Collision-uniform kernel asymptotics for Báez–Duarte zero packets

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Abstract

We determine the high-height kernel of the harmonic-step space underlying the Báez–Duarte approximation problem. For packets with unequal real parts in a fixed compact rectangle inside the critical strip, an exact arithmetic gauge gives an explicit limiting Mellin kernel. Convergence holds with every fixed number of mixed derivatives. Complex Newton–Hermite divided differences then give a relative Gram comparison uniform through arbitrary collisions, without a minimum-separation condition. A literal finite-height error and an explicit separation-free Gram floor make the comparison effective. A regularized variational certificate also applies at every real height $T \geq 4$, without a small-error onset or an independence assumption on the native functionals. The resulting native zero-jet projection bound retains the exact unequal powers and oscillatory phases of the target. We derive a two-point interference formula and a multiple-zero lower bound with a logarithmic multiplicity factor. The proof works directly with the arithmetic step kernel and pays both ends of its infinite series.

1 The arithmetic packet problem

Let S be the closed subspace of $L^2(0,1)$ consisting of functions constant on every interval

$$I_n = (1/(n+1), 1/n], \qquad n \geq 1.$$

We use the Mellin transform $\widehat{f}(s) = \int_0^1 f(t)t^{s-1}\,\mathrm{d}t$ and identify S with its Mellin image. It is a subspace of $H^2(\Omega)$, where $\Omega = \{\Re s > 1/2\}$ and the squared boundary norm is $(2\pi)^{-1}\int_{\mathbb{R}} |f(1/2+it)|^2\,\mathrm{d}t$. The inner product is linear in its first argument. The orthogonal basis and its squared norms are

$$e_n(s) = \frac{n^{-s} - (n+1)^{-s}}{s}, \qquad \|e_n\|^2 = \frac{1}{n(n+1)}. \tag{1}$$

Thus the evaluation kernel of S is

$$K_S(s,w) = \sum_{n\geq 1} n(n+1)e_n(s)\overline{e_n(w)}. \tag{2}$$

This kernel retains the harmonic-step restriction, rather than replacing S by the full half-plane Hardy space.

The Báez–Duarte approximation problem [1, 2] concerns

$$E(s) = \frac{1}{s}, \qquad G_l(s) = \frac{(l^{-s} - l^{-1})\zeta(s)}{s} \quad (l \geq 2), \qquad V = \overline{\mathrm{span}}\{G_l : l \geq 2\} \subset S. \tag{3}$$

Write $D_\infty = \mathrm{dist}(E, V)$. Bounded zero-jet functionals on S give lower bounds for this distance. At high height their conditioning depends on the relative positions of the zeros, their multiplicities, and the phase of the target $1/s$.

The step-space zero-jet projection framework and the sharp diagonal asymptotic were developed in [3, Sections 2–5]. Weighted finite-cutoff conjugate-pair kernels and their monotonicity were studied in [4, Theorem 4]. Our question is the full packet limit, including unequal real parts and offsets that may collide as the height grows. Entrywise kernel asymptotics alone are insufficient in that regime: an evaluation Gram matrix can become arbitrarily ill-conditioned. We obtain a comparison of quadratic forms after a collision-compatible change of basis.

1.1 Main theorem

Fix an integer $m \geq 1$ and a compact rectangle

$$Q = \{q \in \mathbb{C} : b_0 \leq \Re q \leq b_1,\ |\Im q| \leq L\}, \qquad \frac{1}{2} < b_0 \leq b_1 < 1, \tag{4}$$

where $L < \infty$. For $T \geq 4$, put $s_q = iT + q$ and define

$$A_T(q,p) = T^{q+\bar{p}-3} s_q \overline{s_p}\, K_S(s_q, s_p), \qquad T^q = \exp(q \log T). \tag{5}$$

The logarithm of T is real. The limiting kernel is

$$\begin{aligned} H(q,p) &= \int_0^\infty x^{2-q-\bar{p}} 4 \sin^2 \frac{1}{2x}\, \mathrm{d}x \\ &= \frac{\pi}{\Gamma(4-q-\bar{p}) \sin(\pi(3-q-\bar{p})/2)}. \end{aligned} \tag{6}$$

For nodes $\mathbf{q} = (q_1, \ldots, q_m) \in Q^m$, let $\Delta_j g = [q_1, \ldots, q_j]g$ be the complex Newton divided difference, with its Hermite interpretation at repeated nodes. In particular, $[a, \ldots, a]g = g^{(j-1)}(a)/(j-1)!$. For a sesquiholomorphic kernel, $\overline{\Delta_k}$ denotes the anti-holomorphic divided difference in the second variable, with conjugate nodes and conjugate coefficients. Set

$$\mathsf{G}_T(\mathbf{q})_{jk} = \Delta_j^{(q)} \overline{\Delta_k^{(p)}} A_T(q,p), \qquad \mathsf{G}(\mathbf{q})_{jk} = \Delta_j^{(q)} \overline{\Delta_k^{(p)}} H(q,p). \tag{7}$$

Theorem 1.1 (Collision-uniform packet kernel)**.** *For every fixed $r, k \geq 0$,*

$$\sup_{q,p \in Q} \left| \partial_q^r \partial_{\bar{p}}^k (A_T - H)(q,p) \right| \longrightarrow 0 \qquad (T \to \infty). \tag{8}$$

The matrices $\mathsf{G}(\mathbf{q})$ are positive definite with a common positive smallest-eigenvalue bound on Q^m. There is $\eta_T \to 0$, depending only on Q, m, such that uniformly for all $\mathbf{q} \in Q^m$,

$$(1 - \eta_T)\mathsf{G}(\mathbf{q}) \preceq \mathsf{G}_T(\mathbf{q}) \preceq (1 + \eta_T)\mathsf{G}(\mathbf{q}). \tag{9}$$

Arbitrary partial and complete collisions are included.

