Degree-dependent minimax laws for Gaussian factorial moments

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Luminary Oddities

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Abstract

We determine the copy-budget hierarchy and an exact critical risk for estimating photon-number factorial moments of an unknown centered single-mode Gaussian state. Occupation and complex anomalous covariance are unknown, and bias is unrestricted. If the occupation floor a tends to zero, the degree n tends to infinity and $Na^2/n^2 \to \lambda \in (0,\infty)$, the collective quantum minimax squared relative risk converges to $1-\exp\{-1/(2\lambda)\}$. A global bound for a shrunk orthogonal-copy average attains the Poisson–Bessel lower bound. The upper follows from a physical rank-one majorant of the ordered Wick cross-contractions and a degree comparison for Gaussian moments. For every $n \geq 2$, quantum procedures and standard heterodyne have risk of order $\min\{1,(1+a^{-1})^2n^2/N\}$, uniformly in degree, copy count and positive floor. Risk tends to zero exactly when this scale vanishes and tends to one exactly when it diverges. Degree one retains a smaller quantum scale. At identical floors and resource scales, a quartic upper and the exact growing-degree limit differ. Thus the universal copy-budget order coexists with degree-dependent critical behavior.

Keywords. Gaussian quantum states; factorial moments; minimax estimation; relative loss; photon statistics.
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1 Introduction

Photon-number factorial moments measure high-order correlations of an optical mode. For a centered Gaussian state, Wick's rule expresses every such moment through occupation and anomalous covariance. The formulas can be finite and explicit while their statistical cost grows rapidly with degree. Our question is how many copies are needed to estimate these moments with uniform relative accuracy when the covariance, including squeezing, is unknown.

The target of degree n is

$$F_n = \langle (b^\dagger)^n b^n \rangle = \mathbb{E}(K)_{\underline{n}}, \qquad K = b^\dagger b,$$

with $[b, b^\dagger] = 1$. We assume the unknown occupation $t = \langle K \rangle$ is at least a known $a > 0$. For a thermal state, $F_n = n!t^n$, and collective quantum measurements have a low-occupation sampling advantage over standard heterodyne [5]. Once anomalous covariance is allowed to vary, the quartic target has the same inverse-quadratic occupation cost for both classes [6]. This raises a structural question: is that change peculiar to the quartic moment, or does it govern every higher degree?

We prove that degree one is the unique exception in the scalar hierarchy. For every $n \geq 2$, including growing degree, both measurement classes have resource scale

$$x = \frac{(1+a^{-1})^2 n^2}{N}. \tag{1}$$

Risk vanishes exactly at $x \to 0$ and collapses to the zero-decision risk one exactly at $x \to \infty$. This conclusion is uniform over all finite Gaussian covariances above the floor. A covariance plug-in attains the upper order. Its proof needs a derivative bound uniform in degree and control of rare observations; fixed-degree error propagation would leave both issues unpaid.

Within the critical range, we determine the exact quantum risk when degree grows and occupation vanishes. If $Na^2/n^2 \to \lambda \in (0, \infty)$, then the risk converges to $1 - e^{-1/(2\lambda)}$. The lower comes from a Poisson experiment with target I_0, the modified Bessel function. To attain it over the entire Gaussian family, we bound the relative variance of an orthogonal-copy average. A physical rank-one majorant controls every ordered Wick cross-contraction, and sphere averaging produces the sharp limiting constant. This global estimate covers the full occupation and squeezing domain before taking a limit.

The exact curve also makes degree dependence explicit. At the same occupation floor and with $x = 1$ at every index, a quartic quantum risk has limiting upper bound $1/17$, whereas a growing-degree quantum risk converges to $1 - e^{-1/2} > 0.39$. A single resource coordinate therefore determines the copy-budget order and endpoints while different degrees have different critical risks.

Prior work and increment. General Gaussian photon moments and cumulants are established moment calculus; Cardin and Quesada [1] give hafnian formulas and associated computational structure. Heinzel and Sondenheimer [3] use higher correlation identities for Gaussian-state characterization and reconstruction. Their normalized correlations F_n/t^n differ from the raw factorial moments estimated here. We use the scalar Wick polynomial as inherited mathematics.

The earlier thermal paper [5] treats vanishing anomalous covariance. The full Gaussian paper [6] gives exact unbiased polynomial estimators and proves the quartic arbitrary-bias order. Here the statistical conclusions concern the entire scalar hierarchy uniformly in n, complete collapse and its converse, and the exact growing-degree quantum critical risk. The all-order observable and Wick variance from [6] supply the estimator. The new global relative-variance bound combines a physical rank-one majorant with a mixed-moment degree comparison, and matches the Poisson–Bessel lower. The resulting exact limit also gives a concrete obstruction to a degree-independent critical curve. The proofs use classical concentration, oscillator symmetries and Poisson decision theory. The harmonic decomposition underlying the quantum boundary reduction is described in [2]; its required probability law is derived below. Poisson factorial identities have classical Charlier–Parseval ancestry [7]. We give the particular minimax and physical-transfer arguments needed here.

