Dimension-dependent holomorphic certificate degrees on zero-sum hyperplanes

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Luminary Oddities

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Abstract

On the complex zero-sum hyperplane in $\mathbb{C}^n$, consider the symmetric quartics $A = n\sum_i |v_i|^4 - |\sum_i v_i^2|^2$ and $Q = \|v\|^4 - |\sum_i v_i^2|^2$. For every odd $n \geq 3$ and every $\alpha, \beta > 0$, we prove that a holomorphic squared-norm representation of $\|v\|^{2m}(\alpha A + \beta Q)$ requires $m \geq n-2$. For each fixed odd n, equality holds for all sufficiently small positive β/α, whereas at $\beta = 0$ the least exponent is zero. The lower bound uses an explicit Fischer annihilator and a positive Hamming-distance correlation identity on the odd-dimensional sign cube. The upper bound uses the classical quadratic complete intersection associated with the zero-sum hyperplane. For $n = 5$, we determine the entire first-degree window: the least exponent is three exactly when $0 < \beta/\alpha \leq \rho_*$, where $305\rho_*^3 + 786\rho_*^2 + 351\rho_* - 162 = 0$. An explicit four-by-four pencil and harmonic decomposition give this sharp endpoint, including its four-dimensional kernel. We also derive the quartic as a coefficient in a centered permanent deficit, which motivates the family without implying the full permanent inequality.

1 The family and its certificate degree

The input quartic degree stays fixed while the number of coordinates grows. The resource that increases is the exponent of the squared norm multiplying that quartic: every positive perturbation forces the lower bound $n-2$, however small its coefficient. The exact five-coordinate window refines this dimension barrier by identifying the positive-semidefinite endpoint, including its kernel. The theorem below separates the statements valid for every positive ratio from the initial interval on which the lower bound is attained.

Let $n \geq 3$ and give

$$H_n = \{v \in \mathbb{C}^n : \textstyle\sum_i v_i = 0\}$$

the Hermitian metric inherited from $\mathbb{C}^n$. Write

$$N(v) = \sum_i |v_i|^2, \qquad p_k(v) = \sum_i v_i^k,$$

$$A(v) = n\sum_i |v_i|^4 - |p_2(v)|^2 = \sum_{i<j} |v_i^2 - v_j^2|^2, \tag{1}$$

$$Q(v) = N(v)^2 - |p_2(v)|^2 = \sum_{i<j} |v_i\overline{v}_j - v_j\overline{v}_i|^2. \tag{2}$$

Thus A is already a squared norm of holomorphic quadrics; Q is nonnegative, but its displayed squares are not holomorphic. For $\rho \geq 0$ put $q_\rho = A + \rho Q$ and define

$$\mu_n(\rho) = \min\{m \in \mathbb{Z}_{\geq 0} : N^m q_\rho = \textstyle\sum_\nu |h_\nu|^2 \text{ on } H_n,\ h_\nu \text{ holomorphic homogeneous of degree } m+2\}. \tag{3}$$

If the set is empty its minimum is $+\infty$. Multiplication by a positive scalar does not change the minimum, so this also treats $\alpha A + \beta Q$ with $\alpha > 0$ and $\rho = \beta/\alpha$.

When n is odd, $A(v) > 0$ for $v \neq 0$ in H_n: equality would force $v_i = \pm a$ for all i, incompatible with an odd number of terms summing to zero unless $a = 0$. The classical stabilization theorem of Quillen [1] and Catlin–D'Angelo [2] therefore guarantees $\mu_n(\rho) < \infty$ for each $\rho \geq 0$. The question here is the *least* norm power, not eventual existence.

Theorem 1.1 (Odd-coordinate degree barrier and sharp small-ratio degree)**.** *For every odd $n \geq 3$ and every $\rho > 0$,*

$$\mu_n(\rho) \geq n - 2. \tag{4}$$

For each fixed odd n, there exists $\varepsilon_n > 0$ such that

$$\mu_n(\rho) = n - 2 \qquad (0 < \rho \leq \varepsilon_n). \tag{5}$$

At $\rho = 0$ one has $\mu_n(0) = 0$. More precisely, for each odd n there is a finite threshold $\rho_n^* > 0$ such that

$$\mu_n(\rho) = n - 2 \quad \Longleftrightarrow \quad 0 < \rho \leq \rho_n^*. \tag{6}$$

The general threshold is characterized spectrally below; an explicit value is obtained here for $n = 5$.

