

Distributional traces and central moment tests for Vámos quartics

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Luminary Oddities

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Abstract

For the stable quartics $h_\lambda = 4e_1e_3 + \lambda a^2b^2$, we establish moving-graph traces of the joint corner expansion of $h_\lambda^{-1/2}$ and finite moving-plane jets at the endpoint $\lambda = 64/27$. Exact inverse equations provide uniform conormal ellipticity; a Bessel annihilator makes the facet symbol identically $4/3$ on its moving plane. The leading facet trace is an explicit positive ellipse measure of mass $\sqrt{3}/2$. We also construct canonical signed central-band moment functionals and prove quantitative convergence to the moments of $2\text{Beta}(1/4, 1/4)$. Every fixed finite Hankel and associated localizer block is then strictly positive in sufficiently narrow bands. An all-order conditional beta law and exact exclusions of independent positive shifts and coordinate-monotone positive transports complete the structural picture.

1 The inverse distribution and the two trace statements

Put $L = 64/27$, $e_1 = a + b + c + d$, $e_3 = abc + abd + acd + bcd$, and

$$h_\lambda(a, b, c, d) = 4e_1e_3 + \lambda a^2b^2, \quad 0 < \lambda \leq L. \quad (1)$$

Throughout, a parameter is fixed before a scale tends to zero. The classification in [5] establishes real stability in this range. Equivalently, h_λ is nonzero on the tube of positive real parts. Its square root is positive on the positive real orthant and continued holomorphically to that tube. Write $\mathcal{F}_\lambda$ for its inverse Laplace distribution, with

$$\mathcal{L}U(a, b, c, d) = \langle U, e^{-aY_a - bY_b - cY_c - dY_d} \rangle.$$

On each vertical contour this means Fourier inversion with a factor $(2\pi)^{-1}$ per variable, followed by removal of the exponential tilt. A homogeneous hyperbolic polynomial satisfies $|h(v + i\xi)| \geq h(v) > 0$ for an interior positive direction v : factor $h(\xi + tv)$ into its real roots and use homogeneity. The same factorization and differentiation give polynomial bounds for the inverse and its derivatives on a fixed contour. Thus the distributional convention is well defined. The inverse is homogeneous of degree -2 .

Our graph result uses the following precise public input. Set

$$F_\lambda(\tau, \beta, \delta, x) = \mathcal{F}_\lambda(\tau + x, \beta, \tau, \delta), \quad F_\varepsilon(\tau, \beta, \delta, z) = \varepsilon F_\lambda(\tau, \varepsilon\beta, \varepsilon\delta, \varepsilon z).$$

These are pullbacks with their distributional Jacobians. Define

$$k_\pm = \frac{4 \pm 2\sqrt{\lambda}}{4 - \lambda}, \quad q_0(B, D) = 4(B + D)^2 - \lambda B^2, \quad (2)$$

$$f(\beta, \delta) = \frac{1}{\pi\sqrt{4 - \lambda}\sqrt{(\delta - k_-\beta)(k_+\beta - \delta)}}, \quad g(\beta, \delta) = \frac{\sqrt{4 - \lambda}}{4\pi\lambda} \sqrt{(\delta - k_-\beta)(k_+\beta - \delta)}. \quad (3)$$

Both functions are defined on $\beta > 0$, $k_-\beta < \delta < k_+\beta$, and extended by zero. They are locally integrable distributions. With $D_s = \partial_\beta + \partial_\delta$, put

$$F_0 = \frac{f}{2\pi\tau}, \quad F_1 = \frac{D_s g - z D_s^2 g - (z^2/2)\partial_\beta \partial_\delta D_s g}{\pi\tau^2}. \quad (4)$$

All binary derivatives include the zero-extended wedge boundary.

Proposition 1.1 (Published joint corner expansion). *For each fixed $0 < \lambda \leq L$,*

$$F_\varepsilon = F_0 + \varepsilon F_1 + o(\varepsilon) \quad \text{in } \mathcal{D}'((0, \infty)_\tau \times \mathbb{R}_{\beta, \delta}^2 \times \mathbb{R}_z). \quad (5)$$

This is precisely the two joint theorems of [6], edition 2, in the displayed coordinates, including $z = 0$. Their inversion uses K_0 on the full real x axis: if $S = b + d$,

$$Q = 4S(\alpha + S) - \lambda b^2, \quad P = 4\alpha S(\alpha + S), \quad H = 4bd\alpha(\alpha + S),$$

then $h(a, b, \alpha - a, d) = -Qa^2 + Pa + H$ and its partial inverse is $e^{mx} K_0(|x|r)/(\pi\sqrt{Q})$, $m = P/(2Q)$, $r^2 = H/Q + m^2$. The roots obey $\text{Re}(m - r) \leq 0$, $\text{Re}(m + r) \geq \text{Re } \alpha$. Differentiating in α removes the logarithmic counterterms at $\tau = 0$. The cited theorem supplies joint distributional remainders; the conormal estimates below supply their graph restrictions.

