

Exact-rank quadratic correlation: a phase-compatible Clique reduction

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Abstract

We give an additive-gap reduction from Clique to correlation maximization over Boolean quadratic phases of prescribed even polar rank $r \geq 4$. For the bounded function obtained by averaging graph-edge characters, an $(r+1)$ -clique gives exact-rank optimum $2^{-r/2} \binom{r+1}{2} / |E|$, while its absence decreases the optimum by at least $2^{-r/2} / |E|$. The proof combines the graphic span bound with control of nonzero affine Fourier supports and a nondegenerate quadratic Gauss witness realizing every clique-edge sign simultaneously. Consequently, an exact-rank additive optimizer with running time $f(r) \text{poly}(n, \varepsilon^{-1})$ would give a fixed-parameter Clique algorithm, with the corresponding randomized implication for randomized optimization. Explicit examples separate exact rank from rank at most a cap and show why the rank-two gap fails. An anchored affine-restriction lemma preserves both an output-rank cap and its rank distance from a known quadratic, providing a positive structural contrast.

1 The exact-rank objective and the reduction

Let $q : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ be a classical quadratic polynomial,

$$q(x) = c + \ell \cdot x + \sum_{i < j} b_{ij} x_i x_j.$$

Its polar form is

$$B_q(x, y) = q(x + y) + q(x) + q(y) + q(0).$$

It is alternating and has even rank $\rho(q)$. Linear and constant terms do not contribute to this rank. Write $\chi_q = (-1)^q$, with the uniform inner product $\langle f, g \rangle = 2^{-n} \sum_x f(x)g(x)$ and Fourier characters $\chi_v(x) = (-1)^{v \cdot x}$.

For real $|h| \leq 1$ define

$$\text{OPT}_{=r}(h) = \max_{\rho(q)=r} |\langle h, \chi_q \rangle|, \quad \text{OPT}_{\leq r}(h) = \max_{\rho(q) \leq r} |\langle h, \chi_q \rangle|. \quad (1)$$

The input model supplies evaluation-query access to h ; a proper output gives the coefficients of a quadratic in the indicated class. The two maxima in (1) impose different promises on that output.

Theorem 1.1 (Exact-rank graph gap). *Let $r \geq 4$ be even. Let G be a simple graph on $n \geq r+1$ vertices with $M = |E(G)| > 0$. Put*

$$h_G(x) = \frac{1}{M} \sum_{ij \in E(G)} (-1)^{x_i + x_j}, \quad a = 2^{-r/2}, \quad C = \binom{r+1}{2}. \quad (2)$$

This is a real 1-bounded function evaluable in polynomial time. If G contains an $(r + 1)$ -clique, then

$$\text{OPT}_{=r}(h_G) = \frac{aC}{M}.$$

If G contains no such clique, then

$$\text{OPT}_{=r}(h_G) \leq \frac{a(C - 1)}{M}.$$

Corollary 1.2 (Parameterized implication). *An additive optimizer that returns a rank-exactly- r quadratic with correlation at least $\text{OPT}_{=r}(h) - \varepsilon$ in time*

$$f(r) \text{ poly}(n, \varepsilon^{-1}, \log(1/\delta))$$

with success probability at least $1 - \delta$ would yield a randomized fixed-parameter algorithm for Clique. A deterministic optimizer with the analogous guarantee would yield a deterministic fixed-parameter algorithm.

The graph-span extremizer used in this theorem is an established mechanism. Kundu, Rai, Sahiba and Saurabh prove parameterized hardness for Maximum Span on graphic matroids [1]. Passing from a support subspace to a quadratic phase requires two further arguments: Fourier supports can be affine rather than linear, and their equal-magnitude signs must be compatible with a nondegenerate quadratic. Sections 3 and 4 pay these two conditions explicitly.

