

Divisor tests and coefficient transfers for sextic character moments

Mathematical supplement

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Luminary Oddities

28 October 2026

Abstract

We study sextic residue-character moments over the Eisenstein integers, with squarefree ideal columns of norm length D and element rows of norm at most H . The selected sixth-power rows yield a positive arithmetic form expressed through totient-weighted divisor means. A smooth Gram formula has an absolute $O(D \log^2(2D))$ remainder and gives a sufficient growing-test criterion with a sharp quadratic cutoff tail. At $D = H^r$ with fixed $r > 0$, scale-dependent bounded coefficients can annihilate any prescribed fixed finite set of these means exactly while their selected sixth-power block retains energy of order $H^{1/6}D$. For the complete row space, we give an exact sixth-power decomposition and spectral criterion. Under a stated structured inverse-moment input, fixed-local coefficient modulation preserves the row-scale estimate; divisor representations with explicit norm costs and coefficient remainders give further sufficient transfers. The arguments specialize classical Selberg diagonalization and combine ideal counting, smooth lattice summation, Euler splitting and Hilbert-space inequalities.

Keywords. Sextic residue characters, divisor means, character moments, coefficient transfers.

Introduction

A moment estimate for prescribed arithmetic coefficients carries information beyond the size of those coefficients. When such an estimate is transferred to another coefficient family, one needs both a way to detect an inadmissible change and a sufficient representation whose full cost can be bounded. Sextic residue characters make this distinction concrete: on a sixth-power row the character becomes a coprimality indicator. Removing the Möbius oscillation from an inverse polynomial then creates a coherent positive contribution across many rows.

This supplement develops that observation into complementary tests and transfer criteria. The principal arithmetic object is the complete family of totient-weighted divisor means. Theorem S3.4 constructs bounded, scale-dependent coefficients annihilating any fixed finite collection of those means while keeping the selected row energy large. Theorem S4.1 gives the positive counterpart: a growing collection, with its aggregate and quadratic cutoff tail explicitly bounded, controls that selected block. The cutoff exponent is sharp in the sequential sense of Proposition S4.2. These statements explain why the number, accuracy and uniformity of the tests matter together.

For sufficient full-row estimates, Theorem S6.1 retains every sixth-power inflation, unit and zero mask in an exact decomposition. Theorem S7.1 proves stability under fixed-local divisibility weights using the stated structured inverse input. Theorem S7.2 extends the algebraic representation to general residual coefficients with every exclusion-dependent constant included in its norm cost. Proposition S7.3 supplies complete coefficient-remainder bounds, and (S6.7) expresses the corresponding finite source-map certificate with its original metric.

The organization follows these two uses. Sections S2–S4 establish the arithmetic form, bounded witnesses and growing tests. Section S5 describes its finite local spectrum and conditioning. Sections S6–S7 return to the complete character matrix and give sufficient coefficient transfers. The hypotheses of each imported estimate are stated explicitly before it is used.

S1 Arithmetic setting and row families

Let $\mathcal{O} = \mathbb{Z}[\omega]$, $\omega = e^{2\pi i/3}$, and let $q_a = Na$ denote absolute ideal norm. For a nonzero element, $q_u = |u|^2$. Fix a finite set S of prime ideals containing the primes above 6. An *admitted* ideal is prime to S ; sums indexed by n, d, a are over admitted integral ideals unless otherwise specified. Each admitted ideal is represented by its primary generator. The ideal Möbius function is μ , and $\tau(n)$ is its number of ideal divisors.

For admitted squarefree n , let $\chi_n(u)$ be the sextic residue character, extended by zero at nonunits modulo n . Multiplicativity retains this extension. In particular,

$$\chi_n(a^6) = \mathbf{1}_{(n,a)=1}, \quad \chi_n(va^6) = \chi_n(v) \mathbf{1}_{(n,a)=1}. \quad (\text{S1.1})$$

The same identities hold after conjugating all characters. Since $\mathcal{O}$ is a principal ideal domain and its units have sixth power one, the element a^6 is independent of the generator chosen for any integral ideal a .

Fix $0 < w_0 < w_1 < \infty$ and a nonnegative, nonzero smooth function W supported in $[w_0, w_1]$. For $D \geq 2$ and $|b(n)| \leq B$, set

$$c_n = \mu(n)b(n)W(q_n/D), \quad (\text{S1.2})$$

$$F_b(u) = D^{-1/2} \sum_n c_n \chi_n(u), \quad \mathcal{E}_b(H, D) = \sum_{\substack{u \in \mathcal{O} \\ 0 < q_u \leq H}} |F_b(u)|^2. \quad (\text{S1.3})$$

The constant B is fixed. A sequence called *fixed* is independent of H, D ; other sequences may depend on D . All column sums are finite.