Theorem 1.2 (Moving corner graph). *For $i_\varepsilon(\beta, \delta) = (2 - \varepsilon\delta, \beta, \delta, \delta - \beta)$, on every fixed compact (β, δ) patch the canonical traces exist for all sufficiently small positive ε and*

$$i_\varepsilon^* F_\varepsilon = \frac{f}{4\pi} + \varepsilon \left\{ \frac{\delta f}{8\pi} + \frac{D_s g - (\delta - \beta) D_s^2 g - ((\delta - \beta)^2/2)\partial_\beta \partial_\delta D_s g}{4\pi} \right\} + o_{\mathcal{D}'}(\varepsilon). \quad (6)$$

The formula includes the distributional wedge boundaries.

In original dual coordinates this graph is $(Y_a, Y_b, Y_c, Y_d) = (2 - \varepsilon\beta, \varepsilon\beta, 2 - \varepsilon\delta, \varepsilon\delta)$.

For the facet result write

$$\begin{aligned} h_\lambda &= a^2 Q + aC + H, \\ Q(b, c, d) &= \lambda b^2 + 4bc + 4bd + 4cd, \\ C(b, c, d) &= 4\{(b + c + d)(bc + bd + cd) + bcd\}, \\ H(b, c, d) &= 4bcd(b + c + d). \end{aligned} \quad (7)$$

For $s \geq 0$, let S_s be the inverse in (b, c, d) of

$$K_s = Q^{-1/2} e^{-s\mu} I_0(s\nu), \quad \mu = C/(2Q), \quad \nu^2 = \mu^2 - H/Q. \quad (8)$$

Here $I_0(z) = \sum_{j \geq 0} z^{2j}/(4^j(j!)^2) = \pi^{-1} \int_0^\pi e^{z \cos \theta} d\theta$. Its entire series makes the sign of ν immaterial. Let $A_j = (\partial_s^j S_s)_{s=0}/j!$.

Theorem 1.3 (Endpoint moving-plane jets). *At $\lambda = L$, for*

$$T_s(u, v) = (u, v, 3s + 3u - v - 1)$$

and every fixed compact plane patch, the actual distributions $T_s^* S_s$ exist for small $s \geq 0$. For every fixed integer $N \geq 0$,

$$T_s^* S_s = T_s^* \left(\sum_{j=0}^N s^j A_j \right) + o_{\mathcal{D}'}(s^N). \quad (9)$$

In particular the first derivative at zero is $T_0^*(\partial_s S_0 + 3\partial_{Y_d} S_0)$.

Theorems 1.2 and 1.3 concern a prescribed finite jet on a fixed compact patch. Section 6 develops a complementary central-band functional and its finite moment matrices.

Method and relation to predecessors

The joint corner coefficients (4) belong to [6]. Here the added step is a uniform pullback argument for the actual inverse and its normalized remainders. The classical normal topology and parameter-dependent pullback theorem are due to the microlocal framework in [2, Sections 5.2–5.3]; elliptic parametrices supply the required uniform estimates. The object-specific ingredients are the two exact annihilating systems and their conormal symbols. Lorentz inverse normalizations follow the classical Riesz framework [4, 1]. The central-band argument combines rational moment multipliers, finite signed dilation mixtures and compact moment uniqueness. Section 7 then gives an explicit stability estimate for every finite Hankel block.

2 A uniform microlocal pullback lemma

A smooth embedding j admits a canonical pullback of a distribution u when $\text{WF}(u)$ misses the nonzero conormal bundle of j . The normal topology supplements strong distributional seminorms with

$$\sup_{\xi \in V} (1 + |\xi|)^N |\widehat{\chi u}(\xi)|,$$

where $(\text{supp } \chi \times V)$ avoids a prescribed closed wavefront cone. This topology is the one used in the following lemma.

Lemma 2.1 (Uniform elliptic trace transfer). *Let $u_t \rightarrow 0$ in $\mathcal{D}'(\Omega)$ as $t \downarrow 0$ and let j_t be smoothly varying embeddings of a fixed compact patch. Suppose a family of differential operators L_t of fixed order has coefficients smooth up to $t = 0$ and is uniformly elliptic in a common conic neighborhood of the conormals of j_t . Suppose $L_t u_t$ is uniformly microlocally smooth there, in every Sobolev order. Then $j_t^* u_t$ exists for small t and converges to zero in distributions on the patch. The same conclusion holds for a fixed-order system whose squared, order-balanced sum is uniformly elliptic there.*

Proof. Work in nested compact coordinate neighborhoods of the patch. A convergent family of distributions is bounded of one finite order, and hence is bounded in a single sufficiently negative local Sobolev space H^{-M} . On the common elliptic cone the inverse principal symbol has uniform symbol bounds: it divides by a common lower bound $c|\xi|^m$. The usual recursive construction of a properly supported microlocal parametrix therefore has uniform symbol seminorms, including those of its smoothing remainder. Nested spatial and conic cutoffs give, for each fixed Sobolev order q ,

$$\|A u_t\|_{H^q} \leq C_q (\|B L_t u_t\|_{H^{q-m}} + \|\chi u_t\|_{H^{-M}}),$$

where A is supported in a smaller normal cone, B equals one on a larger one, and all constants are uniform in t . Cutoff errors whose microsupports are separated are smoothing; the negative-order bound pays them. This proves arbitrary uniform microlocal Sobolev bounds near the conormals.