2 Model and principal results

For a centered single-mode Gaussian state ϱ, write

$$t = \operatorname{Tr} \varrho b^\dagger b > 0, \qquad c = \operatorname{Tr} \varrho b^2, \qquad r = |c|, \qquad r^2 \le t(t+1).$$

Every finite pair in this physical domain is included, with unknown phase of c. Wick expansion gives

$$F_n(t, r) = n! t^n p_n(r/t), \qquad p_n(q) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(n)_{2j}}{4^j (j!)^2} q^{2j}. \tag{2}$$

In particular $F_1 = t$, $F_2 = 2t^2 + r^2$, and $F_3 = 6t^3 + 9tr^2$. The positive coefficients imply $F_n \ge n! t^n > 0$.

For N independent copies, let $R_Q(N,n,a)$ be the infimum over all collective POVMs with real decisions Y of

$$\sup_{t\geq a,\,|c|^2\leq t(t+1)} \mathbb{E}_{t,c}\left(\frac{Y}{F_n(t,|c|)}-1\right)^2. \tag{3}$$

Bias is unrestricted. Let R_H restrict the measurement to standard heterodyne on each copy with arbitrary processing. A record Z is centered complex Gaussian with $\mathbb{E}|Z|^2=t+1$, $\mathbb{E}Z^2=c$. Thus $R_Q\leq R_H\leq 1$. Put $\gamma_a=1+a^{-1}$.

Theorem 2.1 (Uniform degree threshold)**.** *There are universal constants $c_0,C_0>0$ such that for all $a>0$, $N\geq 1$ and integers $n\geq 2$, with $x=\gamma_a^2 n^2/N$,*

$$c_0\min(1,x)\leq R_Q(N,n,a)\leq R_H(N,n,a)\leq \min(1,C_0x). \tag{4}$$

At degree one the respective orders are

$$R_Q(N,1,a)\asymp \min(1,\gamma_a/N),\qquad R_H(N,1,a)\asymp \min(1,\gamma_a^2/N), \tag{5}$$

with universal comparison constants. One may take $C_0=18432$ in (4).

Theorem 2.2 (Complete endpoint classification)**.** *Along arbitrary sequences $a>0$, $N\geq 1$, $n\geq 2$,*

$$\begin{aligned} R_Q\to 0 &\iff R_H\to 0 \iff x\to 0,\\ R_Q\to 1 &\iff R_H\to 1 \iff x\to\infty. \end{aligned}$$

For every finite $L>0$ there is $\epsilon(L)>0$ such that $x\leq L$ implies $R_H\leq 1-\epsilon(L)$. Hence if x stays between two positive finite constants, both risks stay uniformly away from zero and one.

Theorem 2.3 (Global quantum bound)**.** *For every $a>0$ and integers $n,N\geq 1$, put*

$$E_{N,n,a}=\frac{n^2(\sqrt{1+a}+\sqrt{a})^4}{2Na^2}.$$

The positive orthogonal-copy average T_n defined in Section 3 is globally unbiased for F_n and satisfies

$$\sup_{t\geq a,\,|c|^2\leq t(t+1)} \frac{\mathrm{Var}_{t,c}(T_n)}{F_n(t,|c|)^2}\leq e^{E_{N,n,a}}-1. \tag{6}$$

The spectral decision $e^{-E_{N,n,a}}T_n$ therefore gives

$$R_Q(N,n,a)\leq 1-e^{-E_{N,n,a}}. \tag{7}$$

Theorem 2.4 (Exact growing-degree risk and degree dependence)**.** *If $a\to 0$, $n\to\infty$ and $Na^2/n^2\to\lambda\in(0,\infty)$, then*

$$R_Q(N,n,a)\longrightarrow 1-\exp\{-1/(2\lambda)\}. \tag{8}$$

There is no function Φ for which $R_Q(N,n,a)\to\Phi(x_0)$ whenever $a\to 0$, $n\geq 2$ and $x\to x_0\in(0,\infty)$. In fact two sequences with the same floor and exactly $x=1$ have disjoint asymptotic risk bounds.

Section 3 proves the global quantum bound. The exact Poisson decision problem in Section 7 supplies the matching physical lower in Section 8, which completes Theorem 2.4. The heterodyne construction and quantum hard alternatives establish the uniform orders and endpoint classification for both measurement classes.

For relative mean-squared error η^2 in the consistency regime, (4) requires and suffices, up to universal constants, that N be of order $\gamma_a^2 n^2/\eta^2$. The low-brightness quantum order advantage for mean occupation is therefore special to degree one. This is a comparison with standard heterodyne, within the centered scalar Gaussian model.

3 A global quantum variance bound

The critical upper requires control over the entire covariance family. We obtain it from the orthogonal-copy average of [6], bounding its ordered Wick coefficients relative to the target itself.

3.1 The observable and its exact variance

For a real unit vector $v \in S^{N-1}$, write $b(v) = \sum_{i=1}^{N} v_i b_i$ and

$$H_v = (b(v)^\dagger)^n b(v)^n, \qquad T_n = \int_{S^{N-1}} H_v \, d\sigma(v),$$

where σ is normalized surface measure. Real copy mixing preserves the covariance of a centered Gaussian product, so $\mathbb{E}H_v = F_n$ and $\mathbb{E}T_n = F_n$. The integral is a positive number-preserving polynomial. On every finite total-photon sector it is a finite Hermitian matrix; the self-adjoint direct sum defines its spectral measurement. Gaussian states at every finite covariance have finite polynomial moments, so its first and second moments are well defined, including pure states. These are the polynomial-domain and averaging properties established in [6].