The structured input used in Section S7 is the raw all-element inverse moment of the pinned OpenAI manuscript [3, Section “The inverse moment with prime factors”]:

$$\sum_{0 < q_u \leq H} \left| D^{-1/2} \sum_{(n,S)=1} \mu(n) \nu(n) \chi_n(u) W(q_n/D) \right|^2 \ll_{S,\nu,c,\varepsilon,W} H(HD)^\varepsilon, \quad H \geq D^{1+c}, \quad (\text{S1.4})$$

where $c > 0$ is fixed and ν is a permitted fixed ray character. Its prescribed inverse and prime-slot coefficient structure is part of that input. The subsequent sixth-power amplification in that source is a separate estimate on rows whose prime valuations are at most five. Here the original energy (S1.3) uses all literal element rows; Section S6 decomposes that complete space exactly.

The divisor form below specializes classical Selberg/GCD diagonalization [2, Section 1.4.1 and Exercise 1.4.3(iii)]. Sixth-power averaging with a totient-weighted mean also appears in [4, Section 6]. The fixed-order sieve scale is given in [1], while [5] treats squarefree lower bounds and structured prime moments. The finite algebra and counting arguments required for the supplement are proved here.

S2 The sixth-power Gram and divisor means

Put

$$\rho(n) = \prod_{p|n} (1 - q_p^{-1}), \quad w(d) = \prod_{p|d} (q_p - 1)^{-1} \quad (d \text{ squarefree}), \quad (\text{S2.1})$$

and $w(d) = 0$ otherwise. Define

$$M_d(c) = D^{-1} \sum_{d|n} c_n \rho(n), \quad \mathcal{Q}_D(c) = \sum_d w(d) |M_d(c)|^2. \quad (\text{S2.2})$$

For $A \geq 1$, the selected pure-power block is

$$P_A(c) = D^{-1} \sum_{q_a \leq A} \left| \sum_{(n,a)=1} c_n \right|^2. \quad (\text{S2.3})$$

The ideals a here are admitted. They give distinct literal rows a^6 , so

$$\mathcal{E}_b(H, D) \geq P_{H^{1/6}}(c). \quad (\text{S2.4})$$

Lemma S2.1 (Exact arithmetic compression). *Write $S_d(c) = \sum_{d|n} c_n$ and $R_A(l) = \#\{a : q_a \leq A, (a, l) = 1\}$. Then*

$$F_b(a^6) = D^{-1/2} \sum_{d|a} \mu(d) S_d(c), \quad (\text{S2.5})$$

$$P_A(c) = D^{-1} \sum_{n,m} c_n \overline{c_m} R_A(\text{lcm}(n, m)), \quad (\text{S2.6})$$

$$\sum_{n,m} c_n \overline{c_m} \rho(\text{lcm}(n, m)) = D^2 \mathcal{Q}_D(c). \quad (\text{S2.7})$$

Both kernels in the latter two identities are positive semidefinite.

Proof. The finite identity $\mathbf{1}_{(n,a)=1} = \sum_{d|n, d|a} \mu(d)$ proves (S2.5). Expanding the square in (S2.3) proves (S2.6); its matrix is the Gram matrix of $a \mapsto \mathbf{1}_{(a,n)=1}$. Only squarefree n, m contribute. Prime by prime,

$$\rho(\text{lcm}(n, m)) = \rho(n) \rho(m) \rho((n, m))^{-1}, \quad \rho((n, m))^{-1} = \sum_{d|n, d|m} w(d).$$

At a common prime the second identity is $1 + (q_p - 1)^{-1} = (1 - q_p^{-1})^{-1}$. Interchanging finite sums proves (S2.7) as a sum of nonnegative squares. $\square$

For a constant $\kappa_S > 0$, elementary ideal counting gives

$$I_S(X) := \#\{a : q_a \leq X\} = \kappa_S X + O_S(\sqrt{X}) \quad (X \geq 1). \quad (\text{S2.8})$$

One ideal corresponds to an orbit of six nonzero Eisenstein lattice points in a disk. The lattice boundary estimate is $O(\sqrt{X} + 1)$; finite inclusion-exclusion imposes S . Partial summation consequently yields

$$\sum_{q_n \leq X} \tau(n) = \sum_{q_d \leq X} I_S(X/q_d) \ll_S X \log(2X). \quad (\text{S2.9})$$

Proposition S2.2 (Sharp and smooth Gram formulas). *Uniformly for $A \geq 1$,*

$$P_A(c) = \kappa_S A D \mathcal{Q}_D(c) + O_{S,W,B}(\sqrt{A} D \log^2(2D)). \quad (\text{S2.10})$$

Let V be fixed, nonnegative, smooth and compactly supported on $[0, \infty)$, and constant near zero. Put $v_0 = \int_0^\infty V(t) dt$. Then

$$P_A^V(c) := D^{-1} \sum_a V(q_a/A) \left| \sum_{(n,a)=1} c_n \right|^2 = \kappa_S A D v_0 \mathcal{Q}_D(c) + O_{S,V,W,B}(D \log^2(2D)). \quad (\text{S2.11})$$

