements, proofs and citations.

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