For evaluating its variance, phase rotation lets us set $c = r \geq 0$. This changes neither the target nor T_n. Define the ordered internal generating function

$$G(x, y) = \exp\{r(x^2 + y^2)/2 + txy\}.$$

For two directions with overlap $s = v^{\mathsf{T}} w$, the ordered Gaussian generating function is

$$G(x, y)G(z, h) \exp\{sC(x, y, z, h)\}, \qquad C = rxz + txh + (t + 1)yz + ryh. \tag{9}$$

The coefficient of $x^n y^n z^n h^n$, multiplied by $(n!)^4$, equals $\mathbb{E}H_v H_w$. In particular the middle annihilator–creator contraction is $t + 1$. Parity eliminates odd cross degrees, giving

$$\mathbb{E}H_v H_w = F_n^2 + \sum_{k=1}^{n} A_k s^{2k}, \qquad A_k \geq 0.$$

The exact real-sphere moment formula from [6] yields

$$\frac{\operatorname{Var} T_n}{F_n^2} = \sum_{k=1}^{n} \frac{A_k}{F_n^2} \frac{(2k-1)!!}{N(N+2)\cdots(N+2k-2)}. \tag{10}$$

It also holds at $N = 1$. We next bound every coefficient without discarding anomalous contractions.

3.2 Mixed insertions and physical majorization

Let $M_{p,q} = \langle (b^\dagger)^p b^q \rangle$, with $F_0 = 1$. Quantum Cauchy–Schwarz gives $|M_{p,q}| \leq \sqrt{F_p F_q}$. The coefficients of p_j in (2) increase with integer degree, hence for $0 \leq d \leq n$,

$$F_n \geq (n)_{\underline{d}} \, t^d F_{n-d}. \tag{11}$$

For integers $0 \leq u, v \leq n$, define the insertion coefficient

$$B_{u,v} = (n!)^2 [x^n y^n] x^u y^v G(x, y) = (n)_{\underline{u}} (n)_{\underline{v}} M_{n-u,n-v}.$$

Combining Cauchy–Schwarz with (11) twice gives

$$\frac{|B_{u,v}|}{F_n} \le \frac{\sqrt{(n)_{\underline{u}}(n)_{\underline{v}}}}{t^{(u+v)/2}} \le (n/t)^{(u+v)/2}. \tag{12}$$

An insertion exponent above n contributes zero.

Put $\epsilon = \sqrt{t/(t+1)}$. Physicality $r \le \sqrt{t(t+1)}$ gives the coefficientwise nonnegative-polynomial inequality

$$C(x,y,z,h) \preceq (t+1)(y+\epsilon x)(z+\epsilon h). \tag{13}$$

The right side has coefficients $\sqrt{t(t+1)}, t, t+1, \sqrt{t(t+1)}$ in the four respective positions of (9). All internal coefficients are nonnegative, so the inequality survives powers, multiplication by the two internal generating functions and coefficient extraction. It is an inequality of formal coefficients; the overlap s need not be positive.

Set

$$B_m(\epsilon) = (n!)^2[x^n y^n](y+\epsilon x)^m G(x,y).$$

Internal symmetry in x, y makes the two extracted factors of the rank-one majorant equal. The $2k$th cross coefficient therefore obeys

$$A_k \le \frac{(t+1)^{2k}}{(2k)!} B_{2k}(\epsilon)^2.$$

Binomial expansion and (12) give

$$\frac{B_{2k}(\epsilon)}{F_n} \le (1+\epsilon)^{2k}(n/t)^k.$$

Here extending the binomial sum to exponents above n only adds nonnegative upper bounds. Consequently

$$\frac{A_k}{F_n^2} \le \frac{1}{(2k)!}\left[\frac{n^2(\sqrt{t+1}+\sqrt{t})^4}{t^2}\right]^k. \tag{14}$$

3.3 Sphere averaging and the global decision

Substitute (14) into (10). The cancellation $(2k-1)!!/(2k)! = 1/(2^k k!)$ gives

$$\begin{aligned}\frac{\operatorname{Var} T_n}{F_n^2} &\le \sum_{k=1}^{n} \frac{1}{k!} \frac{[n^2(\sqrt{t+1}+\sqrt{t})^4/(2t^2)]^k}{N(N+2)\cdots(N+2k-2)} \\ &\le \exp\left\{\frac{n^2(\sqrt{t+1}+\sqrt{t})^4}{2Nt^2}\right\} - 1.\end{aligned}$$

The occupation factor is

$$\frac{(\sqrt{t+1}+\sqrt{t})^4}{t^2} = [2 + t^{-1} + 2\sqrt{1+t^{-1}}]^2,$$

which decreases for $t > 0$. Its maximum on $t \ge a$ occurs at a, proving (6). Every operator moment used above comes from a finite polynomial; the exponential is a scalar bound for a finite sum.

For an unbiased statistic with relative variance at most v, division by $1+v$ gives relative risk at most $v/(1+v)$. Taking $v = e^{E_{N,n,a}} - 1$ proves (7). The decision depends only on the known N, n, a. Finally,

$$a \to 0, \quad Na^2/n^2 \to \lambda > 0 \quad \Longrightarrow \quad \limsup R_Q \le 1 - e^{-1/(2\lambda)}, \tag{15}$$

since $(\sqrt{1+a}+\sqrt{a})^4 \to 1$. Section 8 supplies the matching lower when $n \to \infty$.