Unrestricted proper quadratic correlation optimization is the setting of the near-optimal quadratic Goldreich–Levin algorithm of Briët and Castro-Silva [2] and its extended algorithmic framework [3]. The polynomial-time Marton framework gives weak quadratic-structure and improper learning applications [4]. The prescribed polar-rank constraint in (1) is a separate output condition. Section 5 gives a rank-preserving operation for a useful anchored class, and Section 6 identifies exact failures of several enlargements.

2 Quadratic Fourier spectra

Lemma 2.1 (Flat affine support). *If $\rho(q) = r$, then $\widehat{\chi_q}$ has support on an affine r -plane and every nonzero coefficient has magnitude $2^{-r/2}$.*

Proof. Let $R = \ker B_q$, of dimension $n - r$. For $u \in R$, the derivative $q(x + u) + q(x) = q(u) + q(0)$ is independent of x . Put $\lambda(u) = q(u) + q(0)$; it is linear on R . Translating the Fourier sum by u shows that a nonzero coefficient at v must satisfy $v \cdot u = \lambda(u)$ for every $u \in R$. The solutions form an affine coset of $R^\perp$, of dimension r .

For a solution v , $q(x) + v \cdot x$ is constant on cosets of R and descends, after subtracting its constant, to a nondegenerate quadratic Q on an r -dimensional quotient. Its Gauss sum has magnitude $2^{r/2}$. Indeed, writing $S = \sum_y (-1)^{Q(y)}$, and changing variables in S^2 gives

$$S^2 = \sum_u (-1)^{Q(u)} \sum_y (-1)^{B_Q(y, u)} = 2^r,$$

where $Q(0) = 0$ and nondegeneracy annihilates every $u \neq 0$ inner sum. Thus the normalized quotient Fourier coefficient has magnitude $2^{-r/2}$ and is nonzero. This proves both the complete support description and flatness. $\square$

The proof also gives an exact coefficient algorithm. Solve the linear equations on the radical. A failed equation gives zero; otherwise enumerate the 2^r quotient points and sum their signs, with an explicitly determined constant phase. Hence Fourier coefficients of an explicitly given rank- r quadratic are computable in $2^{O(r)} \text{ poly}(n)$ time using integer arithmetic.

3 Counting graph edges in Fourier supports

Associate the edge ij with $e_i + e_j \in \mathbb{F}_2^n$. For an affine space S let $G[S]$ contain exactly the edges of G with edge vectors in S . Distinct graph edges have distinct vectors.

Lemma 3.1 (Linear support bound). *If S is linear of dimension r , then $|E(G[S])| \leq \binom{r+1}{2}$. Equality holds only if G has an $(r+1)$ -clique.*

Proof. Let the nontrivial connected components of $G[S]$ have orders $v_1, \dots, v_k \geq 2$. In a connected component, edge vectors span its full even-weight space: spanning-tree edges are independent and generate every pair by summing the path. Its rank is $v_i - 1$. Disjoint components have disjoint coordinate supports, so

$$R := \sum_i (v_i - 1) \leq r.$$

Write $t_i = v_i - 1 \geq 1$. The number of edges is at most

$$\sum_i \binom{t_i + 1}{2} \leq \binom{R + 1}{2} \leq \binom{r + 1}{2}.$$

The middle difference is $\sum_{i < j} t_i t_j$, positive when at least two nontrivial components occur. Equality at the final threshold therefore requires one component on $r + 1$ vertices and all its possible edges. The empty-edge case is immediate. $\square$

Lemma 3.2 (Nonzero affine support bound). *If S is an affine r -plane with $0 \notin S$, then*

$$|E(G[S])| \leq \left\lfloor \frac{(r+2)^2}{4} \right\rfloor. \quad (3)$$

For even $r \geq 4$ this is strictly below $\binom{r+1}{2}$.

Proof. Write $S = v + L$, $v \notin L$. Choose a binary functional φ vanishing on L with $\varphi(v) = 1$. If $e_i + e_j \in S$, then $\varphi(e_i) + \varphi(e_j) = 1$. Thus $G[S]$ is bipartite, partitioned by the vertex labels $\varphi(e_i)$.