The native target vector is

$$z_T(\mathbf{q})_j = [q_1, \ldots, q_j] T^q. \tag{10}$$

Corollary 1.2 (Zero-packet distance bound)**.** *Suppose the points $iT + q_j$ are zeros of ζ, with each repeated point listed at most its multiplicity. For all sufficiently large T,*

$$D_\infty^2 \geq T^{-3} z_T^* \mathsf{G}_T^{-1} z_T \geq \frac{T^{-3}}{1 + \eta_T} z_T^* \mathsf{G}^{-1} z_T. \tag{11}$$

The same η_T works for every such packet in Q^m.

All parameters in Q and m are fixed in these statements. In particular, no separation is imposed between the nodes, whereas the size of the packet and its compact real-part range remain specified.

1.2 A certificate at every finite height

The quantitative statements use an expanded rectangle $Q_e = \{b \le \Re q \le c,\ |\Im q| \le L_e\}$, with $1/2 < b \le c < 1$. For any $B \ge \sup_{Q_e} |q|$, set

$$C = 1 + B, \quad g = 2b - 1, \quad h = 2 - 2c, \quad \kappa = \tfrac{3}{2} - c, \quad J = C^2(64 + 16/g + 16/h). \tag{12}$$

Choose $\rho > 0$ such that the closed radius-ρ disks about points of Q lie in Q_e. Define

$$E_0 = JT^{-\kappa}, \qquad a_j = \rho^{-(j-1)}, \qquad \epsilon = E_0 \sum_{j=1}^{m} a_j^2.$$

Theorem 1.3 (Effective native packet certificate)**.** *For every real $T \ge 4$ and $q, p \in Q$,*

$$|\partial_q^r \partial_{\bar p}^k (A_T - H)(q,p)| \le r!k!\rho^{-r-k} JT^{-\kappa} \qquad (r, k \ge 0). \tag{13}$$

For every tuple in Q^m, including all collisions,

$$|(\mathsf{G}_T - \mathsf{G})_{jk}| \le E_0 a_j a_k, \qquad \mathsf{G}_T \preceq \mathsf{G} + \epsilon I. \tag{14}$$

If the tuple is a zeta-zero packet as in Corollary 1.2, then, at every such height,

$$D_\infty^2 \ge T^{-3} z_T^* (\mathsf{G} + \epsilon I)^{-1} z_T. \tag{15}$$

The inverse in this formula exists without a small-error condition.

Sections 5–7 prove this theorem, an adaptive derivative estimate retaining the exponent $3/2 - b_1$, and an explicit positive floor for G. The floor gives a literal onset for the relative law. The additive inverse in (15) avoids that onset altogether. Section 9 connects these bounds to finite arithmetic primal and dual certificates in the same Hilbert norm.

1.3 Related analytic interfaces

The kernel-translation operator of Manzur–Noor [5, Sections 4–5] provides a different quantitative zero obstruction and an invariant-subspace criterion. The shifted-Hardy approximation of Manzur–Noor–Quintero [6, Section 3] also concerns the Báez–Duarte generators, with a different approximation norm. Our result addresses the high-height geometry of the exact kernel (2). The one-point exponent and constant, together with an elementary diagonal power-error argument, are inherited from [3, Section 5, Remark 4]. The full complex kernel, literal mixed-derivative constants, separation-free Gram floor, relative comparison and exact regularized packet receiver are developed below. The divided-difference integral and the positive moment-matrix argument are classical tools.

2 The native kernel and finite zero-jet projection

For $\rho \in \Omega$, the Mellin evaluation functional is represented in $L^2(0,1)$ by $t^{\bar\rho - 1}$, of squared norm $1/(2\Re\rho - 1)$. Its projection onto S is the restricted representer r_ρ. This proves boundedness of evaluation. Higher jets are bounded as well, since $t^{\bar\rho-1}(\log t)^j \in L^2(0,1)$. Expanding in the orthogonal basis (1) proves (2), with local convergence of all required derivatives. Alternatively, these follow directly by projecting the bounded full-space jet functionals.

For completeness, G_l belongs to S. In the original variable it is the negative Mellin transform of

$$\rho_{1/l}(t) = \{1/(lt)\} - l^{-1}\{1/t\}.$$

If $1/t = n + \theta$, $0 \le \theta < 1$, then $\rho_{1/l}(t) = \{n/l\}$, so it is constant on each I_n and bounded. The Mellin identity follows first where the defining zeta integrals converge and then by continuation. Each zero of ζ therefore forces the corresponding jets of every G_l to vanish.

We use the following elementary projection principle, with the evaluation-row convention of (2).

Lemma 2.1. *Let $\Lambda_1, \dots, \Lambda_d$ be independent bounded linear functionals on a complex Hilbert space, annihilating a closed subspace V. If r_j is the Riesz representer of Λ_j, put $\Gamma_{jk} = \langle r_k, r_j \rangle$ and $c_j = \Lambda_j(E)$. Then*

$$\operatorname{dist}(E, V)^2 \ge c^* \Gamma^{-1} c.$$

Proof. The representers lie in $V^\perp$. If the projection of E onto their span is $\sum_k a_k r_k$, its functional values give $\Gamma a = c$. Its squared norm is $a^* \Gamma a = c^* \Gamma^{-1} c$. Projection onto this subspace cannot exceed projection onto $V^\perp$. □

This argument uses only finitely many bounded functionals. It does not require a description of the entire orthogonal complement of V.