4 A degree-uniform covariance estimator

4.1 The sensitivity inequality

Lemma 4.1. *For all formal $t > 0, r \geq 0$ and $n \geq 1$,*

$$0 \leq \partial_t \log F_n(t,r) \leq n/t, \qquad 0 \leq \partial_r \log F_n(t,r) \leq n/t.$$

Proof. The first bound follows from the positive monomial expansion (2). Write its coefficient as a_j and set $v_j = \sqrt{a_j} q^j$. Then

$$p_n'(q) = \sum_{j=0}^{\lfloor n/2 \rfloor - 1} \sqrt{(n-2j)(n-2j-1)}\, v_j v_{j+1} \leq n \sum_j v_j^2 = n p_n(q),$$

using $2uv \leq u^2 + v^2$. The identity follows from the ratio of adjacent coefficients and extends to $q = 0$ by continuity. Divide by t to obtain the second bound. □

The inequality applies even when an empirical pair is outside the physical covariance cone. Define

$$\widehat{t} = \max\left(a, \frac{1}{N}\sum_i |Z_i|^2 - 1\right), \qquad \widehat{r} = \left|\frac{1}{N}\sum_i Z_i^2\right|, \qquad H = F_n(\widehat{t}, \widehat{r}).$$

Let D_0, D_1, D_2 be the centered empirical errors in $|Z|^2, \Re Z^2, \Im Z^2$, and put $e = \sum_j |D_j|$, $z = e/t$. Projection gives $|\widehat{t} - t| \leq |D_0|$ and $|\widehat{r} - r| \leq |D_1| + |D_2|$. If $z \leq 1/2$, integrate Lemma 4.1 along the line segment to get $|\log(H/F_n)| \leq 2nz$. For every observation, monotonicity gives $H \leq F_n(t+e, r+e) \leq F_n e^{2nz}$. A negative relative error has magnitude at most one. These facts imply globally

$$|H/F_n - 1| \leq e^{4nz} - 1 \leq 4nz e^{4nz}. \tag{16}$$

4.2 Global moment control

Let $V = t+1$. Each observed quadratic is $X^T A_j X$ for a real Gaussian vector with covariance of trace V and $\|A_j\| \leq 1$. Diagonalizing the quadratic form and expanding its exact moment-generating function gives

$$\log \mathbb{E} e^{sD_j} \leq 4V^2 s^2/N, \qquad |s| \leq N/(4V).$$

Indeed every eigenvalue has magnitude at most V, and their squared sum is at most $2V^2$; the centered logarithmic series has the stated bound in that range. Absolute values and Hölder's inequality yield

$$\mathbb{E} e^{sz} \leq 2\exp(36\gamma_a^2 s^2/N), \qquad 0 \leq s \leq N/(12\gamma_a). \tag{17}$$

Convexity in the two covariance eigenvalues gives $\mathbb{E}|Z|^8 \leq 105V^4$. Thus the fourth centered moment of each observed quadratic is at most $848V^4$, its variance at most $8V^2$, and the fourth moment of its empirical average at most $1040V^4/N^2$. Using $(x_0 + x_1 + x_2)^4 \leq 27\sum x_j^4$ gives

$$\mathbb{E} z^4 \leq 84240\gamma_a^4/N^2. \tag{18}$$

If $N \geq 18432\gamma_a^2 n^2$, Cauchy–Schwarz, (16) and (17) at $s = 16n$ give

$$\mathbb{E}(H/F_n - 1)^2 \leq 16\sqrt{168480}\frac{\gamma_a^2 n^2}{N} e^{4608\gamma_a^2 n^2/N} \leq 18432\frac{\gamma_a^2 n^2}{N}.$$

The moment-generating-function range holds under this sample condition. Otherwise use the zero decision. This proves the upper in (4), including all rare observations.

5 Quantum hard alternatives and degree one

For data laws P_0, P_1 and positive targets f_0, f_1, every decision has maximum relative risk at least

$$\left(\frac{f_1 - f_0}{f_1 + f_0}\right)^2 (1 - \mathrm{TV}(P_0, P_1)). \tag{19}$$

Use prior weights $f_i/(f_0 + f_1)$ and integrate the pointwise minimum loss against the common part of the two densities. For quantum states, trace distance, with factor $1/2$, bounds outcome total variation for every measurement.

5.1 A squeezed pair at small occupation

Suppose $a \le 1/4$. Squeeze a thermal state of occupation a by

$$S(v) = e^{ivG}, \quad G = \frac{b^2 - (b^\dagger)^2}{2i}, \qquad t_v = a + (2a+1)\sinh^2 v, \quad r_v = (2a+1)\sinh v \cosh v.$$

Its occupation stays above a. Compare $v_0 = a/n$ and $v_1 = v_0 + h$, where $h = \frac{1}{4}\min(a/n, N^{-1/2})$. On this interval $v \le 5/32$; the elementary bounds $\sinh v \le 6v/5$, $\cosh v \le 6/5$ give

$$t_v/a \le 1 + 27/(32n^2), \qquad n r_v/t_v \le 27/10.$$

Since $p_n(q) \le \exp(n^2 q^2/4)$, (2) implies $F_n(t_v, r_v) \le e^3 n! a^n$. The $j = 1$ Wick term alone and $r_v \ge v$, $r_v' \ge 1$ give

$$\frac{d}{dv} F_n(t_v, r_v) \ge n! n a^{n-1}/4.$$

The harmonic target separation in (19) is therefore at least $nh/(8e^3 a)$.