After compact localization, $u_t \rightarrow 0$ in a sufficiently negative Sobolev norm. Indeed its Fourier transform converges pointwise to zero and is bounded by one common polynomial; an integrable polynomial weight gives dominated convergence. Interpolation with the preceding high-order bounds gives convergence at every finite microlocal Sobolev order in the smaller normal cone. Applying these bounds also after multiplication by compactly supported coordinate monomials controls frequency derivatives. Sobolev embedding in frequency space then gives convergence of every rapidly decaying normal Fourier seminorm. Strong convergence on bounded smooth test sets follows from the common distribution order and compactness of such test sets in the requisite finite

differentiability norm. Thus the family converges to zero in the normal topology for a common closed wavefront cone disjoint from the graph conormals.

A smooth family of local coordinate changes straightens j_t to a fixed embedding. Their pullbacks preserve distributional convergence by the common finite-order bound and smooth dependence of the transformed tests; the symbol estimates remain uniform. The normal-topology pullback theorem of [2, Proposition 5.1 and Section 5.3] now gives the assertion. For a system one first forms the stated scalar elliptic sum, whose source has the same uniform microlocal bounds. $\square$

3 Proof of the graph jet

The full Fourier–Laplace symbol of F_ε is

$$\mathcal{H}_\varepsilon(\alpha, B, D, Z) = h_\lambda(Z, B, \varepsilon\alpha - Z, D)^{-1/2}.$$

The factor ε^{-2} from the three scaled dual variables and the amplitude ε is canceled by quartic homogeneity of the inverse symbol. Write

$$h_\varepsilon(\partial) = h_\lambda(\partial_z, \partial_\beta, \varepsilon\partial_\tau - \partial_z, \partial_\delta).$$

Differentiating the symbol in α and inverting gives $h_\varepsilon(\partial)(\tau F_\varepsilon) = (\varepsilon/2)(h_c)_\varepsilon(\partial)F_\varepsilon$. Commuting τ through the polynomial differential operator gives

$$P_\varepsilon F_\varepsilon = 0, \quad P_\varepsilon = \tau h_\varepsilon(\partial) + (\varepsilon/2)(h_c)_\varepsilon(\partial), \quad E F_\varepsilon = 0, \quad (10)$$

where $E = \tau\partial_\tau + \beta\partial_\beta + \delta\partial_\delta + z\partial_z + 2$. Both F_0, F_1 satisfy $E = 0$.

On i_ε , every conormal is $(v, u, \varepsilon v - u, u)$ in (τ, β, δ, z) order. Its Euler symbol is $2v$. At $v = 0$ the fourth-order symbol of P_ε is

$$\tau h_\lambda(u, u, -u, -u) = \tau \lambda u^4.$$

On a fixed compact patch $\tau = 2 - \varepsilon\delta$ stays positive. Thus these two symbols have no common nonzero conormal zero, uniformly for small ε . The scalar operator

$$P_\varepsilon^* P_\varepsilon + E^*(1 - \Delta)^3 E$$

has order eight and principal symbol $|p_\varepsilon|^2 + |\xi|^6|e|^2$, uniformly elliptic on a common normal cone.

Let $U_\varepsilon = (F_\varepsilon - F_0 - \varepsilon F_1)/\varepsilon$. By Proposition 1.1, $U_\varepsilon \rightarrow 0$ in distributions. Expand $P_\varepsilon = \sum_j \varepsilon^j P_j$. Passing to the limit in (10), then subtracting and dividing by ε , gives $P_0 F_0 = 0$ and $P_1 F_0 + P_0 F_1 = 0$. Hence $P_\varepsilon U_\varepsilon$ is a finite polynomial family in ε with coefficients that are differential operators applied to F_0, F_1 , and it is $O(\varepsilon)$. Those coefficients have wavefront covectors with $\xi_\tau = \xi_z = 0$, since they are smooth in $\tau > 0$ and polynomial in z . That cone misses every nonzero graph conormal. Also $EU_\varepsilon = 0$. The squared system therefore has uniformly microlocally smooth source. Lemma 2.1 applies. One can straighten the graph explicitly by $\tau' = \tau + \varepsilon\delta$, $z' = z - \delta + \beta$. Thus

$$i_\varepsilon^* F_\varepsilon = i_\varepsilon^* F_0 + \varepsilon i_\varepsilon^* F_1 + o_{\mathcal{D}'}(\varepsilon).$$

Expanding $(2 - \varepsilon\delta)^{-1}$ and evaluating the polynomial z factors proves (6).

4 The facet Bessel family and its actual Taylor jets

Lemma 4.1. *Formula (8) is the actual inverse in a of $h_\lambda^{-1/2}$ on $s = Y_a \geq 0$. After a fixed exponential tilt in the transverse dual variables, S_s is a C^∞ right family in $\mathcal{S}'(\mathbb{R}^3)$. All its prescribed finite Taylor remainders hold in that space.*

Proof. The elementary transform

$$\int_0^\infty e^{-as} e^{-\mu s} I_0(\nu s) ds = ((a + \mu)^2 - \nu^2)^{-1/2}$$

follows from the angular integral for I_0 and initially large real a . It gives (8) for positive real transverse parameters and then by holomorphic continuation on their tube. The roots $-\mu \pm \nu$ in a have nonpositive real parts, since stability excludes a root with positive real part while the other three real parts are positive. Thus $\operatorname{Re} \mu \geq |\operatorname{Re} \nu|$. The angular integrand $e^{s(-\mu + \nu \cos \theta)}$ has modulus at most one for $s \geq 0$. Its j th s derivative is bounded by $(|\mu| + |\nu|)^j$.