3 The exact arithmetic gauge and its smooth limit

For $x = n/T$, define

$$F_T(q, x) = \sqrt{1 + 1/n}\, x^{1-q} \big[1 - (1 + 1/n)^{-iT-q}\big]. \tag{16}$$

Expansion of (2) gives the exact identity

$$A_T(q, p) = \frac{1}{T} \sum_{n \ge 1} F_T(q, n/T) \overline{F_T(p, n/T)}. \tag{17}$$

In particular, the factors $s_q \overline{s_p}$ in (5) have not been approximated by T^2.

Lemma 3.1. *For every fixed integer $r \ge 0$, constants depending on Q, r give*

$$\begin{aligned} |\partial_q^r F_T(q, x)| &\le C_r x^{1-b_1} (1 + |\log x|)^r && (x \le 1), \\ |\partial_q^r F_T(q, x)| &\le C_r x^{-b_0} (1 + \log x)^r && (x \ge 1), \end{aligned} \tag{18}$$

uniformly for $q \in Q$ and $T > 2L + 2$.

Proof. Put $a_n = \log(1 + 1/n)$. The bracket in (16) has modulus at most two, since $\Re(iT + q) > 0$, and its positive derivatives have modulus at most a_n^j. On the small-x range use $a_n \le \log 2$ and $\sqrt{1 + 1/n} \le \sqrt{2}$. On the large-$x$ range, the integral identity for $1 - e^{-z}$, $\Re z \ge 0$, yields

$$|1 - e^{-(iT+q)a_n}| \le |iT + q| a_n \le CT/n = C/x.$$

For $j \ge 1$, $a_n^j \le C_j/n \le C_j/x$ there. Differentiating x^{1-q} supplies only powers of $\log x$. Finally use $x^{1-\Re q} \le x^{1-b_1}$ for $x \le 1$ and $x^{-\Re q} \le x^{-b_0}$ for $x \ge 1$. □

Proof of (8). On any fixed $\varepsilon \le x \le M$, uniformly in $q \in Q$,

$$F_T(q, x) \longrightarrow F(q, x) = x^{1-q}(1 - e^{-i/x}),$$

with every fixed number of q derivatives. Indeed $T \log(1 + 1/n) \to 1/x$ uniformly on this range, while the derivatives of the bracket of positive order introduce powers of $\log(1 + 1/n) = O(1/T)$.

Uniform Riemann sums on the compact interval therefore converge, including mixed derivatives in the two kernel slots.

We verify the infinite tails separately. By Lemma 3.1, a mixed derivative of orders r, k of the summand product is bounded at zero by

$$Cx^{2-2b_1}(1+|\log x|)^{r+k},$$

and at infinity by

$$Cx^{-2b_0}(1+\log x)^{r+k}.$$

Both are integrable. On a sufficiently small interval the first envelope is increasing, so its mesh sum through ε is bounded by its integral through 2ε; an empty mesh contributes zero. On a sufficiently late tail the second envelope is decreasing, and its mesh sum is bounded by an integral beginning at $M/2$. These bounds tend to zero uniformly as $\varepsilon \downarrow 0$, $M \to \infty$. They justify the termwise derivatives of (17) and the differentiated limiting integral. Choosing the tails first and then taking the compact Riemann limit proves (8).

To evaluate the integral, put $u = 1/x$ and $\alpha = 3 - q - \bar{p}$, so $1 < \Re\alpha < 2$. The Mellin identity

$$2\int_0^\infty (1-\cos u)u^{-1-\alpha}\,\mathrm{d}u = \frac{\pi}{\Gamma(1+\alpha)\sin(\pi\alpha/2)}, \qquad 0 < \Re\alpha < 2,$$

gives (6). One derivation integrates by parts and uses the sine Mellin integral, first on an overlapping convergence strip and then by analytic continuation. Absolute convergence of the displayed $1-\cos u$ integral throughout this strip also identifies the same analytic continuation directly. □

4 Collisions and relative inversion

The complex Hermite–Genocchi formula is

$$[q_1,\dots,q_j]g = \int_{\substack{\theta_l \geq 0 \\ \sum \theta_l = 1}} g^{(j-1)}\left(\sum_l \theta_l q_l\right)\,\mathrm{d}\theta, \tag{19}$$

where simplex measure has total mass $1/(j-1)!$. Repeated fundamental-theorem integration proves the formula for complex nodes as well as real ones; continuity gives every Hermite endpoint. The convexity of Q keeps all arguments inside Q.

Applying (19) in both slots of (8), with orders at most $m-1$, gives

$$\sup_{\mathbf{q}\in Q^m} \|\mathsf{G}_T(\mathbf{q}) - \mathsf{G}(\mathbf{q})\| \longrightarrow 0. \tag{20}$$

The conjugation of the second divided-difference coefficients is essential here.

Lemma 4.1. *The limiting Gram matrix $\mathsf{G}(\mathbf{q})$ has a common positive eigenvalue floor on Q^m.*

Proof. Set $\psi_j(x) = [q_1,\dots,q_j]x^{-q}$. Then

$$\mathsf{G}(\mathbf{q})_{jk} = \int_0^\infty \psi_j(x)\overline{\psi_k(x)}x^2 4\sin^2\frac{1}{2x}\,\mathrm{d}x. \tag{21}$$

The base measure has infinite mass, but all the displayed products are integrable by the preceding envelopes. In $y = \log x$, the expansion of e^{-qy} gives

$$\psi_j(e^y) = \frac{(-y)^{j-1}}{(j-1)!} + O(y^j).$$

An order-$j-1$ divided difference kills q^k for $k < j-1$ and sends q^{j-1} to one, including at collisions. The triangular leading orders therefore make $\psi_1, \dots, \psi_m$ linearly independent for every tuple. A nontrivial linear combination is an entire exponential polynomial in y and cannot vanish on a set of positive measure. Since the weight in (21) is positive off a discrete set, the matrix is positive definite.