For positive constants depending only on fixed data,

$$c_1 A D \mathcal{Q}_D(c) - C_1 D \log^2(2D) \leq P_A(c) \leq c_2 A D \mathcal{Q}_D(c) + C_2 D \log^2(2D). \quad (\text{S2.12})$$

Proof. Möbius inversion and (S2.8) give, uniformly in l ,

$$R_A(l) = \kappa_S A \rho(l) + O_S \left(\sqrt{A} \sum_{d|\text{rad } l} q_d^{-1/2} \right).$$

The omitted main terms with $q_d > A$ obey $A/q_d \leq \sqrt{A}/\sqrt{q_d}$. Thus the error is at most $C\sqrt{A}\tau(\text{rad } l)$ even for large l . In (S2.6), use $\tau(\text{lcm}(n, m)) \leq \tau(n)\tau(m)$ and $\sum_n |c_n| \tau(n) \ll D \log(2D)$. This proves (S2.10) with an absolute error.

For the smooth count, planar Poisson summation gives

$$I_{S,V}(X) := \sum_a V(q_a/X) = \kappa_S X v_0 + O_{S,V}(1) \quad (X > 0). \quad (\text{S2.13})$$

For $X \geq 1$, the zero dual frequency is the volume term, and rapid Fourier decay bounds the sum of nonzero frequencies by $O_V(1)$. Removing the original zero point changes the sum by a constant. Divide by six associates and impose S by finite inclusion–exclusion. For $0 < X < 1$, compact support bounds both terms uniformly.

Inclusion–exclusion over $d \mid \text{rad } l$ now gives

$$R_A^V(l) := \sum_{(a,l)=1} V(q_a/A) = \kappa_S A v_0 \rho(l) + O_{S,V}(\tau(\text{rad } l)).$$

If V is supported in $[0, C_V]$, divisors beyond $C_V A$ have zero actual contribution and an omitted main term of size $O_V(1)$ each. This accounts for the full divisor range. Expanding the column square as before proves (S2.11). Choose fixed $V_+ \geq 1$ on $[0, 1]$ and a nonzero $0 \leq V_- \leq 1$ supported inside $[0, 1]$. These give the upper and lower comparisons in (S2.12). $\square$

The errors are absolute quadratic-form errors, and the two constants in (S2.12) come from different smooth tests.

Corollary S2.3 (A smoothed one-mean test). *For fixed positive constants depending on the fixed data,*

$$P_A(c) \geq c A D \left(|M_1(c)| - C \frac{\log(2D)}{A} \right)_+^2. \quad (\text{S2.14})$$

Consequently a raw bound $\mathcal{E}_b(H, D) \ll H(HD)^\varepsilon$ requires, for $A = H^{1/6}$,

$$|M_1(c)| \ll \sqrt{H/(AD)} (HD)^{\varepsilon/2} + \log(2D)/A.$$

Proof. Choose the fixed minorant V_- used above. Summing the unsquared coprime columns with its weights and applying the smooth count gives $\kappa_S A v_0 D M_1(c) + O(D \log(2D))$. Weighted Cauchy bounds its squared modulus by $D P_A^{V_-}(c) \sum_a V_-(q_a/A)$. The final count is $O(A)$ and $P_A^{V_-} \leq P_A$. The triangle inequality and division prove (S2.14); combine it with (S2.4) for the stated necessary rate. $\square$

S3 Densities and bounded resonant coefficients

Lemma S3.1 (Weighted squarefree density). *For each fixed $\sigma \in (1/2, 1)$,*

$$D^{-1} \sum_n \mu(n)^2 \rho(n) W(q_n/D) = C_{S,W} + O_{S,W,\sigma}(D^{\sigma-1}), \quad C_{S,W} > 0. \quad (\text{S3.1})$$

For fixed squarefree admitted d , the same sum restricted by $d \mid n$ has limiting constant $C_{S,W} \alpha_d$, where

$$\alpha_d = \prod_{p \mid d} \alpha_p, \quad \alpha_p = \frac{q_p - 1}{q_p^2 + q_p - 1}. \quad (\text{S3.2})$$

The normalized error is $O_{S,W,d,\sigma}(D^{\sigma-1})$.