Theorem 1.2 (Sharp five-coordinate window)**.** *Let ρ_* be the unique positive root of*

$$305\rho^3 + 786\rho^2 + 351\rho - 162 = 0. \tag{7}$$

Then

$$\mu_5(\rho) = 3 \quad \Longleftrightarrow \quad 0 < \rho \leq \rho_*. \tag{8}$$

Moreover $0.274637 < \rho_* < 0.274638$. The Hermitian coefficient matrix of $N^3 q_\rho$ is positive definite for $0 \leq \rho < \rho_*$, is positive semidefinite with kernel dimension four at ρ_*, and is indefinite for $\rho > \rho_*$. At $\rho = 0$, its positive definiteness does not change the smaller minimum $\mu_5(0) = 0$.

The dimension-growing obstruction in Theorem 1.1 holds at fixed bidegree $(2, 2)$ and is independent of the positive perturbation size. It is the additional conclusion beyond eventual stabilization. The five-coordinate result identifies the full parameter window for the first admissible degree; the conclusion concerns that window. Its endpoint kernel is a coefficient-space kernel: q_{ρ_*} remains strictly positive at every nonzero point of H_5.

1.1 Relation to previous work

The stabilization theorem and its quantitative versions are classical; see [1, 2, 3]. The $n = 3$ member of our family is exactly the two-variable quartic example in Catlin–D'Angelo [2, Example 2] after a unitary change of coordinates. We give its parameter specialization in Section 7 as a comparison, not as a new family. Halfpap–Lebl [4] prove signature constraints and strict hierarchy examples, including examples that persist under arbitrarily small perturbations. Their high-depth construction increases the input degree. The present conclusion concerns a fixed quartic family with increasing dimension and a uniform-in-$\rho > 0$ lower bound. The general phenomenon of high certificate degrees and perturbative hierarchy jumps has the cited predecessors.

The quadratic ideal used for the upper bound is also classical: it is the $r = 2$ type-A ideal in Chmutova–Etingof [5]. Its Hilbert series supplies the full-span conclusion needed for the small-ratio plateau. It does not supply the negative Fischer energy needed for the lower bound; that sign is established in Sections 3–4. The real invariant-quartic geometry on the same hyperplane is studied

in Debus–Goel–Kuhlmann–Riener [6]; the present obstruction concerns holomorphic squared norms rather than real sums of squares.

The family was suggested by permanent-deficit calculations related to the occupation and cofactor framework of [7]. Section 8 derives the relevant coefficient here and states the boundary of that connection.

2 Coefficient matrices and Fischer forms

Choose real orthonormal coordinates $z = (z_1, \dots, z_{n-1})$ on H_n. The Fischer Hermitian product on holomorphic polynomials is defined by

$$\langle z^a, z^b \rangle_F = a!\, \mathbf{1}_{a=b}.$$

For homogeneous degree ℓ, use the orthonormal basis $e_a = z^a/\sqrt{a!}$. A real-valued Hermitian polynomial of bidegree (ℓ, ℓ) has a unique Hermitian coefficient matrix C in this basis. It is a holomorphic squared norm exactly when $C \succeq 0$. We identify C with an operator on the Fischer space $\mathcal{P}_\ell$. The matrix for N^ℓ is $\ell! I$.

Multiplication by a holomorphic polynomial h has Fischer adjoint $\overline{h}(\partial)$, where the bar conjugates its coefficients. In particular, with $p = p_2 = \sum_{a=1}^{n-1} z_a^2$, the adjoint of multiplication by p is $\Delta = \sum_a \partial_{z_a}^2$. Therefore the coefficient operator of $N^{\ell-2}|p|^2$ is $(\ell-2)!p\Delta$, and the operator $C_{Q,\ell}$ of $N^{\ell-2}Q$ satisfies

$$\frac{\langle f, C_{Q,\ell} f \rangle_F}{\ell!\, \langle f, f \rangle_F} = 1 - \frac{\|\Delta f\|_F^2}{\ell(\ell-1)\|f\|_F^2}. \tag{9}$$

Here and below either consistent convention for the conjugate-linear argument of the Hermitian product gives the same quadratic forms.