The simplex formula and the uniform integrable logarithmic envelopes make the matrix continuous on Q^m. Its least eigenvalue thus has a positive minimum on this compact set. □

Completion of Theorem 1.1. Let $c > 0$ be the floor in Lemma 4.1, and let ϵ_T be the supremum in (20). For every vector v,

$$|v^*(\mathsf{G}_T - \mathsf{G})v| \le \epsilon_T \|v\|^2 \le (\epsilon_T/c) v^* \mathsf{G} v.$$

Taking $\eta_T = \epsilon_T/c$ proves (9). □

Proof of Corollary 1.2. Use the native functionals

$$\Lambda_j(f) = [q_1, \dots, q_j](s_q T^q f(s_q)). \tag{22}$$

Their Gram matrix is exactly $T^3 \mathsf{G}_T$, by (5). Their values on $E = 1/s$ are exactly z_T from (10). The gauge $s_q T^q$ is analytic and nonzero on a neighborhood of Q, and multiplication by it is triangular and invertible on each finite jet space. The derivative of s_q with respect to q is one. Thus repeated nodes in (22) are invertible combinations of the native Mellin jets, with the stated Hermite factorials.

Under the zero-packet hypothesis these functionals annihilate all G_l and hence V. The positive definiteness from (9) makes them independent for large T. Lemma 2.1 gives the first inequality of (11). Inverting the upper inequality of (9) gives the second. The same proof applies if the complete off-line zero set is infinite, since only the listed finite packet enters the projection. □

5 An explicit arithmetic kernel error

We first prove a scalar estimate on the whole expanded rectangle Q_e. The derivative estimates will then follow from complex Cauchy bounds. This separates the arithmetic error from the conditioning of any packet.

Proposition 5.1. *With the constants in* (12)*, for every real* $T \ge 4$*,*

$$\sup_{q,p \in Q_e} |A_T(q,p) - H(q,p)| \le J T^{-\kappa}. \tag{23}$$

Proof. Put $z = q + \bar{p}$, $a = \Re z \in [2b, 2c] \subset (1,2)$, and

$$b_q(n) = 1 - (1 + 1/n)^{-iT-q}, \qquad b_0(n) = 1 - e^{-iT/n}.$$

Both exponents in this comparison have nonpositive real part. The line-segment integral formula for the exponential, and $0 \le u - \log(1+u) \le u^2/2$, give

$$|b_q(n) - b_0(n)| \le B/n + T/(2n^2) = d_n. \tag{24}$$

Also $|b_q(n)| \le C \min(2, T/n)$ and $|b_0(n)| \le \min(2, T/n)$. Let

$$R_T = \frac{1}{T} \sum_{n \ge 1} f_z(n/T), \qquad f_z(x) = x^{2-z} 4 \sin^2(1/(2x)).$$

The discrepancy D_n between the brackets in the exact sum (17) and this limiting mesh is bounded by

$$D_n \le 2C \min(2, T/n) d_n + \frac{C^2}{n} \min(2, T/n)^2.$$

In particular,

$$D_n \le \begin{cases} 8C^2/n + 2CT/n^2, & n \le T, \\ 4C^2 T/n^2, & n \ge T. \end{cases} \tag{25}$$

For all n, the separate exact and limiting products give $D_n \le 12$.

Set $N = \lceil \sqrt{T} \rceil$ and $x_0 = N/T$. Then $\sqrt{T} \le N \le (3/2)\sqrt{T}$ and $x_0 \le 3/4$. The terms $n < N$, using the last bound, contribute at most $12(N/T)^{3-a} \le 27T^{-(3-a)/2}$. On the range $N \le n \le T$, the first and second terms of (25) contribute, respectively, at most

$$\frac{8C^2}{(2-a)T}, \qquad \frac{4C}{a-1} T^{-(3-a)/2}.$$

Here we used $\sum_{n \le T} n^{1-a} \le T^{2-a}/(2-a)$ and $\sum_{n \ge N} n^{-a} \le 2N^{1-a}/(a-1)$. The remaining range $n \ge \lceil T \rceil$ contributes at most $8C^2/((a-1)T)$. A duplicated endpoint when T is an integer only enlarges these positive bounds. Thus

$$|A_T - R_T| \le (27 + 4C/g)T^{-\kappa} + (8C^2/h + 8C^2/g)T^{-1}. \tag{26}$$

For completeness, the whole quadrature error is paid as follows. Writing $D = |2 - z| \le 2C$, differentiation gives

$$f_z'(x) = (2-z)x^{1-z} 4\sin^2(1/(2x)) - 2x^{-z}\sin(1/x),$$

and hence

$$|f_z'(x)| \le \begin{cases} 4Dx^{1-a} + 2x^{-a}, & x \le 1, \\ (D+2)x^{-a-1}, & x \ge 1. \end{cases}$$

The head mesh and head integral through x_0 together cost at most $8x_0^{3-a} \le 18T^{-(3-a)/2}$. On all remaining mesh cells, left-endpoint quadrature costs at most $T^{-1}\int_{x_0}^{\infty} |f_z'|$. Consequently

$$|R_T - H| \le (18 + 2/g)T^{-\kappa} + (8C/h + 2C + 2)T^{-1}. \tag{27}$$

Since $T^{-1} \le T^{-\kappa}$, the total constant is at most

$$47 + 2C + (8C^2 + 4C + 2)/g + (8C^2 + 8C)/h \le C^2(49 + 14/g + 16/h) \le J.$$

The last inequalities use $C \ge 1$. This proves (23). □

This square-root split and variation estimate extend the diagonal rate argument of [3, Section 5, Remark 4]; the constants above are uniform in both complex slots. The sums at zero and infinity are both included in the estimate.