The quadratic Q is Lorentzian with matrix

$$G = \begin{pmatrix} \lambda & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \quad \det G = 4(4 - \lambda) > 0, \quad (11)$$

and signature $(1, 2)$. On fixed future real parts, $|Q(x + i\xi)| \geq Q(x) > 0$. To verify this standard Lorentz bound, put $x = (t, 0)$ in a Lorentz rest frame. Writing $\xi = (u, w)$ gives

$$|Q(x + i\xi)|^2 - t^4 = (u^2 - |w|^2)^2 + 2t^2(u^2 + |w|^2) \geq 0.$$

The rational formulas for μ, ν^2 now give polynomial frequency bounds for all the preceding derivatives. Taylor's integral remainder, tested against Schwartz functions on the contour, proves right C^∞ regularity and finite-order remainders. Fourier inversion is continuous. The transform identity also proves that the full inverse is obtained by integrating the section S_s over $s \geq 0$. $\square$

Put $q(Y) = Y^T G^{-1} Y$ on the future dual Lorentz cone and $c_G = 1/(\pi \sqrt{\det G})$.

Lemma 4.2 (Lorentz normalization and coefficient wavefronts). *If $\mathcal{R}_\alpha = \mathcal{L}^{-1} Q^{-\alpha}$, then*

$$\mathcal{R}_{1/2} = c_G \delta(q), \quad \mathcal{R}_{n+1/2} = c_G \frac{n}{(2n)!} q_+^{n-1} \quad (n \geq 1),$$

with the future component understood. In particular,

$$A_j = \frac{1}{j!} \sum_{r+2t=j} \frac{(-1)^{r+t} (1/2)_{r+t}}{r!t!} C(\partial)^r H(\partial)^t \mathcal{R}_{r+t+1/2}. \quad (12)$$

Away from the vertex, $\operatorname{WF}(A_j)$ is contained in the conormal bundle of $q = 0$.

Proof. On the standard cone $q = t^2 - |x|^2$, $t > 0$, polar integration gives $\mathcal{L}[\delta(q)](a, 0, 0) = \pi/a$. For $n \geq 1$, radial integration first gives $\pi t^{2n}/n$, and its transform is $\pi(2n)!/(na^{2n+1})$. Lorentz invariance and a linear change of variables give the determinant factor in the displayed formulas. Expanding $a^{-1} Q^{-1/2} (1 + C/(aQ) + H/(a^2 Q))^{-1/2}$ at large a and using $\mathcal{L}^{-1} a^{-j-1} = s^j/j!$ yields (12). Lemma 4.1 identifies these as the actual finite Taylor coefficients. The displayed Riesz distributions are smooth off their null boundary and vertex, and constant-coefficient differentiation preserves the asserted wavefront containment. $\square$

4.1 An ordered fourth-order annihilator

The scalar Bessel equation gives the exact distribution identity

$$sQ(\partial)S_s'' + (Q(\partial) + sC(\partial))S_s' + (C(\partial)/2 + sH(\partial))S_s = 0.$$

Homogeneity gives $s\partial_s S_s = -(E_Y + 2)S_s$, $E_Y = Y \cdot \partial_Y$. Multiply the identity by s and eliminate its s derivatives. The result is

$$\mathcal{L}_s S_s = 0, \quad \mathcal{L}_s = Q(\partial)(E_Y + 2)^2 - sC(\partial)(E_Y + 3/2) + s^2 H(\partial). \quad (13)$$

The order of the factors displayed here is part of the formula. Its order-four symbol is

$$\ell_s(Y, \xi) = Q(\xi)(Y \cdot \xi)^2 - sC(\xi)(Y \cdot \xi) + s^2 H(\xi). \quad (14)$$

At $\lambda = L$, the moving-plane normal $n = (3, -1, -1)$ satisfies

$$Q(n) = 4/3, \quad C(n) = -8, \quad H(n) = 12, \quad T_s(u, v) \cdot n = 1 - 3s. \quad (15)$$

Consequently $\ell_s(T_s(u, v), n) = 4/3$ identically in s .

Proof of Theorem 1.3. On a fixed compact plane patch the identity just proved gives a common conic ellipticity neighborhood of all plane normals. Such patches stay away from $Y = 0$ for small s . Write $\mathcal{L}_s = \mathcal{L}_0 + s\mathcal{L}_1 + s^2\mathcal{L}_2$, $A_{-1} = 0$, and $W_s = s^{-N}(S_s - \sum_{j=0}^N s^j A_j)$. Then $W_s \rightarrow 0$ in distributions. Taylor expansion of the actual equation (13) gives precisely

$$\mathcal{L}_s W_s = -s(\mathcal{L}_1 A_N + \mathcal{L}_2 A_{N-1}) - s^2 \mathcal{L}_2 A_N. \quad (16)$$