For a common thermal purification, $\mathrm{Var}\, G = a^2 + a + 1/2 \le 13/16$. The characteristic-function overlap satisfies $|\operatorname{Tr} \varrho_a e^{ihG}|^2 \ge 1 - h^2 \operatorname{Var} G$. Tensoring purifications and contracting trace distance bounds every outcome total variation by $h\sqrt{N \operatorname{Var} G} < 1/4$. Hence

$$R_Q \ge \frac{3}{4096 e^6} \min\left(1, \frac{n^2}{Na^2}\right). \tag{20}$$

Since $\gamma_a^2 \le 25/(16a^2)$ on this range, this is the required universal lower in terms of x.

5.2 Thermal pairs and the first-degree exception

For a thermal geometric law of mean a, its relative entropy to mean ae^h has derivative

$$\frac{d}{dh} D(P_a \| P_{ae^h}) = \frac{a(e^h - 1)}{1 + ae^h} \le \frac{eh}{\gamma_a} \quad (0 \le h \le 1). \tag{21}$$

Take $h = \frac{1}{4}\min(1/n, \sqrt{\gamma_a/N})$. Then $ND \le e/32$, outcome total variation is at most $\sqrt{e}/8$, and the target ratio is e^{nh}. Since $nh \le 1/4$, its harmonic separation is at least $nh/4$. Thus for every $n \ge 1$,

$$R_Q(N, n, a) \ge c_Q \min(1, \gamma_a n^2/N), \qquad c_Q = (1 - \sqrt{e}/8)/256. \tag{22}$$

Thermal products commute, so their quantum experiment is a randomization of the geometric counts used in this bound. For $a \ge 1/4$, $\gamma_a \le 5$, and (22) bounds (4) from below within another universal factor. Together with (20), this completes Theorem 2.1 for $n \ge 2$.

At $n = 1$, mean photon counts have relative variance at most $2\gamma_a/N$, since $\operatorname{Var} K = t(t+1) + |c|^2 \le 2t(t+1)$. The heterodyne mean $N^{-1}\sum |Z_i|^2 - 1$ has relative variance at most $2\gamma_a^2/N$. Shrinking each by its known variance envelope gives the stated upper orders. The quantum lower is (22). For the heterodyne thermal subexperiment, with $t_1 = ae^h$, the one-copy divergence has derivative

$$\frac{t_1(t_1 - a)}{(1+t_1)^2} \le e^2 h/\gamma_a^2 \quad (0 \le h \le 1).$$

Choose $h = \frac{1}{4}\min(1, \gamma_a/\sqrt{N})$ and apply (19). The total variation is at most $e/8$ and the resulting lower is $c_H \min(1, \gamma_a^2/N)$, with $c_H = (1 - e/8)/256$. This proves (5) without a local-unbiasedness assumption.

6 Complete collapse and its converse

6.1 Diverging scale forces collapse

For $a \le 1/4$ and $x \to \infty$, put $z_0 = n/(a\sqrt{N}) \to \infty$ and compare the thermal state to its squeeze by $h = \sqrt{a/(n\sqrt{N})}$. Positivity of the first two Wick terms gives

$$\frac{F_n(t_h, r_h)}{n! a^n} \ge 1 + \frac{n(n-1)h^2}{4a^2} \ge 1 + \frac{z_0}{8} \longrightarrow \infty.$$

Their outcome total variation is at most $h\sqrt{N \operatorname{Var} G} \le \sqrt{13/16}\, z_0^{-1/2} \to 0$. Equation (19) proves collapse.

For $a \ge 1/4$, $x \to \infty$ implies $y = n^2/N \to \infty$. Use thermal means a, ae^h with $h = (n\sqrt{N})^{-1/2}$. The exact derivative in (21) also obeys $D' \le 1 - e^{-h} \le h$ for every $h \ge 0$. Thus $ND \le Nh^2/2 = 1/(2\sqrt{y}) \to 0$, while the target ratio e^{nh} diverges. Again (19) gives risk tending to one. Splitting arbitrary sequences into the two floor ranges proves $x \to \infty \Rightarrow R_Q, R_H \to 1$.

6.2 Every bounded scale admits a strict improvement

Fix $L > 0$. We prove the uniform upper below one asserted in Theorem 2.2. Let $W_N = N^{-1}\sum_i G_i G_i^T$ for standard real two-dimensional normals. For every $b > 0$,

$$p(b) = \inf_{N \ge 2} \Pr\{\|\sqrt{N}(W_N - I_2)\|_{\mathrm{op}} \le b\} > 0. \tag{23}$$

For each fixed $N \ge 2$, Wishart density is positive near I_2. As $N \to \infty$, the three symmetric entries obey a multivariate central limit theorem with nondegenerate Gaussian limit (diagonal variances two, off-diagonal variance one). Its operator-norm ball has positive probability and boundary measure zero. Convergence to that positive probability, combined with the finite initial values, proves (23).

Whitening a heterodyne covariance of trace V gives $e \le 6V\|W_N - I_2\|_{\mathrm{op}}$. If $x \le L$, the event in (23) with $b = 1/(24\sqrt{L})$ implies $z \le 1/(4n)$ and hence $H/F_n \ge e^{-1}$. Define $b_L = e^{-1}p(1/(24\sqrt{L})) > 0$; then

$$\mathbb{E}(H/F_n) \ge b_L \qquad (N \ge 2). \tag{24}$$

Let $K = \lceil \max(2, 48L) \rceil$. If $n\gamma_a > K$, then $N \ge n^2\gamma_a^2/L > 48n\gamma_a$. The global bound $H/F_n \le e^{2nz}$ and (17) at $s = 4n$ give

$$\mathbb{E}(H/F_n)^2 \le M_L = 2e^{576L}.$$

The admissible decision $b_L H/M_L$ therefore has risk at most $1 - b_L^2/M_L$.