Let $P = I - \mathbf{1}\mathbf{1}^T/n$ be the real orthogonal projection onto $H_n \cap \mathbb{R}^n$, and write ∂_i for differentiation in the tangent direction Pe_i. A polynomial on H_n may equivalently be extended to $\mathbb{C}^n$ by composing with P. This extension is translation invariant in the $\mathbf{1}$ direction; on it the ambient derivatives agree with these tangent derivatives and the ambient Laplacian is Δ.

Lemma 2.1 (Ideal kernel)**.** *Let I be the ideal of $\mathbb{C}[H_n]$ generated by $v_i^2 - v_j^2$. If $f \in \mathcal{P}_\ell$ is annihilated by every $\partial_i^2 - \partial_j^2$, then f is in the kernel of the coefficient operator of $N^{\ell-2}A$.*

Proof. Expanding $N^{\ell-2}$ gives a positive weighted sum of all monomial squares of that degree. By (1), the resulting holomorphic factors span I_ℓ. The Fischer adjoint identity makes f orthogonal to every one of these factors, and hence to the range of their coefficient matrix. □

3 An explicit annihilator

Fix an odd $n \geq 3$, and put $d = n - 1$. For a sign vector $\epsilon \in \{1, -1\}^n$, set

$$c(\epsilon) = \frac{\prod_i \epsilon_i}{\sum_i \epsilon_i}, \qquad G(x) = \sum_{\epsilon \in \{1,-1\}^n} c(\epsilon)(\epsilon \cdot x)^d. \tag{10}$$

The denominators are nonzero because n is odd. In the next proof, ∂_i temporarily denotes the ambient coordinate derivative $\partial/\partial x_i$. After translation invariance has been established, these derivatives agree with the tangent derivatives used elsewhere in the paper.

Lemma 3.1. *The polynomial G is nonzero, translation invariant in the* $\mathbf{1}$ *direction, and satisfies* $(\partial_i^2 - \partial_j^2)G = 0$ *for all* i, j.

Proof. Equality of the second derivatives follows from $\epsilon_i^2 = 1$. Furthermore

$$\sum_i \partial_i G = d \sum_\epsilon \Big(\prod_i \epsilon_i\Big)(\epsilon \cdot x)^{d-1} = 0.$$

Indeed a monomial of degree $d - 1 = n - 2$ cannot contain every one of the n sign coordinates to an odd power. Summing over signs therefore kills every term. This proves translation invariance. The coefficient of $x_1 \cdots x_{n-1}$ is

$$d! \sum_\epsilon \frac{\epsilon_n}{\sum_i \epsilon_i} = \frac{d!\, 2^n}{n} \neq 0.$$

For the equality, the n coordinate sums are equal by permutation symmetry and add to 2^n. Thus $G \neq 0$, also when regarded as a polynomial on H_n. □

For even $k \geq 0$, define

$$S_k = \sum_{\epsilon,\eta} c(\epsilon)c(\eta)(\epsilon \cdot \eta)^k. \tag{11}$$

The Fischer identity $\langle (\epsilon \cdot x)^d, (\eta \cdot x)^d \rangle_F = d!(\epsilon \cdot \eta)^d$ gives

$$\|G\|_F^2 = d! S_d, \qquad \|\Delta G\|_F^2 = [d(d-1)n]^2 (d-2)! S_{d-2}. \tag{12}$$

These ambient calculations give the intrinsic norms because G is translation invariant. In particular $S_d > 0$. Equations (9) and (12) show that its Q Rayleigh quotient relative to N^d is

$$1 - \frac{n^2 S_{d-2}}{S_d}. \tag{13}$$

We next prove that this is negative in every odd dimension.

4 A positive sign-cube correlation

For $0 \leq k \leq n$ write

$$W_k = \sum_{\mathrm{Ham}(\epsilon,\eta)=k} c(\epsilon)c(\eta). \tag{14}$$

Individual summands can have either sign.