Proof of (13) *and* (14). The arithmetic series is locally normally convergent for $\Re q, \Re p > 1/2$. The limiting integral is locally normally convergent when both real parts are in $(1/2, 1)$. Thus $A_T - H$ is holomorphic in q and in the independent variable $\bar{p}$. Cauchy's formula on the two radius-ρ circles proves (13). Apply (19) in both slots. The simplex masses cancel $r!k!$, so the resulting entry bound is $E_0\rho^{-(j+k-2)} = E_0 a_j a_k$. For every vector v,

$$|v^*(\mathsf{G}_T - \mathsf{G})v| \le E_0 \Big(\sum_j a_j |v_j|\Big)^2 \le \epsilon \|v\|^2. \tag{28}$$

This proves the stated Loewner comparison without node separation. □

Corollary 5.2 (Adaptive native exponent)**.** *For the rectangle Q of* (4)*, define*

$$\begin{aligned} d &= \min(b_0 - 1/2, 1 - b_1), & g_0 &= 2b_0 - 1, & h_0 &= 2 - 2b_1, \\ \kappa_0 &= 3/2 - b_1, & \bar{C} &= 1 + b_1 + L + d, & \bar{J} &= \bar{C}^2(64 + 32/g_0 + 32/h_0), \\ M(T) &= \max(2/d, \log T). \end{aligned}$$

For every real $T \geq 4$,

$$\sup_{q,p \in Q} |\partial_q^r \partial_p^k (A_T - H)| \leq 3\bar{J} r! k! T^{-\kappa_0} M(T)^{r+k}. \tag{29}$$

In (14) *one may instead take $E_0 = 3\bar{J}T^{-\kappa_0}$ and $a_j = M(T)^{j-1}$.*

Proof. Choose $\rho = 1/M(T) = \min(d/2, 1/\log T)$. The expanded rectangle has $g \geq g_0/2$, $h \geq h_0/2$ and $C \leq \bar{C}$, so $J \leq \bar{J}$. Its exponent is $\kappa_0 - \rho$, and $T^\rho \leq e < 3$. Cauchy's formula and the same factorial cancellation give the result. □

6 A literal floor for every colliding tuple

The compactness proof of Lemma 4.1 does not specify a numerical floor. The following formula supplies one. For $m \geq 1$ and $R > 0$, define

$$\begin{aligned} c_m &= \frac{1}{(2m)^{m-1}} \prod_{k=0}^{m-1} \frac{2^{2k+1}}{(2k+1)\binom{2k}{k}^2}, & b_m &= \frac{1}{\sqrt{m}(1+R)^{m-1}(m-1)!}, \\ K_m(R)^2 &= \sum_{j=0}^{m-1} \left(\frac{R^{m-j}}{j!(m-j)!} \right)^2, & a &= \min\left(\frac{1}{4}, \frac{\sqrt{c_m} b_m}{2\sqrt{2} e^R K_m(R)} \right), \\ \delta_m(R) &= \frac{c_m b_m^2 a^{2m-1}}{8}. \end{aligned} \tag{30}$$

Theorem 6.1 (Explicit separation-free floor)**.** *If $|q_j| \leq R$ and the nodes are in the strip $1/2 < \Re q < 1$, then $\mathsf{G}(\mathbf{q}) \succeq \delta_m(R) I$. The same formula covers arbitrary complex collisions.*

Proof. Index the limiting features by $0 \leq j < m$: $\psi_j(e^y) = [q_1, \ldots, q_{j+1}] e^{-qy}$. The complete-homogeneous identity for divided differences gives

$$\psi_j(e^y) = \sum_{k \geq j} \frac{(-y)^k}{k!} h_{k-j}(q_1, \ldots, q_{j+1}).$$

For a coefficient vector v, truncate its feature combination at degree $m-1$, obtaining $P_v(y) = \sum_{k<m} (Bv)_k y^k$. Here $B = DH$, where $D_{kk} = (-1)^k/k!$ and $H_{kj} = h_{k-j}(q_1, \ldots, q_{j+1})$ for $j \leq k$. The inverse Newton conversion is

$$(H^{-1})_{kj} = (-1)^{k-j} e_{k-j}(q_1, \ldots, q_k).$$

Indeed expansion of the Newton products $\prod_{i=1}^j (z - q_i)$ and of $z^k = \sum_{j=0}^k h_{k-j}(q_1, \ldots, q_{j+1}) \prod_{i=1}^j (z - q_i)$ proves both conversion formulas. Thus each row of H^{-1} has absolute sum at most $(1+R)^k$, and $\|B^{-1}\| \leq 1/b_m$.

The monomial moment matrix on $[-1, 1]$ has trace at most $2m$. Its determinant is the product in (30), by monic Legendre orthogonalization: the degree-k Legendre polynomial has leading coefficient

$2^{-k}\binom{2k}{k}$ and squared norm $2/(2k+1)$. Its smallest eigenvalue is therefore at least c_m, by bounding each of the other eigenvalues by the trace. For $0 < a \le 1$, rescaling $y = at$ yields

$$\|P_v\|_{L^2(-a,a)} \ge \sqrt{c_m} a^{m-1/2} \|Bv\| \ge \sqrt{c_m} b_m a^{m-1/2} \|v\|. \tag{31}$$

Since $|h_{k-j}| \le \binom{k}{j} R^{k-j}$ and $\sum_{l \ge d} u^l/l! \le e^u u^d/d!$, the discarded tail of feature j on this interval is at most $e^{Ra} R^{m-j} a^m/(j!(m-j)!)$. The L^2 norm of the combined tail is consequently at most $\sqrt{2} e^R K_m(R) a^{m+1/2} \|v\|$. The choice of a makes this at most half the lower bound (31).