If $n\gamma_a \le K$, use a single heterodyne record and the unbiased statistic

$$h_n(Z) = \sum_{j=0}^{n} \frac{(-1)^j ((n)_{\underline{j}})^2}{j!} |Z|^{2(n-j)}. \tag{25}$$

Its mean is F_n: differentiate $e^{\xi Z + \eta \overline{Z} - \xi\eta}$ n times in each source and take Gaussian expectation, which replaces the cross covariance $t+1$ by t. This formal finite-coefficient argument applies also to improper Gaussian records. Since $\mathbb{E}|Z|^{4m} \le (4m-1)!! V^{2m}$ by convexity in covariance eigenvalues, the square-of-sum bound gives

$$\mathbb{E}[h_n(Z)^2/F_n^2] \le D_n \gamma_a^{2n}, \qquad D_n = \frac{n+1}{(n!)^2} \sum_{j=0}^{n} \frac{((n)_{\underline{j}})^4}{(j!)^2} (4(n-j)-1)!!,$$

where $(-1)!! = 1$ and $V \ge 1$. Let $M_K = \max(2, \max_{1 \le j \le K} D_j K^{2j})$. Then $h_n(Z_1)/M_K$ has risk at most $1 - 1/M_K$, covering $N = 1$ as well. Thus

$$\epsilon(L) = \min(1/M_K, b_L^2/M_L) > 0$$

proves the required strict bound. These constants establish existence of a nontrivial uniform risk at bounded scale; the explicit construction in Section 4 gives the rate bound in the consistency regime.

Theorem 2.1 already proves consistency exactly at $x \to 0$. If a risk tended to one while x failed to diverge, a subsequence with $x \le L$ would contradict the strict upper just proved. This completes Theorem 2.2.

7 The Poisson–Bessel decision problem

Theorem 7.1. *Fix $s > 0$. For $J \sim \mathrm{Pois}(v)$, $v \ge 0$, estimate $g(v) = I_0(2\sqrt{sv})$ under squared relative loss. Its arbitrary-bias minimax risk is $1 - e^{-s}$. An attaining decision is $e^{-s} L_J(-s)$, where*

$$L_j(-s) = \sum_{r=0}^{j} \frac{s^r}{(r!)^2} (j)_{\underline{r}}.$$

Upper bound. Write $h(j) = L_j(-s)$. Poisson factorial moments give $\mathbb{E}h(J) = g(v)$. All moments are finite because $h(j) \le (1+s)^j$. The overlap identity

$$\mathbb{E}[(J)_{\underline{r}} (J)_{\underline{l}}] = \sum_{k \le \min(r,l)} \binom{r}{k} \binom{l}{k} k! v^{r+l-k}$$

and Tonelli yield

$$\operatorname{Var} h(J) = \sum_{k \ge 1} \frac{v^k}{k!} [g^{(k)}(v)]^2 = \sum_{k \ge 1} \frac{s^k}{k!} I_k(2\sqrt{sv})^2. \tag{26}$$

This is the factorial form of the classical Poisson orthogonal expansion [7]. The series and angular integral for I_k give $0 \le I_k(w) \le I_0(w)$ [4, §10.32]. Hence $\operatorname{Var} h / g^2 \le e^s - 1$, and shrinking by e^{-s} gives risk at most $1 - e^{-s}$. For fixed k, $I_k(w)/I_0(w) \to 1$ as $w \to \infty$; dominated convergence in (26) shows that this rule's supremum risk is the stated value. A lower against every decision is proved next. □

7.1 A proper-prior lower

Use a gamma prior on v with shape $\sqrt{X}$ and rate $X^{-1/2}$, where $X \to \infty$. It has mean X and variance $X^{3/2}$; predictively $J/X \to 1$ in probability. The posterior is Gamma(U, R) with $U = J + \sqrt{X}$ and $R = 1 + X^{-1/2}$. Let $\mu = U/R$.

We claim, for $j = 1, 2$, uniformly along deterministic sequences $\mu \to \infty$, $R \to 1$,

$$A_j = g(\mu)^j \mathbb{E}[g(v)^{-j} \mid J] \longrightarrow e^{j^2 s/2}. \tag{27}$$

The fixed-order Bessel expansion [4, §10.40] and $g'(v) = \sqrt{s/v}\, I_1(2\sqrt{sv})$ give

$$d_\mu = (\log g)'(\mu) = \sqrt{s/\mu} + O(\mu^{-1}).$$

The exact differential equation $vg'' + g' - sg = 0$ then gives $(\log g)'' = s/v - (\log g)'/v - ((\log g)')^2 = O(v^{-3/2})$. On $|v - \mu| \le \mu^{5/8}$, Taylor's remainder is $O(\mu^{-1/4})$. The normalized inverse power is consequently $e^{-jd_\mu(v-\mu)}(1 + o(1))$ there. The exact gamma transform has logarithm

$$\log \mathbb{E} e^{-jd_\mu(v-\mu)} = U\left[\frac{jd_\mu}{R} - \log\left(1 + \frac{jd_\mu}{R}\right)\right] \longrightarrow j^2 s/2.$$