Proposition 4.1. *For odd n, every W_k is strictly positive.*

Proof. Use the normalized Walsh coefficients

$$a_S = 2^{-n} \sum_\epsilon c(\epsilon) \prod_{i \in S} \epsilon_i.$$

They depend only on $j = |S|$; denote the value by a_j. Since $c(-\epsilon) = c(\epsilon)$, odd a_j vanish. Taking Walsh coefficients in the identity $(\sum_i \epsilon_i)c(\epsilon) = \prod_i \epsilon_i$ gives

$$j a_{j-1} + (n-j) a_{j+1} = 0 \quad (j < n \text{ odd}), \qquad n a_{n-1} = 1. \tag{15}$$

Consequently $a_0 \neq 0$ and

$$\frac{a_{j+2}}{a_j} = -\frac{j+1}{n-j-1} \qquad (j < n-1 \text{ even}). \tag{16}$$

Let T have the symmetric probability density $b_n(1+t^2)^{-(n+2)/2}$ on $\mathbb{R}$, with normalizing constant $b_n > 0$. Its absolute moments through degree n are finite. Integration by parts, applied to $t^{j+1}(1+t^2)^{-n/2}$, gives

$$\frac{\mathbb{E}T^{j+2}}{\mathbb{E}T^j} = \frac{j+1}{n-j-1} \qquad (j < n-1 \text{ even}). \tag{17}$$

Thus $a_{2r} = a_0(-1)^r \mathbb{E}T^{2r}$. For an independent copy U of T, this implies

$$a_S^2 = a_0^2 \mathbb{E}(TU)^{|S|}. \tag{18}$$

For odd $|S|$ both sides are zero.

Fix a sign vector σ with k negative entries. Walsh orthogonality and (18) yield

$$\begin{aligned} \sum_{\epsilon} c(\epsilon)c(\epsilon\sigma) &= 2^n \sum_S a_S^2 \prod_{i \in S} \sigma_i \\ &= 2^n a_0^2 \mathbb{E}\big[(1+TU)^{n-k}(1-TU)^k\big]. \end{aligned} \tag{19}$$

Multiplying by $\binom{n}{k}$ gives W_k.

To determine the sign of this expectation, put $R = |TU|$ and let h be its density on $(0, \infty)$. The density g of $|T|$ satisfies

$$\frac{g(1/t)}{t^2} = t^n g(t).$$

Inverting both independent factors gives the corresponding product identity

$$\frac{h(1/r)}{r^2} = r^n h(r). \tag{20}$$

For example, substitute $t = 1/u$ in $h(r) = \int_0^\infty g(t)g(r/t)\,\mathrm{d}t/t$ to verify the identity directly. The sign of TU is uniform and independent of R. Set

$$F_k(r) = (1+r)^{n-k}(1-r)^k + (1-r)^{n-k}(1+r)^k.$$

The expectation in (19) equals $\frac{1}{2}\int_0^\infty F_k(r)h(r)\,\mathrm{d}r$. Splitting at one and using (20), the integral without the prefactor $1/2$ is

$$\int_1^\infty \big[F_k(r) + r^n F_k(1/r)\big] h(r)\,\mathrm{d}r. \tag{21}$$

For $r > 1$, write $a = (1+r)^{n-k}(r-1)^k$ and $b = (r-1)^{n-k}(1+r)^k$. Since n is odd,

$$F_k(r) = (-1)^k(a-b), \qquad r^n F_k(1/r) = a+b.$$

Their sum is $2a$ for even k and $2b$ for odd k, strictly positive in both cases. All integrals converge: the integrands are polynomials of degree at most n in TU, and $\mathbb{E}|TU|^n < \infty$. Also $h(r) > 0$ for $r > 0$. Thus (21), the expectation in (19), and W_k are all positive. $\square$

Proof of the lower bound in Theorem 1.1. By (11) and (14),

$$n^2 S_{d-2} - S_d = \sum_{k=0}^{n} W_k [n^2 - (n-2k)^2](n-2k)^{d-2} > 0. \tag{22}$$

Every term is nonnegative, and every interior k gives a strictly positive term: $d-2$ is even, $n-2k \neq 0$, and $n^2 - (n-2k)^2 = 4k(n-k) > 0$. Thus (13) is negative. Lemmas 2.1 and 3.1 put G in the kernel of the A operator at degree d. The coefficient operator of $N^{n-3}(A + \rho Q)$ is therefore indefinite for every $\rho > 0$. In particular it is not a holomorphic squared norm. If a smaller nonnegative power worked, multiplication by the remaining power of N, itself a holomorphic squared norm, would give a certificate at $n-3$. This is impossible, so $\mu_n(\rho) \geq n-2$. □