Let its nontrivial component orders again be $v_i = t_i + 1$. Their edge-vector span has dimension $R = \sum_i t_i$. Since every included vector belongs to S , its span is contained in $\text{span}(S)$ of dimension $r + 1$, so $R \leq r + 1$. A bipartite component has at most $\lfloor v_i^2/4 \rfloor$ edges. Furthermore

$$\sum_i (t_i + 1)^2 \leq (R + 1)^2,$$

because the right minus the left is $2 \sum_{i < j} t_i t_j - (k - 1) \geq 0$ for $t_i \geq 1$. Taking floors yields

$$\sum_i \left\lfloor \frac{(t_i + 1)^2}{4} \right\rfloor \leq \left\lfloor \frac{(R + 1)^2}{4} \right\rfloor \leq \left\lfloor \frac{(r + 2)^2}{4} \right\rfloor.$$

For even r , the unfloored difference from $\binom{r+1}{2}$ is $(r^2 - 2r - 4)/4$, which is positive for $r \geq 4$. $\square$

By Fourier orthogonality, for any rank- r quadratic whose support is S ,

$$|\langle h_G, \chi_q \rangle| = \frac{1}{M} \left| \sum_{ij \in E(G)} \widehat{\chi_q}(e_i + e_j) \right| \leq \frac{a}{M} |E(G[S])|. \quad (4)$$

The two lemmas therefore prove the global upper bound aC/M and the required $a(C-1)/M$ bound for every NO instance. Integrality pays the unit edge gap in both cases.

4 A phase-compatible clique witness

The upper support bound is useful only if its signs can be realized by one quadratic phase. This is the role of the following construction.

Lemma 4.1. *For a clique on $r + 1$ vertices, with r even, there is a rank- r quadratic phase whose Fourier coefficient at every clique-edge vector is $+2^{-r/2}$, and whose Fourier support contains no other edge vectors outside that clique.*

Proof. Let K be the clique vertex set and let W be the even-weight space on K , zero on the other coordinates. Because $|K| = r + 1$ is odd, $\dim W = r$. On W put

$$Q(y) = \sum_{i < j \in K} y_i y_j = \frac{\text{wt}(y)}{2} \pmod{2}.$$

The last formula is valid for even weight. Its polar form is the dot product on W : expanding gives $(\sum y_i)(\sum z_i) + y \cdot z = y \cdot z$. The radical is $W \cap \text{span}(\mathbf{1}_K) = \{0\}$, because $\mathbf{1}_K$ has odd weight. Thus Q is nondegenerate.

Define a Fourier array by

$$\widehat{f}(y) = a(-1)^{Q(y)+1} \quad (y \in W), \quad \widehat{f}(y) = 0 \quad (y \notin W). \quad (5)$$

For each x , nondegeneracy gives a unique $v_x \in W$ such that $x \cdot y = B_Q(v_x, y)$ for all $y \in W$. The map $x \mapsto v_x$ is linear and onto, since every linear functional on W extends to the ambient space. Completing the quadratic gives

$$f(x) = -a \sum_{y \in W} (-1)^{Q(y)+x \cdot y} = -a(-1)^{Q(v_x)} \sum_{y \in W} (-1)^{Q(y)}.$$

The Gauss sum is $\pm 2^{r/2}$ by the proof of Lemma 2.1. Hence $f(x) = \pm(-1)^{Q(v_x)}$ is a real quadratic phase. Its polar rank is r : the pullback of the nondegenerate form through a surjective map has radical equal to the map's kernel. Every clique edge has weight two, so $Q(e_i + e_j) = 1$ and its coefficient in (5) is $+a$. The only weight-two vectors of W are edges within K . $\square$

Combining Lemma 4.1 with (4) proves Theorem 1.1.