The null conormals of the finite source set satisfy $Q(\nabla q) = 4q = 0$. They are separated from the plane normal because $Q(n) = 4/3 > 0$. Lemma 4.2 therefore makes the right side uniformly microlocally smooth on the common normal cone. Apply Lemma 2.1, straightening the plane by replacing Y_d with $Y_d - 3s$. This proves (9). Differentiating the finite polynomial terms gives the stated drift $3\partial_{Y_d} S_0$. $\square$

5 The explicit ellipse trace

At the endpoint put $z = (u, v)$,

$$m = (7/3, 3)^T, \quad A = \begin{pmatrix} 11/3 & 11/2 \\ 11/2 & 9 \end{pmatrix}, \quad \gamma(z) = 1 - (z - m)^T A^{-1} (z - m).$$

Then $\det A = 11/4$ and direct substitution in (11) gives $q(T_0 z) = 3\gamma(z)/4$. The matrix A is positive definite. The ellipse has $Y_b \geq 7/3 - \sqrt{11/3} > 0$, $Y_c \geq 0$, $Y_d \geq 0$, so it belongs to the future component. Lemma 4.2 gives

$$T_0^* S_0 = \frac{\sqrt{3}}{\pi\sqrt{11}} \delta(\gamma). \quad (17)$$

This pullback is defined by the noncharacteristic distributional restriction, without an additional surface-area factor. Changing $z = m + A^{1/2}x$ gives $\int \delta(\gamma) dz = \pi\sqrt{\det A}$, hence total mass $\sqrt{3}/2$. Thus for every compact smooth ψ equal to one near the entire ellipse,

$$\lim_{s \downarrow 0} \langle T_s^* S_s, \psi \rangle = \sqrt{3}/2.$$

The first jet is obtained from (9) with $A_0 = c_G \delta(q)$ and $A_1 = -c_G C(\partial) \mathbf{1}_{q>0}/4$. The two points $(Y_b, Y_c, Y_d) = (1/2, 0, 1/2), (1/2, 1/2, 0)$ are included in this same moving-plane trace.

6 Canonical central-band functionals

Let $X \sim \text{Beta}(1/2, 1/2)$ and let ϕ be uniform on $[0, 2\pi]$, independently. Put

$$M = 8/\sqrt{\lambda}, \quad D = MX \cos \phi, \quad \mu = \text{Law}(D).$$

Then μ is compactly supported, symmetric, and has no atom at zero;

$$m_k^D = \int D^{2k} d\mu = \frac{(4k)!}{k!^2 (2k)! \lambda^k}. \quad (18)$$

For every integer $n \geq 0$, let $R_n(k)$ be the explicitly defined rational function (29) in Appendix A. The proof there establishes that it is bounded at infinity, its poles lie among negative half-integers, and

$$R_n(-1/4) = b_n := 2^n \frac{(1/4)_n}{(1/2)_n}. \quad (19)$$

Write its finite partial-fraction expansion as

$$R_n(k) = c_{n,0} + \sum_{a,j} \frac{c_{n,a,j}}{(k+a)^j}, \quad a \in \{1/2, 1, 3/2, \dots\}, \quad j \geq 1. \quad (20)$$

Every sum below is finite for fixed n . Define

$$K_{a,j}(r) = \frac{2^j}{(j-1)!} r^{2a-1} (-\log r)^{j-1}, \quad 0 < r < 1, \quad (21)$$

$$\sigma_n = c_{n,0}\mu + \sum_{a,j} c_{n,a,j} \int_0^1 (\text{dil}_r \mu) K_{a,j}(r) dr, \quad (22)$$

$$r_n(\varepsilon) = \frac{\sigma_n((-\varepsilon, \varepsilon))}{\mu((-\varepsilon, \varepsilon))}, \quad 0 < \varepsilon < M. \quad (23)$$

Here dil_r is pushforward by multiplication by r . These are canonical finite signed measures and their band functionals; $\sigma_0 = \mu$, so $r_0 = 1$. The interpretation under a bounded positive realization is given in Appendix B.

Theorem 6.1 (Quantitative central convergence). *For every fixed n and $0 < \lambda \leq L$, $r_n(\varepsilon) \rightarrow b_n$. More explicitly, let*

$$E_{a,j}(x) = \int_0^x K_{a,j}(r) r^{-1/2} dr + \frac{x}{\sqrt{2}} \int_x^1 K_{a,j}(r) r^{-3/2} dr. \quad (24)$$

Then $E_{a,j}$ is nondecreasing, tends to zero at zero, and

$$|r_n(\varepsilon) - b_n| \leq \sum_{a,j} |c_{n,a,j}| E_{a,j}(\varepsilon/M). \quad (25)$$

Proof. In the dimensionless variable $x = |D|/M$, the positive-side density is

$$m(x) = \frac{1}{\pi^2} \int_x^1 \frac{dt}{\sqrt{t(1-t)} \sqrt{t^2 - x^2}} = x^{-1/2} F(x),$$

where substitutions $u = x/t$, then $u = x + (1-x)v$, give

$$F(x) = \frac{1}{\pi^2} \int_0^1 \frac{dv}{\sqrt{v(1-v)} \sqrt{1+x+(1-x)v}}. \quad (26)$$

The beta weight is integrable and gives

$$F(x) \geq \frac{1}{\pi\sqrt{2}}, \quad |F'(x)| \leq \frac{1}{2\pi} \quad (0 \leq x \leq 1).$$

For $r \geq x$, the density ratio of rD to D is $r^{-1/2}F(x/r)/F(x)$; its difference from $r^{-1/2}$ has magnitude at most $(x/\sqrt{2})r^{-3/2}$. For $0 < r < x$ the dilation density vanishes. Integrating against (21) gives exactly the error $E_{a,j}(x)$. The constant part in (20) cancels. Also

$$\int_0^1 r^{-1/2} K_{a,j}(r) dr = (a - 1/4)^{-j}.$$

Thus the limiting density ratio is $R_n(-1/4)$.