Proof. For $f(n) = \mu(n)^2 \rho(n)$, its series is

$$\sum_n \frac{f(n)}{q_n^s} = \zeta_S(s) G_S(s), \quad G_S(s) = \prod_{p \notin S} (1 - q_p^{-s})(1 + \rho(p) q_p^{-s}).$$

The local correction is $1 - q_p^{-s-1} - \rho(p) q_p^{-2s}$. Its Dirichlet coefficients g obey $\sum_d |g(d)| q_d^{-\sigma} < \infty$ for $\sigma > 1/2$. Convolve g with (S2.8): both the tail of the main term and the summed counting error are $O(X^\sigma)$. Thus

$$\sum_{q_n \leq X} f(n) = \kappa_S G_S(1) X + O_{S,\sigma}(X^\sigma).$$

Partial summation gives (S3.1), with $C_{S,W} = \kappa_S G_S(1) \int W > 0$, since the positive Euler factors at one have summable defects. At a fixed prime p , the divisible local term has relative density $[\rho(p)/q_p]/[1 + \rho(p)/q_p] = \alpha_p$. Delete that fixed factor first and apply the same convolution to its two local terms. Repeating over primes of d gives (S3.2) and the error. $\square$

Theorem S3.2 (Log-free obstruction and stability). *For fixed $r > 0$ and $D = H^r$, the choice $b = \mu$ satisfies*

$$\mathcal{E}_\mu(H, D) \geq P_{H^{1/6}}(c) \gg_{S,W,r} H^{1/6} D. \quad (\text{S3.3})$$

The same lower order holds when $|b(n) - \mu(n)| \leq \delta < 1$ on the squarefree columns. Consequently a bound $\mathcal{E}_b(H, D) \ll H(HD)^\varepsilon$ uniform over bounded row-independent b fails under the strict margin $H \geq D^{1+c}$ if

$$\frac{5}{6} < r < 1, \quad r(1+c) < 1, \quad 0 < \varepsilon < \frac{r - 5/6}{1+r}. \quad (\text{S3.4})$$

Proof. For $b = \mu$, Lemma S3.1 gives $M_1(c) \rightarrow C_{S,W} > 0$. With $A = H^{1/6}$, the error inside (S2.14) tends to zero. For the perturbation, nonnegativity of W gives

$$\Re M_1(c) \geq (1 - \delta) D^{-1} \sum_n \mu(n)^2 \rho(n) W(q_n/D),$$

and the same argument applies. The lower exponent is $r + 1/6$, while the proposed upper exponent is $1 + (1+r)\varepsilon$. Their difference is positive under (S3.4). Counting $O(D)$ bounded terms per row gives the elementary upper bound $P_A(c) \ll B^2 AD$, so the selected block has order AD for these witnesses. $\square$

Proposition S3.3 (Fixed coefficients and fixed tests). *For every fixed finite collection $\mathcal{T}$ of square-free admitted ideals, there is a fixed sequence b_p , $|b_p| \leq 1$, such that, at each fixed-power scale $D = H^r$, $r > 0$,*

$$M_d(c) = O_{d,p,\sigma}(D^{\sigma-1}) \quad (d \in \mathcal{T}), \quad P_{H^{1/6}}(c) \gg H^{1/6}D, \quad (\text{S3.5})$$

for every fixed $\sigma \in (1/2, 1)$.

Proof. Choose a fixed admitted prime p dividing no member of $\mathcal{T}$, and set $b_p(n) = \mu(n)(\mathbf{1}_{p|n} - \alpha_p)$. In each prescribed test the two density constants in Lemma S3.1 cancel. In contrast, $M_p(c) \rightarrow C_{S,W}\alpha_p(1 - \alpha_p) > 0$. Keep this term in the positive aggregate and apply (S2.12); its remainder is $o(AD)$. $\square$

Theorem S3.4 (Exact finite-test annihilation). *For every fixed finite collection $\mathcal{T}$ as above, there are scale-dependent sequences b_D , $|b_D| \leq 1$, for which*

$$M_d(c) = 0 \quad (d \in \mathcal{T}), \quad P_{H^{1/6}}(c) \asymp H^{1/6}D \quad (\text{S3.6})$$

when $D = H^r$, $r > 0$ is fixed, and H is sufficiently large.

Proof. Let Q be the product of primes in $\mathcal{T}$ and choose a fixed admitted prime $p \nmid Q$. For $f_D(n) = \mu(n)^2 \rho(n) W(q_n/D)$ and $t \mid Q$, define

$$F_t = \sum_{(n,Q)=t} f_D(n), \quad G_t = \sum_{\substack{(n,Q)=t \\ p|n}} f_D(n), \quad \alpha_{t,D} = \begin{cases} G_t/F_t, & F_t > 0, \\ 0, & F_t = 0. \end{cases}$$

Set $b_D(n) = \mu(n)(\mathbf{1}_{p|n} - \alpha_{(n,Q),D})$. Then $|b_D| \leq 1$. Every cell has weighted mean exactly zero, including empty cells. A condition $d \mid n$ with $d \mid Q$ is a union of cells, proving all exact test identities.