In logarithmic coordinates the positive weight in (21) is $w(y) = 4e^{3y} \sin^2(e^{-y}/2)$. For $|y| \le 1/4$, $t = e^{-y}/2 \le 2/3$, $\sin t \ge 5t/6$, and $e^y \ge 3/4$. Hence $w(y) \ge 25/48 > 1/2$. Restricting the integral to $[-a, a]$, and squaring the half-amplitude estimate, proves the stated floor. □

A rational alternative can be useful. For any integer $R \ge 1$ bounding the nodes, put

$$b = \frac{1}{m(1+R)^{m-1}(m-1)!}, \quad d_* = \min(1, c_m), \quad K = \frac{(1+R)^m}{m!}, \quad a_0 = \min\left(\frac{1}{4}, \frac{d_* b}{4 \cdot 3^R K}\right).$$

Then

$$\mathsf{G} \succeq \delta_0 I, \qquad \delta_0 = c_m b^2 a_0^{2m-1}/8 > 0. \tag{32}$$

All these constants are rational. To check the same proof, note $b \le b_m$, $d_* \le \sqrt{c_m}$, $K_m \le \sum_j R^{m-j}/(j!(m-j)!) \le K$, $e^R \le 3^R$ and $2\sqrt{2} \le 4$.

Corollary 6.2 (Explicit relative onset)**.** *For any proved floor $\delta > 0$ from the preceding theorem, set*

$$\eta_T = \frac{J \sum_{j=0}^{m-1} \rho^{-2j}}{\delta T^\kappa}.$$

Then (9) *holds for all $T \ge 4$. For a prescribed $0 < \eta < 1$, the literal sufficient onset is*

$$T \ge \max\left(4, \left[\frac{J \sum_{j=0}^{m-1} \rho^{-2j}}{\eta \delta}\right]^{1/\kappa}\right).$$

The adaptive constants of Corollary 5.2 give the alternative $\eta_T = 3\bar{J} T^{-\kappa_0} \sum_{j=0}^{m-1} M(T)^{2j}/\delta$.

Proof. Apply (28) and $\|v\|^2 \le v^* \mathsf{G} v/\delta$. Both displayed errors tend to zero for fixed m, Q. □

7 All-height regularized receiving bounds

Proof of (15)*.* The native functionals (22) have Gram $\Gamma = T^3 \mathsf{G}_T$ and target z_T, exactly, at every $T \ge 4$. The zero-packet hypothesis makes their representers r_j belong to $V^\perp$. With the inner product linear in its first argument, $\Gamma_{jk} = \langle r_k, r_j \rangle$. For any vector u set $h = \sum_j u_j r_j$. Positivity of $\|(E - v) - h\|^2$, for $v \in V$, and then taking the infimum give

$$D_\infty^2 \ge 2\Re(u^* z_T) - u^* \Gamma u \ge 2\Re(u^* z_T) - T^3 u^* (\mathsf{G} + \epsilon I) u.$$

The limiting Gram is positive semidefinite by its integral, and $\epsilon > 0$. The concave quadratic on the right is maximized at $u = T^{-3}(\mathsf{G} + \epsilon I)^{-1} z_T$, proving the result. This argument does not invert Γ and remains valid if its functionals are dependent. □

If $\mathsf{G} \succeq \delta I$, then $\mathsf{G}+\epsilon I \preceq (1+\epsilon/\delta)\mathsf{G}$. Thus the additive-inverse bound is at least as strong as the scalar relative bound with error ϵ/δ. It remains available before the relative comparison establishes positive definiteness by perturbation.

Corollary 7.1 (Weighted and chosen-vector certificates)**.** *For any positive numbers ω_j, put*

$$R_\omega = E_0 \left(\sum_j a_j^2/\omega_j \right) \operatorname{diag}(\omega_1, \ldots, \omega_m).$$

Under the zero-packet hypothesis, at every real $T \geq 4$,

$$D_\infty^2 \geq T^{-3} z_T^* (\mathsf{G} + R_\omega)^{-1} z_T, \tag{33}$$

$$D_\infty^2 \geq T^{-3} \left[2\Re(v^* z_T) - v^* \mathsf{G} v - E_0 \left(\sum_j a_j |v_j| \right)^2 \right] \quad (v \in \mathbb{C}^m). \tag{34}$$

Proof. Weighted Cauchy–Schwarz bounds the first expression in (28) by $v^* R_\omega v$. Repeat the preceding maximization. Retaining the original absolute-value penalty and choosing $u = T^{-3} v$ gives (34) directly. □

The choices $\omega_j = 1$ and $\omega_j = a_j$ are explicit examples. Alternatively, (34) permits a certified test vector without computing an inverse. Every formula retains the exact unequal powers and oscillatory phases in z_T.

8 Phase interference and multiple zeros

For a shared real part, write $q = \beta + i\tau$ and

$$H_\beta(a) = \frac{\pi}{\Gamma(4 - 2\beta - ia) \sin(\pi(3 - 2\beta - ia)/2)}. \tag{35}$$

Then $H(q, p) = H_\beta(\tau - \sigma)$ and $T^q = T^\beta e^{i\tau \log T}$. For distinct offsets τ_j, cancellation of the invertible Newton change of basis in (11) gives

$$D_\infty^2 \geq \frac{T^{2\beta-3}}{1 + \eta_T} z^* \mathsf{H}^{-1} z, \quad \mathsf{H}_{jk} = H_\beta(\tau_j - \tau_k), \quad z_j = e^{i\tau_j \log T}. \tag{36}$$

The relative error is uniform even when the minimum separation tends to zero. At coincident nodes, (11), rather than the inverse of a singular evaluation matrix, defines the endpoint.