5 The first degree for small positive ratios

We use a classical complete-intersection fact, spelling out its application. The ideal I in Lemma 2.1 is generated by the $n-1$ quadrics

$$v_i^2 - v_n^2 \qquad (1 \leq i < n).$$

For odd n they have no nonzero common affine zero on H_n, by the same balanced-sign argument used in Section 1. They form a homogeneous system of parameters in the polynomial ring on H_n. The standard complete-intersection criterion for a polynomial ring therefore makes them a regular sequence. Equivalently, this is the $r = 2$ specialization of the type-A ideal in [5, Theorem 2.3, Proposition 3.1, Theorem 3.2 and Corollary 3.3]. The generators there specialize to $v_i^2 - p_2/n$, which span the same quadratic ideal. Its Hilbert series is

$$\mathrm{Hilb}(\mathbb{C}[H_n]/I, t) = \frac{(1-t^2)^{n-1}}{(1-t)^{n-1}} = (1+t)^{n-1}. \tag{23}$$

In particular $I_n = \mathcal{P}_n$.

Expanding $N^{n-2}A$ as in Lemma 2.1 now gives a positive weighted sum of squares whose holomorphic factors span all of $\mathcal{P}_n$. Its coefficient operator M_A is consequently positive definite. Let M_Q be the operator of $N^{n-2}Q$ in the same orthonormal Fischer basis. For example, the choice

$$\varepsilon_n = \frac{\lambda_{\min}(M_A)}{2(1 + \|M_Q\|_{\mathrm{op}})} > 0 \tag{24}$$

makes $M_A + \rho M_Q$ positive definite for $0 \leq \rho \leq \varepsilon_n$. Combined with the lower bound, this proves (5). Finally (1) gives $\mu_n(0) = 0$, completing the small-ratio and zero-parameter assertions. The Hilbert-series input is classical; the independent negative Q energy in the ideal kernel is what supplies the lower bound.

To obtain the finite first-degree window in (6), let h_1 be any nonzero real linear polynomial on H_n, and set $f = p_2^{(n-1)/2} h_1 \in \mathcal{P}_n$. The elementary identity for its Laplacian gives

$$p_2 \Delta f = 2(n-1)^2 f.$$

Indeed in $d = n-1$ orthonormal coordinates, $\Delta(p^j h_1) = 2j(d+2j)p^{j-1}h_1$, with $j = (n-1)/2$. By (9), the M_Q Rayleigh quotient on f relative to N^n is $1 - 2(n-1)/n = -(n-2)/n < 0$. Thus the Hermitian matrix $M_A^{-1/2} M_Q M_A^{-1/2}$ has a negative least eigenvalue λ_n. The finite positive number

$$\rho_n^* = -\lambda_n^{-1} \tag{25}$$

is exactly the last nonnegative ρ for which $M_A + \rho M_Q \succeq 0$. This follows by diagonalizing the congruent matrix $I + \rho M_A^{-1/2} M_Q M_A^{-1/2}$. Combining with the lower bound proves (6). This is an implicit spectral characterization, not an explicit all-dimensional threshold formula.

6 The sharp four-by-four pencil for five coordinates

In this section $n = 5$. Write M_A and M_Q for the coefficient operators of $N^3 A$ and $N^3 Q$ on $\mathcal{P}_5(H_5)$, of dimension 56. The preceding ideal argument proves $M_A \succ 0$.

6.1 The negative index of the Q operator

Let $p = p_2$. By Section 2,

$$\frac{1}{120} M_Q = I - \frac{1}{20} p\Delta. \tag{26}$$

The Fischer harmonic decomposition in four real orthonormal coordinates is

$$\mathcal{P}_5 = \mathcal{H}_5 \oplus p\mathcal{H}_3 \oplus p^2\mathcal{H}_1, \qquad \mathcal{H}_k = \ker(\Delta|_{\mathcal{P}_k}). \tag{27}$$

The summands are orthogonal. For completeness, multiplication by p is adjoint to Δ, and repeated splitting by their range and kernel gives this decomposition; the following identity also verifies its directness:

$$\Delta(p^j h_k) = 2j(2k + 4 + 2j - 2)p^{j-1} h_k \qquad (h_k \in \mathcal{H}_k).$$

Thus $p\Delta$ has eigenvalues $0, 20, 32$ on the three summands of (27). Their dimensions are $36, 16, 4$, respectively, from $\dim \mathcal{H}_k = \binom{k+3}{3} - \binom{k+1}{3}$. Consequently M_Q has 36 positive, 16 zero, and exactly four negative eigenvalues. Relative to the coefficient operator $120I$ of N^5, the three eigenvalues are $1, 0, -3/5$.