The p channel is the positive conditional-variance expression

$$M_p(c) = D^{-1} \sum_{t|Q} F_t \alpha_{t,D} (1 - \alpha_{t,D}).$$

Lemma S3.1 and finite inclusion–exclusion give

$$F_t/D \rightarrow C_{S,W} \prod_{q|t} \alpha_q \prod_{q|Q/t} (1 - \alpha_q) > 0, \quad \alpha_{t,D} \rightarrow \alpha_p.$$

Hence $M_p(c) \rightarrow C_{S,W}\alpha_p(1 - \alpha_p) > 0$. The smooth lower comparison and $P_A \ll AD$ prove (S3.6). $\square$

S4 Growing tests and the quadratic cutoff tail

Theorem S4.1 (Growing divisor tests). *Uniformly for $K \geq 1$,*

$$\sum_{q_d > K} w(d) |M_d(c)|^2 \ll_{S,W,B} K^{-2}, \quad (\text{S4.1})$$

$$P_A(c) \ll AD \sum_{q_d \leq K} w(d) |M_d(c)|^2 + AD/K^2 + D \log^2(2D). \quad (\text{S4.2})$$

For fixed $5/6 < r < 1$, $D = H^r$ and $A = H^{1/6}$, a sufficient condition for $P_A(c) = O(H)$ is

$$K \gg H^{(r-5/6)/2}, \quad \sum_{q_d \leq K} w(d) |M_d(c)|^2 = O\left(\frac{H}{AD}\right). \quad (\text{S4.3})$$

The number of tests is $O_S(K)$.

Proof. Counting multiples in the annular support gives $|M_d(c)| \ll 1/q_d$, with zero contribution beyond $w_1 D$. The primewise identity $1/\rho(d) = \sum_{l|d} w(l)$ on squarefree ideals implies

$$\sum_{\substack{q_d \leq X \\ d \text{ sf}}} \rho(d)^{-1} \leq \sum_l w(l) I_S(X/q_l) \ll X \sum_l w(l)/q_l \ll_S X.$$

The final Euler product is $\prod_{p \notin S} (1 + [q_p(q_p - 1)]^{-1}) < \infty$. Partial summation gives

$$\sum_{q_d > K} w(d)/q_d^2 = \sum_{\substack{q_d > K \\ d \text{ sf}}} [q_d^3 \rho(d)]^{-1} \ll K^{-2}.$$

This proves (S4.1). Split the smooth upper comparison at K to obtain (S4.2). Under (S4.3) its first two terms are $O(H)$ and $D \log^2(2D) = o(H)$. $\square$

Proposition S4.2 (Sharpness of the uniform exponent). *For $b = \mu$ there is $C_t > 0$ such that, in the sequential limits indicated,*

$$\lim_{D \rightarrow \infty} \sum_{q_d > K} w(d) |M_d(c)|^2 = C_{S,W}^2 \left(\frac{C_t}{2K^2} + O_\sigma(K^{\sigma-3}) \right), \quad \frac{1}{2} < \sigma < 1. \quad (\text{S4.4})$$

Thus an exponent $2 + \delta$, $\delta > 0$, cannot replace 2 uniformly in D, K in (S4.1).

Proof. For fixed d , $M_d(c) \rightarrow C_{S,W} \alpha_d$. The summable majorant $Cw(d)/q_d^2$ permits dominated convergence. Set $t(d) = q_d^3 w(d) \alpha_d^2$ on squarefree ideals, zero otherwise. Its local value is

$$t(p) = \frac{q_p^3(q_p - 1)}{(q_p^2 + q_p - 1)^2} = 1 + O(q_p^{-1}) > 0.$$

The series of t is $\zeta_S(s)$ times a correction with absolutely summable coefficients at $\Re s = \sigma > 1/2$: the local defects are $O(q_p^{-s-1}) + O(q_p^{-2s})$. The convolution proof of Lemma S3.1 yields $\sum_{q_d \leq X} t(d) = C_t X + O_\sigma(X^\sigma)$, with $C_t > 0$. Partial summation of $\sum_{q_d > K} t(d) q_d^{-3}$ proves (S4.4). A uniform improved exponent would survive the first limit and contradict its positive K^{-2} term. $\square$

S5 Local tensor structure and relative conditioning

Let Q be squarefree and admitted with s prime factors, and index the full local column set by $n, m \mid Q$. Put $K_Q(n, m) = \rho(\text{lcm}(n, m))$. In this section the smooth test V is nonzero, so $v_0 > 0$.