Corollary 8.1 (Two-point interference)**.** *Put $h = H_\beta(0)$ and $\ell = \log T$. For two distinct zeros $\beta + iT$, $\beta + i(T + \delta)$ in a bounded-offset packet, the limiting factor in* (36) *is*

$$z^* \mathsf{H}^{-1} z = \frac{2h - 2\Re(H_\beta(-\delta) e^{i\delta\ell})}{h^2 - |H_\beta(\delta)|^2}. \tag{37}$$

The denominator is positive for $\delta \neq 0$.

Proof. The matrix is $\left(\begin{smallmatrix} h & H_\beta(-\delta) \\ H_\beta(\delta) & h \end{smallmatrix}\right)$, positive definite by independence of its two limiting features. Direct Hermitian inversion gives (37). □

The target phase in (37) is part of the receiving formula. In particular, close distinct zeros are not replaced by a multiple zero merely from their geometric proximity.

For a fixed multiplicity m, set $r = m - 1$ and

$$\mu_j(\beta) = \int_0^\infty (\log x)^j x^{2-2\beta} 4 \sin^2 \frac{1}{2x} \, \mathrm{d}x, \qquad M_\beta = (\mu_{a+b}(\beta))_{a,b=0}^r. \tag{38}$$

Every moment exists and M_β is positive definite.

Corollary 8.2 (Multiplicity factor)**.** *Along a sequence of zeros $\rho = \beta + iT$ of multiplicity at least m, with $T \to \infty$ and β in a fixed compact subinterval of $(1/2, 1)$,*

$$D_\infty^2 \geq (1 - o(1))(M_\beta^{-1})_{r,r} T^{2\beta-3} (\log T)^{2m-2}. \tag{39}$$

The error is uniform in that real-part interval.

Proof. At fully repeated offset zero, the limiting Newton Gram and target are

$$\mathsf{G}_\beta = A M_\beta A^*, \qquad A_{aa} = \frac{(-i)^a}{a!}, \qquad z_a = \frac{(i\ell)^a}{a!}.$$

Consequently

$$z^* \mathsf{G}_\beta^{-1} z = (M_\beta^{-1})_{r,r} \ell^{2r} + O(\ell^{2r-1})$$

for $m \geq 2$; for $m = 1$ it equals $1/\mu_0(\beta)$. The factorials cancel in the highest-degree term. The moment matrix and its inverse vary continuously on a compact real-part interval and have positive eigenvalue floors there, so the estimate and its positive leading coefficient are uniform. Apply (11) in the shared-beta chart. □

The coefficient $(M_\beta^{-1})_{r,r}$ is also the reciprocal squared norm of the monic degree-r polynomial orthogonal to lower powers for the positive measure in (38). This follows from the Schur complement of the lower-order moment block. When $m = 1$, (39) recovers the sharp single-zero constant of [3, Section 5, Remark 4]. Monic square norms also retain collision geometry in the signed-measure packet estimates of [7]. Here the measure is the native positive logarithmic measure in (38), and the receiving functionals are Mellin jets; the signed-measure recovery theorem is not used as an input.

9 Finite certificates in the same arithmetic norm

The lower bounds above can be compared with a trial approximation only when its full norm is certified in the same space. We give the relevant finite formulas, including their infinite-tail payments.

9.1 A periodic primal residual

In the step realization, $E = 1$ and $G_k|_{I_n} = -\{n/k\}$, as follows by taking the Mellin transform. For rational coefficients b_k let

$$r_n = 1 - \sum_{k=2}^N b_k \{n/k\}, \qquad U(b) = \sum_{n \geq 1} \frac{|r_n|^2}{n(n+1)}.$$

Then $D_\infty^2 \le d_N^2 \le U(b)$, where $d_N = \mathrm{dist}(E, \mathrm{span}\{G_2, \dots, G_N\})$. For real coefficients the sequence $f_n = r_n^2$ is periodic. Its mean is

$$\mu = 1 - \sum_k b_k \frac{k-1}{k} + \sum_{j,k} b_j b_k \left[\frac{(j-1)(k-1)}{4jk} + \frac{\gcd(j,k)^2 - 1}{12jk} \right]. \tag{40}$$

To see the pair formula, average the residues modulo $\mathrm{lcm}(j,k)$; their covariance after scaling by j,k is $(\gcd(j,k)^2-1)/(12jk)$. The single means are $(k-1)/(2k)$. This is the periodic-mean mechanism of [4, Theorem 3] in the native norm.

One explicit bound on every interval sum of $f_n - \mu$ is

$$B_f = \sum_k \frac{|b_k| \lfloor k/2 \rfloor \lceil k/2 \rceil}{k} + \sum_{j,k} \frac{|b_j b_k| \,\mathrm{lcm}(j,k)}{4}. \tag{41}$$

Indeed the centered sawtooth has prefix-sum range $\lfloor k/2 \rfloor \lceil k/2 \rceil/(2k)$. For a sequence in $[0,1]$ of period P and mean m_0, the total positive centered mass is at most $Pm_0(1-m_0) \le P/4$, bounding its prefix-sum range. Expansion of r_n^2 now proves (41). Abel summation with the decreasing weights $1/(n(n+1))$ yields

$$\left| \sum_{n>M} \frac{f_n}{n(n+1)} - \frac{\mu}{M+1} \right| \le \frac{B_f}{(M+1)(M+2)}. \tag{42}$$

Thus the finite head, rational mean and tail error supply a full-norm primal certificate without traversing the entire period.