The tails require control because the inverse moment is unbounded after normalization. For $R \in [1/2, 2]$, Stirling's bound gives posterior density at most

$$C\mu^{-1/2} \exp[-c(v - \mu)^2/\mu] \quad \text{on } [\mu/2, 2\mu].$$

The log-target derivative on this interval is bounded by $C/\sqrt{\mu}$, so both the normalized inverse power and the linear exponential are at most $\exp(C|v - \mu|/\sqrt{\mu})$. Completing the square bounds their integrals outside the central window by $O(e^{-c'\mu^{1/4}})$. On $v < \mu/2$ they are at most $\exp(C\sqrt{\mu})$, since $g(v) \ge 1$, while gamma Chernoff probability is $e^{-c\mu}$. On $v > 2\mu$ both integrands are at most one and the upper-tail probability is also $e^{-c\mu}$. This proves (27) with all tails included.

The posterior minimum relative risk equals $1 - A_1^2/A_2$, which lies in $[0, 1]$. Predictive concentration gives convergence in probability and hence in expectation to $1 - e^{-s}$. Every prior is proper, so the minimax risk dominates these Bayes risks. This completes Theorem 7.1.

Its lower can be approached by compactly supported priors. For any fixed prior π, its Bayes risk is

$$1 - \sum_j \frac{\left[\int p_j(v) g(v)^{-1} \pi(dv)\right]^2}{\int p_j(v) g(v)^{-2} \pi(dv)}.$$

Each summand is at most the predictive probability of j. Truncating π to expanding compact intervals makes individual terms converge and predictive distributions converge in total variation; their uniformly tight tails justify passage through the sum. This supplies precisely the compact lower transfer used below.

8 The exact physical critical risk and degree dependence

8.1 The common-phase invariant experiment

For completeness we derive the probability reduction, using standard oscillator harmonics [2]. On the N copies put $K = \sum b_r^\dagger b_r$, $Q = \sum b_r^2$. The kernel of Q in the ℓ-photon sector has dimension

$$d_{\ell,N} = \binom{N + \ell - 1}{\ell} - \binom{N + \ell - 3}{\ell - 2} \quad (N \ge 2),$$

with the second term zero for $\ell < 2$. Applying $Q^\dagger$ repeatedly gives radial ladders with $K = \ell + 2j$ and

$$Q^\dagger Q = 4j(\ell + N/2 + j - 1), \qquad k_\ell = N/4 + \ell/2.$$

These formulas follow from $[Q, Q^\dagger] = 4K + 2N$ and the orthogonal splitting of homogeneous Bargmann polynomials into the kernel of $\sum \partial_r^2$ and the image of multiplication by $\sum z_r^2$. Thus (ℓ, j) is a known joint spectral record, with an additional harmonic multiplicity on which Gaussian products are scalar.

Write the scalar Gaussian state as squeezed thermal with

$$\nu = \sqrt{(t + 1/2)^2 - |c|^2} - 1/2, \quad q = \nu/(1 + \nu), \quad z = \frac{c}{1 + t + \nu}, \quad p = |z|^2.$$

The thermal harmonic weight is

$$w_\ell(q) = d_{\ell,N} \frac{(1 - q)^N q^\ell}{1 - q^2}, \tag{28}$$

and its radial state is $(1-q^2)\sum_{v\geq 0} q^{2v}|v\rangle\langle v|$. Normalization follows from $\sum_\ell d_{\ell,N} q^\ell = (1-q^2)/(1-q)^N$. Collective squeezing preserves ℓ and acts on the radial ladder. The commutators of $K_+ = Q^\dagger/2$, $K_- = Q/2$ and $K_0 = (K + N/2)/2$ give its vacuum probabilities

$$\Pr\{j \mid \ell, \text{radial vacuum}\} = (1 - p)^{N/2+\ell} \frac{(N/2 + \ell)^{\overline{j}}}{j!} p^j. \tag{29}$$

For example, this follows by applying the disentangled squeeze $e^{zK_+}(1 - p)^{K_0} e^{-\overline{z}K_-}$ to the radial vacuum. The actual internal thermal state differs from that vacuum in trace distance q^2, so (29) approximates its conditional law with that error.

Averaging any decision over the common phase $e^{i\phi K}$ cannot increase maximum risk for the phase-invariant F_n. The averaged source is diagonal in the harmonic/radial basis: common phase removes radial off-diagonals, and Gaussian products already have no coherence between harmonic labels. Multiplicity is uniform conditional on ℓ. Every POVM on this commuting family is a classical randomization of (ℓ, j), and conditional averaging reduces squared loss. Consequently this record gives the exact minimax reduction for the phase-invariant target. The averaging retains the common phase relation between copies.

8.2 Growing degree produces the Bessel target

Assume $a \to 0$, $n \to \infty$, $Na^2/n^2 \to \lambda > 0$ and restrict to

$$t = a, \qquad c = (a/n)\kappa, \qquad |\kappa| \leq C.$$

These are physical for all sufficiently large indices. The normalized target obeys

$$\frac{F_n}{n! a^n} = p_n(|\kappa|/n) \longrightarrow I_0(|\kappa|) \tag{30}$$

uniformly on this compact set: $(n)_{\underline{2j}}/n^{2j} \leq 1$ tends to one at fixed j, and the dominating Bessel series is summable.