Proposition S5.1 (Local Gram and preconditioner). *For $r_p = 1 - q_p^{-1}$,*

$$K_Q = \bigotimes_{p|Q} K_p, \quad K_p = \begin{pmatrix} 1 & r_p \\ r_p & r_p \end{pmatrix}, \quad K_p^{-1} = \begin{pmatrix} q_p & -q_p \\ -q_p & q_p/r_p \end{pmatrix}. \quad (\text{S5.1})$$

Its local eigenvalues and safe global bounds are

$$\lambda_{p,\pm} = \frac{1 + r_p \pm \sqrt{(1 - r_p)^2 + 4r_p^2}}{2} > 0, \quad (\text{S5.2})$$

$$\lambda_{\min}(K_Q) \geq \frac{\rho(Q)}{2^s q_Q}, \quad \lambda_{\max}(K_Q) \leq 2^s. \quad (\text{S5.3})$$

For $G_{A,Q}^V(n, m) = A^{-1} R_A^V(\text{lcm}(n, m))$,

$$\left\| K_Q^{-1/2} (G_{A,Q}^V - \kappa_S v_0 K_Q) K_Q^{-1/2} \right\| \ll_{S,V} \frac{8^s q_Q}{A \rho(Q)}. \quad (\text{S5.4})$$

Proof. The local occurrence patterns of a prime give K_p . Its determinant is $r_p/q_p > 0$ and its trace is $1+r_p$. The larger eigenvalue is at most 2; the smaller, being the determinant divided by the larger, is at least $r_p/(2q_p)$. Tensor multiplication proves the spectrum and (S5.3). The smooth counting error per entry is at most $C2^s/A$. Multiplying this by the dimension 2^s bounds its operator norm by $C4^s/A$. Finally $\|K_Q^{-1/2}\|^2 \leq 2^s q_Q/\rho(Q)$ proves (S5.4). A principal column submatrix inherits the quadratic-form bound by zero extension. $\square$

After diagonal normalization by $\text{diag}(1, r_p^{-1/2})$, the local matrix has diagonal 1 and off-diagonal $\sqrt{r_p}$. Its condition number is $(1 + \sqrt{r_p})/(1 - \sqrt{r_p}) \sim 4q_p$. Thus $8^s q_Q/[A\rho(Q)] \rightarrow 0$ is an explicit growing-block budget for relative Gram control.

S6 Complete row decomposition and operator criteria

Here c may be any finite coefficient vector on admitted squarefree columns $q_n \leq w_1 D$. Let $\mathcal{V}_6$ be the literal nonzero elements whose ideal prime valuations are in $\{0, 1, \dots, 5\}$, including the unit of the element. For every integral ideal a , including those meeting S , put

$$(P_a c)_n = c_n \mathbf{1}_{(n,a)=1}, \quad \mathcal{M}_6(c; X, D) = D^{-1} \sum_{\substack{v \in \mathcal{V}_6 \\ q_v \leq X}} \left| \sum_n c_n \chi_n(v) \right|^2.$$

Theorem S6.1 (Complete sixth-power return). *The complete energy has the exact finite identities*

$$\mathcal{E}(c; H, D) = \sum_{\substack{a \text{ integral} \\ q_a \leq H^{1/6}}} \mathcal{M}_6(P_a c; H/q_a^6, D), \quad (\text{S6.1})$$

$$= D^{-1} \sum_{\substack{v \in \mathcal{V}_6 \\ q_v \leq H}} \sum_{\substack{a \text{ integral} \\ q_a \leq (H/q_v)^{1/6}}} \left| \sum_n c_n \chi_n(v) \mathbf{1}_{(n,a)=1} \right|^2. \quad (\text{S6.2})$$

If the a th term in (S6.1) is at most $L_a(H/q_a^6)(HD)^\varepsilon$ and $\sum_a L_a/q_a^6 \leq L$, then $\mathcal{E}(c; H, D) \leq LH(HD)^\varepsilon$.

Proof. At each prime, write the valuation of (u) uniquely as $6t + s$, $0 \leq s \leq 5$. The t define a . Its sixth-power generator is independent of units, and $v = u/a^6$ is a unique literal element of $\mathcal{V}_6$. Conversely $u = va^6$ recovers the row, with $q_u = q_v q_a^6$. Use (S1.1) and regroup the same finite nonnegative sum. The sufficient estimate follows by summing the stated bounds. $\square$

Let $T_6(X)$ be the actual matrix with rows $v \in \mathcal{V}_6$, $q_v \leq X$, the specified columns, and entries $\chi_n(v)$. Denote its squared operator norm by $B_6(X, D)$. Then

$$\mathcal{E}(c; H, D) \leq D^{-1} \|c\|_2^2 \sum_{\substack{a \text{ integral} \\ q_a \leq H^{1/6}}} B_6(H/q_a^6, D), \quad (\text{S6.3})$$

since each puncture P_a is a contraction.