The function $K_{a,j}(r)r^{-1/2}$ is integrable because $a \geq 1/2$. Dominated convergence, using $x/r \leq 1$ on $r \geq x$, proves $E_{a,j}(x) \rightarrow 0$. Differentiating its two integral terms gives

$$E'_{a,j}(x) = (1 - 1/\sqrt{2})K_{a,j}(x)x^{-1/2} + \frac{1}{\sqrt{2}} \int_x^1 K_{a,j}(r)r^{-3/2} dr \geq 0.$$

Integrate the pointwise density-ratio estimate over the band with respect to the positive measure μ . Monotonicity permits its replacement by the endpoint bound at $x = \varepsilon/M$, proving (25). $\square$

7 Strict positivity of every fixed finite band test

For a fixed integer $N \geq 1$, define

$$\begin{aligned} H_N(\varepsilon) &= [r_{i+j}(\varepsilon)]_{i,j=0}^{N-1}, & L_N(\varepsilon) &= [r_{i+j+1}(\varepsilon)]_{i,j=0}^{N-1}, \\ U_N(\varepsilon) &= [2r_{i+j}(\varepsilon) - r_{i+j+1}(\varepsilon)]_{i,j=0}^{N-1}. \end{aligned}$$

The three limiting matrices H_N^0, L_N^0, U_N^0 are obtained by replacing r_n by b_n . Set

$$\gamma_N = \min_{A \in \{H_N^0, L_N^0, U_N^0\}} \frac{\det A}{(\operatorname{tr} A)^{N-1}}. \quad (27)$$

Theorem 7.1 (Finite central-band positivity). *For every fixed N , γ_N is a positive rational number. If*

$$\max_{0 \leq n \leq 2N-1} |r_n(\varepsilon) - b_n| < \gamma_N/(6N),$$

then $H_N(\varepsilon), L_N(\varepsilon), U_N(\varepsilon) \succeq (\gamma_N/2)I$. In particular this holds for all sufficiently small $\varepsilon > 0$.

Proof. The numbers b_n are the moments of $Z = 2B$ with $B \sim \text{Beta}(1/4, 1/4)$. Its density is positive on $(0, 2)$. For any nonzero polynomial P of degree less than N , all three integrals

$$\mathbb{E}P(Z)^2, \quad \mathbb{E}[ZP(Z)^2], \quad \mathbb{E}[(2-Z)P(Z)^2]$$

are strictly positive. Thus the limiting matrices are positive definite. Their entries are rational. If A is positive definite, its least eigenvalue is at least $\det A/(\operatorname{tr} A)^{N-1}$, since every other eigenvalue is at most the trace. An entrywise error δ gives operator norm error at most $N\delta$ for H_N, L_N and $3N\delta$ for U_N . The displayed hypothesis therefore proves the common floor. Finally Theorem 6.1 supplies the hypothesis for every fixed N . $\square$

For $N = 2$, the exact matrices are

$$H_2^0 = \begin{pmatrix} 1 & 1 \\ 1 & 5/3 \end{pmatrix}, \quad L_2^0 = \begin{pmatrix} 1 & 5/3 \\ 5/3 & 3 \end{pmatrix}, \quad U_2^0 = \begin{pmatrix} 1 & 1/3 \\ 1/3 & 1/3 \end{pmatrix}.$$

Their determinants are respectively $2/3, 2/9, 2/9$, and $\gamma_2 = 1/18$. Thus a moment error below $1/216$ guarantees the common floor $1/36$.

The construction applies at every interior parameter as well as at L . Combined with the interior inverse-halfpower obstruction in [7], it proves a precise finite-test limitation: any prescribed finite list of these central-band Hankel and localizer tests passes in sufficiently narrow bands even for those nonpositive full inverses. The quantifiers are fixed N followed by sufficiently small band. A finite-degree polynomial-square certificate at $\lambda = 1$ is given in [8]. The upper localizer uses the interval $[0, 2]$ of the limiting beta model; for an actual finite-band positive realization its role as a support test has the additional hypothesis $S \leq 2$.

A The complete rational multiplier calculation

This appendix defines and derives the R_n used in (20). Use $P = ab$, $s = a + b$, $U = c + d$, $V = c - d$. Expanding the original quartic,

$$h_\lambda = \lambda P^2 + 4sPU + (4P + s^2)U^2 + sU^3 - s^2V^2 - sUV^2. \quad (28)$$

Substitute $U = -2q$, $V = -2t$. The constant part is $H_0 = \lambda P^2 - 4s^2t^2$ and the q -dependent part is

$$-8s(P - t^2)q + 4(4P + s^2)q^2 - 8sq^3.$$

The normalized double inverse in a, b is evaluated at $(1, 1)$ and multiplied by $\sqrt{\lambda}$; the constant term $(\sqrt{\lambda}ab)^{-1}$ then has mass one. Coefficients of $q^n t^{2k}$ are multiplied by $n!(2k)!$ to specify the associated mixed-moment functional. Its difference moments are (18).