6.2 An invariant four-dimensional slice

Consider the four degree-five polynomials

$$\begin{aligned} f_1 &= p_4 v_5, \qquad & f_2 &= p_2^2 v_5, \\ f_3 &= p_3(v_5^2 - p_2/5), \qquad & f_4 &= p_2(v_5^3 - p_3/5). \end{aligned} \tag{28}$$

Their M_A and M_Q forms are explicit:

$$\left(\langle f_i, M_A f_j\rangle_F\right)_{ij} = \frac{1152}{125}\mathsf{A}, \qquad \left(\langle f_i, M_Q f_j\rangle_F\right)_{ij} = \frac{1152}{125}\mathsf{Q}, \tag{29}$$

where

$$\mathsf{A} = \begin{pmatrix} 1932 & 2280 & 756 & 1332 \\ 2280 & 3200 & 840 & 1980 \\ 756 & 840 & 1368 & 756 \\ 1332 & 1980 & 756 & 1377 \end{pmatrix}, \qquad \mathsf{Q} = \begin{pmatrix} -24 & -960 & 18 & -384 \\ -960 & -2400 & -180 & -960 \\ 18 & -180 & 459 & -72 \\ -384 & -960 & -72 & -384 \end{pmatrix}. \tag{30}$$

The differentiation formulas in Appendix A provide a direct rational computation of every entry. The leading principal minors of A are

$$1932, \quad 984000, \quad 1049760000, \quad 62985600000. \tag{31}$$

In particular the f_i are independent. Direct determinant expansion gives

$$\det(\mathsf{A} + \rho\mathsf{Q}) = -388800000(305\rho^3 + 786\rho^2 + 351\rho - 162). \tag{32}$$

Let R average over the permutations fixing coordinate 5, and S average over all five coordinates. These are orthogonal projections for the Fischer product and $RS = SR = S$. The space

$$\mathcal{W} = \operatorname{im} R \cap \ker S$$

has dimension $6 - 2 = 4$. Indeed, invariants of the subgroup on four independent coordinates have generators of degrees $1, 2, 3, 4$; there are six monomials of weighted degree five. Full symmetric invariants on H_5 have generators of degrees $2, 3, 4, 5$; their degree-five component has dimension two, with possibilities 5 and $3 + 2$. Each f_i is fixed by R, and replacing its distinguished index 5 by every j and summing gives zero. Thus $Sf_i = 0$, and their independence shows that they form a basis of $\mathcal{W}$.

6.3 Why the slice gives the full endpoint

Both coefficient operators commute with the real orthogonal permutation action. Therefore $T = M_A^{-1} M_Q$ commutes with that action, preserves $\mathcal{W}$, and is self-adjoint in the positive M_A inner product. Its restriction to $\mathcal{W}$ is the generalized eigenvalue problem for the pencil (30).

The cubic in (7) is strictly increasing on $[0, \infty)$, negative at zero, and positive for large ρ. By (31), $\mathsf{A} \succ 0$. Hence the pencil remains positive definite until its unique positive singularity ρ_*. That determinant root is simple, so the endpoint kernel on $\mathcal{W}$ has dimension one. In particular T has a nonzero eigenvector $h \in \mathcal{W}$ for $\lambda_- = -1/\rho_* < 0$.

Because h is fixed by the subgroup fixing 5, its permutation orbit consists of at most five vectors h_j. Their sum is $5Sh = 0$. The map from the zero-sum permutation representation in $\mathbb{C}^5$ to their span is nonzero and equivariant. That representation is irreducible: subtracting the image of a transposition from a nonconstant coordinate vector yields a nonzero coordinate difference, whose permutations span the zero-sum space. The orbit span therefore has dimension four. Since T commutes with the group, it is entirely a λ_- eigenspace.

The operator T is similar to the Hermitian matrix $M_A^{-1/2} M_Q M_A^{-1/2}$. By Sylvester inertia it has exactly four negative eigenvalues, counted with multiplicity, because M_Q does. They are therefore precisely the four copies of λ_- just exhibited. Every other eigenvalue of T is nonnegative. By congruence, for $\rho \geq 0$,

$$M_A + \rho M_Q \succeq 0 \quad \Longleftrightarrow \quad 1 + \rho\lambda_- \geq 0 \quad \Longleftrightarrow \quad \rho \leq \rho_*.$$

All the remaining eigenvalues of the congruent matrix are strictly positive at ρ_*. This proves the endpoint kernel dimension and the full positivity assertions in Theorem 1.2. Theorem 1.1 excludes all exponents below three for $\rho > 0$; above ρ_* exponent three fails. The rational bracket follows by substituting $274637/10^6$ and $274638/10^6$ into the cubic, with opposite signs.