Corollary S6.2 (A generic norm input and its return). *Suppose the literal matrix just defined satisfies*

$$B_6(X, D) \leq C_\varepsilon (XD)^\varepsilon (X + D + X^{5/6} D^{1/3} + X^{1/3} D^{5/6}), \quad X, D \geq 1. \quad (\text{S6.4})$$

Then

$$\mathcal{E}(c; H, D) \ll_{\varepsilon} (HD)^{\varepsilon} \frac{\|c\|_2^2}{D} \underbrace{(H + H^{1/6}D + H^{5/6}D^{1/3} + H^{1/3}D^{5/6})}_{\kappa(H,D)}. \quad (\text{S6.5})$$

Proof. Insert (S6.4) in (S6.3). All four sums here range over every integral ideal, including ideals meeting S . They use $\sum_a q_a^{-6} < \infty$, $I(H^{1/6}) \ll H^{1/6}$, $\sum_a q_a^{-5} < \infty$ and $\sum_a q_a^{-2} < \infty$, respectively, where I counts all integral ideals. Adjust the preliminary power loss. $\square$

The ideal-indexed theorem of [1, Theorem 1.1] has the scale in (S6.4); that equation specifies the literal unit, ray and support form of the input to Corollary S6.2. The dominating order of $\mathcal{K}$ is H for $D \leq H^{1/2}$, $H^{5/6}D^{1/3}$ for $H^{1/2} \leq D \leq H$, and $H^{1/6}D$ for $D \geq H$. The current squarefree Gauss-vector lower bounds in [5] give the distinct middle-range resonance context.

An exact spectral form retains all mixed source columns. Define

$$G_H = \sum_{\substack{a \text{ integral} \\ q_a \leq H^{1/6}}} P_a T_6(H/q_a^6)^* T_6(H/q_a^6) P_a. \quad (\text{S6.6})$$

For a finite source map Z and positive coordinate Gram J ,

$$\mathcal{E}(Zx; H, D) \leq Lx^* Jx \text{ for every } x \iff Z^* G_H Z \leq DLJ. \quad (\text{S6.7})$$

This is the expansion of the complete quadratic form. Entry enclosures and a Loewner inequality therefore supply a finite certificate preserving the source metric and every mixed term.

S7 Sufficient coefficient classes and controlled transfers

For a finite excluded prime set $T \supseteq S$ and $X > 0$, write explicitly

$$M_{T,u}(X; W) = X^{-1/2} \sum_{(m,T)=1} \mu(m) \chi_m(u) W(q_m/X).$$

Theorem S7.1 (Fixed-local modulation). *Let P be a fixed squarefree admitted ideal and β a bounded function on its divisors. Set $b(n) = \beta((n, P))$ on squarefree admitted ideals and $S' = S \cup \{p : p \mid P\}$. Under the raw input (S1.4) with excluded set S' , for fixed $c > 0$ and $H \geq D^{1+c}$,*

$$\mathcal{E}_b(H, D)^{1/2} \leq C_{S',c,\varepsilon,W} H^{1/2} (HD)^{\varepsilon/2} \mathcal{A}_P(\beta), \quad \mathcal{A}_P(\beta) = \sum_{d \mid P} |\beta(d)| q_d^{-1/2}. \quad (\text{S7.1})$$

Moreover $\mathcal{A}_P(\beta) \leq \|\beta\|_{\infty} \prod_{p \mid P} (1 + q_p^{-1/2})$.

Proof. Split each squarefree $n = dm$ uniquely by $d = (n, P)$ and $(m, P) = 1$. Multiplicativity with all zero masks gives

$$F_b(u) = \sum_{d \mid P} \mu(d) \beta(d) q_d^{-1/2} \chi_d(u) M_{S',u}(D/q_d; W), \quad (\text{S7.2})$$

where $M_{S',u}$ is the centrally normalized inverse with exclusions S' . For $D \geq q_P$, all new scales are at least one and retain the scale margin. Apply (S1.4) with the common fixed data and use the triangle inequality in the complete row Hilbert space; multiplication by $\chi_d(u)$ is a contraction. The bounded range $2 \leq D < q_P$ is covered by fixed finite-column counting in the P -dependent constant. The final inequality is the finite Euler expansion. A permitted fixed ray character ν is retained by inserting its factors throughout this identity. $\square$

Fixed local divisibility modulation thus preserves the inverse estimate. For example, the weight $b(n) = \mathbf{1}_{p|n} - \alpha_p$ is covered by Theorem S7.1. Multiplying it by $\mu(n)$ produces the resonant weight of Proposition S3.3, with prime-parity dependence outside every fixed finite set.