For each triple of nonnegative integers n_1, n_2, n_3 with $n_1 + 2n_2 + 3n_3 = n$, put $j = n_1 + n_2 + n_3$. For $0 \leq i \leq n_1$ and $0 \leq v \leq n_2$, put $\ell = n_1 + n_2 + 2n_3 - i + v$. Define $R_n(k)$ to be the sum of the following terms:

$$\begin{aligned} & \frac{n!(-1)^j(-8)^{n_1+n_3}4^{n_2}}{n_1!n_2!n_3!} \binom{n_1}{i} (-1)^i \binom{n_2}{v} 4^{n_2-v} \\ & \times \frac{(k + 1/2)_{j-i} (k - i + 1)_i \lambda^{i-j}}{4^i} \frac{(4k + 1)_{2\ell}}{(2k + 1)_\ell^2 (2k + 1)_n}. \end{aligned} \quad (29)$$

Every rising-factorial index is nonnegative. For integer $k \geq n$, this rational function is the exact ratio of the mixed $S^n D^{2k}$ moment functional to (18).

To verify the calculation, use the multinomial expansion of the inverse halfpower, choose i copies of $-t^2$ from $(P - t^2)^{n_1}$ and v copies of s^2 from $(4P + s^2)^{n_2}$. The remaining t^2 count is $k - i$. The powers for the double inverse are

$$N = 2k + n_1 + n_3 + 2v - 2i, \quad m = 1 + 2k + \ell.$$

For these indices, expanding $(a + b)^N$ and using Vandermonde's identity in the elementary inverse of $(ab)^{-m}$ gives

$$I(N, m) = \frac{(2m - 2)!}{(m - 1)!^2 (2m - N - 2)!}, \quad 2m - N - 2 = 2k + n \geq 0.$$

Dividing by the $n = 0$ expression gives $(4k+1)_{2\ell}/((2k+1)_\ell^2(2k+1)_n)$. The remaining binomial halfpower factor divided by $(1/2)_k$ gives $(k+1/2)_{j-i}(k-i+1)_i$, with the powers of $\lambda, 4$ displayed in (29). This proves the ratio for $k \geq n$.

Each summand has numerator degree $j+2\ell$ and denominator degree $n+2\ell$, with $j \leq n$. Thus R_n is bounded at infinity and its possible poles are negative half-integers. At $k = -1/4$, every term with $\ell > 0$ vanishes. The condition $\ell = 0$ forces $n_2 = n_3 = v = 0$, $i = n_1 = n$. The sole remaining term is

$$(-2)^n \frac{(k-n+1)_n}{(2k+1)_n} \Big|_{k=-1/4} = 2^n \frac{(1/4)_n}{(1/2)_n},$$

which proves (19) for every n .

For completeness, (21) has moments $\int_0^1 r^{2k} K_{a,j}(r) dr = (k+a)^{-j}$. Thus σ_n has moments $m_k^D R_n(k)$ for all integer $k \geq 0$. This identity is a property of the defined signed measure; its matching to a bounded positive realization uses only the already derived range $k \geq n$, as the next appendix shows.

B An all-order conditional beta law

Theorem B.1. *Suppose a positive probability law of (S, D) has S bounded and nonnegative, has the difference marginal μ above, and realizes*

$$\mathbb{E}[S^n D^{2k}] = m_k^D R_n(k) \quad (n \geq 0, k \geq n).$$

Then the law of S conditioned on $|D| < \varepsilon$ converges weakly to $2\text{Beta}(1/4, 1/4)$ as $\varepsilon \downarrow 0$.

Proof. Reflecting D and averaging makes the representing law symmetric in D while preserving every band-conditioned law of S . Let μ_n be its S^n -weighted D marginal. It is finite and compactly supported. For $k \geq n$ its even moments agree with those of σ_n , and odd moments vanish for both. Thus $D^{2n}(\mu_n - \sigma_n)$ has every moment zero. Polynomial approximation on their common compact support proves that this measure is zero. Their difference is supported at zero. Both have no atom there: for μ_n this follows from bounded S and $\mu(\{0\}) = 0$, and for σ_n from its finite dilation construction. Hence $\mu_n = \sigma_n$.