The joint harmonic/radial law converges in total variation, uniformly on the same set, to

$$w_\ell\left(\frac{a}{1 + a}\right) \otimes \text{Pois}(\lambda|\kappa|^2/2). \tag{31}$$

Here are the bounds paying that assertion. Rationalization gives $|\nu - a| = O(a^2/n^2)$. The nearby geometric thermal divergence is $O((\nu-a)^2/a) = O(a^3/n^4)$; multiplying by N gives $O(a/n^2)$. Data processing and Pinsker therefore replace the harmonic weight by that in (31) at vanishing cost. The radial thermal replacement costs $q^2 = O(a^2)$.

The negative binomial (29) is a gamma mixture of Poisson laws, with shape $r_\ell = N/2+\ell$, mean $r_\ell p/(1-p)$ and mixing variance $r_\ell p^2/(1-p)^2$. Coupling gives total variation at most $\sqrt{r_\ell}p/(1-p)$ from the Poisson law of its mean. Since $p = O(a^2/n^2)$ and $\mathbb{E}\ell \le N\nu = O(Na)$, its averaged error is $O(a/n)$. Moreover $p/(1-p) = (a^2|\kappa|^2/n^2)(1+O(a))$, and $\mathbb{E}\ell/N = O(a)$, so the random mean converges in L^1 to $\lambda|\kappa|^2/2$. Another Poisson coupling proves (31). Conditional radial errors are averaged under the original harmonic weights before replacing those weights.

The first factor is ancillary for κ, since occupation is fixed in this subexperiment. Normalized targets are positive and uniformly bounded on compact κ sets; project normalized decisions to their target range. Total variation and (30) then transfer bounded losses and Bayes risks. With $v = \lambda|\kappa|^2/2$, the limiting target is $I_0(\sqrt{2v/\lambda})$. Theorem 7.1 applies with $s = 1/(2\lambda)$, and its compact-prior exhaustion gives

$$\liminf R_Q(N,n,a) \ge 1 - e^{-1/(2\lambda)}. \tag{32}$$

Together with the global upper (15), this proves the exact limit (8).

8.3 The same scale, different critical risks

For the quartic target write $u = |c|^2$. The published unbiased observable [6, Theorem 2.2] is

$$T = \frac{Q^\dagger Q + 2K(K-1)}{N(N+2)}.$$

Its exact variance implies the uniform bound

$$\frac{\operatorname{Var} T}{F_2^2} \le \frac{1}{4Na^2} + \frac{29/2 + 13/(2a)}{N} + \frac{3(7 + 4/a + 1/a^2)}{N(N+2)} =: v_N. \tag{33}$$

Indeed the leading numerator is $2u$, with $2u/(2t^2+u)^2 \le 1/(4t^2)$. The other exact first-order numerator terms divide by F_2^2 to at most $29/2 + 13/(2t)$; the second-order numerator divides to at most $7 + 4/t + 1/t^2$. These follow from $F_2^2 \ge 4t^4, 8t^2u, u^2$. The decision $T/(1+v_N)$ has risk at most $v_N/(1+v_N)$. Thus $a \to 0$, $Na^2 \to 4$ imply $\limsup R_Q(N,2,a) \le 1/17$.

For an explicit comparison let $k \ge 2$ be an integer tending to infinity, and use the common floor $a_k = 1/(k^2-1)$, so $\gamma_{a_k} = k^2$:

degree n	copies N	$x = \gamma_{a_k}^2 n^2/N$	quantum risk bound
2	$4k^4$	1	$\limsup R_Q \le 1/17$
k	k^6	1	$R_Q \to 1 - e^{-1/2} > 0.39$

The first row has $Na_k^2 \to 4$; the second has $Na_k^2/n^2 \to 1$. The floor and scalar resource scale agree exactly at every index, but their risk envelopes are disjoint. This proves the final assertion of Theorem 2.4. The comparison uses the published quartic variance upper and the exact growing-degree law.

9 Interpretation

The scalar hierarchy has a sharp degree threshold in its copy-budget order. Mean occupation retains the quantum inverse-linear brightness scale. Every higher factorial moment incurs the inverse-quadratic scale once anomalous covariance is unknown, uniformly even when degree grows. The complete collapse theorem and bounded-scale upper show that this order genuinely organizes the entire consistency transition.

Within that transition, the growing-degree quantum risk has the exact curve $1 - e^{-1/(2\lambda)}$. A global spectral decision attains the lower from the Poisson–Bessel experiment, so the limiting benchmark is also achievable throughout the physical covariance family. For a fixed desired limiting relative risk $\delta \in (0, 1)$, the corresponding copy coefficient is

$$\lambda = \frac{1}{-2\log(1-\delta)}, \qquad N \sim \lambda n^2/a^2.$$

For example, $\lambda = 1$ gives limiting risk $1 - e^{-1/2} \simeq 0.393469$. The same-scale quartic comparison shows that degree remains essential even after the copy-budget order is fixed. These results quantify estimation costs for raw factorial moments in the centered single-mode family and measurement classes defined in Section 2.

Reproducibility. The supplied Python programs use exact rational arithmetic to check finite Wick coefficients, sensitivity inequalities, Hermite unbiasedness and moment bounds, Poisson factorial-product identities, and the new mixed-insertion and cross-contraction bounds. The analytic arguments establish the uniform statistical and asymptotic conclusions.

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