Theorem S7.2 (A divisor representation with explicit costs). *For bounded b on squarefree admitted ideals, put*

$$g_b(d) = \sum_{e|d} \mu(d/e)b(e). \quad (\text{S7.3})$$

Fix a loss $\delta > 0$ and margin $c > 0$. For each fixed squarefree d , let $C_d(\delta) \geq 1$ be an actually valid constant in the raw inverse row-norm estimate with exclusions $S \cup \{p : p \mid d\}$, loss δ , the same W , and scales $D' \geq 1/w_1$. Then, when $H \geq \max(1, D^{1+c})$,

$$\mathcal{E}_b(H, D)^{1/2} \leq H^{1/2} (HD)^{\delta/2} \mathcal{A}_D(b; \delta), \quad \mathcal{A}_D(b; \delta) = \sum_{\substack{d \text{ sf} \\ q_d \leq w_1 D}} |g_b(d)| q_d^{-1/2} C_d(\delta). \quad (\text{S7.4})$$

Proof. Finite divisor inversion gives $b(n) = \sum_{d|n} g_b(d)$. Interchange finite sums and set $n = dm$; squarefreeness imposes $(d, m) = 1$. The exact whole-row identity is

$$F_b(u) = \sum_{\substack{d \text{ sf} \\ q_d \leq w_1 D}} \mu(d) g_b(d) q_d^{-1/2} \chi_d(u) M_{S \cup p(d), u}(D/q_d; W). \quad (\text{S7.5})$$

Each new scale is at least $1/w_1$. Those between $1/w_1$ and 1 involve finitely many fixed possible m and are bounded by direct counting within $C_d(\delta)$. Larger scales retain the margin. Apply each bound and the complete row-space triangle inequality. Finally $(HD/q_d)^{\delta/2} \leq (HD)^{\delta/2}$. $\square$

For each target $\varepsilon > 0$, choosing a smaller preliminary $\delta > 0$ and proving $\mathcal{A}_D(b; \delta) \ll (HD)^{(\varepsilon-\delta)/2}$ gives the target loss ε . In particular, subpower growth of $\mathcal{A}_D(b; \delta)$ for *every* preliminary $\delta > 0$ gives the every- ε raw conclusion. All exclusion-dependent constants remain in this arithmetic coefficient criterion.

For $b(n) = \beta((n, P))$, pairing divisors with and without a prime outside P proves $g_b(d) = 0$ unless $d \mid P$. For the resonant coefficient,

$$g_\mu(d) = \mu(d) 2^{\omega(d)} \quad (d \text{ squarefree}), \quad (\text{S7.6})$$

because each of its $2^{\omega(d)}$ summands equals $\mu(d)$. The exact representation and its norm cost are separate quantities.

Proposition S7.3 (Remainder bounds for coefficient transfer). *Let $c = s + e$ in the same coefficient space. If $\mathcal{E}(s; H, D)^{1/2} \leq B_s \sqrt{H}$, then*

$$\mathcal{E}(c; H, D)^{1/2} \leq \sqrt{H} (B_s + C \|e\|_2). \quad (\text{S7.7})$$

Under (S6.4), the sharper sufficient inequality is

$$\mathcal{E}(c; H, D)^{1/2} \leq B_s \sqrt{H} + C_\varepsilon (HD)^{\varepsilon/2} \sqrt{\mathcal{K}(H, D)/D} \|e\|_2. \quad (\text{S7.8})$$

Proof. There are $O(D)$ admitted column ideals in the support and $O(H)$ literal element rows. Cauchy on the finite column sum gives $\mathcal{E}(e; H, D) \leq CH \|e\|_2^2$. The row-space triangle inequality proves (S7.7); Corollary S6.2 applied to e proves (S7.8). $\square$

For example, use $s_n = \mu(n)\beta((n, P))W(q_n/D)$ and the coefficient remainder $e_n = \mu(n)[b(n) - \beta((n, P))]W(q_n/D)$. Theorem S7.1 supplies B_s . Under the generic input, a row-scale sufficient residual budget, with preliminary losses allocated, is

$$\frac{\|e\|_2}{\sqrt{D}} \ll \sqrt{H/\mathcal{K}(H, D)}. \quad (\text{S7.9})$$

For $D = H^r$, $1/2 < r < 1$, this scale is $H^{1/12}D^{-1/6}$. For several source columns retain the remainder map R and coordinate Gram J : the complete condition is $D^{-1}R^*G_H R \leq LJ$.

The resonant witnesses also quantify representation cost. If row-space atoms A_j satisfy $\|A_j\| \leq CH^{1/2}(HD)^{\varepsilon/2}$ and $F_\mu = \sum_j \lambda_j A_j + R$ is a finite or norm-absolutely-convergent sum, then

$$\sum_j |\lambda_j| \geq \frac{(\|F_\mu\| - \|R\|)_+}{CH^{1/2}(HD)^{\varepsilon/2}}. \quad (\text{S7.10})$$

This is the triangle inequality. If $D = H^r$, $5/6 < r < 1$, and $\|R\| \leq \theta \|F_\mu\|$ with fixed $\theta < 1$, Theorem S3.2 makes the right side at least $c(1 - \theta)H^{[r-5/6-(1+r)\varepsilon]/2}$. For nonuniform atom bounds B_j , the exact statement is $\sum_j |\lambda_j|B_j \geq (\|F_\mu\| - \|R\|)_+$. These inequalities measure the full row-space cost alongside the arithmetic divisor tests.

Acknowledgment. AI assistants supported mathematical exploration and manuscript preparation.

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