9.2 A finite dual and a continuous-tail majorant

There is also an elementary head-supported annihilator for every finite approximation space. With μ_{Mob} denoting the Möbius function, put

$$t_n = \frac{1}{n} \sum_{r \le N/n} \frac{\mu_{\mathrm{Mob}}(r)}{r} \quad (1 \le n \le N), \qquad t_{N+1} = 0, \qquad \lambda_n = t_n - t_{n+1}.$$

Möbius inversion gives $\sum_{j \le N/k} t_{jk} = 1/k$ for $1 \le k \le N$. Summation by parts consequently gives

$$\sum_{n \le N} \lambda_n \lfloor n/k \rfloor = 1/k, \qquad \sum_{n \le N} n\lambda_n = 1, \qquad \sum_{n \le N} \lambda_n \{n/k\} = 0 \quad (2 \le k \le N).$$

The functional $L(f) = \sum_{n \le N} \lambda_n f_n$ annihilates the first-N span and has squared norm $\sum_{n \le N} n(n+1)\lambda_n^2$. Therefore, for all complex approximation coefficients,

$$d_N^2 \ge H_N := \frac{|\sum_{n \le N} \lambda_n|^2}{\sum_{n \le N} n(n+1)\lambda_n^2}. \tag{43}$$

The denominator is nonzero by the normalization identity. A positive lower bound requires a nonzero numerator, which is checked at the chosen N rather than assumed universally.

For a finite point packet $s_1, \dots, s_m$ in Ω, define

$$(A_M)_{jk} = \sum_{n \le M} n(n+1) e_n(s_j) \overline{e_n(s_k)}, \qquad (C_M)_{jk} = \frac{(M+1)^{1-s_j-\bar{s}_k}}{s_j + \bar{s}_k - 1}.$$

The continuous evaluation representers on $(0, 1/(M+1))$ have Gram C_M. Their orthogonal step projections give the remaining native tail Gram. Projection is a contraction for every linear combination, so the exact native Gram satisfies

$$\Gamma \preceq A_M + C_M. \tag{44}$$

For distinct nodes C_M is positive definite, by independence of distinct exponentials on an interval. The variational argument used in Section 7, with target $c_j = 1/s_j$, gives

$$D_\infty^2 \geq c^*(A_M + C_M)^{-1}c$$

whenever the nodes are zeta zeros. Derivative functionals are treated by differentiating the continuous-tail Gram, with the appropriate Hermite factorials. This extends the conjugate-pair projection payment of [4, Theorem 4] by the same Hilbert-space argument.

9.3 The fixed-packet approximation budget

The complex simplex formula gives $|[q_1, \ldots, q_{j+1}]T^q| \leq T^{b_1}(\log T)^j/j!$. For a proved floor $\mathsf{G} \succeq \delta I$, the scalar relative certificate obeys the uniform upper ceiling

$$\frac{T^{-3}z_T^*\mathsf{G}^{-1}z_T}{1+\eta_T} \leq \delta^{-1}T^{2b_1-3}\sum_{j=0}^{m-1}\frac{(\log T)^{2j}}{(j!)^2} \longrightarrow 0. \tag{45}$$

The additive-inverse certificate has the same ceiling, since $(\mathsf{G}+\epsilon I)^{-1} \preceq \mathsf{G}^{-1}$. Thus any fixed first-$N$ space with $H_N > 0$ eventually cannot produce a full-norm upper smaller than this particular fixed-m, fixed-Q certificate. This follows directly from (43); it is an accounting constraint on these certificates, not an upper bound on D_∞ or on all possible native dual tests.

An exact one-point version is available without an asymptotic error. For $s = \beta + iT$, $1/2 < \beta \leq 1$, $T \geq 1$, let $M = \lceil T \rceil$. On $M \leq n \leq 2M-1$, $\theta = T\log(1+1/n) \leq 1$ and $a = (1+1/n)^{-\beta} \geq 1/2$. Since $1 - \cos\theta \geq \theta^2/4$ and $\log(1+1/n) \geq 1/(2n)$,

$$|1 - ae^{-i\theta}|^2 \geq T^2/(16n^2), \qquad |s|^2K_S(s,s) \geq \frac{T^{3-2\beta}}{16\cdot 4^{2\beta}} \geq \frac{T^{3-2\beta}}{256}.$$

The exact native one-point requirement consequently satisfies

$$\frac{1}{|s|^2K_S(s,s)} \leq 256T^{2\beta-3} \leq 256/T. \tag{46}$$

In particular, if $H_N > 256/T$, no full-norm trial in that fixed approximation space can beat this exact one-point requirement. The comparison always uses the same arithmetic norm: a small head residual alone is not a bound for the full residual.

A complementary whole-line recipient. The accompanying mathematical supplement gives a height-adapted Hardy norm with a complete arithmetic upper, and proves why an unweighted compact boundary band cannot replace a full norm. It includes the exact exterior-weight payment and a prefix-Schur signal identity. Those localized criteria use their own recipients; they are not applications of the native kernel theorem with an upper computed in a different norm.

Algebraic controls. The accompanying standard-library program checks the leading Newton identities at rational complex nodes, mixed and complete collisions, the native-jet factorial convention and the two-point Hermitian inverse identity. Its exact finite controls supplement the analytic proof; the derivative estimates, explicit floor and variational receiver are established above independently of computation. Additional exact controls check the scalar error payments, Newton factorial cancellation, rational floor constants and completed-square identities, including singular Grams.

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