Proof of Corollary 1.2. Request additive error $\varepsilon = a/(4M)$ and, for a randomized optimizer, failure probability $\delta = 1/3$. In a YES instance the returned correlation is at least $a(C - 1/4)/M$ on success. In a NO instance every rank- r output has correlation at most $a(C - 1)/M$. Compute the returned correlation exactly using its M edge Fourier coefficients and the quotient algorithm following Lemma 2.1; compare to $a(C - 1/2)/M$.

Since $M \leq n^2$ and $\varepsilon^{-1} = 4M2^{r/2}$, the total time is a function of r times a fixed polynomial in n . The bounded function (2) is simulated exactly for each evaluation query. Clique restricted to odd requested sizes at least five suffices: when a requested size is even, add one universal vertex and increase that size by one. Small requested sizes are polynomial-time decidable. Empty-edge and too-small graph instances can be handled directly. Randomization is inherited exactly from the optimizer. $\square$

5 Rank-preserving anchored restrictions

The exact-rank obstruction has a useful positive contrast. Affine restriction contracts polar rank. If a known anchor is retained by that restriction, it simultaneously contracts the rank distance from the anchor.

Proposition 5.1 (Two-cap restriction). *Let a, q be quadratics, let V_a be the linear span of the Fourier support of χ_a , and choose a subspace $W \supseteq V_a$. Put $K = W^\perp$. Choose any complement L of K and let T be the projection onto L along K . For $u \in K$ define $q_u(x) = q(Tx + u)$. Then*

$$\rho(q_u) \leq \rho(q), \quad \rho(q_u + a) \leq \rho(q + a). \quad (6)$$

Moreover

$$\mathbb{E}_{u \in K} \chi_{q_u}(x) = \sum_{v \in W} \widehat{\chi_q}(v) \chi_v(x). \quad (7)$$

Proof. Fourier expansion shows that χ_a is K -invariant. Hence $a(Tx + u) = a(x)$ in $\mathbb{F}_2$ for all $u \in K$, because $x - Tx \in K$. Therefore

$$q_u + a = (q + a) \circ (x \mapsto Tx + u).$$

An affine pullback transforms a polar matrix B to $T^\top B T$, proving both rank inequalities. For (7), the random vector $Tx + u$ is uniform on $x + K$. Averaging a Fourier character on that coset keeps it if $v \in K^\perp = W$ and annihilates it otherwise. $\square$

A quantitative consequence follows directly. Put $p = q + a$ and $f = h\chi_a$, and take $D_0 \geq 0$ and $\theta \geq 0$ with $\rho(p) \leq D_0$. If W also contains every frequency in $\text{supp } \widehat{\chi_p}$ for which $|\widehat{f}(v)| > \theta$, then the loss in correlation after projection is at most

$$|\langle h, \chi_q \rangle - \mathbb{E}_u \langle h, \chi_{q_u} \rangle| \leq 2^{D_0/2} \theta. \quad (8)$$

Indeed, $\chi_{q_u} = \chi_a \chi_{p \circ (Tx + u)}$, so the difference is the sum of $\widehat{f}(v) \widehat{\chi_p}(v)$ outside W . At most 2^d terms of magnitude at most $\theta 2^{-d/2}$ occur, where $d = \rho(p) \leq D_0$. If the original correlation is oriented nonnegative by a constant term, one translation achieves the projected average or more. This gives a candidate preserving both caps and the stated additive loss. A heavy-frequency routine may supply directions, while the proposition specifies the exact structural requirement on the resulting subspace.

6 Exact examples and failed enlargements

6.1 Exact rank and rank at most

On $G = K_9$, $M = 36$, Theorem 1.1 at $r = 8$ gives $\text{OPT}_{=8}(h_G) = 1/16$. A rank-six witness on a seven-vertex clique has correlation

$$\frac{2^{-3} \binom{7}{2}}{36} = \frac{7}{96} > \frac{1}{16}.$$

Thus a rank-at-most-eight optimizer can correctly prefer a lower-rank output. Its guarantee does not answer the exact-rank decision used in the reduction.