Theorem 6.1 now gives every band-conditioned moment b_n . Boundedness of S makes these probability laws supported in one fixed compact interval. Every subsequential weak limit therefore has the moments of $2\text{Beta}(1/4, 1/4)$; polynomial approximation identifies that measure uniquely. This proves the convergence. Its first five moments are $1, 5/3, 3, 39/7, 221/21$. $\square$

C Two exact positive-construction exclusions

At $\lambda = L$, use the boundary direction $(3, 3, -1, -1)$ and coordinates

$$a = 3t + u, \quad b = 3t + v, \quad c = -t + w, \quad d = -t.$$

The corresponding dual coordinates are $\eta = 3Y_a + 3Y_b - Y_c - Y_d$, $y = (Y_a, Y_b, Y_c)$; the linear change has absolute determinant one. The classification [5] places that direction in the closure of the hyperbolicity cone. Indeed, with $e = (1, 1, 0, 0)$ one has

$$h_L((3, 3, -1, -1) + te) = \frac{4}{27}(6 + 2t)(2t)^2(9/2 + 2t),$$

whose roots are nonpositive. The transverse point $(u, v, w, 0)$ is in its interior for $u, v, w > 0$, by continuity from the positive orthant and strict positivity of h there. Addition of the boundary ray stays inside the cone. Direct expansion is

$$\begin{aligned} h_L &= t^2 \widehat{Q} + t \widehat{C} + \widehat{H}, \\ \widehat{Q} &= \frac{4}{3}(u^2 + 10uv + 18uw + v^2 + 18vw + 9w^2), \\ \widehat{C} &= \frac{8}{9}(7u^2v + 9u^2w + 7uv^2 + 27uvw + 9uw^2 + 9v^2w + 9vw^2), \\ \widehat{H} &= \frac{4}{27}uv(16uv + 27uw + 27vw + 27w^2). \end{aligned} \tag{30}$$

The same elementary Bessel inversion as in Lemma 4.1 gives a continuous right distribution family $\widehat{R}_\eta$ with Laplace transform

$$\widehat{F}_\eta = \widehat{Q}^{-1/2} \widehat{G}_\eta, \quad \widehat{G}_\eta = e^{-\eta \widehat{\mu}} I_0(\eta \widehat{\nu}), \quad \widehat{\mu} = \widehat{C}/(2\widehat{Q}), \quad \widehat{\nu}^2 = \widehat{\mu}^2 - \widehat{H}/\widehat{Q}. \tag{31}$$

The actual inverse has $y_i \geq 0$ support, since its full hyperbolic inverse is supported in the dual cone [1, 3] and the original positive orthant lies in the primal cone. Continuity of the section family preserves this support for every $\eta \geq 0$. The central measure $\widehat{R}_0$ is the positive Lorentz inverse of $\widehat{Q}^{-1/2}$, with the origin in its support.

Proposition C.1 (Independent shifts). *There is no positive Borel measure σ on $\mathbb{R}^3$ whose positive convolution $\widehat{R}_0 * \sigma$ equals $\widehat{R}_1$.*

Proof. Such an equality makes $\widehat{R}_1$ positive. Since the origin belongs to $\text{supp } \widehat{R}_0$, positivity of the convolution and its orthant support force $\text{supp } \sigma \subset [0, \infty)^3$. Taking Laplace transforms gives $\mathcal{L}\sigma = \widehat{G}_1$. At the origin along positive rays this tends to one, so monotone convergence gives total mass one. On each positive coordinate axis $\widehat{C} = \widehat{H} = 0$, so the axial transforms of σ are identically one. Each coordinate is consequently zero almost surely, and $\sigma = \delta_0$. But exact evaluation at $(1, 1, 1)$ gives

$$\widehat{G}_1(1, 1, 1) = e^{-77/171} I_0(20/171) < 1,$$

since $I_0(x) \leq e^{|x|}$. This contradiction proves the proposition. $\square$

Proposition C.2 (Coordinate-monotone transports). *There is no positive Borel coupling of $\widehat{R}_0$ and $\widehat{R}_1$ for which each coordinate has a fixed direction of motion almost everywhere. The chosen direction may differ between coordinates.*

Proof. A coupling makes $\widehat{R}_1$ positive. Let the two unused Laplace coordinates decrease to zero. Monotone convergence and (31) show that the three coordinate marginals of both measures are

$$(\pi_1)_* \widehat{R}_i = (\sqrt{3}/2) dy, \quad (\pi_2)_* \widehat{R}_i = (\sqrt{3}/2) dy, \quad (\pi_3)_* \widehat{R}_i = (1/(2\sqrt{3})) dy \quad (y \geq 0).$$

Indeed the three axial values of $\widehat{Q}$ are respectively $4u^2/3, 4v^2/3, 12w^2$. In particular each finite initial interval has finite marginal mass. If $Y_j \geq X_j$ under the coupling, then $\{Y_j \leq t\} \subset \{X_j \leq t\}$ and these events have the same finite mass; hence the crossing set $\{X_j \leq t < Y_j\}$ has zero mass. The rational cuts imply $X_j = Y_j$ almost everywhere. Reverse the roles for a coordinate with the opposite direction. All coordinates agree, so the two full marginals coincide. The strict transform difference at $(1, 1, 1)$ in Proposition C.1 contradicts this. $\square$

D Portable verification

The accompanying `verify.py` checks the original quartic expansions, ordered annihilator symbol, graph conormal identities, ellipse normalization, the rational central values through a specified finite order, and exact limiting Hankel/localizer arithmetic. It also includes altered coefficient controls. All acceptance tests raise explicit exceptions, so the same checks run under ordinary Python and `python -O`. The analytic finite-to-all-order arguments are the proofs above; the finite calculations provide independently reproducible algebra and normalization checks. The reader instructions specify dependencies, commands and the complete public file inventory.

AI assistance. OpenAI’s Codex assisted with exploration, proof checking, code, literature comparison and manuscript preparation. The author takes responsibility for the mathematics and the accompanying files.

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