7 The classical three-coordinate example

The endpoint $n = 3$ of Theorem 1.1 is consistent with the familiar example of [2]. To specify the comparison, take real orthonormal coordinates z_1, z_2 on H_3, then put $w_\pm = (z_1 \pm iz_2)/\sqrt{2}$. One has

$$N = |w_+|^2 + |w_-|^2, \qquad p_2 = 2w_+w_-, \qquad A = N^2 - \tfrac{1}{2}|p_2|^2.$$

Thus $q_\rho = (1+\rho)N^2 - (2+4\rho)|w_+w_-|^2$. At $\ell = m+2$ its stabilized coefficient matrix is diagonal, with entries in the ordinary monomial basis $w_+^j w_-^{\ell-j}$ given by

$$(1+\rho)\binom{\ell}{j} - (2+4\rho)\binom{\ell-2}{j-1}.$$

Their signs are controlled by the maximum of $j(\ell-j)$. For even ℓ all entries are nonnegative exactly when $\ell \geq 2+2\rho$; for odd ℓ exactly when $\ell \geq 1+2\rho$. Consequently

$$\mu_3(\rho) = 2\lceil\rho\rceil - 1 \qquad (\rho > 0). \tag{33}$$

This records the semidefinite endpoint of the same classical diagonal example; the all-odd dimensional obstruction and the five-coordinate pencil are the additional conclusions of the present paper.

8 A coefficient of the centered permanent deficit

This section explains the origin of the family. It is not a reduction of a full permanent inequality to the preceding certificate-degree results. Let K be Hermitian positive semidefinite of order n with $K\mathbf{1} = 0$, let $J = \mathbf{1}\mathbf{1}^*$, and put $s = \sqrt{2}$. Define

$$F_K(u) = \operatorname{per}(K + uJ), \quad P_n = F_K(1) + F_K'(1), \quad D_n(K) = P_n^2 - \operatorname{per}(|K + sJ|^2). \tag{34}$$

The absolute values and square in the last term are entrywise. This deficit is related to the occupation/cofactor framework in [7]. Write $[D_n]_3$ for its homogeneous degree-three part in K, and use

$$S_3 = \sum_{a,b} K_{ab}^3, \qquad B_3 = \sum_{a,b} K_{ab}|K_{ab}|^2.$$

These are real for Hermitian K; S_3 is not the spectral trace $\operatorname{tr} K^3$.

Proposition 8.1. *For $n \geq 3$,*

$$\begin{aligned} [D_n]_3 &= c_n S_3 - e_n B_3, \\ c_n &= \frac{4}{3}(n-3)!\big[(n+1)(n-2)n! - s^{2n-3}\big], \\ e_n &= 2n(n-3)!s^{2n-3}. \end{aligned} \tag{35}$$

For $v \in H_n$ and $W = nI - J$, the coefficient of $t^2 a$ in $D_n(tvv^* + aW)$ is

$$\alpha_n A(v) + \beta_n Q(v), \qquad \alpha_n = 3(c_n - e_n) > 0, \quad \beta_n = 2e_n > 0. \tag{36}$$

Proof. The degree-k residual term of $F_K(u)$ is $(n-k)!u^{n-k}/k!$ times the ordered sum of k entries over distinct rows and columns. Enforce distinctness by partition-lattice inclusion–exclusion. By centering, a singleton row or column block vanishes. For $k = 2, 3$ the only surviving partitions without singletons are the single blocks, with Möbius factors -1 and 2. The linear term is zero, and the next terms are

$$[F_K(u)]_2 = \tfrac{1}{2}(n-2)!u^{n-2}\sum_{a,b} K_{ab}^2, \quad [F_K(u)]_3 = \tfrac{2}{3}(n-3)!u^{n-3}S_3.$$

Since the constant P_n is $(n+1)n!$, its squared cubic contribution is $\frac{4}{3}(n+1)(n-2)n!(n-3)!S_3$.