6.2 Why rank two has a different graph gap

The four edge vectors of a $K_{2,2}$ lie in a nonzero affine two-plane, so their support count is four rather than the triangle threshold three. Even after signs are paid, a triangle-free graph can meet the threshold. Take the path 12, 23, 34 on four vertices and set

$$u = e_1 + e_2, \quad v = e_2 + e_3, \quad w = e_3 + e_4, \quad z = u + v + w = e_1 + e_4.$$

These are a nonzero affine two-plane. Assign Fourier coefficients $+1/2$ at u, v, w and $-1/2$ at z . The inverse is a genuine rank-two phase. For an explicit verification, translate the support by u and use its independent direction vectors $v + u, w + u$. The sign array is $[+, +, +, -]$, equal to $(-1)^{ab}$ on the two direction coordinates. The two-variable Gauss transform is again $\pm 1/2$ at each dual point, so its inverse is a quadratic phase with nondegenerate rank-two polar form. All three path-edge coefficients are $+1/2$, giving correlation $1/2$, exactly the triangle formula $2^{-1} \binom{3}{2}/3$. A rank-two analogue of the NO-instance unit gap is therefore false.

6.3 Correction-only projection can break a rank cap

Use variables u, v, x, y , with $q = uv$, anchor $a = uv + xy$, and correction $p = q + a = xy$. The target has rank two and the correction has rank two, while the anchor has rank four. Project the correction onto the x coordinate and translate the discarded coordinates. It becomes either 0 or x . Multiplying its phase by χ_a gives a or $a + x$, both of rank four. Retaining the anchor's full Fourier span, as in Proposition 5.1, pays the missing first cap.

6.4 A stabilizer-state enlargement changes the objective

There is another concrete distinction between a classical phase optimizer and a more general quantum optimization. For nonzero real h put

$$Vh = 2^{-n} \sum_{x, y \in \mathbb{F}_2^n} \frac{h(x + y)}{\|h\|_2} |x, y\rangle, \quad \|h\|_2^2 = \mathbb{E}h^2.$$

This preserves normalized uniform inner products. For $h = \chi_q$, the Schmidt coefficients across the two registers have squared values $|\widehat{\chi_q}(v)|^2$, as diagonalizing the convolution matrix by the Walsh transform shows. Lemma 2.1 therefore gives entanglement entropy $\rho(q)$ bits.

For $h = \mathbf{1}_{\{0\}}$, however,

$$Vh = 2^{-n/2} \sum_x |x, x\rangle$$

is an EPR stabilizer state of entropy n and fidelity one with itself. Every normalized classical quadratic phase has squared overlap 2^{-n} with the normalized function h . At $n = 2$ this is the literal gap 1 versus $1/4$, even for exact rank two. The EPR state belongs to the linear image of V , so imposing that image condition still leaves the flat-amplitude phase constraint unpaid. Fixed-entanglement stabilizer optimization occurs as a tool in Clifford-structured process learning [5]; the example identifies the extra constraint required for this particular classical reduction.

7 Verification and mathematical organization

The supplied verifier uses exact integer finite-field arithmetic. It reconstructs the five-vertex clique chirp and its complete Fourier transform, checks the affine-support counting bound in a finite

dimension, reconstructs the rank-two path example, and checks the anchored restriction identity and both rank contractions on deterministic finite fixtures. It also verifies the exact fractions in the K_9 comparison. These calculations illustrate and challenge the distinct hypotheses; the general proofs above establish the reduction for every even $r \geq 4$.

The central result is one exact-rank obstruction. The affine support and Gauss witness are its two necessary payments; the anchor restriction and explicit enlargements explain how output constraints interact with familiar Fourier and quantum optimization tools.

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