For the paired permanent, write $|K+sJ|^2 = s^2J + X + Y$ with $X = s(K+\overline{K})$ and $Y = |K|^2$. The matrix X is centered. The cubic contribution of three X entries is

$$\tfrac{4}{3}(n-3)!s^{2n-3}(S_3 + 3B_3).$$

For one X and one Y, the ordered distinct-row/column sum reduces by centering to $\sum_{a,b} X_{ab}Y_{ab} = 2sB_3$. Its contribution is $2(n-2)!s^{2n-3}B_3$. Combining gives the cubic paired term $\frac{4}{3}(n-3)!s^{2n-3}S_3 + 2n(n-3)!s^{2n-3}B_3$, and subtraction proves (35).

In $K = tvv^* + aW$, direct expansion gives $[t^2a]S_3 = 3A$ and $[t^2a]B_3 = 3A - 2Q$. This proves (36). Positivity of β_n is immediate. For α_n, write

$$c_n - e_n = \tfrac{2}{3}(n-3)![2(n+1)(n-2)n! - (3n+2)s^{2n-3}].$$

Use $n! \geq 3 \cdot 2^{n-2}$, $s^{2n-3} = 2^{n-2}s$, and $s < 2$. After division by 2^{n-2} the bracket is at least $6(n+1)(n-2) - (3n+2)s > 6(n-1)^2 - 22 > 0$ for $n \geq 3$. □

For $n = 5$, the coefficients are $\alpha_5 = 17280 - 544\sqrt{2}$ and $\beta_5 = 320\sqrt{2}$. One has $0 < \beta_5/\alpha_5 < 1/12 < \rho_*$, so the least norm exponent for this quartic is three. The first inequality follows from $4384\sqrt{2} < 17280$; the second follows by evaluating the increasing cubic at $1/12$, where it is negative. This concerns a single permanent-energy coefficient.

A Reproducing the small pencil

All entries of (30) can be checked by short rational differentiation, independently of the spectral argument. For the polynomials in (28), one has

$$\begin{aligned}\langle f, M_A g\rangle_F &= 6\sum_{i<j}\langle(\partial_i^2 - \partial_j^2)f, (\partial_i^2 - \partial_j^2)g\rangle_F,\\ \langle f, M_Q g\rangle_F &= 120\langle f, g\rangle_F - 6\langle \Delta f, \Delta g\rangle_F.\end{aligned} \tag{37}$$

These follow directly from multiplication–differentiation adjointness. Replace v_i by $x_i - (\sum_j x_j)/5$, expand in five ambient variables, and use $\langle x^a, x^b\rangle_F = a!\mathbf{1}_{a=b}$. This is an entirely rational recipe. Equivalently, the pairing of products of linear forms is

$$\left\langle \prod_{r=1}^{\ell}(a_r \cdot x), \prod_{r=1}^{\ell}(b_r \cdot x)\right\rangle_F = \sum_{\sigma\in S_\ell}\prod_{r=1}^{\ell}(\overline{a}_r \cdot b_{\sigma(r)}).$$

For the physical coordinates the relevant linear-form products are $P_{ij} = \delta_{ij} - 1/5$.

As a third normalization check, the matrix $(120\langle f_i, f_j\rangle_F)_{ij}$ is $(1152/125)\mathsf{N}$, where

$$\mathsf{N} = \begin{pmatrix} 1040 & 1600 & 270 & 740\\ 1600 & 4000 & 300 & 1600\\ 270 & 300 & 585 & 270\\ 740 & 1600 & 270 & 890\end{pmatrix}.$$

Its leading principal minors are $1040, 1600000, 810000000, 162000000000$. For instance $f_2 = p^2v_5$ is in $p^2\mathcal{H}_1$, so its Q form is $-3/5$ of its N^5 form: the $(2,2)$ entries indeed give $-2400/4000 = -3/5$. The four-by-four determinant (32), its principal minors, and the two rational bracket signs then involve only integer arithmetic.

The accompanying `verify_certificates.py` reconstructs the physical Fischer tables and independently checks them against a direct coefficient expansion in four independent coordinates. It also verifies the full coefficient-matrix inertia, permutation covariance, a cyclic relation and finite sign-cube identities as redundant checks. These computations support the displayed finite identities; the all-odd theorem is established by the analytic proof, not by finite sampling.

Assistance and reproducibility

AI-assisted tools were used in mathematical exploration and manuscript preparation. The proofs and their classical dependencies are stated in the paper. The accompanying exact-arithmetic verifier uses only the Python standard library and does not require numerical eigenvalue tolerances.

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